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On categoricity questions for universal unars and undirected graphs under semantic Jonsson quasivariety

The article is devoted to the study of semantic Jonsson quasivarieties of universal unars and undirected graphs. The first section of the article consists of basic necessary concepts from Jonsson model theory. The following two sections are results of using new notions of semantic Jonsson quasivariety of Robinson unars $J\mathbb{C}_{\mathfrak{U}}$ and semantic Jonsson quasivariety of Robinson undirected graphs $J\mathbb{C}_{\mathfrak{G}}$, its elementary theory and semantic model. In order to prove two main results of the paper, Robinson spectra $RSp(J\mathbb{C}_{\mathfrak{U}})$ and $RSp(J\mathbb{C}_{\mathfrak{G}})$ and their partition onto equivalence classes $[\Delta]_{\mathfrak{U}}$ and $[\Delta]_{\mathfrak{G}}$ by cosemanticness relation were considered. The main results are presented in the form of theorems 11 and 13 and imply following useful corollaries: countably categorical Robinson theories of unars are totally categorical; countably categorical Robinson theories of undirected graphs are totally categorical. The obtained results can be useful for continuation of the various Jonsson algebras' research, particularly semantic Jonsson quasivariety of S-acts over cyclic monoid.

Keywords: Jonsson theory, unar, graph, undirected graph, universal theory, Robinson theory, quasivariety, semantic Jonsson quasivariety, Jonsson spectrum, Robinson spectrum, cosemanticness, categoricity, countable categoricity.

Introduction

This paper and focuses on the study of model-theoretic properties of well-known and sufficiently simple classes in the sense of the signature of algebras, namely unars and undirected graphs. One can note that this paper is a continuation of works [1–4].

At one time, the famous mathematician-logician H.J. Keisler, in his review article "Fundamentals of Model Theory" in the four-volume monograph "Reference Book on Mathematical Logic" (edited by J. Barwise), defined the basic concepts and directions of the development of model theory. H.J. Keisler identified two historical trends in the development of model theory. They are called "western" and "eastern" model theory. This division is due to the fact that A. Tarski lived on the west coast from 1940, and A. Robinson lived on the east coast from 1967 until his premature death in 1975. This distinction has long lost its geographical significance, but it is useful from a mathematical point of view.

"Western" model theory develops in the traditions of Skulem and A. Tarski. It was mostly motivated by problems in number theory, calculus and set theory, it uses all the formulas of first-order logic.

"Eastern" model theory develops in the traditions of A.I. Mal'tsev and A. Robinson. It was motivated by problems in abstract algebra, where the formulas of theories usually have at most two blocks of quantifiers. It emphasizes a set of quantifier-free and existential formulas.

Jonsson theories as an object of research were first considered in the works of Jonsson [5] and Morley, Voot [6]. In the mid-80s of the twentieth century, the works of T.G. Mustafin identified a new direction in the study of Jonsson theories. In particular, he defined a natural subclass of Jonsson theories, which

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he called perfect. The main method of his research was the following: the study of the properties of arbitrary Jonsson theories by transferring the properties of the central completion of this Jonsson theory. In the early 90s of the twentieth century A.R. Yeshkeyev obtained a criterion for the perfection of the Jonsson theory [7]. In particular, there was obtained a complete description of the Jonsson universal unars in the works [2,3] by A.R. Yeshkeyev, T.G. Mustafin, as well as the relationship between the theory of unars and their center in the language of stability. On the other hand, one of the weak points in the study of Jonsson theories within the framework of the method proposed by T.G. Mustafin was the presence of an additional axiom about the existence of a strongly inaccesible cardinal to the axioms of Zermelo-Frenkel set theory in the definition of a semantic model. It should be noted that during the talk of R.M. Ospanov at the "5th Kazakh-French colloquium on model theory well-known experts in the field of model theory Ye.A. Palyutin and B. Poizat pointed out the need to change this definition. The realization of this remark was the output of the work of Ye.T. Mustafin [8], in which he redefines the concept of k-homogeneity and semantic model. Accordingly, the modified definition of the perfection of the Jonsson theory appeared in [9], in which the main results obtained earlier in [10] were re-demonstrated within the framework of the new definition.

The results discussed here relate in their content to the "eastern" model theory. Various properties of unars from the perspective of "western" model theory (the case when the complete theory of some unar is considered) were obtained in the works of Yu.E. Shishmarev [11], A.N. Ryaskin [12].

The notion of countable categoricity in "western" model theory distinguishes probably the narrowest class of theories, and it is well studied. In the case of "eastern" model theory (meaning studies of Jonsson theories), it should be noted that Vought theorem on the relationship between completeness and categoricity of the theory does not hold, since Jonsson theories, generally speaking, are not complete and have finite models. The following question of Ye.A. Palyutin is well-known: is there an ω-categorical universal K that is not ω_1-categorical? If this question is projected into the framework of research on the Jonsson theories, then one can notice some interesting connections between the Jonsson theory itself and its center appear.

In this regard, A.R. Yeshkeyev [13] obtained the following results:

Theorem 1. If the Jonsson theory T is ω-categorical, then T is perfect.

Theorem 2. If the Jonsson theory T is k-categorical, then the $\#$-companion of the theory T is k-categorical, where $k \geq \omega$.

Theorem 3. In the case of a negative answer to question of Ye.A. Palyutin for a Jonsson theory that satisfies the conditions of the question, the center of the Jonsson theory cannot be finitely axiomatized.

There is considered a class of existentially closed models of an arbitrary universal theory in the work of A. Pillai [14], and for this class he develops a forking theory with a suitable concept of the simplicity of the theory. S. Shelah [15], E. Hrushovski [16] studied classes of existentially closed models of Robinson theory. A theory is called a Robinson theory if it is universal and admits AP and JEP. From here it is easy to see that any Robinson theory is a special case of the Jonsson theory. And if we take into account that unars and undirected graphs are Jonsson universals, then obtaining a description of their existentially closed models within the framework of the above topic is an urgent task. This article discusses the description of an existentially closed model of a countably categorical universal of unars, as well as undirected graphs.

All definitions that were not given in the current article can be extracted from [7, 17–27].

