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Asymptotic behavior of solutions of sum-difference equations

In this study, we present an investigation of the asymptotic behavior of solutions of sum-difference equations. Based on some mathematical inequalities, we have obtained our results. The obtained results can apply to some fractional type difference equations as well.

Keywords: asymptotic behavior, oscillation, nonoscillation, difference equation, Caputo fractional difference operator.

Introduction

In alignment with the extensive interest in the research of difference/differential equations which has demonstrated high potential for real life applications, the determination of qualitative behavior of solutions of them have also received significant attention amongst researchers [1–16].

In [15], the authors have investigated the positive solutions of the following equation

$$\begin{aligned} {}_C\Delta^{\alpha} y(t) &= d(t+\alpha) + f(t+\alpha, x\,(t+\alpha)), \\ y(0) &= c_0, \end{aligned}$$

where $0 < \alpha \leq 1$, ${}_C\Delta^{\alpha}$ is Caputo–like delta fractional difference operator, d is a positive sequence. The authors consider the some particular cases of $y(t)$ for the above equation. In [16], the authors have studied the nonoscillatory solutions of the following fractional difference equations

$$\begin{aligned} &{}_C\Delta^{\alpha} y(t) = e(t+\alpha) + f(t+\alpha, x\,(t+\alpha)), \\ &y(0) = c_0, \end{aligned}$$

where $0 < \alpha \leq 1$, ${}_C\Delta^{\alpha}$ is Caputo–like delta fractional difference operator. Considering some particular cases of $y(t)$, they have obtained some nonoscillatory solutions for the equation.

Motivated by the idea in [12–16], in this article, we study the oscillatory behavior of the following difference equations of the form

$$\begin{cases} \Delta y(t) = e(t+\alpha) - \sum_{s=1-\alpha}^{t-\alpha} (t-s-1)^{(\alpha-1)} k(t+\alpha, s+\alpha) f(s+\alpha, y(s+\alpha)) \\ y(0) = c_0, \end{cases} \tag{1}$$

where $t \in \mathbb{N}_{1-\alpha}$, $0 < \alpha \leq 1$, $\mathbb{N}_t = \{t, t+1, t+2, \ldots\}$, $f : \mathbb{N}_1 \times \mathbb{R} \to \mathbb{R}$, k and e are sequence. By a solution $y(t)$ of Equation (1), we mean a real-valued sequence y satisfying Equation (1) for $t \in \mathbb{N}_{t_0}$ with $t_0 \in \mathbb{N}_1$. A solution y of Equation (1) is called oscillatory if it has arbitrarily large zeros, otherwise it is called nonoscillatory. Equation (1) is called oscillatory if all of its solutions are oscillatory.

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1 Background materials

In this section, we present some background materials.

Definition 1. [17] The generalized falling function is defined by

$$t^{(r)} = \frac{\Gamma(t+1)}{\Gamma(t-r+1)},$$

for any t, $r \in \mathbb{R}$ for which the right hand-side is defined. Here Γ denotes the Euler's gamma function. We also use the standard extensions of the domain of this rising function by defining it to be zero whenever the numerator is well defined, but the denominator is not defined.

Lemma 1. [18] Assume that $\beta > 1$ and $\gamma > 0$, then

$$\left[t^{(-\gamma)}\right]^{\beta} < \frac{\Gamma(1+\beta\gamma)}{\Gamma^{\beta}(1+\gamma)} t^{(-\beta\gamma)}, \qquad t \in \mathbb{N}_1.$$

Definition 2. [19] Assume $y : \mathbb{N}_a \to \mathbb{R}$ and $\nu > 0$. Delta fractional sum of y is defined by

$$\Delta_a^{-\nu} y(t) = \frac{1}{\Gamma(\nu)} \sum_{s=a}^{t-\nu} (t-s-1)^{(\nu-1)} y(s), \quad t \in \mathbb{N}_{a+\nu}.$$

