DOI 10.31489/2023M2/4-20

UDC 517.9

N. Adil, A.S. Berdyshev*

Abai Kazakh National Pedagogical University, Almaty, Kazakhstan;
Institute of Information and Computational Technologies, Almaty, Kazakhstan
(E-mail: nauryzbai_ adil@mail.ru, berdyshev@mail.ru)

Spectral properties of local and nonlocal problems for the diffusion-wave equation of fractional order

The paper investigates the issues of solvability and spectral properties of local and nonlocal problems for the fractional order diffusion-wave equation. The regular and strong solvability to problems stated in the domains, both with characteristic and non-characteristic boundaries are proved. Unambiguous solvability is established and theorems on the existence of eigenvalues or the Volterra property of the problems under consideration are proved.

Keywords: diffusion-wave equations, fractional order equations, boundary value problems, strong solution, Volterra property, eigenvalue.

1 Introduction

The theory of derivatives and integrals of non-integer (fractional) order, called fractional calculus, is becoming increasingly important both for the development of modern mathematics and for applications in various fields of natural science. Both ordinary and partial differential equations of fractional order have been used over the past few decades to model many physical and chemical processes and in engineering [1–7].

Fractional partial differential equations have become especially important for modeling the so-called anomalous diffusion processes in nature and the theory of complex systems [1]. Such equations are also associated with fractional Brownian motions, the continuous random walk in time (CTRW) method, stable Levy distributions, etc. [2, 7]. Fractional differential equations also make it possible to study the long-term and nonlocal dependence of many anomalous processes.

Since the fractional order equation generalizes the integer order equation, as well as a relatively small number of systematized analytical and numerical methods for such equations, make this direction a priority in the general theory of differential equations.

The mathematical theory of fractional differential equations is more or less fully investigated for ordinary equations [1], whereas for partial differential equations it differs from the situation for the equation of one variable. In the scientific literature, analogs of the initial data problem and initial boundary value problems for the simplest partial differential equations of fractional order were considered mainly. Methods for solving such problems are considered in [1, 8–10].

The issues of solvability of local and non-local problems for various fractional order equations are considered in [11–16].

Spectral properties, including Volterra property and the existence of eigenvalues, for a mixed fractional order equation, as far as we know, are almost not studied. Note that the solvability issues and spectral properties of local and nonlocal problems for a mixed parabolic-hyperbolic equation of the second and third orders are studied in [17–24].

*Corresponding author.
E-mail: berdyshev@mail.ru

The work is devoted to the study of the solvability and spectral properties of local and nonlocal problems for the diffusion-wave equation of fractional order. The regular and strong solvability of the tasks set in the domains with both characteristic and non-characteristic boundaries of the domain is proved. The unambiguous solvability of the problem is established, theorems on the existence of eigenvalues are proved, or the Volterra nature of the problems under consideration.

Consider equation

$$Lu(x,y) = f(x,y), \tag{1}$$

where

$$Lu(x,y) = \begin{cases} {}_cD_{0x}^{\alpha}u(x,y) - u_{yy}(x,y), \ y > 0, \\ u_{xx}(x,y) - u_{yy}(x,y), \ y < 0, \end{cases} \tag{2}$$

$${}_cD_{0x}^{\alpha}u\,(x,y) = \frac{1}{\Gamma\,(1-\alpha)}\int\limits_0^x \frac{u_x\,(t,y)}{(x-t)^{\alpha}}dt, \ \ 0<\alpha<1.$$

$\Gamma(x)$ is Euler's gamma-function, (2) is an integral-differential operator of fractional order α in the sense of Caputo [1; 92], $f(x,y)$ is a given function.

2 Solvability and Volterra property of local and nonlocal problems for the diffusion-wave equation

Let $\Omega = \Omega_0 \cup \Omega_1 \cup AB$ be a domain, where Ω_0 is a rectangle ABB_0A_0 with vertices $A\,(0,0)$, $B\,(1,0)$, $B_0\,(1,1)$, $A_0\,(0,1)$, Ω_1 is a domain bounded by segments AB and smooth curve $AD: y = -\gamma\,(x), \ 0 < x < l$, where $0,5 < l \le 1; \ \gamma\,(0) = 0, \ l + \gamma\,(l) = 1$, and characteristic $BD: x - y = 1$ of equation (1), if $l < 1$ and $\gamma(l) = 0$, if $l = 1$ (when $D = B$), located inside the characteristic triangle $0 < x + y \le x - y < 1$.

With respect to the curve $\gamma\,(x)$, we suppose that $\gamma\,(x)$ is twice continuously differentiable function and $x \pm \gamma\,(x)$ are monotonically increasing functions, and $0 < \gamma'(x) < 1, \ \gamma(x) > 0, \ x > 0$.

Problem M_1A. Find a solution to equation (1) satisfying conditions:

$$u(0,y) = 0, \ 0 \le y \le 1, \tag{3}$$

$$u(x,1) = 0, \ 0 \le x \le 1, \tag{4}$$

$$(u_x - u_y)\,|_{AD} = 0. \tag{5}$$

Definition 1. The regular solution to the problem M_1A in the domain Ω will be called the function $u(x,y) \in V$, where

$$V = \left\{u(x,y): \ u(x,y) \in C(\bar{\Omega}) \cap C^{1.1}(\Omega \cup AC), \ D_{0x}^{\alpha}u(x,y), \ u_{yy}(x,y) \in C(\Omega_0), \ u(x,y) \in C^{2.2}(\Omega_1)\right\},$$

satisfying the equation (1) in $\Omega_0 \cup \Omega_1$ and conditions (3)–(5).

In domain $\Omega_0$consider the following auxiliary problem:

Problem C_1. Find a solution to equation (1) for $y > 0$ satisfying conditions (3), (4) and

$$u_x(x,0) - u_y(x,0) = \delta(x), \ 0 < x < 1, \tag{6}$$

where $\delta(x)$ is a given function.

Lemma 1. Let be $\delta(x) \in C^1[0,1]$. Then for any function $f(x,y) \in C^1(\bar{\Omega}_0)$ is a solution to problem C_1 allows a priori estimates.

$$D_{0x}^{\alpha-1}\left\|u(x,y)\right\|^2_{L_2(0,1)} + 2\int\limits_0^x \left\|u_y(t,y)\right\|^2_{L_2(0,1)}dt \le C\left[\int\limits_0^x \left\|f(t,y)\right\|^2_{L_2(0,1)}dt + \int\limits_0^x \delta^2(t)dt\right], \tag{7}$$

where $\|f(x,y)\|^2_{L_2(0,1)} = \int\limits_0^1 f^2(x,y)dy$. Hereinafter symbol will denote a positive constant that does not depend on $u(x,y)$, not necessarily the same.