1 *Necessary concepts of Jonsson model theory*

Let us recall the conditions, that should be satisfied in order for a theory to be Jonsson.

Definition 1. [5] A theory T is said to be Jonsson, if:

1) T has at least one infinite model;
2) T is $\forall\exists$-axiomatising;
3) T has JEP property;
4) T has AP property.

For example, the following theories are Jonsson: unars, graphs and their various subclasses, groups, abelian groups, Boolean algebras, linear order, fields of characteristic p (p is a prime number or zero), ordered fields. In addition to these natural examples and rather broad classes of algebras, we may also notice that for an arbitrary theory T its scolemization and morleization are also examples of Jonsson theories.

$\forall$-axiomatizing Jonsson theory is called the Robinson theory.

By virtue of theorem of Morley and Vaught [6] an arbitrary Jonsson theory T has T-universal, T-homogeneous model $\mathfrak{C}$ in some inaccessible cardinality. Let us consider elementary theory $Th(\mathfrak{C})$. We denote it as T^*, i.e. $T^* = Th(\mathfrak{C})$.

The next definitions belong to T.G. Mustafin.

Definition 2. [7] 1) Let T be a Jonsson theory. A model $\mathfrak{C}_T$ of power $2^{|T|}$ is called to be a semantic model of the theory T if $\mathfrak{C}_T$ is a $|T|^+$-homogeneous $|T|^+$-universal model of the theory T.

2) The elementary theory of a semantic model of the Jonsson theory T is called the central completion or center of this theory. The center is denoted by T^*, i.e. $Th(C) = T^*$.

In the "west" model theory, when isomorphic embedding in the definitions of universal and homogeneous model changes to elementary embedding, and also the definition of the homogeneous model changes, then the following theorem is true:

Theorem 4. [7] A system $\mathfrak{A}$ is saturated iff it is homogeneous and universal.

Unfortunately, in the "east" model theory T-universal, T-homogeneous model does not have to be saturated model. The following notions are required for proofing the main theorems of this paper.

Definition 3. [7] Jonsson theory T is called perfect theory, if its semantic model $\mathfrak{C}_T$ is saturated.

Theorem 5. [7] Let T be arbitrary Jonsson theory, then the following conditions are equivalent:
1) Theory T is perfect,
2) T^* is model completion of theory T.

The following criterium is nedded for clarification of constructing semantic Jonsson quasivariety.

Theorem 6. [7] T is Jonsson iff it has a semantic model $\mathfrak{C}_T$.

Since we will work with Robinson theories of unars and undirected graphs, let us recall the definition of universal.

Definition 4. [2] If $T = T_\forall$, then T is said to be universal.

The next two notions of κ-categorical Jonsson theory and existentially closed model of theory T are nedded for obtaining main theorems of this paper.

Definition 5. [7] Let $\kappa \geq \omega$. Jonsson theory T is called κ-categorical, if any two models of power κ of theory T are isomorphic to each other.

Definition 6. [7] Model A of theory T is called existentially closed model of theory T, if for any model B of theory T such that $A \subseteq$ B, for any $\exists$ - formula $\exists x\varphi(x, \overline{y})$, for any $\overline{a}$ from A $(l(\overline{a})) = (l(\overline{y}))$ from $B \models \exists x\varphi(x, \overline{a})$ follows that $A \models \exists x\varphi(x, \overline{a})$

We will denote a class of existentially closed models of theory T as E_T.

Since the current research is connected with consideration of Robinson spectrum for classes of algebras, let us give the following conditions of Jonsson theories cosemanticness.

Definition 7. [7] Let T_1 and T_2 be Jonsson theories, $\mathfrak{C}_{T_1}$ and $\mathfrak{C}_{T_2}$ be their semantic models, respectively. T_1 and T_2 are said to be cosemantic Jonsson theories (denoted by $T_1 \bowtie T_2$), if $\mathfrak{C}_{T_1} = \mathfrak{C}_{T_2}$.

Theorem 7. [7] Let T_1 and T_2 be Jonsson theories, $\mathfrak{C}_{T_1}$ and $\mathfrak{C}_{T_2}$ be their semantic models, respectively. Then the next conditions are equivalent:
1) $\mathfrak{C}_{T_1} \bowtie \mathfrak{C}_{T_2}$;
2) $\mathfrak{C}_{T_1} \equiv_J \mathfrak{C}_{T_2}$;
3) $\mathfrak{C}_{T_1} = \mathfrak{C}_{T_2}$.

Let K be a class of models of fixed signature σ. Then we can consider Jonsson spectrum for K, which can be defined as follows.

Definition 8. [28] A set $JSp(K)$ of Jonsson theories of signature σ, where

$$JSp(K) = \{T \mid T \text{ is Jonsson theory and } K \subseteq Mod(T)\},$$

is called the Jonsson spectrum for class K.

Hence, in the particular case, when the Jonsson theory is $\forall$-axiomatising we get the concept of the Robinson theory, respectively, the notion of the Jonsson spectrum allows us to consider the Robinson spectrum.

Definition 9. [4] A set $RSp(K)$ of Robinson theories of signature σ, where

$$RSp(K) = \{T \mid T \text{ is Robinson theory and } \forall A \in K, A \models T\},$$

is called the Robinson spectrum for class K.

Based on theorem 7, we can consider the cosemanticity relation on Jonsson spectrum $JSp(K)$ and obtain a partition of $JSp(K)$ onto equivalence classes. we get a factor-set, denoted as $JSp(K)_{/\bowtie}$. The factor-set $RSp(K)_{/\bowtie}$ will be obtained correspondingly.

Let K be a class of quasivariety in the sense of [29] of first-order language L, $L_0 \subset L$, where L_0 is the set of sentences of language L. Let us consider the elementary theory $Th(K)$ of such class K. By adding to $Th(K)$ $\forall\exists$ sentences of language L, that are not contained in the $Th(K)$, we can consider the set of Jonsson theories $J(Th(K))$ defined as follows.

Denotation 1. [4] A set $J(Th(K)) = \{\Delta \mid \Delta - \text{Jonsson theory}, \ \Delta = Th(K) \cup \{\varphi^i\}\}$, where $\varphi^i \in \forall\exists(L_0)$ and $\varphi^i \notin Th(K)$, $i \in \{0,1\}$, $Th(K)$ is elementary theory of class of quasivariety K, $\forall\exists(L_0)$ is a set of all $\forall\exists$ sentences of language L.