Lemma 2. [20] Let $\mu \in \mathbb{R} \setminus \{..., -2, -1, 0\}$, $a \in \mathbb{R}$, $\nu > 0$ and $(t-a)^{(\mu)} : \mathbb{N}_{a+\mu} \to \mathbb{R}$. Then,

$$\Delta_{a+\mu}^{-\nu} (t-a)^{(\mu)} = \frac{\Gamma(\mu+1)}{\Gamma(\mu+\nu+1)} (t-a)^{(\mu+\nu)} \text{ for } t \in \mathbb{N}_{a+\mu+\nu}.$$

Lemma 3. [16] Let $0 < \alpha \leq 1$, $p > 1$, $p(\alpha-1)+1 > 0$ and $\gamma = 2 - \alpha - \frac{1}{p}$. Then one has

$$(t-s-1)^{(p\alpha-p)} \geq (t-s-1+\alpha+p(1-\alpha)-1)^{(p\alpha-p)}$$

and

$$(s)^{(p\gamma-p)} \geq (s+p(\alpha-1)+1)^{(p\gamma-p)},$$

where

$$t \in \mathbb{N}_1 \text{ and } s \in \{1-(p\alpha-p), 2-(p\alpha-p), \cdots, t-2-(p\alpha-p)\}.$$

Lemma 4. [21] Assume that X and Y are nonnegative real numbers, then

$$X^k - (1-k)Y^k - kXY^{k-1} \leq 0, \ \ \textit{for} \ \ 0 < k < 1,$$

where the equality holds if and only if $X = Y$.

Lemma 5. [22] Assume that m and x be nonnegative sequences and c be a nonnegative constant. If

$$x(t) \leq c + \sum_{s=0}^{t} m(s)\, x(s) \quad \text{for } t \geq 0.$$

Then, the following inequality holds

$$x(t) \leq c \exp\left(\sum_{s=0}^{t} m(s)\right), \qquad \text{for } t \geq 0.$$

2 Main results

We assume that there exist positive sequences a, h, m and $\gamma > 0$, $0 < \delta < 1$ are real numbers such that

$$0 \le k(t,s) \le a(t)h(s) \text{ for all } t \ge s \ge 0 \tag{2}$$

and

$$0 < yf(t,y) \le t^{\gamma-1}m(t)|y|^{\delta+1} \text{ for all } y \ne 0 \text{ and } t \ge 0. \tag{3}$$

Furthermore, there exist real numbers $M_1 > 0$ and M_2 such that

$$|a(t)| \le M_1 \tag{4}$$

and for every $T \ge 0$

$$-M_2 \le \liminf_{t\to\infty} \frac{1}{t} \sum_{s=t_1}^{t-\alpha} e\,(s+\alpha) \le \limsup_{t\to\infty} \frac{1}{t} \sum_{s=t_1}^{t-\alpha} e\,(s+\alpha) \le M_2. \tag{5}$$

For the sake of convenience, we denote

$$g_1(t) := \sum_{s=1-\alpha}^{t-1-\alpha} (t-s-1)^{(\alpha-1)} (s+\alpha)^{(\gamma-1)} \upsilon^{\delta/1-\delta}(s+\alpha)\, m^{1/1-\delta}(s+\alpha)\, h^{1/1-\delta}(s+\alpha),$$

where υ is a postive sequence.

Theorem 1. Assume that q be a conjugate number of $p > 1$, $p < 1/(1-\alpha)$, $\gamma = 1-\alpha+1/q$ and the conditions (2)–(5) and

$$\limsup_{t\to\infty} g_1(t) < \infty$$

hold. Then every nonoscillatory solution of Equation (1) satisfies

$$\limsup_{t\to\infty} \frac{|y(t)|}{t} < \infty.$$

Proof. Assume that y be a nonoscillatory solution of (1), say $y(t) > 0$ for all $t \in \mathbb{N}_{t_0}$ where t_0 is a postive integer. Let

$$k_1 := \max\{|f(t,y(t))| : t \in \mathbb{N}_{t_0}\} \ge 0 \text{ and } k_2 := k_1 \sum_{s=1-\alpha}^{t_1-1-\alpha} (t_1-s-1)^{(\alpha-1)} h(s+\alpha) \ge 0.$$

