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# An optimal control problem for the systems with integral boundary conditions

In this paper, we consider an optimal control problem with a «pure», integral boundary condition. The Green's function is constructed. Using contracting Banach mappings, a sufficient condition for the existence and uniqueness of a solution to one class of integral boundary value problems for fixed admissible controls is established. Using the functional increment method, the Pontryagin's maximum principle is proved. The first and second variations of the functional are calculated. Further, various necessary conditions for optimality of the second order are obtained by using variations of controls.

*Key words:* integral boundary conditions, singular control, optimal control problem, existence and uniqueness of the solution.

## *Introduction*

Boundary value problems with integral conditions last few decades became one of the intensively studied classes of the problems of mathematical physics. These problems included different problems with two-, three-, multiple and non-local boundary value problems [1–3]. One of the reasons that make these problems so actual is that they have a strong relation with various fields of applications (see, for example [4, 5] and references therein).

There exist many works devoted to investigation of the systems with local conditions and finding necessary optimality conditions of first and second orders [6–10]. For such problems with integral conditions we refer to [11–15].

Various type optimal control problems for the systems with boundary conditions are considered in [16–22] and with integral boundary condition in [16, 17], where the first order necessary conditions are obtained. In some cases, when the first order optimality conditions are "degenerated", i.e. are fulfilled trivially one has to try to obtain second order conditions.

Another direction in investigation of the optimal control problems with multipoint and integral boundary conditions is developing the numerical methods. For the first-order ordinary differential equations such problems are studied in [23, 24].

In this paper, optimal control problem is investigated, when the state of the system is described by differential equations with integral boundary conditions. The existence and uniqueness of solutions to the boundary value problem is investigated. The first and second variations of the corresponding functional are calculated. Optimality conditions of first and second order are obtained applying the method of variations of the controls.

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*Problem Statement*

Consider the following system of differential equations with an integral boundary condition

$$\frac{dx}{dt} = f(t, x, u(t)), 0 \leq t \leq T, \quad (1)$$

$$\int_0^T m(t)\, x(t)\, dt = C, \quad (2)$$

$$u(t) \in U, t \in [0, T], \quad (3)$$

where $x(t) \in R^n$; $f(t, x, u)$ is $n$-dimensional continuous function; $C \in R^n$ is a given constant vector and $m(t) \in R^{n\times n}$ is $n \times n$ matrix function; $u$ is a control parameter; $U \in R^r$ is bounded set.

The problem is: to minimize the functional

$$J(u) = \varphi(x(0), x(T)) + \int_0^T F(t, x, u)\, dt \quad (4)$$

on the solutions of problem (1)–(3).

The following assumption is accepted: the scalar functions $\varphi(x, y)$ and $F(t, x, u)$ are continuous with respect to their own arguments and have continuous and bounded first order partial derivatives with respect to $x, y$. As a solution of problem (1)-(3) corresponding to the fixed control $u(t)$ we consider absolutely continuous on $[0, T]$ function $x(t) : [0, T] \to R^n$. The space of such functions is denoted as $AC([0, T], R^n)$. $C([0, T], R^n)$ stands for the space of continuous functions on $[0, T]$ which gets values from $R^n$. It is obvious that this is a Banach with the norm $\|x\|_{C[0,T]} = \max\limits_{[0,T]} |x(t)|$, where $|\cdot|$ is the norm in space $R^n$.

As admissible controls we consider the functions from the class of bounded measurable functions with the values from the set $U \subset R^r$. We call the pair consisting of admissible control and the corresponding solution of (1), (2) an admissible process.

Thus the admissible process $\{u(t), x(t, u)\}$ that is a solution to (1)-(4), subject to (1)-(3), is said to be an optimal process, and $u(t)$ – an optimal control.

The existence of an optimal control in problem (1)–(4) is also assumed.

*Existence of solutions of boundary value problem (1)–(3)*

Let's set the following conditions.

H1) Let $\det B \neq 0$, where $B = \int_0^T m(t)\, dt$.