Proof of Lemma 1. We multiply equation (1) for $y > 0$ by $u(x,y)$ and integrating from 0 to 1 over y and taking into account conditions (3),(4) after some transformations we have

$$\int\limits_0^1 u(x,y)D^{\alpha}_{0x}u(x,y)dy + \int\limits_0^1 u^2_y(x,y)dy + \tau(x)\nu(x) = \int\limits_0^1 f(x,y)u(x,y)dy, \tag{8}$$

where

$$\tau(x) = u(x,0), \;\; 0 \le x \le 1, \tag{9}$$

$$\nu(x) = u_y(x,0), \;\; 0 < x < 1. \tag{10}$$

It is known [10], that

$$\int\limits_0^1 u(x,y) \cdot D^{\alpha}_{0x}u(x,y)dy \ge \frac{1}{2}\int\limits_0^1 D^{\alpha}_{0x}u^2(x,y)dy.$$

By virtue of the last inequality, from (8), taking into account (6) and the notations (9), (10) we obtain

$$\int\limits_0^1 D^{\alpha}_{0x}u^2(x,y)dy + 2\int\limits_0^1 u^2_y(x,y)dy + 2\tau(x)\tau'(x) \le 2\int\limits_0^1 u(x,y)f(x,y)dy + 2\tau(x)\delta(x). \tag{11}$$

Integrating (11) over t from 0 to x, taking into account $\tau(0) = 0$ and using known inequalities we have

$$D^{\alpha-1}_{0x}\|u(x,y)\|^2_{L_2(0,1)} + 2\int\limits_0^x \|u_y(t,y)\|^2_{L_2} dt + \tau^2(x) \le \int\limits_0^x \left[\|u(t,y)\|^2_{L_2(0,1)} + \|f(t,y)\|^2_{L_2(0,1)} + \tau^2(t) + \delta^2(t)\right]dt. \tag{12}$$

In the left part of (12), omitting the first two terms and applying the Gronwall-Bellman inequality, we will have

$$\int\limits_0^x \tau^2(t)dt \le C\int\limits_0^x \left[\|u(t,y)\|^2_{L_2(0,1)} + \|f(t,y)\|^2_{L_2(0,1)} + \delta^2(t)\right]dt.$$

Taking into account the last from (12) we have

$$D^{\alpha-1}_{0x}\|u(x,y)\|^2_{L_2(0,1)} + 2\int\limits_0^x \|u_y(t,y)\|^2_{L_2(0,1)} \le C\int\limits_0^x \left[\|u(t,y)\|^2_{L_2(0,1)} + \|f(t,y)\|^2_{L_2(0,1)} + \delta^2(t)\right]dt. \tag{13}$$

Similarly as above, omitting the second term of the left part in (13) and applying Lemma 1 in [10] we have

$$D^{-\alpha-1}_{0x}\|f(x,y)\|^2_{L_2(0,1)} \le \frac{x^{\alpha}}{\Gamma(1+\alpha)}\int\limits_0^x \|f(t,y)\|^2_{L_2(0,1)} dt$$

we have

$$\int\limits_0^x \|u(t,y)\|^2_{L_2(0,1)} dt \le C\int\limits_0^x \left[\|f(t,y)\|^2_{L_2(0,1)} + \delta^2(t)\right]dt. \tag{14}$$

From (12)–(14) it is followed the validity of the a priori estimate (7). *Lemma 1 is proved.*

Now consider equation (1) in the domain Ω_1. By virtue of the unambiguous solvability of the Cauchy problem (1), (9), (10) for the wave equation, any regular solution of the в problem in the domain Ω_1 is represented as

$$u(x,y)=\frac{1}{2}\left[\tau(\xi)+\tau(\eta)-\int\limits_{\xi}^{\eta}\nu(t)\,dt\right]-\int\limits_{\xi}^{\eta}d\xi_1\int\limits_{\xi_1}^{\eta}f_1(\xi_1,\eta_1)\,d\eta_1, \tag{15}$$

where $\xi=x+y,\ \ \eta=x-y,\ \ \ 4f_1(\xi,\eta)=f\left(\frac{\xi+\eta}{2},\frac{\xi-\eta}{2}\right)$. Due to the conditions imposed on the function $\gamma(x)$, equation of the curve AD in characteristic variables ξ, η allows representation

$$\xi=\lambda(\eta),\ \ 0\le\eta\le1,\ \ and\ \lambda(\eta)<\eta.$$

In (15) satisfying condition (5) after some simple transformations we have

$$\nu(x)=\tau'(x)-2\int\limits_{\lambda(x)}^{x}f_1(\xi,x)d\xi,\ \ 0<x<1. \tag{16}$$

The ratio (16) is the main functional relationship between $\tau(x)$ and $\nu(x)$ brought to the segment AB from hyperbolic domain Ω_1.

Substituting obtained expression $\nu(x)$ into (15), after some transformations we get presentation of the solution $u(\xi,\eta)$ in domain Ω_1.

$$u(x,y)=\tau(\xi)+\int\limits_{\xi}^{\eta}d\eta_1\int\limits_{\lambda(\eta_1)}^{\xi}f_1(\xi_1,\eta_1)\,d\xi_1. \tag{17}$$

Now in (7) assuming that $\delta(x)=2\int\limits_{\lambda(x)}^{x}f_1(\xi,x)d\xi$ it is not difficult to establish the validity of the following lemma.

Lemma 2. For any function $f(x,y)\in C^1(\bar{\Omega}),\ f(0,0)=0$ the solution to problem M_1B allows a priori estimate

$$D_{0x}^{\alpha-1}\left\|u(x,y)\right\|_{L_2(0,1)}^2+\int\limits_0^x\left\|u_y(t,y)\right\|_{L_2(0,1)}^2dt\le C\left[\int\limits_0^x\left\|f(t,y)\right\|_{L_2(0,1)}^2dt+\int\limits_0^x d\xi\int\limits_{\xi}^{x}\left|f(\xi,t)\right|^2dt\right]. \tag{18}$$

Lemma 2 implies the validity of the following estimate

$$\left\|u(x,y)\right\|_{L_2(\Omega_0)}+\left\|u_y(x,y)\right\|_{L_2(\Omega_0)}\le C\left\|f(x,y)\right\|_{L_2(\Omega)}, \tag{19}$$

where $L_2(\Omega)$ is quadratically summable functions in Ω.

Consider the following auxiliary problem C_2. In domain Ω_0 find a solution of equation (1), satisfying conditions (3), (4) and (9).

The solution of equation (1), satisfying conditions (3), (4) and (9) in domain Ω_0 can be presented in a form [8]

$$u(x,y)=\int\limits_0^x E_{y_1}(x-x_1,y,0)\,\tau(x_1)\,dx_1+\int\limits_0^x dx_1\int\limits_0^1 E(x-x_1,y,y_1)\,f(x_1,y_1)\,dy_1, \tag{20}$$

where

$$E(x,y,y_1)=\frac{x^{\beta-1}}{2}\sum_{n=-\infty}^{+\infty}\left[e_{1,\beta}^{1,\beta}\left(-\frac{|y-y_1+2n|}{x^\beta}\right)-e_{1,\beta}^{1,\beta}\left(-\frac{|y+y_1+2n|}{x^\beta}\right)\right],\ \beta=\frac{\alpha}{2},$$

$e_{1,\beta}^{1,\beta}(t)=\sum\limits_{n=0}^{\infty}\frac{t^n}{n!\Gamma(\beta-\beta n)}$ is Wright type function [8]. Differentiating (20) over y we have

$$u_y(x,y)=\int_0^x E_{y_1y}(x-x_1,y,0)\,\tau(x_1)\,dx_1+\int_0^x dx_1\int_0^1 E_y(x-x_1,y,y_1)\,f(x_1,y_1)\,dy_1 \tag{21}$$

and using known formulas [8], [18] after some calculations, going to limit in (21) for $y\to 0$ we have:

$$\nu(x)=-\int m(x-x_1)\,\tau'(x_1)dx_1+\int_0^x dx_1\int_0^1 E_y(x-x_1,0,y_1)\,f(x_1,y_1)\,dy_1, \tag{22}$$

where

$$m(x)=\sum_{n=-\infty}^{+\infty}x^{-\beta}e_{1,\beta}^{1,1-\beta}\left(-\frac{|2n|}{x^\beta}\right)=\frac{1}{\Gamma(1-\beta)}x^{-\beta}+2x^{-\beta}\sum_{n=1}^{+\infty}e_{1,\beta}^{1,1-\beta}\left(-\frac{2n}{x^\beta}\right). \tag{23}$$

Note that (22) is the main functional rate between $\tau'(x)$ and $\nu(x)$, brought to the segment from domain Ω_0.