From Equation (1) and our conditions, we have

$$\begin{aligned}
\Delta y(t) &= e(t+\alpha) - \sum_{s=1-\alpha}^{t_1-1-\alpha} (t-s-1)^{(\alpha-1)} k(t+\alpha, s+\alpha) f\left(s+\alpha, y(s+\alpha)\right) \\
&- \sum_{s=t_1-\alpha}^{t-1-\alpha} (t-s-1)^{(\alpha-1)} k(t+\alpha, s+\alpha) f\left(s+\alpha, y(s+\alpha)\right) \\
&\leq e(t+\alpha) + \sum_{s=1-\alpha}^{t_1-1-\alpha} (t_1-s-1)^{(\alpha-1)} k(t+\alpha, s+\alpha) \left|f\left(s+\alpha, y(s+\alpha)\right)\right| \\
&+ \sum_{s=t_1-\alpha}^{t-1-\alpha} (t-s-1)^{(\alpha-1)} k(t+\alpha, s+\alpha) \left|f\left(s+\alpha, y(s+\alpha)\right)\right| \\
&\leq e(t+\alpha) + k_1 a\left(t+\alpha\right) \sum_{s=1-\alpha}^{t_1-1-\alpha} (t_1-s-1)^{(\alpha-1)} h\left(s+\alpha\right) \\
&+ a\left(t+\alpha\right) \sum_{s=t_1-\alpha}^{t-1-\alpha} (t-s-1)^{(\alpha-1)} \left(s+\alpha\right)^{(\gamma-1)} h\left(s+\alpha\right) m\left(s+\alpha\right) y^{\delta}\left(s+\alpha\right) \\
&\leq e(t+\alpha) + k_2 a\left(t+\alpha\right) \\
&+ a\left(t+\alpha\right) \sum_{s=t_1-\alpha}^{t-1-\alpha} \Big\{ (t-s-1)^{(\alpha-1)} \left(s+\alpha\right)^{(\gamma-1)} \\
&\qquad \times \Big(h\left(s+\alpha\right) m\left(s+\alpha\right) y^{\delta}\left(s+\alpha\right) - \upsilon\left(s+\alpha\right) y\left(s+\alpha\right) \Big) \Big\} \\
&+ a\left(t+\alpha\right) \sum_{s=t_1-\alpha}^{t-1-\alpha} (t-s-1)^{(\alpha-1)} \left(s+\alpha\right)^{(\gamma-1)} \upsilon\left(s+\alpha\right) y\left(s+\alpha\right).
\end{aligned}$$

Setting $X = h^{1/\delta}\left(s+\alpha\right) m^{1/\delta}\left(s+\alpha\right) y\left(s\right)$, $Y = \left(\frac{\upsilon(s+\alpha)}{\delta h^{1/\delta}(s+\alpha) m^{1/\delta}(s+\alpha)}\right)^{1/(\delta-1)}$ and $\beta = \delta$, then using the above lemma, we deduce that

$$h\left(s+\alpha\right) m\left(s+\alpha\right) y^{\delta}\left(s+\alpha\right) - \upsilon\left(s+\alpha\right) y\left(s+\alpha\right) \leq \lambda_1 \upsilon^{\delta/1-\delta}\left(s+\alpha\right) m^{1/1-\delta}\left(s+\alpha\right) h^{1/1-\delta}\left(s+\alpha\right),$$

where $\lambda_1 = (1-\delta)\,\delta^{\delta/(1-\delta)}$. Hence we have

$$\begin{aligned}
\Delta y(t) &\leq e(t+\alpha) + k_2 M_1 \\
&+ \lambda_1 M_1 \sum_{s=t_1-\alpha}^{t-1-\alpha} (t-s-1)^{(\alpha-1)} \left(s+\alpha\right)^{(\gamma-1)} \upsilon^{\delta/1-\delta}\left(s+\alpha\right) m^{1/1-\delta}\left(s+\alpha\right) h^{1/1-\delta}\left(s+\alpha\right) \\
&+ M_1 \sum_{s=t_1-\alpha}^{t-1-\alpha} (t-s-1)^{(\alpha-1)} \left(s+\alpha\right)^{(\gamma-1)} \upsilon\left(s+\alpha\right) y\left(s+\alpha\right).
\end{aligned}$$

