DOI 10.31489/2023M2/104-115
UDC 519.6

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Numerical solution of differential – difference equations having an interior layer using nonstandard finite differences

This paper addresses the solution of a differential-difference type equation having an interior layer behaviour. A difference scheme is suggested to solve this equation using a non-standard finite difference method. Finite differences are derived from the first and second order derivatives. Using these approximations, the given equation is discretized. The discretized equation is solved using the algorithm for the tridiagonal system. The method is examined for convergence. Numerical examples are illustrated to validate the method. Maximum errors in the solution, in contrast to the other methods are organized to justify the method. The layer behaviour in the solution of the examples is depicted in graphs.

Keywords: Differential-difference equation, Boundary layer, Nonstandard finite difference, Convergence.

Introduction

Differential equations are ones in which the time evolution of a state variable is inconsistently dependent on a particular past. This means that the rate of change of a physical system is dependent not only on its current state but also on its previous history. The layer behavior differential difference equations have been extensively used in control theory for a number of years.

Subsequently, these equations play an important part in predator-prey models [1], population dynamics [2] and models of the red blood cell system [3] and models of neuronal variability [4]. Bender and Orszag [5], Doolan et al. [6] just are a few of the authors who have produced papers and books in recent years explaining various methods for solving differential-difference equations with singular perturbations, El'sgol'ts and Norkin [7], Mickens [8], Driver [9], Kokotovic et al. [10], Miller et al [11], O'Malley [12] are the authors who have produced books explaining various methods for solving delay differential equations and singularly perturbed differential-difference equations. In [13], authors developed an asymptotic analysis for a class of singularly perturbed problems with negative and positive shifts. In [14], the authors concentrate on problems with solutions that display layer behaviour at either one of the boundaries or both of the boundaries. The Laplace transforms used to the analysis of the layer equations produce new and interesting findings. The authors in [15] designed non-standard fitted finite difference methods based on the methods given in El-Mistikawy–Werle exponential finite difference scheme for differential–difference equations with negative and positive shifts. Rai and Sharma [16] developed numerical schemes using some modifications in El-Mistikawy–Werle exponential finite difference scheme. Sirisha et al. [17] devised a mixed difference scheme to solve the same problem. Salama and Al-Amery [18] constructed a mixed asymptotic solution for SPDDE using the composite expansion method. This work deals with constant shift arguments, which are independent of perturbation parameter. Swamy et al. [19] constructed a computational method of order four to solve SPDDE with mixed arguments. Bestehorn and Grigorieva [20] solved coupled nonlinear partial differential equations and single diffusion equation with an additional nonlinear delay term. Kadalbajoo and Sharma [21] solved a mathematical model arising from a model of neuronal variability and mathematical modelling

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for the determination of the expected time for generation of action potentials in nerve cells by random synaptic inputs in dendrites. Kadalbajoo and Sharma [22] exponentially fitted method based on finite difference to solve boundary-value problem for a singularly perturbed differential-difference equation with small shifts of mixed type.

The rest of the paper is organized as: In Section 1, problem description is given. In Section 2, a maximum principle and some important properties of the exact solution and its derivatives are established. The proposed numerical scheme is described in Section 3. Error estimate is derived in Section 4. Section 5 presents numerical examples to support the theoretical findings. Finally, Section 6 concludes with a summary and discussion.

1 Description of the problem

Consider a differential-difference equation with layer behaviour consisting a small delay and advanced terms of the form:

$$\varepsilon z''(u) + P(u)z'(u) + Q(u)z(u-\delta) + R(u)z(u) + V(u)z(u+\eta) = F(u), \tag{1}$$

on $(-1, 1)$, with the boundary conditions

$$z(u) = \phi(u), \quad -1-\delta \le u \le -1, z(u) = \gamma(u) \quad 1 \le u \le 1+\eta, \tag{2}$$

where $0 \le \varepsilon \ll 1$ is a perturbation parameter, $P(u), Q(u), R(u), V(u), F(u), \phi(u)$ and $\gamma(u)$ are smooth functions and $0 < \delta = o(\varepsilon)$ is the delay or negative shift and $0 < \eta = o(\varepsilon)$ is the advance or positive shift parameter. If $(P(u)-\delta Q(u)+\eta V(u)) > 0$, the solution of problem (1) with conditions (2) exposes layer at the left end of the interval and if $(P(u) - \delta Q(u) + \eta V(u)) < 0$ then the layer at the right-end of the interval. If $P(u) = 0$, the problem has either an oscillatory solution or two layers, depending on whether $Q(u) + R(u) + V(u)$ is positive or negative.