Excluding from the functional relations (16) and (22) the function $\nu(x)$, with respect to $\tau'(x)$ we obtain the equation

$$\tau'(x)+\int_0^x m(x-t)\,\tau'(t)dt=Q(x),\ \ 0\le x\le 1, \tag{24}$$

where

$$Q(x)=2\int_{\lambda(x)}^x f_1(\xi,x)d\xi+\int_0^x dx_1\int_0^1 \mathrm{E}_y(x-x_1,0,y_1)f(x_1,y_1)dy_1. \tag{25}$$

Lemma 3. [8] Let be $0<\theta\le 1$. Then for functions $\mathrm{E}(x,y,y_1)$ and $\mathrm{E}_y(x,y,y_1)$ the following estimates take place

$$|\mathrm{E}(x,y,y_1)|\le Cx^{(2+\theta)\beta-1},\ \ 0<x\le 1,\ \ 0\le y_1<y\le 1,\ \ 0<\theta\le 1, \tag{26}$$

$$|\mathrm{E}_y(x,y,y_1)|\le Cx^{\beta(1+\theta)-1},\ \ 0<x\le 1,\ \ 0\le y_1<y\le 1,\ \ 0<\theta\le 1. \tag{27}$$

The proof of Lemma 3 is carried out using the inequality

$$\left|y^{p-1}t^{\delta-1}e_{\omega,\tau}^{p,\delta}(-y^\omega t^{-\tau})\right|<Cy^{p-\omega\theta-1}\cdot t^{\delta+\theta\tau-1},\ \ 0<\theta\le 1.$$

By virtue of Lemma 3 and $\gamma(x)\in C^2[0,l]$, $f(x,y)\in C^1(\bar\Omega)$, $f(0,0)=0$ from (25) it is not difficult to establish that

$$Q(x)\in C^1[0,1]\ \text{ and }\ Q(0)=0. \tag{28}$$

Thus, by virtue of (23), the problem M_1A is equivalently (in the sense of unambiguous solvability) reduced to a Volterra type integral equation of the second kind with a weak singularity (24). Therefore, by virtue of (28), there is a unique solution of equation (24) from the class $C^1[0,1]$ and it is representable as

$$\tau'(x) = Q(x) + \int\limits_0^x R(x-t)\,Q(t)\,dt, \tag{29}$$

where $R(x)$ is the resolvent of the integral equation (24)

$$R(x) = \sum_{n=1}^{\infty} (-1)^n m_n(x)\,,\ m_1(x) = m(x),\ m_{n+1}(x) = \int\limits_0^x m_1(x-t)\,m_n(t)\,dt.$$

From (29) taking into account $\tau(0) = 0$, we have

$$\tau(x) = \int\limits_0^x R_1(x-t)\,Q(t)\,dt,\ where\ R_1(x) = 1 + \int\limits_0^x R(t)\,dt. \tag{30}$$

Substituting (30) in (17) and (20), taking into account (25) after some transformations we have

$$u(x,y) = \iint\limits_{\Omega} M_1(x,y,x_1,y_1) f(x_1,y_1)dxdy, \tag{31}$$

where

$$M_1(x,y,x_1,y_1) = \theta(x-x_1)\left[\theta(y)\,M_{01}(x,y,x_1,y_1) + \theta(-y)\,M_{11}(x,y,x_1,y_1)\right], \tag{32}$$

$$M_{01}(x,y,x_1,y_1) = \theta(y_1)\left[E(x-x_1,y,y_1) + \int\limits_{x_1}^{x} dz \int\limits_{x_1}^{z} E_{y_1}(x-z,y,0)\,R_1(z-t)\,E_y(t-x_1,0,y_1)\,dt\right] +$$

$$+\theta(-y_1)\int\limits_{\eta_1}^{x} E_{y_1}(x-t,y,0)\,R_1(t-\eta_1)\,dt,$$

$$M_{11}(x,y,x_1,y_1) = \theta(y_1)\int\limits_0^{\xi} R_1(\xi-t)\,E_y(t-x_1,0,y_1)\,dt +$$

$$+\frac{1}{2}\theta(-y_1)\left[\theta(\xi-\eta_1)\,R_1(\xi-\eta_1) + \theta(-y_1)\,\theta(\eta-\eta_1)\,\theta(\eta_1-\xi)\,\theta(\xi-\xi_1)\right],$$

where $\xi_1 = x_1 + y_1,\ \eta_1 = x_1 - y_1,\ \xi = x + y,\ \eta = x - y,\ \theta(y) = 1,\ y > 0$ and $\theta(y) = 0, y < 0$.

Taking into account explicit types of functions

$$M_{01}(x,y,x_1,y_1)\,,\ M_{11}(x,y,x_1,y_1)$$

it is not difficult to establish that in (32) all terms are bounded, with the exception of the first – $M_{01}(x,y,x_1,y_1)$, in which by virtue of Lemma 3, the summand may not be limited $E(x-x_1,y,y_1)$. Therefore, it is enough to show that

$$\theta(x-x_1)\,\theta(y_1)\,\theta(y)\,E(x-x_1,y,y_1) \in L_2(\Omega\times\Omega)\,.$$

By virtue of Lemma 3 from estimation (26) by direct calculation we have

$$\|\theta(x-x_1)\mathrm{E}(x-x_1,\,y,\,y_1)\|^2_{L_2(\Omega\times\Omega)} \le C\{(2+\theta)\beta\,[1+(2+\theta)\beta]\}^{-1}.$$

Therefore, $M_1(x,y,x_1,y_1) \in L_2(\Omega\times\Omega)$.

Lemma 4. If $f(x,y) \in L_2(\Omega)$, then $Q(x) \in L_2[0,1]$ and $\|Q(x)\|^2_{L_2(0,1)} \le C\,\|f(x,y)\|^2_{L_2(\Omega)}$.

Proof of Lemma 4 taking into account (25), (27) It is carried out by direct calculation using the well-known Cauchy-Bunyakovsky inequality. From (29) we have

$$\left\|\tau'(x)\right\|_{L_2(0,1)} \le C\|Q(x)\|_{L_2(0,1)} \le C\|f(x,y)\|_{L_2(\Omega)}. \tag{33}$$

From (17) by virtue (33) by direct calculation it is not difficult to establish that

$$\|u(x,y)\|_{W_2^1(\Omega_1)} \le C\,\|f(x,y)\|_{L_2(\Omega)}, \tag{34}$$

where $W_2^1(\Omega)$ is S.L. Sobolev's space. From (18) and (34) we have

$$\begin{aligned} &D_{0x}^{\alpha-1}\,\|u(x,y)\|^2_{L_2(0,1)} + \int\limits_0^x \|u_y(t,y)\|^2_{L_2(0,1)}\,dt + \|u(x,y\|^2_{W_2^1(\Omega_2)} \le \\ &\le C\left[\int\limits_0^x \|f(t,y)\|^2_{L_2(0,1)} + \int\limits_0^x d\xi \int\limits_\xi^1 |f(\xi,x)|^2 dt + \|f(x,y)\|^2_{L_2(\Omega)}\right]. \end{aligned} \tag{35}$$

Thus, summarizing the above statements, the following theorem is proved.

