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On the solvability of a nonlinear optimization problem with boundary vector control of oscillatory processes

In the paper, the solvability of the nonlinear boundary optimization problem has been investigated for the oscillation processes, described by the integro-differential equation in partial derivatives with Fredholm integral operator. It has been established that the components of the boundary vector control are defined as a solution to a system of nonlinear integral equations of a specific form, and the equations of this system have the property of equal relations. An algorithm for constructing a solution to the problem of nonlinear optimization has been developed.

Keywords: General solution, nonlinear optimization, boundary vector control, functional, optimal conditions, property of equal relations.

Introduction

There are plenty of works [1-12], that are devoted to the study of nonlinear optimization problems described by systems with distributed parameters. However, methods for solving nonlinear optimization problems while minimizing a piecewise linear functional have not been sufficiently developed. This article deals with the solvability of the problem of optimal boundary control of oscillatory processes described by partial integro-differential equations with an integral Fredholm operator, while minimizing a piecewise linear functional.

Consider the following nonlinear optimization problem where it is required to minimize the piecewise linear functional:

$$J\left[u_1(t,x),...u_m(t,x)\right]=\int_0^T\int_Q\left\{\left[V\left(T,x\right)-\xi_1\left(x\right)\right]^2+\left[V_t\left(T,x\right)-\xi_2\left(x\right)\right]^2\right\}dx+ \tag{1}$$

$$+\beta\int_0^T\int_Q\sum_{k=1}^m\left|u_k(t,x)\right|dxdt\rightarrow\min,\quad\beta>0,$$

on the set of generalized solutions to the boundary value problem

$$V_{tt}-AV=\lambda\int_0^T K\left(t,\tau\right)V\left(\tau,x\right)d\tau,\ x\in Q\subset R^n,\quad 0<t<T, \tag{2}$$

$$V(0,x)=\psi_1(x),\ V_t(0,x)=\psi_2(x),\ \ x\in Q, \tag{3}$$

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$$GV(t,x) \equiv \sum_{i,j}^{n} a_{ij}(x) V_{x_j}(t,x) \cos(\nu, x_i) + a(x) V(t,x) = \tag{4}$$
$$= f(t,x,u_1(t,x),...,u_m(t,x)), \quad x \in \gamma,\ 0 < t \leq T.$$

Here A is the elliptic operator, ν is a normal vector, emanating from the point $x \in \gamma$; $K(t,\tau)$ is a given function of $H(D), D = \{0 \leq t \leq T, 0 \leq \tau \leq T\}, \psi_1(x) \in H_1(Q), \psi_2(x) \in H(Q), \xi_1(x) \in H(Q), \xi_2(x) \in H(Q)$ are given functions; $f[t,x,u_1(t,x),...,u_m(t,x)] \in H(Q_T)$ is a boundary source function; $f_{u_i}[t,x,u_1(t,x),...,u_m(t,x)] \neq 0, \forall (t,x) \in (Q_T), u_i(t,x) \in H(Q_T), i = 1,2,3,...,m$; is a control function, λ is a parameter, and T is a fixed moment of time. Q is a region of the space R^n bounded by a piecewise smooth surface γ; $Q_t = Q \times (0,T]$.

The boundary value problem (2)–(4) has not a classical solution, with the above conditions imposed on the given functions. Therefore, we use the concept of a generalized solution to the boundary value problem.

With respect to the methodology of work [1], we give a definition.

Definition 1. The generalized solution to the boundary value problem (2)–(4) is called the function $V(t,x) \in H(Q_T)$ that satisfies the integral identity

$$\int_Q [(V_t(t,x)\Phi(t,x)) - (V(t,x)\Phi_t(t,x))]_{t_1}^{t_2} dx \equiv$$

$$\equiv \int_{t_1}^{t_2} \int_Q \left[-V(t,x)\Phi_{tt}(t,x) - \sum_{i,j=1}^{n} a_{ij}(x) V_{x_j}(t,x) \Phi_{x_i}(t,x) - c(x)V(t,x)\Phi(t,x) \right] dxdt +$$

$$+ \int_{t_1}^{t_2} \int_\gamma \left(f[t,x,u_1(t,x),...,u_m(t,x)] - a(x)V(t,x) \right) \Phi(t,x) dxdt +$$