H2) $f : [0, T] \times R^n \times R^r \to R^n$ is a continuous function and there exists the constant $K \geq 0$

$$|f(t, x, u) - f(t, y, u)| \leq K |x - y|, \quad t \in [0, T], \quad x, y \in R^n, u \in U.$$

H3) $L = KTM < 1$,

where $M = \max\limits_{0\leq t,s\leq T} \|M(t, s)\|$, $M(t, s) = \begin{cases} B^{-1} \int_0^s m(\tau)\, d\tau, 0 \leq t \leq s \\ -B^{-1} \int_s^T m(\tau)\, d\tau, s \leq t \leq T. \end{cases}$

*Theorem 1.* Under the condition H1) the function $x(\cdot) \in AC([0, T], R^n)$ is an absolutely continuous solution to problem (1)-(3) iff

$$x(t) = B^{-1}C + \int_0^T M(t, \tau) f(\tau, x(\tau), u(\tau))d\tau. \quad (5)$$

Here $M(t, s) = \begin{cases} B^{-1} \int_0^s m(\tau)\, d\tau, & 0 \leq t \leq s, \\ -B^{-1} \int_s^T m(\tau)\, d\tau, & s < t \leq T. \end{cases}$

*Proof.* It is obvious that if $x = x(\cdot)$ is a solution to (1), then

$$x(t) = x(0) + \int_0^t f(s, x(s), u(s))ds, \tag{6}$$

for $t \in (0, T)$, where $x(0)$ is an arbitrary constant. To determine $x(0)$ we suppose that the function given by (6) satisfy (2), i.e.

$$Bx(0) = C - \int_0^T m(t) \int_0^t f(\tau, x(\tau), u(\tau))d\tau dt.$$

Since $\det B \neq 0$ we have

$$x(0) = B^{-1}C - B^{-1} \int_0^T m(t) \int_0^t f(\tau, x(\tau), u(\tau))d\tau dt. \tag{7}$$

Considering in (6) the value of $x(0)$ determined by equality (7) we get

$$x(t) = B^{-1}C + \int_0^T M(t, \tau) f(\tau, x(\tau), u(\tau))d\tau.$$

By this way we reduced boundary value problem (1)–(3) to the integral equation (5). It is easy to check that the solution of integral equation (5) also satisfies (1)–(3). Theorem 1 is proved.

Introduce the operator $P : C([0, T], R^n) \to C([0, T], R^n)$ as

$$(Px)(t) = B^{-1}C + \int_0^T M(t, \tau) f(\tau, x(\tau), u(\tau))d\tau. \tag{8}$$

*Theorem 2.* Within the conditions H1)-H3) for any $C \in R^n$ and for each fixed admissible control, problem (1)–(3) has a unique solution that satisfies the following relation

$$x(t) = B^{-1}C + \int_0^T M(t, \tau) f(\tau, x(\tau), u(\tau))d\tau. \tag{9}$$

*Proof.* Let $C \in R^n$ and $u(\cdot) \in U$ be fixed. Consider the mapping $P : C([0, T], R^n) \to C([0, T], R^n)$ defined by (8). It is obvious that the fixed points of the operator $(Px)(t)$ are the solutions of (1)–(2). To prove that the mapping $P$ has a fixed point we apply the Banach contraction principle. For any $v, w \in C([0, T], R^n)$ we have

$$|(Pv)(t) - (Pw)(t)| \leq \int_0^T |M(t, s)| \cdot |f(s, v(s), u(s)) - f(s, w(s), u(s))| \, ds \leq$$

$$\leq KTN \left\| v(\cdot) - w(\cdot) \right\|_{C[0,T]}, \quad t \in [0, T],$$

or

$$\left\| Pv - Pw \right\|_{C[0,T]} \leq L \left\| v - w \right\|_{C[0,T]}.$$

The last relation shows that $P$ is the contraction in the space $C([0, T], R^n)$. Thus, based on the principle of contraction operators one can state that $P$ has a unique fixed point at $C([0, T], R^n)$. It means that integral equation (9) or boundary value problem (1)–(3) has a unique solution.

Theorem 2 is proved.

*Derivation of Pontryagin's maximum principle*

Here we assume that $U$ is closed set in $R^r$. To obtain the necessary conditions for optimality one should analyze the variation of the objective functional caused by some control impulse [7] i.e. one must calculate the increment formula that obtained from Taylor's series expansion. It is important to give a definition of the conjugate system that allows one to determine the dominant term that leads to the necessary condition for optimality. For the sake of simplicity, it is expedient to construct a linearized model of system (8), (9) in some small neighborhood.