Theorem 1. For any function $f(x,y) \in C^1(\bar{\Omega})$, $f(A) = 0$ there is a unique regular solution to the problem M_1A (1), (3)-(5) and it is represented in the form (31) and satisfies the inequality (35). From (35) or (19) and (34) it is followed the the validity of the estimate

$$\|u(x,y)\|_{L_2(\Omega_0)} + \|u_y(x,y)\|_{L_2(\Omega_0)} + \|u(x,y)\|_{W_2^1(\Omega_1)} \le C\|f(x,y)\|_{L_2(\Omega)}. \tag{36}$$

Definition 2. The function $u(x,y) \in L_2(\Omega)$ is called a strong solution to problem M_1A, if there is a sequence of functions $\{u_n(x,y)\}$, $u_n(x,y) \in V$, satisfying conditions (3)–(5), such that

$$\|u_n(x,y) - u(x,y)\|_{L_2(\Omega)} \to 0, \;\; \|Lu_n(x,y) - f(x,y)\|_{L_2(\Omega)} \to 0 \;\text{ for }\; n \to \infty.$$

Theorem 2. For any function $f(x,y) \in L_2(\Omega)$ there is a unique strong solution $u(x,y)$ to the problem M_1A. This solution can be represented as (31) and satisfies the estimate (36).

The proof of Theorem 2 in the presence of a representation of the solution (31) and the estimate (36) is proved in the same way as in [22–24].

By B_1 we denote a closure in space $L_2(\Omega)$, of fractional differential operator given at the set of functions V, satisfying conditions (3)–(5), with expression (2).

According to the definition of a strong solution to the problem M_1A, $u(x,y)$ is a strong solution to the problem M_1A only and only then, when $u(x,y) \in D(B_1)$, where $D(B_1)$ is a definition domain of operator B_1.

From theorem 2 it follows that operator B_1 is closed and its definition domain is dense in $L_2(\Omega)$; there exists an inverse operator B_1^{-1}, it is defined in all $L_2(\Omega)$ and quite continuous.

In this regard, a natural question arises: is there an eigenvalue of the operator B_1^{-1}, and therefore to the problem? The main result is the theorem on the absence of eigenvalues of the operator B_1^{-1}.

Theorem 3. Integral operator

$$B_1^{-1} f(x,y) = \iint\limits_\Omega M_1(x,y,x_1,y_1) f(x_1,y_1)\,dx_1 dy_1, \tag{37}$$

where $M_1(x,y,x_1,y_1) \in L_2(\Omega \times \Omega)$ is Volterra in $L_2(\Omega)$.

Proof. To prove Theorem 3, we need to show that the operator B_1^{-1} defined by formula (37) is completely continuous and quasinilpotent. Since the complete continuity of this operator follows from the fact that $M_1(x,y,x_1,y_1) \in L_2(\Omega \times \Omega)$, show that B_1^{-1} is quasinilpotent, i.e.

$$\lim_{n\to\infty} \left\|B_1^{-1}\right\|_{L_2(\Omega)\to L_2(\Omega)}^{\frac{1}{n}} = 0, \tag{38}$$

where $B_1^{-n} = B_1^{-1}\left[B_1^{-(n-1)}\right],\ n = 1.2...$

From (37) by direct calculation, taking into account (32) is not difficult to obtain that

$$B_1^{-n} f(x,y) = \iint\limits_{\Omega} M_n(x,y,x_1,y_1) f(x_1,y_1) dx_1 dy_1, \tag{39}$$

where

$$M_n(x,y,x_1,y_1) = \iint\limits_{\Omega} M_1(x,y,x_2,y_2) M_{(n-1)}(x_2,y_2,x_1,y_1) dx_2 dx_1,\ \ n = 2.3...$$

Lemma 5. For iterated kernels $M_n(x,y,x_1,y_1)$ there is an assessment

$$|M_n(x,y,x_1,y_1)| \leq \left(\frac{3}{2}\right)^{n-1} N^n \frac{\Gamma^n(\gamma)}{\Gamma(n\gamma)} (x-x_1)^{n\gamma-1}, \tag{40}$$

where $\gamma = (2+\theta)\beta,\ \ N = Cd$, C is coefficient from the assessment (26),

$$d = \max_{\substack{(x,y)\in\Omega \\ (x_1,y_1)\in\Omega}} \left|(x-x_1)^{1-\gamma} M_1(x,y,x_1,y_1)\right|,\ if\ \gamma < 1.$$

$$d = \max_{\substack{(x,y)\in\Omega \\ (x_1,y_1)\in\Omega}} |M_1(x,y,x_1,y_1)|,\ if\ \gamma \geq 1.$$

The proof of Lemma 5 we carry out by induction method over n.

For $n = 1$ the inequality

$$|M_1(x,y,x_1,y_1)| \leq N(x-x_1)^{\gamma-1}$$

follows from representation (32) taking into account estimate (26).

Let be (40) valid for $n = k-1$. Let 's prove the validity of this formula for $n = k$.

Using inequality (40) for $n = 1$ and $n = k-1$ we have

$$|M_k(x,y,x_1,y_1)| = \left|\iint\limits_{\Omega} M_1(x,y,x_2,y_2) \cdot M_{(k-1)}(x_2,y_2,x_1 y_1) dx_2 dy_2\right| \leqslant$$

$$\leqslant \iint\limits_{\Omega} |M_1(x,y,x_2,y_2)| \cdot \left|M_{(k-1)}(x_2,y_2,x_1,y_1)\right| dx_2 dy_2 \leqslant$$

$$\leqslant \iint\limits_{\Omega} \theta(x-x_2) N (x-x_2)^{\gamma-1} \theta(x_2-x_1) \left(\frac{3}{2}\right)^{k-2} N^{k-1} \frac{\Gamma^{k-1}(\gamma)}{\Gamma\left[(k-1)\gamma\right]} (x_2-x_1)^{(k-1)\gamma-1} dx_2 dy_2 \leqslant$$

$$\leq \left(\frac{3}{2}\right)^{k-1} N^k \frac{\Gamma^{k-1}(\gamma)}{\Gamma\left[(k-1)\gamma\right]} \int\limits_{x_1}^{x} (x-x_2)^{\gamma-1} (x_2-x_1)^{(k-1)\gamma-1} dx_2 =$$

$$= \left(\frac{3}{2}\right)^{k-1} N^k \frac{\Gamma^{k-1}(\gamma)}{\Gamma\left[(k-1)\gamma\right]}(x-x_1)^{k\gamma-1}\int\limits_0^1 \sigma^{\gamma-1}(1-\sigma)^{(k-1)\gamma-1}d\sigma = \left(\frac{3}{2}\right)^{k-1} N^k \frac{\Gamma^{k}(\gamma)}{\Gamma(k\gamma)}(x-x_1)^{k\gamma-1},$$

which proves Lemma 5.

Using the consistently known Schwarz inequality and Lemma 5 from the representation (39) we have

$$\left\|B_1^{-n}f(x,y)\right\|^2_{L_2(\Omega)} = \iint\limits_{\Omega}\left|B_1^{-n}f(x,y)\right|^2 dxdy = \iint\limits_{\Omega}\left[\iint\limits_{\Omega} M_n(x,y,x_1,y_1)f(x_1,y_1)dx_1dy_1\right]^2 dxdy \leqslant$$

$$\leqslant \iint\limits_{\Omega}\left[\left(\iint\limits_{\Omega}\left|M_n(x,y,x_1,y_1)\right|^2 dx_1dy_1\right)\left(\iint\limits_{\Omega}\left|f(x_1,y_1)\right|^2 dx_1dy_1\right)\right]dxdy \leqslant$$

$$\leq \left(\frac{3}{2}N\right)^{2n}\frac{\Gamma^{2n}(\gamma)}{\left[(2n\gamma-1)\right](2n\gamma)\Gamma^2(n\gamma)}\left\|f(x,y)\right\|^2_{L_2(\Omega)}.$$

From here we get

$$\left\|B_1^{-n}\right\|_{L_2(\Omega)\to L_2(\Omega)} \leq \left(\frac{3N}{2}\right)^n\left(4-\frac{2}{n\gamma}\right)^{-\frac{1}{2}}\frac{\Gamma^n(\gamma)}{\Gamma(1+n\gamma)}.$$

From the latter it is not difficult to establish equality (38). *Theorem 3 is proved.*

Corollary 1. Problem M_1A is Volterra nature problem.