Summing both sides from t_1 to $t-1$ and interchanging the order of the summation, we get

$$\begin{aligned}
y(t) &\leq y\left(t_1\right) + k_2 M_1 \left(t-t_1\right) + \sum_{s=t_1}^{t-1} e(s+\alpha) + \frac{\lambda_1 M_1 t}{\alpha} g_1\left(t\right) \\
&+ \frac{M_1 t}{\alpha} \sum_{s=t_1-\alpha}^{t-1-\alpha} (t-s-1)^{(\alpha-1)} \left(s+\alpha\right)^{(\gamma-1)} \upsilon\left(s+\alpha\right) y\left(s+\alpha\right).
\end{aligned}$$

That is

$$y\left(t\right)\leq A_1+\frac{M_1t}{\alpha}\sum_{s=t_1-\alpha}^{t-1-\alpha}\left(t-s-1\right)^{(\alpha-1)}\left(s+\alpha\right)^{(\gamma-1)}\upsilon\left(s+\alpha\right)y\left(s+\alpha\right), \tag{6}$$

where $A_1=y\left(t_1\right)+k_2M_1\left(t-t_1\right)+\sum_{s=t_1}^{t-1}e(s+\alpha)+\frac{\lambda_1M_1t}{\alpha}g_1\left(t\right)$. By applying the Holder's inequality and lemmas, we have

$$\begin{aligned}
&\sum_{s=t_1-\alpha}^{t-\alpha-1}(t-s-1)^{(\alpha-1)}(s+\alpha)^{(\gamma-1)}\upsilon(s+\alpha)y(s+\alpha)\\
&\leq\Big[\sum_{s=t_1-\alpha}^{t-\alpha-1}\Big((t-s-1)^{(\alpha-1)}\Big)^p\Big((s+\alpha)^{(\gamma-1)}\Big)^p\Big]^{1/p}\Big[\sum_{s=t_1-\alpha}^{t-\alpha-1}\upsilon^q(s+\alpha)y^q(s+\alpha)\Big]^{1/q}\\
&<\left[\Big(\frac{\Gamma(1-p\alpha+p)}{\Gamma^p(2-\alpha)}\Big)\Big(\frac{\Gamma(1-p\gamma+p)}{\Gamma^p(2-\gamma)}\Big)\right]^{1/p}\Big[\sum_{s=-1-(p\alpha-p)}^{t-(p\alpha-p+1)}\Big((t-s-1)^{(p\alpha-p)}\Big)\Big((s)^{(p\gamma-p)}\Big)\Big]^{1/p}\\
&\times\Big[\sum_{s=t_1-\alpha}^{t-\alpha-1}\upsilon^q(s+\alpha)y^q(s+\alpha)\Big]^{1/q}
\end{aligned}$$

and then we have

$$\begin{aligned}
&\sum_{s=t_1-\alpha}^{t-\alpha-1}(t-s-1)^{(\alpha-1)}(s+\alpha)^{(\gamma-1)}\upsilon(s+\alpha)y(s+\alpha)\\
&\leq\left[\Big(\frac{\Gamma(1-p\alpha+p)}{\Gamma^p(2-\alpha)}\Big)\left(\frac{\Gamma(1-p\gamma+p)}{\Gamma^p(2-\gamma)}\right)\right]^{1/p}\Big[\Delta_{-1-(p\alpha-p)}^{-(p\alpha-p+1)}(t)^{(p\gamma-p)}\Big]\Big[\sum_{s=t_1-\alpha}^{t-\alpha-1}\upsilon^q(s+\alpha)y^q(s+\alpha)\Big]^{1/q}\\
&=\left[\Big(\frac{\Gamma(1-p\alpha+p)}{\Gamma^p(2-\alpha)}\Big)\left(\frac{\Gamma(1-p\gamma+p)}{\Gamma^p(2-\gamma)}\right)\right]^{1/p}\left[\frac{\Gamma[p(\gamma-1)+1]}{\Gamma[p(\gamma-1)+p(\alpha-1)+2]}(t)^{(p\alpha-p+1+p\gamma-p)}\right]^{1/p}\\
&\qquad\Big[\sum_{s=t_1-\alpha}^{t-\alpha-1}\upsilon^q(s+\alpha)y^q(s+\alpha)\Big]^{1/q},
\end{aligned}$$