Let us consider the set of such semantic models and denote it as $J\mathbb{C}$.

Denotation 2. [4] A set $J\mathbb{C} = \{\mathfrak{C}_\Delta \mid \Delta \in J(Th(K)), \mathfrak{C}_\Delta \text{ is semantic model of } \Delta\}$.

We will call the set $J\mathbb{C}$ semantic Jonsson quasivariety of class K if its elementary theory $Th(J\mathbb{C})$ is Jonsson theory.

2 Countable categoricity of semantic Jonsson quasivarieties of universal unars

Let $\mathfrak{A}$ be some unar, i.e. the model of signature $\sigma = \{f\}$, where f is a unary functional symbol. Let $f^0(x) = x$, $f^{n+1}(x) = f(f^n(x))$, $n \in \omega$. Elements $a, b \in \mathfrak{A}$ are called $\mathfrak{A}$-connected in X if there exist natural numbers m and n such that $(f^m(a) = f^n(b))$ and $f^0(a) = f^m(a), f^0(b), ..., f^n(b) \in X$.

A set $X \subseteq \mathfrak{A}$ is called $\mathfrak{A}$-connected if any two elements from X are $\mathfrak{A}$-connected. A subsystem $\mathfrak{B} \subseteq \mathfrak{A}$ carrier of which is the maximal $\mathfrak{A}$-connected subset of carrier $\mathfrak{A}$ is called a component in $\mathfrak{A}$. If $\mathfrak{B}$ is a component in system $\mathfrak{A}$, then the set $\{a \in \mathfrak{B} : \mathfrak{A} \models (f^n(a) = a)$ for some $n \in \omega$ is called a cycle of component. By $K(a, \mathfrak{A})$ we denote the restriction of $\mathfrak{A}$ to the set $\{b \in \mathfrak{A} : \mathfrak{A} \models (f^n(b) = a)$ for some

$n \in \omega\}$ and we call it the root of the element a in the unar $\mathfrak{A}$, while the element a is called the vertex of the root $K(a, \mathfrak{A})$.

We will write down the special connections between the elements of the unar in the form of $\exists$-formulas:

1) the property of the elements to be at "the beginning of the cycle":

$\Phi_0^n(z) = \Phi^n(z) \& \exists y \neg \Phi^(y) \& f(y) = z$, where $\Phi^n(z) = (f^n(z) = z) \& (f(z) \neq z) ... (f^{n-1}(z) \neq z)$;

2) "x has no less than k different immediate representatives":

$\Theta(x) = \exists x_1, ..., \exists k (\wedge_{i \neq j < x} x_i \neq x_j \wedge \wedge_{i=1}^k f(x_i) = x)$;

3) "there are exactly k different elements between x and the beginning of the cycle":

$\Psi_k(x) = \exists z \exists y_1 ... \exists y_k (\wedge_{i \neq j < x} (y_i \neq y_j) \wedge f^i(x) = y_i \wedge \wedge_{i=1}^{k-1} f(y_i) \neq f(y_{i+1}) \wedge \Phi_0^n(z) \wedge f(y_k) = z)$.

By virtue of works [2, 4] we can use the conclusion that $\forall$-axiomatisability of elemantary theory of unars, $Th_{\forall}(\mathfrak{U})$ is the Robinson theory of unars.

Thus, we consider a set $J\mathbb{C}_{\mathfrak{U}} = \{\mathfrak{C}_{\Delta_{\mathfrak{U}}} \mid \Delta_{\mathfrak{U}} \in J(Th(K)), \mathfrak{C}_{\Delta_{\mathfrak{U}}}$is a semantic model $\Delta_{\mathfrak{U}}\}$ of signature $\sigma_{\mathfrak{U}} =< f >$, where $\Delta_{\mathfrak{U}}$ is a Robinson theory of unars, f is unary functional symbol. Such $J\mathbb{C}_{\mathfrak{U}}$ defines semantic Jonsson quasivariety of Robinson unars as in [4].

We are using the definition of the Robinson spectrum of the set $J\mathbb{C}_{\mathfrak{U}}$ [4].

Definition 10. [4] A set $RSp(J\mathbb{C}_{\mathfrak{U}})$ of Robinson theories of signature $\sigma_{\mathfrak{U}}$, where

$$RSp(J\mathbb{C}_{\mathfrak{U}}) = \{\Delta_{\mathfrak{U}} \mid \Delta_{\mathfrak{U}} \text{ is Robinson theory of unars and } \forall \mathfrak{C}_{\Delta_{\mathfrak{U}}} \in J\mathbb{C}_{\mathfrak{U}}, \mathfrak{C}_{\Delta_{\mathfrak{U}}} \models \Delta_{\mathfrak{U}}\},$$

is called the Robinson spectrum for class $J\mathbb{C}_{\mathfrak{U}}$, where $J\mathbb{C}_{\mathfrak{U}}$ is semantic Jonsson quasivariety of Robinson unars.

Further we obtain a factor-set, denoted as $RSp(J\mathbb{C}_{\mathfrak{U}})_{/\bowtie}$ and consisted of equivalence classes parted by cosemanticness relation $[\Delta_{\mathfrak{U}}] \in RSp(J\mathbb{C}_{\mathfrak{U}})_{/\bowtie}$.

Remark 1. Everywhere in this section $[\Delta_{\mathfrak{U}}]$ denotes an equivalence class of Robinson theories of unars parted by cosemanticness relation on Robinson spectrum $RSp(J\mathbb{C}_{\mathfrak{U}})$. $\mathfrak{C}_{\Delta_{\mathfrak{U}}}$ denotes semantic model and $E_{\Delta_{\mathfrak{U}}}$ denotes a class of existentially closed models of class $[\Delta_{\mathfrak{U}}]$.

Further we obtained two useful theorems, concerning the equivalence class $[\Delta_{\mathfrak{U}}]$ of Robinson theories of unars parted by cosemanticness relation on Robinson spectrum $RSp(J\mathbb{C}_{\mathfrak{U}})$.