Since the solution $z(u)$ of problem (1) is sufficiently differentiable, the terms $z(u-\delta)$ and $z(u-\eta)$ can be expanded using Taylor series, then we have

$$z(u-\delta) = z(u) - \delta z'(u) + \frac{\delta^2}{2} z''(u), \tag{3}$$

$$z(u+\eta) = z(u) + \eta z'(u) + \frac{\eta^2}{2} z''(u). \tag{4}$$

Using formula (3) and formula (4) in problem (1), we get

$$C_\varepsilon z''(u) + a(u)z'(u) + b(u)z(u) = F(u), -1 < u < 1. \tag{5}$$

Problem (5) is a convection-diffusion problem. Here $C_\varepsilon = (\varepsilon + Q\frac{\delta^2}{2} + V\frac{\eta^2}{2}), \quad a(u) = P(u) - \delta(u) +$ $+ \eta V(u), \quad b(u) = Q(u) + R(u) + V(u)$. We solve problem (5) subject to the boundary constraints

$$z(-1) = \phi(-1), \quad z(1) = \gamma(1), \tag{6}$$

where the solution of the problem (5) with conditions (6) is taken as the approximation to the solution of the problem (1) with conditions (2). Let $a(u)$ vanishes at some $l_i \in (-1, 1)$. Let $N_i = [l_i - \xi, l_i - \xi]$ be a neighborhood of the turning point l_i such that it does not contain any other turning point. Also, it is assumed that

$$|a'(u)| \ge |\frac{a'(l_i)}{2}| \quad \text{for} \quad u \in N_i.$$

The transformation $u = \xi^{-1}(u - l_i)$ reduces the study of the behaviour of $z(u)$ near a given turning point l_i to the case when $a(u)$ has only one zero located at $u = 0$. Thus, we consider problems (5) with conditions (6) under the following hypothesis:

(i) $a(u) \in C^2[-1, 1]$, $F(u)$ and $b(u) \in C^1[-1, 1]$,

(ii) $b(u) \geq b_0 \geq 0$ on $[-1, 1]$, where b_0 is a positive constant,

(iii) $a(u)$ has simple zero at $u = 0$ and no other zeros in [-1,1],

(iv) $|a'(u)| \geq |\frac{(a'(0))}{2}|$ for $-1 \leq u \leq 1$,

(v) $\beta = \frac{b(0)}{a'(0)}$, and β_l,β_s be positive constants such that $\beta_l \leq 1 \leq \beta_s$ and $\beta_l \leq |\beta| \leq \beta_s$.

For a given function $g(u) \in C^k[-1, 1]$, let $||g||_k$ denote $\sum_{i=0}^{k} \max_{-1\leq u\leq 1} |g^i|$, where g^i denote i^{ih} derivative of $g(u)$, $C_\varepsilon(u) = \left(\varepsilon + Q\frac{\delta^2}{2} + V\frac{\eta^2}{2}\right)$. C_ε is taken as constant part of $C_\varepsilon(u)$ when $a(u)$ depends on u.

2 Analytical results

Lemma 1. (Continuous maximum principle): Let $\psi(u)$ be any sufficiently smooth function satisfying $\psi(-1) \geq 0$ and $\psi(1) \geq 0$. Then, $L\psi(u) \geq 0 \quad \forall\ u \in (-1, 1)$ implies that $\psi(u) \in 0\ \forall\ \ u \in [-1, 1]$.

Proof. Let u^* be such that $\psi(u^*) = \min_{u\in[-1,1]} \psi(u)$. Let us assure that $\psi(u^*) \leq 0$.
Clearly $u^* \notin (-1, 1)$. Since u^* is the point of minima therefore $\psi'(u^*) = 0$ and $\psi''(u^*) > 0$.

Now
$$L\psi(u^*) = C_\varepsilon z''(u^*) + a(u^*)z'(u^*) + b(u^*)z(u^*)$$
$$= C_\varepsilon z''(u^*) + b(u^*)z(u^*) < 0,$$
which is contradiction. This follows $\psi(u^*) \geq 0$ and since u^* is chosen arbitrarily therefore $\psi(u) \geq 0$, for $u \in [-1, 1]$.