or

$$\sum_{s=t_1-\alpha}^{t-\alpha}(t-s-1)^{(\alpha-1)}(s+\alpha)^{(\gamma-1)}m(s+\alpha)x^{\delta}(s+\alpha)\leq N_1\left[\sum_{s=t_1-\alpha}^{t-\alpha-1}\upsilon^q(s+\alpha)y^q(s+\alpha)\right]^{1/q},$$

where

$$N_1=\left[\Big(\frac{\Gamma(1-p\alpha+p)}{\Gamma^p(2-\alpha)}\Big)\left(\frac{\Gamma(1-p\gamma+p)}{\Gamma^p(2-\gamma)}\right)\right]^{1/p}\left[\frac{\Gamma[p(\gamma-1)+1]}{\Gamma[p(\gamma-1)+p(\alpha-1)+2]}\right]^{1/p}.$$

Thus (6) becomes

$$y\left(t\right)\leq A_2t+\frac{K_1t}{\alpha}\left[\sum_{s=t_1-\alpha}^{t-\alpha-1}\upsilon^q(s+\alpha)y^q(s+\alpha)\right]^{1/q},$$

where, in view of the hypothesis of the theorem, A_2 is an upper bound for

$$\frac{y\left(t_1\right)}{t}+\frac{k_2M_1\left(t-t_1\right)}{t}+\frac{1}{t}\sum_{s=t_1}^{t-1}e(s+\alpha)+\frac{\lambda_1M_1}{\alpha}g_1\left(t\right)$$

and $K_1 = M_1 N_1$. Then we have

$$\frac{y(t)}{t} \le A_2 + \frac{K_1}{\alpha}\left[\sum_{s=t_1-\alpha}^{t-\alpha-1} \upsilon^q(s+\alpha) y^q(s+\alpha)\right]^{1/q}.$$

Hence we get

$$\left(\frac{y(t)}{t}\right)^q \le \left(A_2 + \frac{K_1}{\alpha}\left[\sum_{s=t_1-\alpha}^{t-\alpha-1} \upsilon^q(s+\alpha) y^q(s+\alpha)\right]^{1/q}\right)^q.$$

By applying the elementary inequality $(A+B)^q \le 2^{q-1}(A^q + B^q)$ for $A, B \ge 0$, we have

$$\begin{aligned}
\omega(t) &\le 2^{q-1} A_2^q + 2^{q-1}\left(\frac{K_1}{\alpha}\right)^q \sum_{s=t_1-\alpha}^{t-\alpha-1} \upsilon^q(s+\alpha) y^q(s+\alpha) \\
&= 2^{q-1} A_2^q + 2^{q-1}\left(\frac{K_1}{\alpha}\right)^q \sum_{s=t_1-\alpha}^{t-\alpha-1} (s+\alpha)^q \upsilon^q(s+\alpha)\left(\frac{y(s+\alpha)}{s+\alpha}\right)^q \\
&= 2^{q-1} A_2^q + 2^{q-1}\left(\frac{K_1}{\alpha}\right)^q \sum_{s=t_1-\alpha}^{t-\alpha-1} (s+\alpha)^q \upsilon^q(s+\alpha)\omega(s+\alpha),
\end{aligned}$$

where $\omega(t) = y^q(t)/t^q$. If we apply the Lemma 5, we have the following result

$$\limsup_{t\to\infty} \frac{y(t)}{t} < \infty.$$

This completes the proof.