We will use the denotations from [2–4].

Theorem 8. Let $[\Delta_{\mathfrak{U}}]$ be a class of Robinson theories of unars, $[\Delta_{\mathfrak{U}}^*]$ its center. Then

1) $[\Delta_{\mathfrak{U}}^*]$ is model completion of $[\Delta_{\mathfrak{U}}]$;

2) $[\Delta_{\mathfrak{U}}^*]$ allows quantifier elimination (i.e. submodel complete);

3) $[\Delta_{\mathfrak{U}}^*]$ is ω-stable.

Proof. 1) Let $\mathfrak{C}$ be semantic model of $[\Delta_{\mathfrak{U}}]$. Then $[\Delta_{\mathfrak{U}}^*] = Th(\mathfrak{C})$. Let $\mathfrak{C}^*$ be saturated model of $[\Delta_{\mathfrak{U}}^*]$. We can assume that $\mathfrak{C}^* \subseteq \mathfrak{C}$. It easy to understand that if $a \in \mathfrak{C}^*$, then $tp^{\mathfrak{C}^*}(a, \varnothing) = tp^{\mathfrak{C}}(a, \varnothing) = \chi(a)$. Hence $C^{\mathfrak{C}}(a) \simeq C^{\mathfrak{C}^*}(a)$, whene $C^{\mathfrak{A}}(b)$ by definition is $\{c \in \mathfrak{A} : \exists n, k < \omega \;\; f^n(c) = f^k(b)\}$. The quantity of pairwise isomorpic components is uniquely defined by $char[\Delta_{\mathfrak{U}}]$. Hence $\mathfrak{C}^* \simeq \mathfrak{C}$. It means that $[\Delta_{\mathfrak{U}}]$ is perfect Jonsson theory and $[\Delta_{\mathfrak{U}}^*]$ is its model completion.

2) follows from 1) and Robinson theorem [3].

3) Let H be arbitrary subunar of $\mathfrak{C}$. From lemma 5 [2] we have

$$|S^{\mathfrak{C}}(H)| \leq (1 + \omega^2) + (1 + \omega) + |H|, \text{ because}$$
$$|\{f(a) : a \in \mathfrak{C}\}| - |\Omega| \leq 1 + \omega^2,$$
$$|\{\rho(a, H) : a \in \mathfrak{C}\}| \leq 1 + \omega,$$
$$|\{\text{enter}(a, H) : a \in \mathfrak{C}\}| \leq |H|.$$

From this, if $|H| \leq$, then $|S^{\mathfrak{C}}(H)| \leq \omega$.

The theorem is proven.

Theorem 9. 1) The quantity of pairwise different $[\Delta_{\mathfrak{U}}]$ classes of Robinson theories of unars is equal to 2^{ω}.

2) The quantity of pairwise different maximal $[\Delta_{\mathfrak{U}}]$ classes of primitive Robinson theories is equal to 2^{ω}.

3) The quantity of pairwise different maximal $[\Delta_{\mathfrak{U}}]$ classes of Robinson theories of unars is equal to ω. Moreover, these are precisely the classes of theories, that have following characteristics: $\pi_{\omega}, \{\pi_{0,m} \ \ 1 \leq m < \omega\}, \{\pi_{n,m} \ \ 1 \leq n, m < \omega\}$, where

$\pi_{\omega} : \Omega = \{\omega\}, \nu(m) = 0 \ \ \forall m < \omega, \mu(\omega) = 1, \varepsilon = \infty;$

$\pi_{0,m} : \Omega = \{(0,m)\}, \nu(m) = \begin{cases} 0, \text{if } \ k \neq m, \\ \infty, \text{if } \ k = m; \end{cases} \quad \mu(0,m) = 0, \varepsilon = 0;$

$\pi_{n,m} : \Omega = \{(0,m), .., (n,m)\}, \nu(k) = \begin{cases} 0, \text{if } \ k \neq m, \\ 1, \text{if } \ k = m, \end{cases}$

$\mu(k,m) = \begin{cases} 1, \text{if } \ k < n-1, \\ \infty, \text{if } \ k = n-1, \\ 0, \text{if } \ k = n, \end{cases} \ \ \varepsilon = 0.$

4) Maximal ∇-complete $[\Delta_{\mathfrak{U}}]$ classes of Robinson theories of unars is the only class, that has characterstic π_{ω}.

Proof. 1) It is easy to note that the quantity of pairwise different characteristics is equal to 2^{ω}. By theorem 3 [4] the quantity of $[\Delta_{\mathfrak{U}}]$ classes of Robinson theories of unars is equal to 2^{ω}.

2) Let $[\Delta_{\mathfrak{U}}]'_{\pi} = (Th(\mathfrak{C}_{\pi}))_{\nabla}$ where $\mathfrak{C}_{\pi}$ is semantic model of class of Robinson theories of unars of characteristic π. Obviously $[\Delta_{\mathfrak{U}}]'_{\pi}$ is ∇-complete primitive. By lemma 1 [3] $[\Delta_{\mathfrak{U}}]'_{\pi}$ is class of Robinson theories of unars. By Proposition 3 [3] $[\Delta_{\mathfrak{U}'_{\pi}}]$ is maximal class of primitive Robinson theories. If $\pi_1 \neq \pi_2$, then $[\Delta_{\mathfrak{U}'_{\pi_1}}] \neq [\Delta_{\mathfrak{U}'_{\pi_2}}]$, since $([\Delta_{\mathfrak{U}'_{\pi_1}}])_{\forall} \neq ([\Delta_{\mathfrak{U}'_{\pi_2}}])_{\forall}$, hence, the quantity of maximal $[\Delta_{\mathfrak{U}}]$ classes of primitive Robinson theories is equal to 2^{ω}.