$$+ \int_{t_1}^{t_2} \int_Q \left(\lambda \int_0^T K(t,\tau) V(\tau,x) d\tau \right) \Phi(t,x) dxdt$$

for any t_1 and t_2, $0 < t_1 \leq t \leq t_2 \leq T$, and for any function $\Phi(t,x) \in C^{2,1}(Q_T), C^{2,1}(Q_T)$ is a space of functions defined on the set Q_T and having a second-order derivative with respect to t, and the first order in the variables x_i, and satisfies the initial and boundary conditions in a weak sense, i.e. for any functions $\phi_0(x) \in H(Q)$, $\phi_1(x) \in H(Q)$ the following relations hold

$$\lim_{t \to +0} \int_Q V(t,x)\phi_0(x) dx = \int_Q \psi_1(x)\phi_0(x) dx,$$

$$\lim_{t \to +0} \int_Q V_t(t,x)\phi_1(x) dx = \int_Q \psi_2(x)\phi_1(x) dx.$$

The solution to problem (2)–(4) is sought in the form

$$V(t,x)=\sum_{n=1}^{\infty} V_n(t) z_n(x).$$

Where $V_n(t)=\langle V(t,x),z_n(x)\rangle=\int\limits_Q V(t,x)z_n(x)dx$ are Fourier coefficients, the symbol $<\cdot,\cdot>$ is used for the scalar product in the Hilbert Space $H(Q)$, $z_n(x)$ are eigenfunctions of the boundary value problem [1]

$$D_n\left(\Phi(t,x),z_k(x)\right)\equiv\int\limits_Q\left(\sum_{i,j=1}^{n}a_{ij}(x)\Phi_{x_j}(t,x)z_{kx_i}(x)+c(x)z_k(x)\Phi\left(t,x\right)\right)dx+$$

$$+\int\limits_\gamma a(x)z_k(x)\Phi(t,x)dx=\lambda_k^2\int\limits_Q z_k(x)\Phi\left(t,x\right)dx;$$

$$Gz_k\left(x\right)=0,\quad x\in\gamma,\quad [k=1,2,3,...].$$

Using Liouville method we easily prove that the Fourier coefficients satisfy the relations [2]

$$V_n(t)=\lambda\int\limits_0^T\left(\frac{1}{\lambda_n}\int\limits_0^t\sin\lambda_n\left(t-\tau\right)K\left(\tau,s\right)d\tau\right)V_n\left(s\right)ds+\psi_{1n}\cos\lambda_n t+\frac{\psi_{2n}}{\lambda_n}\sin\lambda_n t+ \tag{5}$$

$$+\frac{1}{\lambda_n}\int\limits_0^t\sin\lambda_n\left(t-\tau\right)\cdot\left[f_n\left[\tau,u_1,...,u_m\right]\right]d\tau;\quad n=1,2,3,...,$$

where

$$f_n\left[\tau,u_1,...,u_m\right]=\int\limits_Q f[t,x,u_1(t,x),...,u_m(t,x)]z_n(x)dx.$$

We can rewrite equation (5) as the following equation:

$$V_n(t)=\lambda\int\limits_0^t K_n\left(t,s\right)V_n\left(s\right)ds+a_n(t), \tag{6}$$

where

$$K_n\left(t,s\right)=\frac{1}{\lambda_n}\int\limits_0^t\sin\lambda_n\left(t-\tau\right)K\left(\tau,s\right)d\tau;$$

$$a_n(t)=\psi_{1n}\cos\lambda_n t+\frac{\psi_{2n}}{\lambda_n}\sin\lambda_n t+\frac{1}{\lambda_n}\int\limits_0^t\sin\lambda_n\left(t-\tau\right)\left[f_n\left[\tau,u_1,...,u_m\right]\right]d\tau.$$