Lemma 2. Let $z(u)$ be the solution of the problem (1) with the conditions (2) then
$$||z||_0 \leq \frac{||f||_0}{b_0} + \max(|\phi(-1)|, |\gamma(1)|).$$

Proof. Let us define
$$\psi^\pm \leq \frac{||f||_0}{b_0} + \max(|\phi(-1)|, |\gamma(1)|) + z(u),$$
then we have
$$\psi^\pm(-1) = \frac{||f||_0}{b_0} + \max(|\phi(-1)|, |\gamma(1)|) + z(-1) =$$
$$= \frac{||f||_0}{b_0} + \max(|\phi(-1)|, |\gamma(1)|) \pm \phi(-1) \geq 0.$$
$$\psi^\pm(1) = \frac{||f||_0}{b_0} + \max(|\phi(-1)|, |\gamma(1)|) + z(1) =$$
$$= \frac{||f||_0}{b_0} + \max(|\phi(-1)|, |\gamma(1)|) \pm \gamma(1) \geq 0.$$
$$L\psi^\pm(u) = C_\varepsilon(\psi^\pm(u))'' + a(u)(\psi^\pm(u))' + b(u)(\psi^\pm(u)) =$$
$$= b(u)\left(\frac{||f||_0}{b_0} + \max(|\phi(-1)|, |\gamma(1)|) + Lz(u)\right) =$$
$$= b(u)\left(\frac{||f||_0}{b_0} + \max(|\phi(-1)|, |\gamma(1)|) + f(u)\right) =$$

$$= (||f||_0 \pm f(u) + b(u)\max(|\phi(-1)|, |\gamma(1)|)) \geq 0 \quad (\text{since} \quad b(u) \geq b_0 \geq 0).$$

Therefore, using maximum principle, we obtain $\psi^{\pm}(u) \geq 0$ for $u \in [-1,1]$, which is the required bound on the solution of the problem (1) with conditions (2). Lemma 3 provides bound on the solution of the problem (5) with conditions (6). We now derive bounds on $z(u)$ and its derivatives on a subinterval $[p,q]$ of $[-1,1]$ which does not contain the turning point.

Lemma 3. Let $z(u)$ be the solution to the problem (5) with conditions (6) and $a(u), b(u), f(u) \in C^j[-1,0],\ j \geq 0$, are sufficiently smooth functions in $[-1,1]$. Then, there exit positive constant C and η such that $|D^i z| \geq C$ for $u \in [-1,1]$.

Proof. See in [23].

Theorem 1. Let $z(u)$ be the solution to the problem (5) with conditions (6) and $a(u), b(u), f(u) \in C^j[-1,0),\ j > 0, |a(u)| \geq v$ (v is a positive constant) are sufficiently smooth functions in $[-1,1]$. Then, there exit positive constant C and η such that

$$|D^i z| \leq C\left(1 + C_\varepsilon^{-i} e^{\left(\frac{vu}{C_\varepsilon}\right)}\right) \quad \text{for} \quad i = 1, 2, ...j+1, \quad u \in [-1,0),$$

and

$$|D^i z| \leq C\left(1 + C_\varepsilon^{-s} e^{\left(\frac{-vu}{C_\varepsilon}\right)}\right) \quad \text{for} \quad i = 1, 2, ...j+1, \quad u \in [0,1].$$

Proof. See in [23].

3 Numerical scheme

In this section, we construct a numerical scheme based on EI-Mistikawy and Werle exponentially fitted operator scheme to approximate the solution $z(u)$ of problem (5). Let uniform partition of the interval $[-1,1]$ be given by $u_i = -1 + ih$ for $i = 0, 1, 2, ..., n$ where $h = \frac{2}{n}$. We construct the numerical scheme as:

$$Lz_j = \begin{cases} C_\varepsilon D^+D^- z_j + a(u)D^- z_j + b(u)z(u) = F(u), & j = 1, 2, 3 \ldots .n/2 - 1, \\ C_\varepsilon D^+D^- z_j + a(u)D^+ z_j + b(u)z(u) = F(u), & j = n/2, n/2+1, n/2+2, \ldots .n-1, \end{cases} \tag{7}$$

where

$$D^- z_j = \frac{z_j - z_{j-1}}{h}, \quad D^+ z_j = \frac{z_{j+1} - z_j}{h}, \quad D^+D^- z_j = \frac{z_{j+1} - 2z_j + z_{j-1}}{\phi_j^2},$$

and

$$\phi_j^2 = \begin{cases} \frac{h\varepsilon}{a_j}\left[e^{\frac{(a_j h)}{\varepsilon}} - 1\right], & j = 1, 2, 3 \ldots .n/2 - 1, \\ \frac{h\varepsilon}{a_j}\left[1 - e^{\frac{(-a_j h)}{\varepsilon}}\right], & j = n/2, n/2+1, n/2+2, \ldots .n-1. \end{cases}$$