Theorem 2. In addition to the hypothesis of Theorem 1, suppose that

$$\lim_{t\to\infty} a(t) = 0.$$

If for every $\mu \in (0,1)$ we have

$$\liminf_{t\to\infty}\left[\mu t + \sum_{s=1-\alpha}^{t-1} e(s+\alpha)\right] = -\infty, \quad \limsup_{t\to\infty}\left[\mu t - \sum_{s=1-\alpha}^{t-1} e(s+\alpha)\right] = \infty,$$

then Equation (1) is oscillatory.

Proof. Let y be a nonoscillatory solution of (1). Then we may assume that $y(t)$ is eventually positive for all $t \in \mathbb{N}_{t_0}$ where t_0 is a postive integer. Proceeding as in the proof of the above theorem, we have the following inequality

$$\begin{aligned}
y(t) &\le y(t_1) + k_2 M_1 (t - t_1) + \sum_{s=t_1}^{t-1} e(s+\alpha) \\
&+ \frac{\lambda_1 M_1 t}{\alpha} g_1(t) + \frac{K_1 t}{\alpha}\left[\sum_{s=t_1-\alpha}^{t-\alpha-1} \upsilon^q(s+\alpha) y^q(s+\alpha)\right]^{1/q}.
\end{aligned}$$

From our conditions, we can make M_1 as small as we please by increasing the size of t_1 if necessary. And also considering Theorem 1, we have

$$y(t) \le y(t_1) - \sum_{s=1-\alpha}^{t_1-\alpha-1} e(s+\alpha) + \sum_{s=1-\alpha}^{t-\alpha} e(s+\alpha) + \frac{t}{T}, \tag{7}$$

where $T > 1$. Taking infimum and limit as $t \to \infty$ in (7), respectively. Then we obtain a contradiction with the fact that $y(t)$ is eventually positive of Equation (1). The proof when $y(t)$ is eventually negative is similar.

3 Conclusions

In this study, we present an investigation of the asymptotic behavior of solutions of sum-difference equations. Based on some features of the discrete calculus and mathematical inequalities, we have obtained our results. The obtained results can apply to some fractional type difference equations as well. If we consider

$$k(t+\alpha, s+\alpha) = \frac{1}{\Gamma(\alpha)},$$

$$e(t+\alpha) = y_0 + \frac{1}{\Gamma(\alpha)} \sum_{s=1-\alpha}^{t-\alpha} (t-s-1)^{(\alpha-1)} g(s+\alpha)$$

we may write from Equation (1) that

$$\Delta y(t) = y_0 + \frac{1}{\Gamma(\alpha)} \sum_{s=1-\alpha}^{t-\alpha} (t-s-1)^{(\alpha-1)} [g(s+\alpha) - f(s+\alpha, y(s+\alpha))], \quad 0 < \alpha \leq 1.$$

It is not difficult to see that this equation is equivalent to the fractional difference equation. Here, one can notice that the obtained results can be rewritten for the fractional difference equations.

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Айырымдардың қосындысы теңдеулерінің шешімдерінің асимптотикасы

Мақалада жұмыста айырымдардың қосындысы теңдеулерінің шешімдерінің асимптотикалық өзгеруін зерттеу ұсынылған. Бұл нәтижелер кейбір математикалық теңсіздіктер негізінде алынған. Алынған нәтижелерді бөлшек типтегі кейбір дифференциалдық теңдеулерге де қолдануға болады.

Кілт сөздер: асимптотика, осцилляция, бейосцилляция, айырымдық теңдеу, Капутоның бөлшекті айырымды операторы.

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Асимптотика решений уравнений суммы разностей

В статье мы представлено исследование асимптотического поведения решений уравнений суммы разностей. На основе определенных математических неравенств нами получены результаты, которые можно применить и к некоторым разностным уравнениям дробного типа.

Ключевые слова: асимптотика, осцилляция, неосцилляция, разностное уравнение, оператор дробной разности Капуто.

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