Let $\{u, x = x(t,u)\}$ and $\{\tilde{u} = u + \Delta u,\ \tilde{x} = x + \Delta x = x(t,\tilde{u})\}$ be two admissible processes. Introduce the boundary value problem for problem (1)–(3):

$$\Delta \dot{x} = \Delta f(t,x,u), \ \ t \in [0,T],$$

$$\int_0^T m(t)\,\Delta x(t)\,dt = 0,$$

where $\Delta f(t,x,u) = f(t,\tilde{x},\tilde{u}) - f(t,x,u)$ stands for the total increment of the function $f(t,x,u)$. Then we can represent the increment of the functional as

$$\Delta J(u) = J(\tilde{u}) - J(u) = \Delta\varphi(x(0), x(T)) + \int_0^T \Delta F(x,u,t)\,dt.$$

Consider the non-trivial vector-function $\psi(t),\ \psi(t) \in R^n$, and numerical vector $\lambda \in R^n$. Then the increment of the functional (4) can be written as

$$\Delta J(u) = J(\tilde{u}) - J(u) = \Delta\varphi(x(0), x(T)) + \int_0^T \Delta F(x,u,t)\,dt +$$

$$+ \int_0^T \langle \psi(t), \Delta\dot{x}(t) - \Delta f(t,x,u)\rangle\,dt + \left\langle \lambda, \int_0^T m(t)\,\Delta x(t)\,dt \right\rangle.$$

Making standard operations for the increment of the functional we obtain the formula

$$\Delta J(u) = -\int_0^T \Delta_{\tilde{u}} H(t,\psi,x,u)\,dt - \int_0^T \left\langle \Delta_{\tilde{u}} \frac{\partial H(t,\psi,x,u)}{\partial x}, \Delta x(t) \right\rangle dt +$$
$$+ \int_0^T \left\langle \frac{\partial H(t,\psi,x,u)}{\partial x} + m'(t)\lambda + \dot{\psi}(t), \Delta x(t) \right\rangle dt + \left\langle \left[\frac{\partial\varphi}{\partial x(0)} - \psi(0)\right], \Delta x(0) \right\rangle +$$
$$+ \left\langle \left[\frac{\partial\varphi}{\partial x(T)} + \psi(T)\right], \Delta x(T) \right\rangle + \eta_u, \qquad (10)$$

where

$$H(t,\psi,x,u) = \langle \psi, f(t,x,u)\rangle - F(t,x,u),$$

$$\eta_{\tilde{u}} = -\int_0^T o_H(\|\Delta x(t)\|)\,dt + o_\varphi(\|\Delta x(t_0)\|, \|\Delta x(t_1)\|).$$

Let the vector function $\psi(t) \in R^n$ and vector $\lambda \in R^n$ be a solution of the following conjugate problem

$$\begin{cases} \dot{\psi}(t) = -\frac{\partial H(t,\psi,x,u)}{\partial x} - m'(t)\lambda, \quad t \in [0,T], \\ \frac{\partial\varphi}{\partial x(0)} - \psi(0) = 0, \ \frac{\partial\varphi}{\partial x(T)} + \psi(T) = 0. \end{cases} \qquad (11)$$

Then, formula (10) takes the form

$$\Delta J(u) = -\int_0^T \Delta_{\tilde{u}} H(t,\psi,x,u)\,dt - \int_0^T \left\langle \Delta_{\tilde{u}} \frac{\partial H(t,\psi,x,u)}{\partial x}, \Delta x(t) \right\rangle dt + \eta_{\tilde{u}}. \tag{12}$$

Taking as parameters the point $\tau \in (0,T]$, number $\varepsilon \in (0,\tau]$, vector $v \in U$ and variation interval $(\tau-\varepsilon,\tau)$ from $[0,T]$ we consider needle-shaped variation of the admissible control. Then needle-shaped variation of the control $u = u(t)$ may be given by the relation

$$\tilde{u} = u_\varepsilon(t) = \begin{cases} v \in U, \ t \in (\tau-\varepsilon,\tau] \subset [0,T], \ \varepsilon > 0, \\ u(t), \quad t \notin (\tau-\varepsilon,\tau]. \end{cases} \tag{13}$$