The system of equations (7) can be written in tridiagonal form as:

$$\left(\frac{\varepsilon}{\phi_j^2} - \frac{a_j}{h}\right) z_{j-1} + \left(\frac{-2\varepsilon}{\phi_j^2} + \frac{a_j}{h} + b_j\right) z_j + \left(\frac{\varepsilon}{\phi_j^2}\right) z_{j+1} = f_j \quad \text{for} \quad j = 1, 2, 3, ..., \frac{n}{2} - 1,$$

$$\left(\frac{\varepsilon}{\phi_j^2}\right) z_{j-1} + \left(\frac{-2\varepsilon}{\phi_j^2} - \frac{a_j}{h} + b_j\right) z_j + \left(\frac{\varepsilon}{\phi_j^2} + \frac{a_j}{h}\right) z_{j+1} = f_j \quad \text{for} \quad j = \frac{n}{2}, \frac{n}{2} + 1, \frac{n}{2} + 2, ..., n-1.$$

The above system of equations can be written as:

$$\begin{cases} A_j z_{j-1} + B_j z_j + C_j z_{j+1} = f_j, & j = 1,2,3,...,\frac{n}{2}-1, \\ A_j z_{j-1} + B_j z_j + C_j z_{j+1} = f_j, & j = \frac{n}{2}, \frac{n}{2}+1, \frac{n}{2}+2, ..., n-1. \end{cases} \tag{8}$$

Here

$$\begin{cases} A_j = \left(\frac{\varepsilon}{\phi_j^2} - \frac{a_j}{h}\right), \ B_j = \left(\frac{-2\varepsilon}{\phi_j^2} + \frac{a_j}{h} + b_j\right), \ C_j = \left(\frac{\varepsilon}{\phi_j^2}\right), \ f_j = f_j, & j = 1,2,3,...,\frac{n}{2}-1, \\ A_j = \left(\frac{\varepsilon}{\phi_j^2}\right), \ B_j = \left(\frac{-2\varepsilon}{\phi_j^2} - \frac{a_j}{h} + b_j\right), \ C_j = \left(\frac{\varepsilon}{\phi_j^2} + \frac{a_j}{h}\right), \ f_j = f_j, & j = \frac{n}{2}, \frac{n}{2}+1, \frac{n}{2}+2, ..., n-1. \end{cases}$$

4 Convergence Analysis

In this section, we analyze the convergence of the difference scheme (8). The analysis will be done on $u \in [-1, 0]$ and similarly the same will be done on $u \in (0, 1]$.
Let us define operator L as:

$$Lz_j = \begin{cases} C_\varepsilon \frac{d^2 z(u)}{du^2} + a(u)\frac{dz(u)}{du} + b(u)z(u) = F(u), & j = 1,2,3,...,\frac{n}{2}-1, \\ C_\varepsilon \frac{d^2 z(u)}{du^2} + a(u)\frac{dz(u)}{du} + b(u)z(u) = F(u), & j = \frac{n}{2}, \frac{n}{2}+1, \frac{n}{2}+2, ..., n-1. \end{cases}$$

The local truncation error of the discretization on $[-1, 0]$ can be given as:

$$L\left(U_j - z_j\right) = \varepsilon z_j'' + a_j z_j - \left[\frac{\varepsilon\left(z_{j+1} - 2z_j + z_{j-1}\right)}{(\phi_j^2)} + a_j \frac{z_j - z_{j-1}}{h}\right] =$$

$$= \varepsilon U_j'' - \frac{\varepsilon}{\phi_j^2}\left[h^2 U_j'' + \frac{h^4}{24}(z^4)(\xi_1) + \frac{h^4}{24}(z^4)(\xi_2)\right] + \frac{(a_j h)}{2} z_j'' - \frac{a_j h^2}{6} z_j''' + \frac{(a_j h^3)}{24}(\xi_3).$$

Using the truncated Taylor expansion of $\frac{1}{(\phi_j^2)} = \frac{1}{h^2} + \frac{a_j}{2\varepsilon h}$, it follows that

$$L(U_j - z_j) = h^2\left\{\frac{-2\varepsilon}{12}(z^4)(\xi_1) - \frac{a_j}{6} z_j'''(\xi_2)\right\} + \frac{h^3}{24}\left\{(z^4)(\xi_1 - \xi_2)\right\} +$$

$$+ h^4\left\{\frac{-\varepsilon}{360}(z^6)(\xi_1) - \frac{a_j}{120} z_j^{vi}(\xi_3)\right\} + h^5\left\{\frac{a_j}{720} z_j^6(\xi_1 - \xi_2)\right\}, \tag{9}$$

where $\xi_i \in (u_j, u_{j+1}), i \in 1,3$ and $\xi_2 \in (u_{j-1}, u_j)$. Using bounds on derivatives of z, for small h, we have