3) Let us consider partial order on set of all characteristics in following form. Let $\pi_i = (\Omega_i, \nu_i, \mu_i, \varepsilon_i)$, $i = 1,2$. Then suppose $\pi_1 \leq \pi_2 \Leftrightarrow \Omega_1 \subseteq \Omega_2 \ \ \& \ \ \forall m < \omega(\nu_1(m)) \leq \nu_2(m)) \ \ \& \ \ \forall \alpha \in \Omega_1(\mu_1(\alpha) \leq \mu_2(\alpha)) \ \ \& \ \ \varepsilon_1 \subseteq \varepsilon_2$. From definition of class $[\Delta_{\mathfrak{U}]_{\pi}}$ in the proof for theorem 3 [4] it easy to see that

$$[\Delta_{\mathfrak{U}]_{\pi_1}} \supseteq [\Delta_{\mathfrak{U}]_{\pi_2}} \Leftrightarrow \pi_1 \leq \pi_2.$$

Case 1. $\varepsilon = \infty$.

Among such characteristics the minimal is the only characteristic π_{ω}.

Case 2. $\varepsilon = 0$.

In this case $\omega \notin \Omega$ and $|\Omega| < \omega$. By condition 10) from definition of characteristic [3] either $\exists 0 < k < \omega \ \ (\nu(k) = \infty)$, either $\exists (k,l) \in \Omega, \ \ (\mu(k,l)) = \infty)$.

Case 2.1. $\exists 1 \leq k < \omega \ \ (\nu(k)) = \infty))$.

Among such characteristics the minimal are characteristics $\pi_{0,m}, 1 \leq n < \omega, 1 \leq m < \omega$.

Case 2.2. $\exists 1 \leq k < \omega, 1 \leq l < \omega \ \ (\mu(k,1) = \infty)$.

In the set of such characteristics the minimal are characteristics $\pi_{n,m}, 1 \leq n < \omega, 1 \leq m < \omega$.

4) Note that the class $[\Delta_{\mathfrak{U}]_{\pi}}$, that has characteristic π_{ω} is complete, in particular ∇-complete. Therefore it is maximal among classes of Robinson theories of unars. Classes $[\Delta_{\mathfrak{U}]_{n,m}}, 0 \leq n < \omega, 1 \leq m < \omega$ are not ∇-complete, since $[\Delta_{\mathfrak{U}]_{n,m}} \cup \exists x_1, ..., x_{m+1}(\wedge_{1 \leq i < j \leq m+1}(x_i \neq x_j))$ and $[\Delta_{\mathfrak{U}]_{n,m}} \cup \forall x_1, ..., x_{m+1}(\vee_{1 \leq i < j \leq m+1}(x_i = x_j))$ are consistent. The theorem is proven.

By consideration of theorems 9 and 10, we can obtain the following result:

Theorem 10. Let $[\Delta_{\mathfrak{U}}]$ be a class of ω-categorical Robinson theories of unars. Then the following conditions are equivalent:

1) $\mathfrak{A} \in E_{\Delta_{\mathfrak{U}}}$, where $\mathfrak{A}$ is a model of class $[\Delta_{\mathfrak{U}}]$;

2) $\mathfrak{A}$ is disjoint union of components with cycles of the same length.

Proof. The proof of this theorem is based on the following theorem, three facts and three lemmas.

Theorem 11. [30] In order for the algebraic system $\mathfrak{A}$ to be some ω-categorical universal, it is necessary and sufficient that the following conditions will be satisfied:

1) $\mathfrak{A}$ is locally finite;

2) there is a function $g : \omega \to \omega$ such that for every $a \in \mathfrak{A}$ and for every finite subset $X \subseteq \mathfrak{A}$ the type $tp(a, X, \mathfrak{A})$ is realized in every subsystem $\mathfrak{B} \subseteq \mathfrak{A}$ that contains X and has a power $\geq g(|X|)$.

Fact 1. [13] If the Jonsson theory T is ω - categorical, then T is perfect.

Fact 2. [30] Let T be a Jonsson theory. Then the following conditions are equivalent:

1) T is perfect;

2) $E(T) = ModT^*$;

3) T^* is a model companion of the theory T.

Fact 3. [31] Let T be $\forall\exists$-complete Jonsson theory. Then the following conditions are equivalent:

1) T is ω-categorical;

2) T^* is ω-categorical.

We get as a consequence of these facts (1 - 3) that, since $[\Delta_{\mathfrak{U}}]$ is ω-categorical, $[\Delta_{\mathfrak{U}}]$ is an equivalence class of perfect Robinson theories, and $E_{\Delta_{\mathfrak{U}}} = Mod(\Delta^*_{\mathfrak{U}})$ is ω-categorical universal. Thus, if $\mathfrak{A} \in E_{\Delta_{\mathfrak{U}}}$, then $\mathfrak{A} \in Mod(\Delta^*_{\mathfrak{U}})$. Consequently, $\mathfrak{A}$ satisfies the conditions of E.A. Palyutin criterion (Theorem 11).

By virtue of these arguments, it is sufficient to prove the following lemmas to prove Theorem 10.

Lemma 1. Let $\mathfrak{A} \in \omega$-categorical Jonsson universal, $x \in \mathfrak{A}$. Then $\exists n, k\ \omega : f^n(x) = f^k(x)$.

Proof. By virtue of E.A. Palyutin criterion, $\mathfrak{A}$ is locally finite. Now suppose that $\forall n, k \in \omega$: $f^n(x) \neq f^k(x)$. This means that there is a set $Y = \{y_1, y_2, ..., y_n, ...\} \subseteq \mathfrak{A}$, where $f(y_i) = y_{i+1}$, and $y_i \neq y_j$ if $i \neq j$, where $i \in \{1, 2, ...\}$. But then an element, for example y_1, generates an infinite (countable) set Y. And this contradicts the local finiteness of unar $\mathfrak{A}$.

Lemma 2. Let $\mathfrak{A} \in \omega$-categorical Jonsson universal. Then for any element $a \in \mathfrak{A}$, the root $K(a, \mathfrak{A})$ is finite.

Proof. Let us assume the opposite. Let there be an element $a \in \mathfrak{A}$ such that the root $K(a, \mathfrak{A})$ is infinite. Then there are two possible cases:

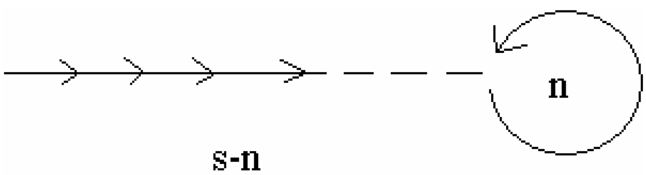
1) $\Psi_k(x) : k \in \omega$ is realized in unar $\mathfrak{A}$;

2) $\Theta_k(x) : k \in \omega$ is realized in unar $\mathfrak{A}$.