To obtain the necessary optimality condition from the increment formula (12) one have to show that on the needle-shaped variation $\tilde{u}(t) = u_\varepsilon(t)$ the state increment $\Delta_\varepsilon x(t)$ has the order $\varepsilon$.
Since,

$$\Delta x(t) = \int_0^T M(t,s)\left[f(s,x(s)+\Delta x(s),\tilde{u}(s)) - f(s,x(s),\tilde{u}(s))\right] ds +$$

$$+ \int_0^T M(t,s)\,\Delta_{\overleftrightarrow{u}} f(s,x(s),u(s))\,ds.$$

The last implies that

$$\|\Delta x(t)\| \le (1-L)^{-1} \int_0^T \|\Delta_{\tilde{u}} f(t,x(t),u(t))\|\,dt,$$

which proves the hypothesis on response of the state increment caused by the needle-shaped variation given by (13)

$$\|\Delta_\varepsilon x(t)\| \le \tilde{L}\varepsilon, \ \ t \in [0,T], \ \tilde{L} = const > 0.$$

This also implies that for $\tilde{u}(t) = u_\varepsilon(t)$ the relation

$$\int_{\tau-\varepsilon}^{\tau} \left\langle \Delta_v \frac{\partial H(t,\psi,x,u)}{\partial x}, \Delta_\varepsilon x(t) \right\rangle dt + \eta_{u_\varepsilon}(\|\Delta_\varepsilon x(t)\|) = o(\varepsilon)$$

holds true, where

$$\Delta_\varepsilon x(t) = x(t,u_\varepsilon) - x(t,u) \sim \varepsilon.$$

It means that according to (12) the variation of the functional caused by the needle-shaped variation (13) can be written

$$\Delta_\varepsilon J(u) = J(u_\varepsilon) - J(u) = -\Delta_v H(s,\psi,x,u).\varepsilon + o(\varepsilon), \ v \in U, \ s \in [0,T]. \tag{14}$$

Note that in the last expression, the mean value theorem was used.
Formula (14) with respect to the estimate for $\|\Delta_\varepsilon x\|$ implies the necessary optimality condition in the form of the maximum principle for the needle-shaped variation of optimal process $\{u^0, x^0 = x(t,u^0)\}$.

*Theorem 3.* (Pontryagin‘s maximum principle). Assume that the admissible process $\{u^0, x^0 = x(t,u^0)$ is optimal for problem (1)–(4) and $\psi^0(t)$ is a solution to problem (11) calculated on the optimal process. Then, inequality

$$\Delta_v H(s,\psi^0,x^0,u^0) \le 0, \ \ for\, every\, v \in U, \tag{15}$$

is valid for all $s \in [0,T]$ .

*Remark.* If the function $f$ is linear with respect to $(x,u)$ and the functions, $F,\varphi$ are convex with respect to $x(0)$, $x(T)$, and $x(t)$, respectively, then maximum principle (15) is both necessary and sufficient optimality condition. This fact can be easily obtained from the formula

$$\Delta J(u) = -\int_0^T \Delta_{\tilde u} H(t,\psi,x,u)\,dt + o_\varphi\left(\|\Delta x(0)\|,\|\Delta x(T)\|\right) + \int_0^T o_F\left(\|x(t)\|\right)dt,$$

where $o_\varphi \geq 0,\ o_F \geq 0$.

*The second order formula for the increment of the functional and variation of the functional*

Let us suppose that the scalar functions $\varphi(x,y)$ and $F(t,x,u)$ are continuous over their own arguments and have continuous and bounded partial derivatives with respect to $x,y$ and $u$ up to second order, inclusively. Let $U$ be an open set in $R^r$ and $\{u, x = x(t,u)\}$, $\{\tilde u = u + \Delta u,\ \tilde x = x + \Delta x = x(t,\tilde u)\}$ be two admissible processes.

Under the above assumptions increment formula (12) turns to

$$\begin{gathered}\Delta J(u) = -\int_0^T \left\langle \frac{\partial H(t,\psi,x,u)}{\partial u}, \Delta u(t)\right\rangle dt - \frac{1}{2}\int_0^T \left\langle \Delta u(t)' \frac{\partial^2 H(t,\psi,x,u)}{\partial u^2}, \Delta u(t)\right\rangle dt - \\ -\int_0^T \left\langle \Delta u(t)' \frac{\partial H^2(t,\psi,x,u)}{\partial x \partial u} + \frac{1}{2}\Delta x'(t)\frac{\partial^2 H(t,\psi,x,u)}{\partial x^2}, \Delta x(t)\right\rangle dt + \\ +\frac{1}{2}\left\langle \Delta x(0)' \frac{\partial^2 \varphi}{\partial x(0)^2} + \Delta x(T)' \frac{\partial^2\varphi}{\partial x(0)\partial x(T)}, \Delta x(0)\right\rangle + \\ +\frac{1}{2}\left\langle \Delta x(0)' \frac{\partial^2\varphi}{\partial x(T)\partial x(0)} + \Delta x(T)' \frac{\partial^2\varphi}{\partial x(T)^2}, \Delta x(T)\right\rangle + \eta_{\tilde u}.\end{gathered} \tag{16}$$