$$L_1(U_j - z_j) \leq Mh^2 \quad \forall j = 1(1)\frac{n}{2} - 1.$$

In a similar way, we can prove that

$$L_2(U_j - z_j) \leq Mh^2 \quad \forall j = \frac{n}{2}(1)n + 1.$$

Theorem 2. Let U_j be the numerical result of the difference scheme (8) along with the condition (9) and z_j is the solution to the problem (1) with condition (2), then a constant M is an independent of ε, h such that

$$\max_{1 \leq j \leq n+1} |U_j - z_j| \leq Mh^2.$$

Proof. Using th triangular inequality $|U_j - u_j| \leq |U_j - z_j| + |z_j - u_j|$, along with the truncation error, the fundamental outcome was generated by a global error.
Therefore,

$$\max_{1 \leq j \leq n+1} |U_j - z_j| \leq Mh^2.$$

5 Numerical examples

Example 1.

$$-\varepsilon z''(u) + 2(1-2u)z'(u) + 4z(u) + 2z(u-\delta) + z(u+\eta) = 0,$$

$$\text{on } u \in (0,1) \;\text{ with } z(u) = 1 \text{ on } -\delta \leq u \leq 0,\; z(1) = 1 \text{ on } 1 \leq u \leq 1+\eta.$$

Example 2.

$$-\varepsilon z''(u) + 2(1-2u)z'(u) + 4z(u) + 2z(u-\delta) + z(u+\eta) = 4(1-4u),$$

$$\text{on } u \in (0,1) \;\text{ with }\; z(u) = 1 \text{ on } -\delta \leq u \leq 0,\; z(1) = 1 \text{ on } 1 \leq u \leq 1+\eta.$$

6 Conclusion and discussions

We have discussed a numerical scheme with nonstandard finite differences for the solution of singularly perturbed differential–difference equations with delay and advance shifts. The domain is divided into two subintervals since the problem under consideration involves internal layer behavior. We constructed numerical scheme in each subinterval to get the solution. The proposed numerical method is analyzed for convergence.

In order to discuss the efficiency of the suggested scheme, some numerical experiments are carried out. The maximum absolute error in the solution of examples is tabulated in the form of Tables 1–4 in comparison to the method given in [16]. The effect of small delay and advance on the interior layer solution is shown by plotting the graphs (Figures 1–4). It is observed that when η is increasing for a fixed delay the width of the interior layer decreases, whereas it increases when δ increases for a fixed η.

Table 1

Maximum absolute errors in the solution of Example 1 for $\eta = 0.8 * \varepsilon$, $\delta = 0.6 * \varepsilon$.

$\varepsilon \setminus N$	**32**	**64**	**128**	**256**	**512**	**1024**
Present method:						
10^{-1}	1.8346e-03	5.6862e-04	1.4517e-04	3.4370e-05	8.0119e-06	1.8466e-06
10^{-2}	7.3498e-04	2.1050e-04	5.8093e-05	1.6472e-05	4.1595e-06	1.0125e-06
10^{-3}	1.1163e-03	2.3226e-04	6.4850e-05	1.7021e-05	4.1640e-06	9.5162e-07
10^{-4}	1.0462e-03	2.3228e-04	6.4875e-05	1.7163e-05	4.4164e-06	1.1201e-06
10^{-5}	1.0384e-03	2.3228e-04	6.4876e-05	1.7164e-05	4.4165e-06	1.1202e-06
10^{-6}	1.0376e-03	2.3228e-04	6.4876e-05	1.7164e-05	4.4165e-06	1.1202e-06
10^{-7}	1.0376e-03	2.3228e-04	6.4876e-05	1.7164e-05	4.4165e-06	1.1202e-06
$10-8$	1.0376e-03	2.3228e-04	6.4876e-05	1.7164e-05	4.4165e-06	1.1202e-06
Results in [16]						
10^{-1}	1.405e-03	4.13e-04	1.124e-04	2.936e-05	7.505e-06	1.897e-06
10^{-2}	3.763e-03	1.706e-03	6.567e-04	2.135e-04	6.170e-05	1.664e-05
10^{-3}	3.985e-03	1.966e-03	9.772e-04	4.849e-04	2.284e-04	9.256e-05
10^{-4}	4.005e-03	1.974e-03	9.813e-04	4.893e-04	2.443e-04	1.221e-04
10^{-5}	4.005e-03	1.974e-03	9.813e-04	4.893e-04	2.443e-04	1.221e-04
10^{-6}	4.007e-03	1.975e-03	9.817e-04	4.895e-04	2.444e-04	1.221e-04
10^{-7}	4.007e-03	1.975e-03	9.817e-04	4.895e-04	2.444e-04	1.221e-04
$10-8$	4.007e-03	1.975e-03	9.817e-04	4.895e-04	2.444e-04	1.221e-04

Table 2

Maximum absolute errors in the solution of Example 1 for $\eta = 0.8 * \varepsilon$, $\delta = 0.6 * \varepsilon$.