By virtue of the Palyutin criterion, there exists a function $\phi : \omega \to \omega$ such that for any $n \in \omega$, for any subunar B [3] of an unar $\mathfrak{A}$ with a power of at least $\phi(n)$, for any type $p \in S^j(\bar{b})$ ($\bar{b} \in B$) from the fact that $\mathfrak{A} \models p(a)$, it follows that there exists a $b \in B$ such that $\mathfrak{A} \models p(b)$. Let $\phi(0) = s$. Then according to the criterion for any subunar B of a unar $\mathfrak{A}$ with a power of at least s for any type $p \in S^j(\bar{b})$ ($\bar{b} \in B$) $\mathfrak{A} \models p(a) \Rightarrow \exists b \in B : \mathfrak{A} \models p(b)$(i.e. any type of element of unar $\mathfrak{A}$ is realized in B).

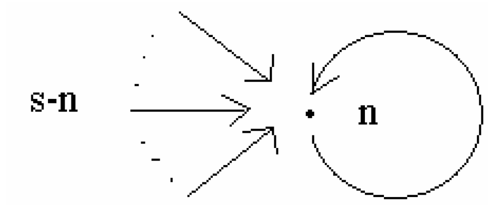
1) Consider the chain Γ. Let Γ_s be a subchain Γ with a cycle, and the number of elements in Γ_s is equal to s.

It has the form:



A type containing the formula $\Psi_{s-n+1}(x)$ cannot be implemented in Γ_s (i.e., there are exactly $s - n + 1$ different elements between x and the beginning of the cycle).

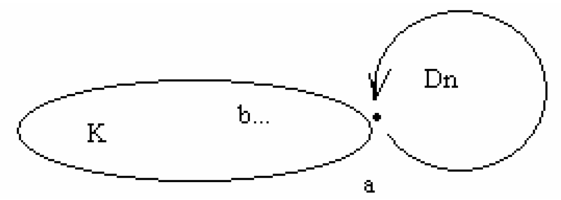
2) Consider a subset where the number of elements of the preimages with a cycle is s:



It is clear that no finite unar realizes the set of formulas $\{\Theta_k(x) : k \in \omega\}$. We get a contradiction.

Lemma 3. Let $\mathfrak{A} \in \omega$-categorical Jonsson universal. Then:
1) each element of $\mathfrak{A}$ enters some cycle;
2) all cycles of unar $\mathfrak{A}$ have the same length.

Proof. By virtue of the previous lemmas, each component of the unar $\mathfrak{A}$ is finite and has the form $D_n \oplus^n K$, where D_n is a cycle of length n, a is an element of the cycle, K is the finite root of a.



Let $b \in K$ and $b \neq a$. b is not included in any cycle. Then there exists k such that $f^k(b) = a$ and $f^s(b) \neq a$ for $s < k$. Consider the formula $\exists y(f^k(y) = a\&_{i<k} f^i(y) \neq a)\& f^k(a) = a\&\&_{i<k} f^i(a) \neq a, k > 1$. It is clear that in the infinite subunar $A' \subseteq \mathfrak{A}$, obtained by combining only the elements included in some cycles, this formula is not realized. Which contradicts condition 2) of the criterion. Thus point 1) of the lemma is proved.

2) Let us assume the opposite: there are at least two cycles of different lengths. Then there are two possible cases:

2.1) For some n there is a finite number of cycles of length n. Then for some n_0 (with a non-empty set of cycles of length n_0), we remove all cycles of length n_0 from unar $\mathfrak{A}$. We get an infinite subunar in which the formula $f^{n_0}(x) = x\&\&_{i<n_0} f^i(x) \neq x$ is not realized.

2.2) (Negation of the first case) Let n_0 be a number for which there is an infinite set of length n_0 in $\mathfrak{A}$. By assumption, there is at least one cycle $k \neq n_0$ in $\mathfrak{A}$. Remove all cycles of length k from $\mathfrak{A}$. We get an infinite subunar in which the formula $f^k(x) = x\&\&_{i<k} f^i(x) = Z$ is not realized.

There is obtained a contradiction to condition 2) of the criterion in each of the two cases.

Let us prove sufficiency. If unar $\mathfrak{A}$ is a disjunctive union of an infinite number of components that are a cycle of the same length, then $\mathfrak{A}$ is ω-categorical universal.

We will show the satisfaction of points 1) and 2) of the criterion.

1) Consider a finite subset of $\{a_1, ..., a_n\} \subseteq \mathfrak{A}$. Each of the elements generates a cycle of length n. Therefore, a subsystem generated by a finite subset of $\{a_1, ..., a_n\}$ contains no more than nk elements.

2) Find the function g, the existence of which is required by the criterion. Consider a finite subset of elements $X_k = \{a_1, ..., a_k\} \subseteq \mathfrak{A}$. It is not difficult to understand that the total number of different types over X_k does not exceed the number $n(k + 1)$. Then any submodel contains cycles "connected" with elements from X_k, and one cycle independent of them realizes all $n(k + 1)$ types. Therefore, $g(k)$ will be equal to $n(k + 1)$.

In connection with the above question by Ye.A. Palyutin, from the description of the existentially closed unar model (Theorem 11.), it can be noted that

Corollary 1. Countably categorical Robinson theories of unars are totally categorical.

3 Countable categoricity of semantic Jonsson quasivarieties of undirected graphs

$\mathfrak{G}$ graph is further understood as an algebraic system of the signature $< R >$, where R is binary symmetric relation, i.e. an undirected graph. Further, the elements of the graph are called vertices, and pairs $< x, y >$ such that $R(x, y)$ are called edges. A graph set of edges of which is empty is called a quite disconnected graph. A path in graph G is an alternating sequence of vertices and edges: $x_i, < x_i, x_{i+1} >, x_{i+1}, < x_{i+1}, x_{i+2} >, \ldots$. A path is called a chain if all its edges are different, and a simple chain if all vertices (and therefore edges) are different. A graph $\mathfrak{G}$ is called connected if any pair of its vertices is connected by a simple chain. A graph is called acyclic if there are no cycles in it. A tree is a connected acyclic graph. The maximal connected subgraph of a graph $\mathfrak{G}$ is called a connectivity component, or simply a graph component. A subgraph of a graph $\mathfrak{G}$ is a graph in which all vertices and edges belong to $\mathfrak{G}$. The degree of a vertex in a graph $\mathfrak{G}$ is the number of edges incident to this vertex. A vertex of degree I is called a pendant (or end point) vertex.