Take $\Delta u(t) = \varepsilon\delta u(t)$, where $\varepsilon > 0$ is small enough number, $\delta u(t)$ is some piecewise continuous function. Then the expression $\Delta J(u) = J(\tilde u) - J(u)$ for the fixed functions $u(t)$, $\Delta u(t)$ will be a function of the parameter $\varepsilon$. If the representation

$$\Delta J(u) = \varepsilon\delta J(u) + \frac{1}{2}\varepsilon^2\delta^2 J(u) + o\left(\varepsilon^2\right) \tag{17}$$

holds true, then $\delta J(u)$ is called the first, $\delta^2 J(u)$ the second variation of the functional. To get an obvious expression for the first and second variations we have to select in $\Delta x(t)$ the principal term with respect to $\varepsilon$.

Let

$$\Delta x(t) = \varepsilon\delta x(t) + o(\varepsilon,t), \tag{18}$$

where $\delta x(t)$ is the variation of the trajectory. Obviously, such a representation exists and for the function $\delta x(t)$ one can obtain an equation in variations. Using the definition of $\Delta x(t)$ we get

$$\Delta x(t) = \int_0^T M(t,\tau)\Delta f(\tau, x(\tau), u(\tau))d\tau.$$

Using the Taylor formula, we get:

$$\varepsilon\delta x(t) + o(\varepsilon,t) = \int_0^T M(t,\tau)\left\{\frac{\partial f(\tau,x,u)}{\partial x}\left[\varepsilon\delta x(\tau) + o(\varepsilon,\tau)\right] + \varepsilon\frac{\partial f(\tau,x,u)}{\partial u}\delta u + o_1(\varepsilon,\tau)\right\}d\tau.$$

If to consider that the last formula is true for any $\varepsilon$ we have

$$\delta x(t) = \int_0^T M(t,\tau)\left\{\frac{\partial f(\tau,x,u)}{\partial x}\delta x(\tau) + \frac{\partial f(\tau,x,u)}{\partial u}\delta u(t)\right\}d\tau. \tag{19}$$

Equation (19) is called the equation in variations. Obviously, (19) is equivalent to the following nonlocal boundary value problem

$$\delta\dot{x}\left(t\right)=\frac{\partial f\left(t,x,u\right)}{\partial x}\delta x\left(t\right)+\frac{\partial f\left(t,x,u\right)}{\partial u}\delta u\left(t\right), \tag{20}$$

$$\int_0^T m\left(t\right)\delta x\left(t\right)dt=0. \tag{21}$$

Any solution of (20) may be written in the form

$$\delta x\left(t\right)=\Phi\left(t\right)\delta x\left(0\right)+\Phi\left(t\right)\int_0^t\Phi^{-1}\left(\tau\right)\frac{\partial f\left(\tau,x,u\right)}{\partial u}\delta u\left(\tau\right)d\tau, \tag{22}$$

where $\Phi\left(t\right)$ is a solution of the equation

$$\frac{d\Phi\left(t\right)}{dt}=\frac{\partial f\left(t,x,u\right)}{\partial x}\Phi\left(t\right),$$

$$\Phi\left(0\right)=E.$$

Let the solution of (20) determined by equality (22) satisfiy (21). Then for the solutions of problem (20), (21) we obtain the explicit formula

$$\delta x\left(t\right)=\int_0^T G\left(t,\tau\right)\frac{\partial f\left(\tau,x,u\right)}{\partial u}\delta\left(\tau\right)d\tau, \tag{23}$$

where

$$G\left(t,s\right)=\begin{cases}\Phi\left(t\right)B_1{}^{-1}\int_0^s m\left(\tau\right)\Phi\left(\tau\right)d\tau\Phi^{-1}\left(\tau\right), & 0\le s\le t\\ -\Phi\left(t\right)B_1{}^{-1}\int_s^T m\left(\tau\right)\Phi\left(\tau\right)d\tau\Phi^{-1}\left(\tau\right), & t\le s\le T\end{cases},$$