Corollary 2. For any complex number λ the equation $B_1u(x,y)-\lambda u(x,y) = f(x,y)$ is unambiguously solvable at all $f(x,y)\in L_2(\Omega)$.

Let now Ω_1 is a domain bounded by segments AB and characteristics $AC: x+y=0$, $BC: x-y=1$ of equation (1) and smooth curve $AD: y=-\gamma(x)$, $0<x<l$, where $0,5<l\leq 1$; $\gamma(0)=0$, $l+\gamma(l)=1$, if $l<1$ and $\gamma(l)=0$, if $l=1$ is located inside the characteristic triangle $0<x+y\leq x-y<1$.

A generalization of the problem in the domain Ω is the following non-local problem for equation (1), where in the hyperbolic part of the mixed domain, the non-local condition pointwise connects the values of the tangent derivative of the desired solution on the characteristic AC with the derivatives in the direction of the characteristic of the desired function on an arbitrary curve AD lying inside the characteristic triangle, with the ends at the origin and on the characteristic BC (at a point B).

Problem M_1B. Find a solution of equation (1) satisfying the conditions (3), (4) and

$$\left[u_x-u_y\right]\left[\theta_0(t)\right]+\mu(t)\left[u_x-u_y\right]\left[\theta^*(t)\right]=0,\ 0<t<1, \tag{41}$$

where $\theta_0(t), (\theta^*(t))$ is an affix of the intersection point of the characteristic AC (curve AD) with the characteristic coming out of the point $(t,0)$, 0<t<1, $\mu(t)$ is a given function.

In the case when $\alpha=1$, the problem M_1B coincides with nonlocal problem for mixed parabolic and hyperbolic equation with non-characteristic line of type change. In this case, regular and strong solvability issues and Volterra property of problem M_1B are investigated in [21–24]. Note that the problem M_1B, when $\mu(x)=0$ coincides with the problem of Tricomi for diffusion and wave equation, and in the case when $\mu(t)=\infty$ coincides with the problem M_1A.

Similarly, as in the case of the problem M_1A, the concept of a regular and strong solution to the problem is introduced. Applying the methodology of proofs of theorems 1–3, the following theorem is proved.

Theorem 4. Let be $\mu(t)\in C^1[0,1]$ and $\mu(x)\neq -1$, $0\leq x\leq 1$. Then :

a) for any function $f(x,y) \in C^1(\bar{\Omega})$, $f(A) = 0$ there is a unique regular solution to the problem M_1B (1), (3), (4), (41) and it is represented in the form (31) and satisfies the inequality (35);

b) for any function $f(x,y) \in L_2(\Omega)$ there exists a unique strong solution $u(x,y)$ to problem M_1B. This solution can be presented in the form (31) and satisfies estimate (36);

c) the problem M_1B is Volterra nature problem.

3 Solvability and existence of eigenvalues of local and nonlocal problems for the diffusion-wave equation

In domain Ω of considered section 2 we investigate the following problem: **Problem** M_2A. Find a solution of equation (1) satisfying the conditions

$$u\,|_{AA_0 \cup A_0B_0} = 0\,, \tag{42}$$

$$u_x + u_y\,|_{AD \cup BD} = 0. \tag{43}$$

Definition 3. The regular solution to the problem M_2A in the domain Ω will be called the function $u(x,y) \in W$, where $W = \left\{(x,y) :\ u(x,y) \in C(\bar{\Omega}) \cap C^{1.1}(\Omega \cup AD \cup BD),\ D^{\alpha}_{0x}u(x,y),\ u_{yy}(x,y) \in C(\Omega_0),\ u(x,y) \in C^{2.2}(\Omega_1)\right\}$, satisfying equation (1) in $\Omega_0 \cup \Omega_1$ and conditions (42)–(43).

Definition 4. The function $u(x,y) \in L_2\,(\Omega)$ is called a strong solution to the problem M_2A, if there exists $\{u_n(x,y)\}$, $u_n(x,y) \in W$, satisfying conditions (42)–(43), such that $\|u_n(x,y) - u(x,y)\|_{L_2(\Omega)} \to 0$, $\|Lu_n(x,y) - f(x,y)\|_0 \to 0$, for $n \to \infty$.

Similarly, as in section 2, the regular solvability of the problem M_2A.

Theorem 5. For any function $f \in L_2\,(\Omega)$ there is a unique strong solution $u(x,y)$ to problem M_2A. This solution can be presented in the form

$$u\,(x,y) = \iint\limits_{\Omega} K\,(x,y;x_1,y_1) f\,(x_1,y_1)\,dx_1dy_1, \tag{44}$$

where $K\,(x,y;x_1,y_1) \in L_2\,(\Omega \times \Omega)$, and satisfies estimate (36).

Similarly as in the problem M_1A, the solution to problem M_2A in domain Ω_1 we seek in the form (15). Based on (43) from (15) we find

$$v\,(\xi) = -\tau'\,(\xi) - 2\int\limits_{\xi}^{\varphi(\xi)} f_1\,(\xi,\eta_1)\,d\eta_1, \quad 0 \le \xi \le 1, \tag{45}$$

where $\eta = \varphi\,(\xi)$, $0 \le \xi \le \xi_0$, $\varphi\,(\xi_0) = 1$ is an equation of the curve AD in characteristic variables ξ,η and $\varphi\,(\xi) \equiv 1$, $\xi_0 \le \xi \le 1$ in the case when $D \ne B$ and $\eta = \varphi\,(\xi)$, $0 \le \xi \le 1$ when $D = B$.

Substituting the resulting expression $v\,(\xi)$ into (15), we obtain

$$u\,(x,y) = \tau\,(\eta) + \int\limits_{\xi}^{\eta} d\xi_1 \int\limits_{\eta}^{\varphi(\xi_1)} f_1\,(\xi_1,\eta_1)\,d\eta_1. \tag{46}$$

The formula (45) gives an integro-differential relation between $\tau\,(x)$ and $\nu\,(x)$, brought to the segment AB from hyperbolic part Ω_1.

Taking into account (22) and (45), it is not difficult to establish that the problem M_2A is equivalent to the following Volterra integral equation of the second kind

$$\tau'(x) - \int_0^x m(x-t)\,\tau'(t)\,dt = \Phi(x), \quad 0 \le x \le 1, \tag{47}$$

where $\Phi(x) = -2 \int_x^{\varphi(x)} f_1(x,\eta_1)\,d\eta_1 - \int_0^x dx_1 \int_0^1 \mathrm{E}_y(x-x_1,0,y_1) f(x_1,y_1) dy_1$.