$\varepsilon \setminus N$	**32**	**64**	**128**	**256**	**512**	**1024**
Present method:						
10^{-1}	4.3167e-03	1.0336e-04	2.4771e-05	6.3523e-06	1.6076e-06	4.0430e-07
10^{-2}	4.5589e-04	9.2210e-05	2.4224e-05	6.2238e-06	1.5747e-06	3.9598e-07
10^{-3}	3.2398e-04	8.9201e-05	1.5556e-05	2.0509e-06	1.5043e-06	3.7861e-07
10^{-4}	3.2274e-04	8.8905e-05	2.3206e-05	5.9299e-06	1.6799e-06	3.7677e-07
10^{-5}	3.2274e-04	8.8905e-05	2.3206e-05	5.9299e-06	1.6799e-06	3.7677e-07
10^{-6}	3.2261e-04	8.8872e-05	2.3197e-05	5.9276e-06	1.4987e-06	3.7676e-07
10^{-7}	3.2261e-04	8.8872e-05	2.3197e-05	5.9276e-06	1.4987e-06	3.7676e-07
$10-8$	3.2261e-04	8.8872e-05	2.3197e-05	5.9276e-06	1.4987e-06	3.7676e-07
Results in [16]						
10^{-1}	1.445e-03	4.246e-04	1.156e-04	3.019e-05	7.716e-06	1.951e-06
10^{-2}	3.779e-03	1.712e-03	6.589e-04	2.142e-04	6.188e-05	1.669e-05
10^{-3}	3.987e-03	1.967e-03	9.776e-04	4.852e-04	2.285e-04	9.259e-05
10^{-4}	4.005e-03	1.974e-03	9.813e-04	4.893e-04	2.443e-04	1.221e-04
10^{-5}	4.007e-03	1.975e-03	9.817e-04	4.895e-04	2.444e-04	1.221e-04
10^{-6}	4.007e-03	1.975e-03	9.817e-04	4.895e-04	2.444e-04	1.221e-04
10^{-7}	4.007e-03	1.975e-03	9.817e-04	4.895e-04	2.444e-04	1.221e-04
$10-8$	4.007e-03	1.975e-03	9.817e-04	4.895e-04	2.444e-04	1.221e-04

Table 3

Maximum absolute errors in the solution of Example 2 for $\eta = 0.8 * \varepsilon$, $\delta = 0.6 * \varepsilon$.

$\varepsilon \setminus N$	**32**	**64**	**128**	**256**	**512**	**1024**
Present method:						
10^{-1}	2.3261e-03	1.2337e-03	4.3756e-04	1.3150e-04	3.6255e-05	9.6086e-06
10^{-2}	3.1095e-03	8.9149e-04	2.2287e-04	5.3497e-05	1.2954e-05	3.5879e-06
10^{-3}	3.1095e-03	8.9149e-04	2.2287e-04	5.3497e-05	1.2954e-05	3.5879e-06
10^{-4}	3.1571e-03	9.8441e-04	2.7495e-04	7.2740e-05	1.8717e-05	4.7473e-06
10^{-5}	3.1569e-03	9.8441e-04	2.7495e-04	7.2741e-05	1.8717e-05	4.7475e-06
10^{-6}	3.1569e-03	9.8441e-04	2.7495e-04	7.2741e-05	1.8717e-05	4.7475e-06
10^{-7}	3.1569e-03	9.8441e-04	2.7495e-04	7.2741e-05	1.8717e-05	4.7475e-06
$10-8$	3.1569e-03	9.8441e-04	2.7495e-04	7.2741e-05	1.8717e-05	4.7475e-06
Results in [16]						
10^{-1}	1.817e-02	5.352e-03	1.459e-03	3.814e-04	9.752e-05	2.466e-05
10^{-2}	4.874e-02	2.212e-02	8.512e-03	2.773e-03	8.017e-04	2.163e-04
10^{-3}	5.148e-02	2.551e-02	1.27e-02	6.302e-03	2.968e-03	1.203e-03
10^{-4}	5.171e-02	2.562e-02	1.275e-02	6.360e-03	3.176e-03	1.587e-03
10^{-5}	5.174e-02	2.563e-02	1.276e-02	6.363e-03	3.178e-03	1.588e-03
10^{-6}	5.174e-02	2.563e-02	1.276e-02	6.363e-03	3.178e-03	1.588e-03
10^{-7}	5.174e-02	2.563e-02	1.276e-02	6.363e-03	3.178e-03	1.588e-03
$10-8$	5.174e-02	2.563e-02	1.276e-02	6.363e-03	3.178e-03	1.588e-03

Table 4

Maximum absolute errors in the solution of Example 2 for $\eta = 0.8 * \varepsilon$, $\delta = 0.6 * \varepsilon$.