$$B_1=\int_0^T m\left(t\right)\Phi\left(t\right)dt.$$

Considering (18) in (16), we obtain

$$\begin{aligned}\Delta J(u)=&-\varepsilon\int_0^T\left\langle\frac{\partial H(t,\psi,x,u)}{\partial u},\delta u(t)\right\rangle dt-\frac{\varepsilon^2}{2}\left\{\int_0^T\left[\left\langle\delta x'(t)\frac{\partial^2H(t,\psi,x,u)}{\partial x^2},\delta x(t)\right\rangle+\right.\right.\\&\left.+2\left\langle\delta u'(t)\frac{\partial^2H(t,\psi,x,u)}{\partial x\partial u},\delta x(t)\right\rangle+\left\langle\delta u'(t)\frac{\partial^2H(t,\psi,x,u)}{\partial u^2},\delta u(t)\right\rangle\right]dt-\\&-\left\langle\delta x'(0)\frac{\partial^2\varphi}{\partial x(0)^2}+\Delta x'(T)\frac{\partial^2\varphi}{\partial x(0)\partial x(T)},\delta x(0)\right\rangle-\\&\left.-\left\langle\delta x'(0)\frac{\partial^2\varphi}{\partial x(T)\partial x(0)}+\delta x'(T)\frac{\partial^2\varphi}{\partial x(T)^2},\delta x(T)\right\rangle\right\}+o(\varepsilon^2).\end{aligned}$$

Using (17) from the last we get

$$\delta J(u)=-\int_0^T\left\langle\frac{\partial H(t,\psi,x,u)}{\partial u},\delta u(t)\right\rangle dt$$

$$\delta^2 J(u) = -\int_0^T \left[\left\langle \delta x'(t) \frac{\partial^2 H(t,\psi,x,u)}{\partial x^2}, \delta x(t) \right\rangle + \right.$$

$$\left. +2\left\langle \delta u'(t) \frac{\partial^2 H(t,\psi,x,u)}{\partial x \partial u}, \delta x(t) \right\rangle + \left\langle \delta u'(t) \frac{\partial^2 H(t,\psi,x,u)}{\partial u^2}, \delta u(t) \right\rangle \right] dt +$$

$$+\left\langle \delta x'(0) \frac{\partial^2 \varphi}{\partial x(0)^2} + \Delta x'(T) \frac{\partial^2 \varphi}{\partial x(0) \partial x(T)}, \delta x(0) \right\rangle +$$

$$+\left\langle \delta x'(0) \frac{\partial^2 \varphi}{\partial x(T) \partial x(0)} + \delta x'(T) \frac{\partial^2 \varphi}{\partial x(T)^2}, \delta x(T) \right\rangle .$$

*Derivation of Legendre-Klebsh conditions*

It follows from (17) that the conditions

$$\delta J(u^0) = 0, \;\; \delta^2 J(u^0) \geq 0 \tag{24}$$

are fulfilled on the optimal control $u^0(t)$.

From (24) it follows that

$$\int_0^T \left\langle \frac{\partial H(t,\psi^0,x^0,u^0)}{\partial u}, \delta u(t) \right\rangle dt = 0.$$

Hence the validity of the equality

$$\frac{\partial H(t,\psi^0,x^0,u^0)}{\partial u} = 0, \quad t \in [0,T] \tag{25}$$

can be proved along the optimal control that indeed is the Euler equation. From (24) we obtain the validity of the following inequality along the optimal control

$$\begin{aligned} \delta^2 J(u) = &-\int_0^T \left[\left\langle \delta x'(t) \frac{\partial^2 H(t,\psi,x,u)}{\partial x^2}, \delta x(t) \right\rangle + \right. \\ &\left. +2\left\langle \delta u'(t) \frac{\partial^2 H(t,\psi,x,u)}{\partial x \partial u}, \delta x(t) \right\rangle + \left\langle \delta u'(t) \frac{\partial^2 H(t,\psi,x,u)}{\partial u^2}, \delta u(t) \right\rangle \right] dt + \\ &+\left\langle \delta x'(0) \frac{\partial^2 \varphi}{\partial x(0)^2} + \Delta x'(T) \frac{\partial^2 \varphi}{\partial x(0) \partial x(T)}, \delta x(0) \right\rangle + \\ &+\left\langle \delta x'(0) \frac{\partial^2 \varphi}{\partial x(T) \partial x(0)} + \delta x'(T) \frac{\partial^2 \varphi}{\partial x(T)^2}, \delta x(T) \right\rangle \geq 0. \end{aligned} \tag{26}$$

Inequality (26) is an implicit necessary optimality condition of first order. Since the verification of the last conditions require heavy calculations their application meets difficulties.