Since $m(x-t)$ is a kernel with a weak feature, then there is a unique strong solution to equation (47), and it is representable as

$$\tau'(x) = \Phi(x) + \int_0^x \Gamma(x-t)\,\Phi(t)\,dt, \tag{48}$$

where $\Gamma(x)$ is a resolvent of equation (48):

$$\Gamma(x) = \sum_{j=1}^{\infty} m_j(x),\ m_1(x) = m(x),\ m_{j+1}(x) = \int_0^x m_1(x-t)\,m_j(t)\,dt.$$

From (48), taking into account $\tau(0) = 0$, we have

$$\tau(x) = -2\int_0^x d\xi_1 \int_{\xi_1}^{\varphi(\xi_1)} \Gamma_1(x-\xi_1)\,f(\xi_1,\eta_1)\,d\eta_1 - \int_0^x dx_1 \int_0^1 E_1(x-x_1,y_1)\,f(x_1,y_1)\,dy_1, \tag{49}$$

where

$$\Gamma_1(x) = 1 + \int_0^x \Gamma(t)\,dt,\ E_1(x,y_1) = \int_0^x E_y(t,0,y_1)\Gamma_1(x-t)\,dt. \tag{50}$$

Substituting (49) into (20) and (46), we obtain

$$u(x,y) = \int_0^x dx_1 \int_0^1 E_2(x-x_1,y,y_1)\,f(x_1,y_1)dy_1 - 2\int_0^x d\xi_1 \int_{\xi_1}^{\varphi(\xi_1)} E_1(x-\xi_1,y)\,f_1(\xi_1,\eta_1)\,d\eta_1,\ y>0, \tag{51}$$

$$\begin{gathered} u(x,y) = \int_\xi^\eta d\xi_1 \int_\eta^{\varphi(\xi_1)} f_1(\xi_1,\eta_1)\,d\eta_1 - 2\int_0^\eta d\xi_1 \int_{\xi_1}^{\varphi(\xi_1)} \Gamma_1(\eta-\xi_1)\,f_1(\xi_1,\eta_1)\,d\eta_1 - \\ - \int_0^\eta dx_1 \int_0^1 E_1(\eta-x_1,y_1)\,f(x_1,y_1)\,dy_1,\ y<0, \end{gathered} \tag{52}$$

where

$$E_2(x,y,y_1) = E(x,y,y_1) - \int_0^x E_y(x,0,y_1)E_1(x-t,y_1)\,dt.$$

From (51) and (52) we get (44), where the kernel has the form

$$\begin{gathered} K\left(x,y;x_1,y_1\right)=\theta\left(y\right)\left\{\theta\left(y_1\right)\theta\left(x-x_1\right)\ E_2\left(x-x_1,y,y_1\right)-\right.\\ -\theta\left(-y_1\right)\theta\left(x-\xi_1\right)E_1\left(x-\xi_1,y\right)\}+\theta\left(-y\right)\left\{-\theta\left(y_1\right)\theta\left(\eta-x_1\right)E_1\left(\eta-x_1,y_1\right)+\right.\\ \left.+\theta\left(-y_1\right)\left[\frac{1}{2}\theta\left(\xi_1-\xi\right)\theta\left(\eta-\xi_1\right)\theta\left(\eta_1-\eta\right)-\theta\left(\eta-\xi_1\right)\Gamma_1(\eta-\xi_1)\right]\right\}. \end{gathered} \tag{53}$$

From (44), (51), (52) and properties of the solution to the first initial boundary value problem for the diffusion equation [8], as in Theorem 2 it follows all statements of Theorem 5.

By B_2 we denote a closure in $L_2\left(\Omega\right)$ of the operator given on a set of functions from W, satisfying conditions (42),(43), with expression (2).

Theorem 6. Let be $\gamma\left(x\right)\not\equiv0$. Then there exists $\lambda\in C$ such that equation $B_2u(x,y)=\lambda u(x,y)$ has non-trivial solution $u(x,y)\in W$.

Proof. From theorem 5 it is followed, that B_2 is inversible and B_2^{-1} is an operator of Hilbert–Schmidt, defined by the formula (44). Then $B_2^{-2}\equiv\left(B_2^{-1}\right)^2$ kernel operator in $L_2\left(\Omega\right)$.

Therefore, for the operator B_2^{-2} we apply the result of V.B. Lidskii [25] on the coincidence of matrix and spectral traces. It is also known that for the kernel operator, represented as the product of two Hilbert-Schmidt operators, the Gaal formula [26] trace calculation takes place. Using formula of Gaal, we calculate the matrix trace B_2^{-2}.

$$SpB_2^{-2}=\iint\limits_{\Omega}dxdy\iint\limits_{\Omega}K\left(x,y;x_1,y_1\right)K\left(x_1,y_1;x,y\right)dx_1dy_1. \tag{54}$$

Taking into account the representation (53) from (54), after simple transformations, we obtain

$$\begin{gathered} SpB_2^{-2}=\int\limits_0^1dx\int\limits_0^1dy\int\limits_0^xE_1\left(\xi_2,y\right)d\xi_2\theta\left[\varphi\left(x-\xi_2\right)-x\right]\int\limits_0^{\varphi(x-\xi_2)-x}E_1\left(\eta_2,y\right)d\eta_2+\\ +\frac{1}{4}\int\limits_0^1d\xi\int\limits_{\xi}^{\varphi(\xi)}d\eta\int\limits_{\xi}^{\eta}d\xi_1\int\limits_{\eta}^{\varphi(\xi_1)}\theta\left(\eta_1-\xi\right)\Gamma_1\left(\eta_1-\xi\right)\left[\Gamma_1\left(\eta-\xi_1\right)-\theta\left(\eta_1-\eta\right)\right]d\eta_1=A+B. \end{gathered}$$

We will show that $A+B>0$. Indeed, taking into account (50) and $\varphi\left(t\right)\not\equiv t$ will take place $A\geq0$, if

$$E_y\left(t,0,y_1\right)>0. \tag{55}$$

We represent the function $E_y\left(t,0,y_1\right)$ in the form

$$E_y\left(t,0,y_1\right)=\frac{1}{2t}\sum_{n=0}^{+\infty}\left[e_{1,\beta}^{1,0}\left(-\frac{|2n+y_1|}{t}\right)+e_{1,\beta}^{1,0}\left(-\frac{|2(n+1)-y_1|}{t}\right)\right].$$

Due to the properties of the Wright function [8; 46] $e_{1,\beta}^{1,0}(-z)>0,\ z>0$, therefore, from the latter we get the justice of inequality (55). Also, from (50) it easily follows that

$$\Gamma_1\left(\eta-\xi_1\right)-\theta\left(\eta_1-\eta\right)\geq0,$$

therefore $B\geq0$.

Thus, $A+B>0$, as an integral in the positive direction of a non-negative and identically non - zero function. From here we get that $SpB_2^{-2}>0$. Further, applying the results of [25], we have

$$\sum_{k=1}^{\infty} \lambda_k \left(B_2^{-2}\right) = \sum_{k=1}^{\infty} \lambda_k^2 \left(B_2^{-1}\right) > 0,$$

where $\lambda_k \left(B_2^{-2}\right)$ are eigenvalues of operator B_2^{-2}. It means that $\sum\limits_{k=1}^{\infty} \frac{1}{\lambda_k^2} > 0$, where λ_k are eigenvalues of the problem (1), (42) and (43). This implies the existence of the eigenvalues of the problem M_2A for the diffusion-wave equation of fractional order. *Theorem 6 is proved.*

In conclusion, we note that the most interesting is the fact that in problems M_1A and M_2A, in the case when point D coincides with point B, the Volterra property or existence of the problems eigenvalues depend on the derivative directions of the desired function given in the non-characteristic curve of the hyperbolic part of the boundary.

Acknowledgments

This research is supported by Ministry of Education and Science of the Republic of Kazakhstan Grant AP09058677.

References

1 Kilbas A.A. Theory and applications of fractional differential equations / A.A. Kilbas, H.M. Srivastava, J.J. Trujillo. — Elsevier, 2006. — 523 p.