Countably categorical graphs were studied in [32]. The main result of this work is the following theorem:

Theorem 12. Let $\mathfrak{G}$ be an arbitrary countable graph in which each component contains a finite number of cycles. Then $\mathfrak{G}$ is ω-categorical if and only if $\mathfrak{G}$ is bounded and a finite number of I-types is realized in it.

By virtue of works [3, 4] we can use the conclusion that $\forall$-axiomatisability of elemantary theory of graphs, $Th_\forall(\mathfrak{G})$ is the Robinson theory of graphs.

Thus, we consider a set $J\mathbb{C}_\mathfrak{G} = \{\mathfrak{C}_{\Delta_\mathfrak{G}} \mid \Delta_\mathfrak{G} \in J(Th(K)), \mathfrak{C}_{\Delta_\mathfrak{G}}$ is a semantic model $\Delta_\mathfrak{G}\}$ of signature $< R >$, where $\Delta_\mathfrak{G}$ is a Robinson theory of unars, R is binary symmetric relation. Such $J\mathbb{C}_\mathfrak{G}$ defines semantic Jonsson quasivariety of Robinson undirected graphs as in [4].

We are using the definition of the Robinson spectrum of the set $J\mathbb{C}_\mathfrak{G}$ as in [4].

Definition 11. A set $RSp(J\mathbb{C}_\mathfrak{G})$ of Robinson theories of signature $< R >$, where

$$RSp(J\mathbb{C}_\mathfrak{G}) = \{\Delta_\mathfrak{G} \mid \Delta_\mathfrak{G} \text{ is Robinson theory of graphs and } \forall \mathfrak{C}_{\Delta_\mathfrak{G}} \in J\mathbb{C}_\mathfrak{G}, \mathfrak{C}_{\Delta_\mathfrak{G}} \models \Delta_\mathfrak{G}\},$$

is called the Robinson spectrum for class $J\mathbb{C}_\mathfrak{G}$, where $J\mathbb{C}_\mathfrak{G}$ is semantic Jonsson quasivariety of Robinson undirected graphs.

Further we obtain a factor-set, denoted as $RSp(J\mathbb{C}_\mathfrak{G})_{/\bowtie}$ and consisted of equivalence classes parted by cosemanticness relation $[\Delta_\mathfrak{G}] \in RSp(J\mathbb{C}_\mathfrak{G})_{/\bowtie}$.

Remark 2. Everywhere in this section $[\Delta_\mathfrak{G}]$ denotes an equivalence class of Robinson theories of undirected graphs parted by cosemanticness relation on Robinson spectrum $RSp(J\mathbb{C}_\mathfrak{G})$. $\mathfrak{C}_{\Delta_\mathfrak{G}}$ denotes semantic model and $E_{\Delta_\mathfrak{G}}$ denotes a class of existentially closed models of class $[\Delta_\mathfrak{G}]$.

Let us compare theorem 12 with the following theorem.

Theorem 13. Let $[\Delta_\mathfrak{G}]$ be a class of ω-categorical Robinson theories of undirected graphs. Then the following conditions are equivalent:
1) $\mathfrak{B} \in E_{\Delta_\mathfrak{G}}$, where $\mathfrak{B}$ is a model of class $[\Delta_\mathfrak{G}]$;
2) $\mathfrak{B}$ is infinite quite disconnected graph.

Proof. To prove this theorem, the same scheme is used as in the proof of Theorem 10 of the previous paragraph, i.e. it is enough for us to prove the following lemmas.

Lemma 4. The following conditions are equivalent:
1) $\mathfrak{G}$ is a countably categorical universal graph;
2) $\mathfrak{G}$ is infinite quite disconnected graph.

Let us prove the necessity.

Let us assume the opposite. Suppose that there is a pair $< x, y >$ in graph $\mathfrak{G}$ such that xRy.

The following statement is known: If $\mathfrak{G}$ is a countably categorical universal graph, then from the fact that $\mathfrak{G}$ has an infinite number of disconnected components follows that $\mathfrak{G}$ is quite disconnected.

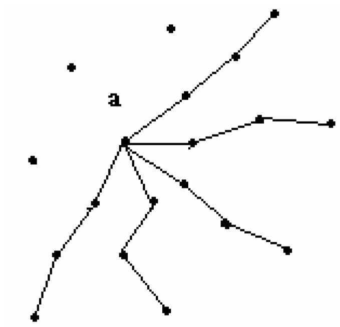
Thus, $\mathfrak{G}$ consists of a finite number of components, but then, due to the infinity of the graph $\mathfrak{G}$, there must be at least one infinite component. Possible cases:

1. There is a bound for the lengths of the chains.

2. There are chains of any given length.

Consider the first case.

Let us take an arbitrary point a from this component. Consider the set of all paths passing through a. The set of all points included in these paths coincides with the component, therefore, is infinite. Since the lengths of the paths are limited, an infinite number of paths pass through a. The ends of these paths are pendant vertices:



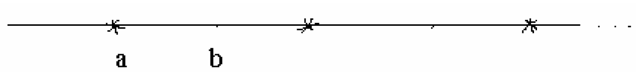
Consider a subgraph Γ consisting only of these pendant vertices.

Obviously, if there are $a \in \mathfrak{G}$ and $b \in \mathfrak{G}$ such that $R(a, b)$, then the type $tp(a, b/\varnothing)$ is not realized in Γ. Which contradicts the criterion of Ye.A. Palyutin.

Consider the second case. To do this, we will prove the following lemma.

Lemma 5. Let $[\Delta_{\mathfrak{G}}]$ be a class of ω-categorical Robinson theories of undirected graphs. If $\mathfrak{G} \models [\Delta_{\mathfrak{G}}]$ and without cycles, then there are no infinite chains in $\mathfrak{G}$.