$\varepsilon \setminus N$	**32**	**64**	**128**	**256**	**512**	**1024**
Present method:						
10^{-1}	3.1747e-03	1.5835e-03	5.6153e-04	1.7120e-04	7.0667e-05	1.3083e-05
10^{-2}	3.1047e-03	8.9098e-04	2.2250e-04	5.3391e-05	1.3766e-05	3.9585e-06
10^{-3}	3.1584e-03	9.8424e-04	2.7479e-04	7.2123e-05	1.7645e-05	4.0325e-06
10^{-4}	3.1571e-03	9.8440e-04	2.7494e-04	7.2738e-05	1.8717e-05	4.7472e-06
10^{-5}	3.1569e-03	9.8441e-04	2.7495e-04	7.2741e-05	1.8717e-05	4.7475e-06
10^{-6}	3.1569e-03	9.8441e-04	2.7495e-04	7.2741e-05	1.8717e-05	4.7475e-06
10^{-7}	3.1569e-03	9.8441e-04	2.7495e-04	7.2741e-05	1.8717e-05	4.7475e-06
$10-8$	3.1569e-03	9.8441e-04	2.7495e-04	7.2741e-05	1.8717e-05	4.7475e-06
Results in [16]						
10^{-1}	1.843e-02	5.462e-03	1.494e-03	3.913e-04	1.002e-04	4.202e-05
10^{-2}	4.855e-02	2.206e-02	8.515e-03	2.775e-03	8.036e-04	2.168e-04
10^{-3}	5.145e-02	2.556e-02	1.269e-02	6.298e-03	2.967e-03	1.203e-03
10^{-4}	5.171e-02	2.562e-02	1.275e-02	6.360e-03	3.176e-03	1.587e-03
10^{-5}	5.171e-02	2.562e-02	1.275e-02	6.360e-03	3.178e-03	1.588e-03
10^{-6}	5.171e-02	2.562e-02	1.275e-02	6.360e-03	3.178e-03	1.588e-03
10^{-7}	5.171e-02	2.562e-02	1.275e-02	6.360e-03	3.178e-03	1.588e-03
$10-8$	5.171e-02	2.562e-02	1.275e-02	6.360e-03	3.178e-03	1.588e-03

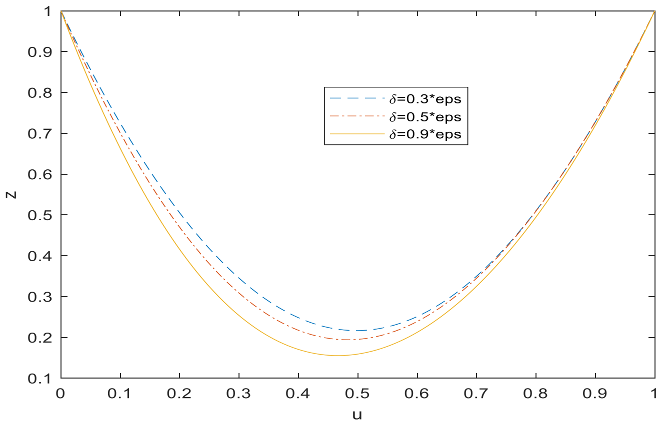

Fig1. Layer profile in Example 1 $\varepsilon = 2^{-2}$, $\eta = 0.5\varepsilon$.

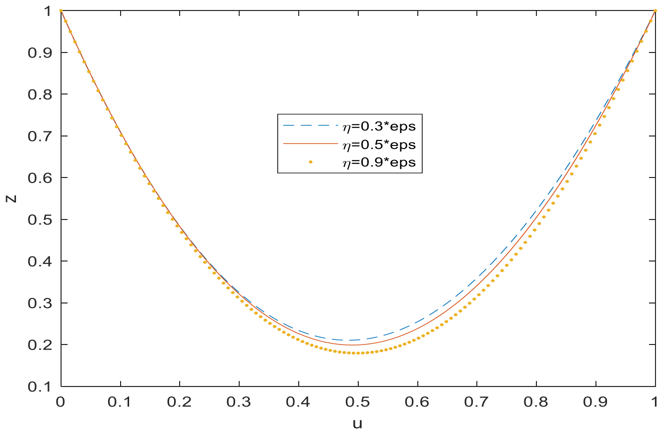

Fig 2. Layer profile in Example 1 $\varepsilon = 2^{-2}$, $\delta = 0.5\varepsilon$.