2 Chechkin A.V. Fractional diffusion in inhomogeneous media / A.V. Chechkin, R. Gorenflo, I.M. Sokolov // Journal of Physics A: Mathematical and General. — 2005. — 38. — No. 42. — P. L679. https://dx.doi.org/10.1088/0305-4470/38/42/L03

3 Freed A. Fractional-Order Viscoelasticity (FOV): Constitutive Development Using the Fractional Calculus / A. Freed, K. Diethelm, Y. Luchko // NASA's Glenn Research Center. — 2002. — 137. https://ntrs.nasa.gov/api/citations/20030014634/downloads/20030014634.pdf

4 Gorenflo R. Random walk models for space-fractional diffusion processes / R. Gorenflo, F. Mainardi // Fractional Calculus and Applied Analysis. — 1998. — 1.

5 Hilfer R. Applications of Fractional Calculus in Physics / R. Hilfer — World Scientific, Singapore, 2000. — 472 p.

6 Mainardi F. Seismic pulse propagation with constant Q and stable probability distributions / F. Mainardi, M. Tomirotti // Annali di Geofisica. — 1997. — 40. — No. 5. — P. 1311–1328. https://doi.org/10.48550/arXiv.1008.1341

7 Metzler R. The random walk's guide to anomalous diffusion: a fractional dynamics approach / R. Metzler, J. Klafter // Physics Reports. — 2000. — 339. — No. 1. — P. 1–77. https://doi.org/10.1016/S0370-1573(00)00070-3

8 Псху А.В. Уравнения в частных производных дробного порядка / А.В. Псху. — М.: Наука, 2005. — 199 с.

9 Luchko Y. Maximum principle and its application for the time-fractional diffusion equations / Y. Luchko // Fractional Calculus and Applied Analysis. — 2011. — 14. — No. 1. — P. 110–124. https://doi.org/10.1016/j.jmaa.2008.10.018

10 Alikhanov A.A. A priori estimates for solutions of boundary value problems for fractional-oreder equations / A.A. Alikhanov // Differential Equations. — 2010. — 46. — No. 5. — P. 660–666. https://link.springer.com/article/10.1134/S0012266110050058

11 Berdyshev A.S. The Samarskii-Ionkin type problem for the fourth order parabolic equation with fractional differential operator / A.S. Berdyshev, A. Cabada A, B.J. Kadirkulov // An Inter. J. Computers and Mathematics with Applications. — 2011. — 62. — No. 10. — P. 3884–3893. https://doi.org/10.1016/j.camwa.2011.09.038

12 Aitzhanov S.E. Solvability issues of a pseudo-parabolic fractional order equation with a nonlinear boundary condition / S.E. Aitzhanov, A.S. Berdyshev, K.S. Bekenayeva // Fractal and Fractional. — 2021. — 5. — No. 4. — P. 134. https://doi.org/10.3390/fractalfract5040134

13 Berdyshev A.S. On a non-local boundary problem for a parabolic-hyperbolic equation involving Riemann-Liouville fractional differential operator / A.S. Berdyshev, A. Cabada, E.T. Karimov // Nonlinear analysis: Real World Applications. — 2012. — No. 75. — P. 3268–3273. https://doi.org/10.1016/j.na.2011.12.033

14 Berdyshev A.S. Boundary value problems for fourth-order mixed type equation with fractional derivative / A.S. Berdyshev, B.E. Eshmatov, B.J. Kadirkulov // EJDE. — 2016. — No. 36. — P. 1–11.

15 Karimov E.T. Unique solvability of a non-local problem for mixed-type equation with fractional derivative / E.T. Karimov, A.S. Berdyshev, N.A. Rakhmatullaeva // Mathematical Methods in the Applied Sciences. — 2017. — 40. — No. 8. — P. 2994–2999. http://dx.doi.org/10.1002/mma.4215

16 Agarwal P. Solvability of a non-local problem with integral transmitting condition for mixed type equation with Caputo fractional derivative / P. Agarwal, A.S. Berdyshev, E.T. Karimov // Results in Mathematics. — 2017. — 71. — No. 3. — P. 1235–1257. https://doi.org/10.1007/s00025-016-0620-1

17 Berdyshev A.S. The Riesz basis property of the system of root functions of a nonlocal boundary value problem for a mixed-composite type equation / A.S. Berdyshev // Siberian Mathematical Journal. — 1997. —38. — No. 2. — P. 213–219. https://doi.org/10.1007/BF02674618

18 Karimov E.T. Boundary value problems with integral transmitting conditions and inverse problems for integer and Fractional order differential equations (dissertation). — Information-resource centre of V.I.Romanovkiy Institute of Mathematics, 2020.

19 Berdyshev A.S. Basisness of system of root functions for boundary value problem with a displacement for parabolic-hyperbolic equations / A.S. Berdyshev // Doklady akademii nauk. — 1999. — 366. — No. 1. — P. 7–9.

20 Berdyshev A.S. The basis property of a system of root functions of a nonlocal problem for a third-order equation with a parabolic-hyperbolic operator / A.S. Berdyshev // Differential Equations. — 2000. — 36. — No. 3. — P. 417–422. https://doi.org/10.1007/BF02754462

21 Berdyshev A.S. The volterra property of some problems with the Bitsadze-Samarskii-type conditions for a mixed parabolic-hyperbolic equation / A.S. Berdyshev // Siberian Mathematical Journal. — 2005. — 46. — No. 3. — P. 386–395. https://doi.org/10.1007/s11202-005-0041-y

22 Berdyshev A.S. On the Volterra property of a boundary problem with integral gluing condition for a mixed parabolic-hyperbolic equation / A.S. Berdyshev, A. Cabada, E.T. Karimov, N.S. Akhtaeva // Boundary Value Problems. — 2013. — No. 94. https://doi.org/10.1186/1687-2770-2013-94

23 Berdyshev A.S. On the existence of eigenvalues of a boundary value problem with transmitting condition of the integral form for a parabolic-hyperbolic equation / A.S. Berdyshev, A. Cabada, E.T. Karimov // Mathematics. — 2020. — 8.— No. 6. — P. 1030. https://doi.org/10.3390/math8061030

24 Бердышев А.С. Краевые задачи и их спектральные свойства для уравнения смешанного параболо-гиперболического и смешанно-составного типов / А.С. Бердышев. — Алматы: Казах. нац. пед. ун-т им. Абая; Ин-т информ. и вычисл. техн., 2015. — 224 с.

25 Лидский В.Б. Несамосопряжённые операторы, имеющие след / В.Б. Лидский // Доклады АН СССР. — 1959. — 125. — № 3. — С. 485–488.

26 Brislawn C. Kernels of trace class operators / C. Brislawn // Proc. Amer. Math. Soc. — 1988. — 104. — No. 4. — P. 1181–1190.

Н. Әділ, А.С. Бердышев

Абай атындағы Қазақ ұлттық педагогикалық университеті, Алматы, Қазақстан;
Ақпараттық және есептеуіш технологиялар институты, Алматы, Қазақстан

Бөлшек ретті диффузиялық-толқындық теңдеу үшін локальді және локальді емес есептердің спектрлік қасиеттері

Мақалада бөлшек ретті диффузиялық-толқындық теңдеу үшін локальді және локальді емес есептердің шешімділік мәселелері мен спектрлік қасиеттері зерттелген. Сипаттауыш және сипаттауыш емес шекаралары бар облыстарда қойылған есептердің регуляр және күшті шешімділігі дәлелденді. Есептердің бірегей шешімділігі дәлелденіп, меншікті мәндердің бар екендігі немесе Вольтерра типіндегі есеп екендігі туралы теоремалар дәлелденген.