To obtain more effective optimality conditions of the second order, we use (23) in (26) and introduce the matrix function

$$R(\tau,s) = -G'(0,\tau) \frac{\partial^2 \varphi}{\partial x(0)^2} G(0,s) - G'(T,\tau) \frac{\partial^2 \varphi}{\partial x(T) \partial x(0)} G(0,s) -$$

$$-G'(0,\tau) \frac{\partial^2 \varphi}{\partial x(0) \partial x(T)} G(T,s) - G'(T,\tau) \frac{\partial^2 \varphi}{\partial x(T)^2} G(T,s) + \int_0^T G'(t,\tau) \frac{\partial^2 H}{\partial x^2} G(t,s)\, dt.$$

It allows us to obtain the following terminal formula for the second variation of the functional

$$\delta^2 J(u) = -\left\{ \int_0^T \int_0^T \left\langle \delta' u(\tau) \frac{\partial' f(\tau,x,u)}{\partial u} R(\tau,s) \frac{\partial f(s,x,u)}{\partial u}, \delta u(s) \right\rangle dt ds \right.$$

$$+ \int_0^T \left\langle \delta' u(t) \frac{\partial^2 H(t,\psi,x,u)}{\partial u^2}, \delta u(t) \right\rangle dt$$

$$\left. + 2\int_0^T \int_0^T \left\langle \delta u'(t) \frac{\partial^2 H(t,\psi,x,u)}{\partial x \partial u} G(t,s) \frac{\partial f(s,x,u)}{\partial u}, \delta u(s) \right\rangle dt ds \right\}.$$

*Theorem 4.* Let the admissible control $u(t)$ satisfy condition (25). Then in order to this function be optimal in problem (1)-(4), the inequality

$$\delta^2 J(u) = -\left\{\int_0^T\int_0^T \left\langle \delta' u(\tau)\frac{\partial' f(\tau,x,u)}{\partial u}R(\tau,s)\frac{\partial f(s,x,u)}{\partial u}, \delta u(s)\right\rangle d\tau ds\right.$$
$$+\int_0^T \left\langle \delta' u(\tau)\frac{\partial^2 H(t,\psi,x,u)}{\partial u^2}, \delta u(t)\right\rangle dt + \tag{27}$$
$$\left. +2\int_0^T\int_0^T \left\langle \delta u'(t)\frac{\partial^2 H(t,\psi,x,u)}{\partial x\partial u}G(t,s)\frac{\partial f(s,x,u)}{\partial u}, \delta u(s)\right\rangle dtds\right\} \geq 0$$

should be fulfilled for all $\delta u(t) \in L_\infty[0,T]$.

The analogy of the Legandre-Klebsh condition for the considered problem follows from condition (28).

*Theorem 5.* The inequality holds true

$$\nu'\frac{\partial^2 H(\theta,\psi(\theta),x(\theta),u(\theta))}{\partial u^2}v \leq 0 \tag{28}$$

over the optimal process $(u(t),x(t))$ for all $\nu \in R^r$ and $\theta \in [0,T]$.

*Proof.* To prove the theorem, we calculate the variation of the control

$$\delta u(t) = \begin{cases} v & t\in[\theta,\theta+\varepsilon) \\ 0 & t\notin[\theta,\theta+\varepsilon) \end{cases}, \tag{29}$$

where $\varepsilon > 0$, $v$ is some $r$-dimensional vector.

By virtue of (23) the variation of the corresponding trajectory is

$$\delta x(t) = a(t)\varepsilon + o(\varepsilon,t), \quad t\in[0,T], \tag{30}$$

where $a(t)$ is a continuous bounded function.