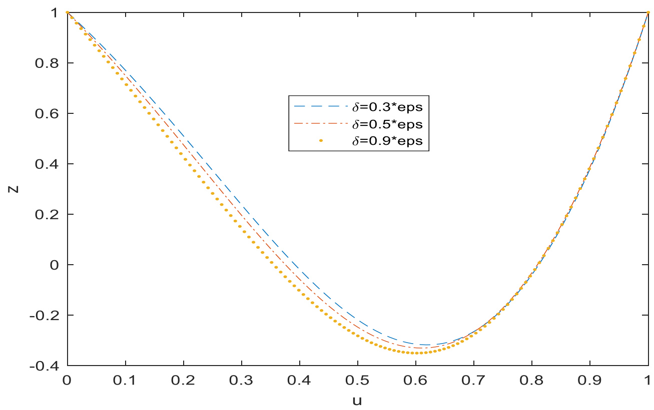

Fig 3. Layer profile in Example 2 $\varepsilon = 2^{-2}$, $\eta = 0.5\varepsilon$.

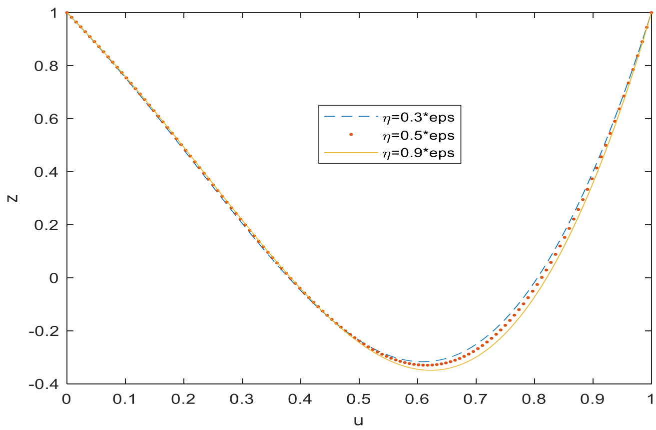

Fig 4. Layer profile in Example 2 with $\varepsilon = 2^{-2}$, $\delta = 0.5\varepsilon$.

Acknowledgments

The authors are very much thankful to the reviewers for their constructive comments/suggestions to improve the standards of the manuscript.

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Ішкі қабаты бар дифференциалдық-айырымдық теңдеулерді стандартты емес шекті айырымдарды қолданып сандық шешу

Мақалада ішкі қабаттың әрекеті бар дифференциалдық-айырымдық типті теңдеудің шешімі қарастырылған. Бұл теңдеуде стандартты емес шекті айырымдық әдісі арқылы шешуге арналған айырымдар схемасы ұсынылған. Шекті айырымдар бірінші және екінші ретті туындылардан алынған. Осы жуықтауларды пайдалана отырып, бұл теңдеу дискреттелген. Дискреттелген теңдеу үш диагональдық жүйенің алгоритмі арқылы шешілген. Әдіс жинақтылыққа тексеріледі. Әдісті тексеру үшін сандық мысалдар келтірілген. Шешімдегі максималды қателер басқа әдістерге қарағанда әдісті негіздеу үшін ұйымдастырылған. Мысалдарды шешудегі қабаттың әрекеті графиктерде көрсетілген.

Кілт сөздер: дифференциалдық-айырымдық теңдеуі, шекаралық қабат, стандартты емес шекті айырым, жинақтылық.

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Численное решение дифференциально-разностных уравнений с внутренним слоем с использованием нестандартных конечных разностей

В статье рассмотрено решение уравнения дифференциально-разностного типа, имеющего поведение внутреннего слоя. Предложена разностная схема решения этого уравнения с использованием нестандартного метода конечных разностей. Конечные разности получены из производных первого и второго порядка. Используя эти приближения, данное уравнение дискретизируется. Дискретизированное уравнение решается с помощью алгоритма для трехдиагональной системы. Метод проверяется на сходимость. Для проверки метода проиллюстрированы численные примеры. Максимальные ошибки в решении, в отличие от других методов, организованы для обоснования метода. Поведение слоя в решении примеров изображено на графиках.

Ключевые слова: дифференциально-разностное уравнение, пограничный слой, нестандартная конечная разность, сходимость.