Proof. Let $\{x_i\}_{i\in\omega}$ be a chain. Consider the subgraph $\{x_i\}_{i\in\omega} \backslash \{x_{3k}\}_{k\in\omega}$, which has the form:



We select a disconnected subgraph Γ in the chain, then the type $tp(a, b/\varnothing)$ is not realized in Γ. By virtue of infinity, Γ contradicts universal categoricity (Palyutin criterion).

The lemma is proved.

Let Γ be a connected component, B_Γ be a set of pendant vertices.

Lemma 6. B_Γ is an infinite set.

Proof. Suppose the opposite: B_Γ is finite. Since the component is infinite, and the set of B_Γ is finite, therefore, there is an infinite set E of Γ vertices that are not pendant. Let $E = \{e_1, e_2, ...\}$. But Γ is a connected component, which means that the set of non-pendant vertices forms an infinite chain, which contradicts the last Lemma 5.

So, we have obtained that if a graph $\mathfrak{G}$ has a pair $< x, y >$ such that xRy, then the graph does not satisfy the assumption condition of Lemma 4 on the countably categorical universality of the graph.

Therefore, if the graph $\mathfrak{G}$ is a countably categorical universal graph, then $\mathfrak{G}$ is a quite disconnected graph.

Let us prove sufficiency.

If the graph $\mathfrak{G}$ is an infinite quite disconnected graph, then $\mathfrak{G}$ is a countably categorical universal graph.

Let us show the satisfaction of the conditions:

1) universality and 2) categoricity.

1) The universality of the class of quite disconnected graphs follows from the fact that it is axiomatized by the universal formula $\forall x \forall y \neg R(x, y)$.

2) Take two subgraphs Γ_1, Γ_2 such that $|\Gamma_1| = |\Gamma_2|$. The set-theoretic mapping of Γ_1 to Γ_2 gives us an isomorphism of Γ_1 and Γ_2 as graphs. The theorem is proved.

Just as in the case of unars with respect to the question of Palyutin, from the description of an existentially closed graph, the following obviously takes place

Corollary 2. Countably categorical Robinson theories of graphs are totally categorical.

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Семантикалық йонсондық квазикөптүрлілік аясында универсал унарлар мен бағытталмаған графтар үшін категориялық мәселелер туралы

Мақала универсалды унарлар мен бағытталмаған графтардың семантикалық йонсондық квазикөптүрліліктерін зерттеуге арналған. Мақаланың бірінші бөлімі йонсондық модельдер теориясының негізгі қажетті ұғымдарынан тұрады. Келесі екі бөлімде робинсондық унарлардың семантикалық йонсондық квазикөптүрліліктерінің $j_{\mathfrak{U}}$ және робинсондық бағытталмаған графтардың семантикалық йонсондық квазикөптүрліліктерінің $J\mathbb{C}_{\mathfrak{G}}$, оның элементар теориясы мен семантикалық моделінің жаңа ұғымдарын қолданудың нәтижелері берілген. Мақаланың екі негізгі нәтижесін дәлелдеу үшін $RSp(J\mathbb{C}_{\mathfrak{U}})$ және $RSp(J\mathbb{C}_{\mathfrak{G}})$ робинсондық спектрлері және олардың косемантты қатынас арқылы $[\Delta]_{\mathfrak{U}}$ және $[\Delta]_{\mathfrak{G}}$ эквиваленттік кластарға бөлінуі қарастырылды. Негізгі нәтижелер 11 және 13 теоремалар ретінде ұсынылған және келесі пайдалы салдарлар туындайды: унарлардың саналымды категориялық робинсондық теориялары тоталды категориялық; бағытталмаған графтардың саналымды категориялық робинсондық теориялары тоталды категориялық. Алынған нәтижелер әртүрлі йонсондық алгебраларды, атап айтқанда циклді моноид арқылы анықталған полигондардың семантикалық йонсондық квазикөптүрліліктерді зерттеуді жалғастыру үшін пайдалы болуы мүмкін.

Кілт сөздер: йонсондық теория, унарлар, графтар, бағытталмаған графтар, универсалды теория, робинсондық теория, квазикөптүрлілік, семантикалық йонсондық квазикөптүрлілік, йонсондық спектр, робинсондық спектр, косеманттылық, категориялық, саналымды категориялық.

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О категоричности универсальных унаров и неориентированных графов с позиции семантического йонсоновского квазимногообразия

Статья посвящена изучению семантических йонсоновских квазимногообразий универсальных унаров и неориентированных графов. Первый раздел статьи состоит из базовых необходимых понятий из йонсоновской теории моделей. Следующие два–это результаты использования новых понятий семантического йонсоновского квазимногообразия робинсоновских унаров $J\mathbb{C}_{\mathfrak{U}}$ и семантического йонсоновского квазимногообразия робинсоновских неориентированных графов $J\mathbb{C}_{\mathfrak{G}}$, их элементарной теории и семантической модели. Для того чтобы доказать главные результаты статьи, были рассмотрены робинсоновские спектры $RSp(J\mathbb{C}_{\mathfrak{U}})$ и $RSp(J\mathbb{C}_{\mathfrak{G}})$ и их разбиение на классы эквивалентности $[\Delta]_{\mathfrak{U}}$ и

$[\Delta]_{\mathfrak{G}}$ с помощью отношения косемантичности. Были проанализированы особенности таких классов эквивалентности $[\Delta] \in RSp(J\mathbb{C}_U)$. Основные результаты представлены в виде теорем 11 и 13 и влекут за собой следующие полезные следствия: счетно категоричные робинсоновские теории унаров — тотально категоричные; счетно категоричные робинсоновские теории неориентированных графов — тотально категоричные. Полученные результаты могут быть полезны в продолжении исследования различных йонсоновских алгебр, в частности, семантического йонсоновского квазимногообразия полигонов над циклическим моноидом.

Ключевые слова: йонсоновская теория, унар, граф, неориентированный граф, универсальная теория, робинсоновская теория, квазимногообразие, семантическое йонсоновское квазимногообразие, йонсоновский спектр, робинсоновский спектр, косемантичность, категоричность, счетная категоричность.

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