Substituting variation (29) into (27) and selecting the principal term with respect to $\varepsilon$ we obtain

$$\delta^2 J(u) = -\int_\theta^{\theta+\varepsilon} v'\frac{\partial^2 H(t,\psi(t),x(t),u(t))}{\partial u^2}vdt + o(\varepsilon) =$$
$$= -\varepsilon v'\frac{\partial^2 H(\theta,\psi(\theta),x(\theta),u(\theta))}{\partial u^2}v + o_1(\varepsilon).$$

From this using condition of (24) the Legandre-Klebsh criterion (28) is obtained.

Condition (30) is the second order optimality condition. It is obvious that when the right hand side of system (1) is linear with respect to control parameters, condition (28) also degenerates, i.e. is fulfilled trivially.

If for all $\theta \in (0,T)$, $v \in R^r$ the relations

$$\frac{\partial H(\theta,\psi(\theta),x(\theta),u(\theta))}{\partial u} = 0, \quad v'\frac{\partial^2 H(\theta,\psi(\theta),x(\theta),u(\theta))}{\partial u^2}v = 0,$$

hold true then the admissible control $u(t)$ is said be a singular control in the classic sense.

*Theorem 6.* Assume that the control $u(t)$ is the singular in the classic sense. Then for optimality of $u(t)$

$$\nu'\left\{\int_0^T\int_0^T \left\langle \frac{\partial f(t,x,u)}{\partial u}R(t,s), \frac{\partial f(s,x,u)}{\partial u}\right\rangle dtds + \right.$$
$$\left. +2\int_0^T\int_0^T \left\langle \frac{\partial^2 H(t,\psi,x,u)}{\partial x\partial u}G(t,s), \frac{\partial f(s,x,u)}{\partial u}\right\rangle dtds\right\} v \leq 0 \tag{31}$$

should be fulfilled for all $v \in R^n$.

Condition (31) is an integral necessary condition of optimality of the controls singular in the classic sense. One can obtain various type necessary optimality conditions by taking the special variation of various forms in formula (30).

*Conclusion*

In this paper, the optimal control problem is considered when the considered system is described by the differential equations with integral boundary conditions. The existence and uniqueness of the solution is proved for the corresponding boundary value problem. The first and second order increment formulas of the functional are obtained. Various necessary conditions of optimality of the first and second order are obtained. Of course, such type existence and uniqueness results and necessary conditions of optimality hold under the same sufficient conditions on nonlinear terms of the system of nonlinear differential equations (1), subject to multi-point nonlocal and integral boundary conditions type of

$$\int_0^T m(t)\,x(t)\,dt + \sum_{j=1}^{J} B_j x(t_j) = C,$$

where $B_j \in R^{n\times n}$ are given matrices and

$$\det\left(B + \sum_{j=1}^{J} B_j\right) \neq 0,$$

here, $0 < t_1 < t_2 < ... < t_J \leq T$ for controls singular in the classic sense. Selecting special variation in different way in formula (30) we can get various necessary optimality conditions.

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## Интегралды шекаралық шарттары бар жүйелер үшін тиімді басқару есебі

Мақалада «таза» интегралды шекаралық шартпен тиімді басқару есебі қарастырылған. Грин функциясы құрылған. Банахтың қысып бейнелеу принципін қолдана отырып, бекітілген рұқсат етілген басқару кезінде интегралды шеттік есептердің бір класының шешімнің бар болуының жеткілікті шарты мен жалғыздығы анықталды. Функционалдың ауытқуы әдісімен Понтрягиннің максимум принципі дәлелденді. Функционалдың бірінші және екінші вариациялары есептелген. Басқарудың вариацияларының көмегімен екінші ретті тиімділіктің әртүрлі қажетті шарттары алынды.

*Кілт сөздер:* интегралды шекаралық шарттар, ерекше басқару, тиімді басқару есебі, шешімнің бар және жалғыз болуы.

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## Задача оптимального управления для систем с интегральными граничными условиями

В статье рассмотрена задача оптимального управления с «чистым» интегральным граничным условием. Построена функция Грина. С помощью принципа сжимающих отображений Банаха установлено достаточное условие существования и единственности решения одного класса интегральных краевых задач при фиксированных допустимых управлениях. Методом приращений функционала доказан принцип максимума Понтрягина. Вычислены первая и вторая вариации функционала. С помощью вариаций управлений получены различные необходимые условия оптимальности второго порядка.

*Ключевые слова:* интегральные граничные условия, особые управления, задача оптимального управления, существование и единственность решения.

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