Кілт сөздер: диффузиялық-толқындық теңдеу, бөлшек ретті теңдеулер, шекаралық есептер, күшті шешім, Вольтерра қасиеті, меншікті мән.

Н. Адил, А.С. Бердышев

Казахский национальный педагогический университет имени Абая, Алматы, Казахстан;
Институт информационных и вычислительных технологий, Алматы, Казахстан

Спектральные свойства локальных и нелокальных задач для диффузионно-волнового уравнения дробного порядка

В статье исследованы вопросы разрешимости и спектральные свойства локальных и нелокальных задач для диффузионно-волнового уравнения дробного порядка. Доказаны регулярная и сильная разрешимости поставленных задач в областях, как с характеристической, так и с нехарактеристической границей области. Установлена однозначная разрешимость задач, и доказаны теоремы о существовании собственных значений либо вольтерровости рассматриваемых задач.

Ключевые слова: диффузионно-волновое уравнение, уравнения дробного порядка, краевые задачи, сильное решение, вольтерровость, собственное значение.

References

1 Kilbas, A.A., Srivastava, H.M., & Trujillo, J.J. (2006). *Theory and applications of fractional differential equations.* Elsevier.

2 Chechkin, A.V., Gorenflo, A.V., & Sokolov, I.M. (2005). Fractional diffusion in inhomogeneous media. *Journal of Physics A: Mathematical and General, 42*(38), L679. https://dx.doi.org/10.1088/0305-4470/38/42/L03

3 Freed, A., Diethelm, K., & Luchko, Y. (2002). *Fractional-Order Viscoelasticity (FOV): Constitutive Development Using the Fractional Calculus.* NASA's Glenn Research Center.

4 Gorenflo, R., & Mainardi, F. (1998). Random walk models for space-fractional diffusion processes. *Fractional Calculus and Applied Analysis,* (38), 1.

5 Hilfer, R. (2000). *Applications of Fractional Calculus in Physics.* World Scientific, Singapore.

6 Mainardi, F., & Tomirotti, M. (1997). Seismic pulse propagation with constant Q and stable probability distributions. *Annali di Geofisica, 40*(5), 1311–1328. https://doi.org/10.48550/arXiv.1008.1341

7 Metzler, R., & Klafter, J. (2000). The random walk's guide to anomalous diffusion: a fractional dynamics approach. *Physics Reports, 339*(1), 1–77. https://doi.org/10.1016/S0370-1573(00)00070-3

8 Pskhu, A.V. (2005). *Uravneniia v chastnykh proizvodnykh drobnogo poriadka [Fractional Partial Differential Equations]*. Moscow: Nauka [in Russian].

9 Luchko, Y. (2011). Maximum principle and its application for the time-fractional diffusion equations. *Fractional Calculus and Applied Analysis, 14*(1), 110–124. https://doi.org/10.1016/j.jmaa.2008.10.018

10 Alikhanov, A.A. (2010). A priori estimates for solutions of boundary value problems for fractional-oreder equations. *Differential Equations, 46*(5), 660–666. https://link.springer.com/article/10.1134/S0012266110050058

11 Berdyshev, A.S., Cabada, A., & Kadirkulov, B.J. (2011). The Samarskii-Ionkin type problem for the fourth order parabolic equation with fractional differential operator *An Inter. J. Computers and Mathematics with Applications, 62*(10), 3884–3893. https://doi.org/10.1016/j.camwa.2011.09.038

12 Aitzhanov, S.E., Berdyshev, A.S., & Bekenayeva, K.S. (2021). Solvability issues of a pseudo-parabolic fractional order equation with a nonlinear boundary condition. *Fractal and Fractional, 5*(4), 134. https://doi.org/10.3390/fractalfract5040134

13 Berdyshev, A.S., Cabada, A., & Karimov, E.T. (2012). On a non-local boundary problem for a parabolic-hyperbolic equation involving Riemann-Liouville fractional differential operator. *Nonlinear analysis: Real World Applications* (75), 3268–3273. https://doi.org/10.1016/j.na.2011.12.033

14 Berdyshev, A.S., Eshmatov, B.E., & Kadirkulov, B.J. (2016). Boundary value problems for fourth-order mixed type equation with fractional derivative. *EJDE,* (36), 1–11.

15 Karimov, E.T., Berdyshev, A.S., & Rakhmatullaeva, N.A. (2017). Unique solvability of a non-local problem for mixed-type equation with fractional derivative. *Mathematical Methods in the Applied Sciences, 40*(8), 2994–2999. http://dx.doi.org/10.1002/mma.4215

16 Agarwal, P., Berdyshev, A.S., & Karimov, E.T. (2017). Solvability of a non-local problem with integral transmitting condition for mixed type equation with Caputo fractional derivative. *Results in Mathematics, 71*(3), 1235–1257. https://doi.org/10.1007/s00025-016-0620-1

17 Berdyshev, A.S. (1997). The Riesz basis property of the system of root functions of a nonlocal boundary value problem for a mixed-composite type equation. *Siberian Mathematical Journal, 38*(2), 213–219. https://doi.org/10.1007/BF02674618

18 Karimov, E.T. (2020). *Boundary value problems with integral transmitting conditions and inverse problems for integer and Fractional order differential equations* (dissertation). Information-resource centre of V.I. Romanovkiy Institute of Mathematics.

19 Berdyshev, A.S. (1999). Basisness of system of root functions for boundary value problem with a displacement for parabolic-hyperbolic equations. *Doklady Akademii Nauk, 366*, (1), 7–9.

20 Berdyshev, A.S. (2000). The basis property of a system of root functions of a nonlocal problem for a third-order equation with a parabolic-hyperbolic operator. *Differential Equations, 36*(3), 417–422. https://doi.org/10.1007/BF02754462

21 Berdyshev, A.S. (2005). The volterra property of some problems with the Bitsadze-Samarskii-type conditions for a mixed parabolic-hyperbolic equation. *Siberian Mathematical Journal, 46*, (3), 386–395. https://doi.org/10.1007/s11202-005-0041-y

22 Berdyshev, A.S., Cabada, A., Karimov, E.T., & Akhtaeva, N.S. (2013). On the Volterra property of a boundary problem with integral gluing condition for a mixed parabolic-hyperbolic equation. *Boundary Value Problems* (94). https://doi.org/10.1186/1687-2770-2013-94

23 Berdyshev, A.S., Cabada, A., & Karimov, E.T. (2020). On the existence of eigenvalues of a boundary value problem with transmitting condition of the integral form for a parabolic-hyperbolic equation. *Mathematics, 8*(6), 1030. https://doi.org/10.3390/math8061030

24 Berdyshev, A.S. (2015). *Kraevye zadachi i ikh spektralnye svoistva dlia uravneniia smeshannogo parabolo-giperbolicheskogo i smeshanno-sostavnogo tipov [Boundary Value Problems and Their Spectral Properties for an Equation of Mixed Parabolic-Hyperbolic and Mixed-Composite Types].* Almaty: Kazakhskii nationalnyi pedagogicheskii universitet; Institut informatsionnykh i kompiuternykh technologii [in Russian].

25 Lidskii, V.B. (1959). Nesamosopriazhennye operatory, imeiushchie sled [Non-self-adjoint operators with trace]. *Doklady Akademii nauk SSSR — Reports of the Academy of Sciences of the USSR, 125*(3), 485–488 [in Russian].

26 Brislawn, C. (1988). Kernels of trace class operators. *Proc. Amer. Math. Soc., 104*(4), 1181–1190.