The solution of the integral equation (6) is defined by the following formula [2]

$$V_n(t)=\lambda\int\limits_0^T R_n\left(t,s,\lambda\right)a_n\left(s\right)ds+a_n(t),$$

where

$$R_n(t,s,\lambda)=\sum_{i=1}^{\infty}\lambda^{i-1}K_{n,i}(t,s)\,,\ n=1,2,3,\dots \tag{7}$$

is the resolvent of the kernel $K_n(t,s)$, the iterated kernels are $K_{n,i}(t,s)$ defined by the formulas

$$K_{n,i+1}(t,s)=\int_0^T K_n(t,\eta)\,K_{n,i}(\eta,s)\,d\eta, i=1,2,3,\dots,$$

for each $n=1,2,3,...$

Now we investigate the convergence of Neumann series (7). According to the following estimates

$$|K_{n,i}(t,s)|^2\le\left(\frac{T}{\lambda_n^2}\right)^i(K_0T)^{i-1}\int_0^T K^2(y,s)dy;\quad \forall t\in(0,T)\,;$$

$$\int_0^T K_{n,i}^2(t,s)\,ds\le\left(\frac{T}{\lambda_n^2}\right)^i(K_0T)^{i-1}\int_0^T\int_0^T K^2(y,s)\,dyds\le\left(\frac{TK_0}{\lambda_n^2}\right)^i(T)^{i-1},$$

we can easily prove that Neumann series converges absolutely for the values of parameter satisfying following condition

$$|\lambda|<\frac{\lambda_n}{T\sqrt{K_0}}\underset{n\to\infty}{\to}\infty,\quad n=1,2,3,...$$

The radius of the convergence increases when n grows. As the sum of an absolutely convergent series, resolvent $R_n(t,s,\lambda)$ is the continuous function. It is easy to check that the following estimates hold

$$|R_n(t,s,\lambda)|\le\frac{\sqrt{T}\sqrt{\int_0^T K^2(y,s)dy}}{\lambda_n-|\lambda|\sqrt{K_0T^2}};$$

$$\int_0^T R_n^2(t,s,\lambda)\,ds=\frac{T}{\left(\lambda_n-|\lambda|\sqrt{K_0T^2}\right)^2}\int_0^T\int_0^T K^2(y,s)dyds=\frac{K_0T}{\left(\lambda_n-|\lambda|\sqrt{K_0T^2}\right)^2}.$$

Note that the Neumann series converges absolutely for values of the parameters satisfying

$$|\lambda|<\frac{\lambda_1}{T\sqrt{K_0}}$$

for each $n=1,2,3,...$ Thus, we find the solution of problem (2)–(4) by formula

$$V(t,x)=\sum_{n=1}^{\infty}V_n(t)z_n(x)=\sum_{n=1}^{\infty}\left(\lambda\int_0^T R_n(t,s,\lambda)\,a_n(s)\,ds+a_n(t)\right)z_n(x)\,. \tag{8}$$

Lemma 1. The generalized solution of problem (2)–(4) which is defined by (7) and its derivatives are elements of the Hilbert space $H(Q_T)$.

Proof. The proof is carried out by direct calculation and does not present any excessive difficulties.

Optimality conditions and a system of nonlinear integral equations

Since each vector control $\overline{u}\,(t,x) = (u_1(t,x),...,u_m(t,x))$ uniquely defines the solution of boundary value problem (2)–(4), the control $\overline{u}(t,x) + \Delta\overline{u}(t,x)$ corresponds to the solution of the problem (2)–(4) of the form $V(t,x) + \Delta V(t,x)$, where $\Delta V(t,x)$ is the increment corresponding to the increment $\Delta\overline{u}(t,x)$. According to the maximum principle [3–6] we calculate the increment of functional (1)

$$\Delta I\,(\overline{u}) = I\,(\overline{u} + \Delta\overline{u}) - I\,(\overline{u}) =$$

$$= -\int_0^T \Delta\Pi\,(t,x,\omega\,(t,x)\,\overline{u}(t,x))\,dt + \int_Q \left\{\Delta V^2(T,x) + \Delta V_t^2(T,x)\right\}dx,$$

where

$$\Pi\,[t,x,V\,(t,x)\,,\omega\,(t,x)\,,\overline{u}\,(t,x)] = \omega\,(t,x)\cdot f\,(t,x,\overline{u}\,(t,x)) - \beta\sum_{k=1}^{m}|u_k(t,x)|. \tag{9}$$

Function $\omega(t,x)$ is the solution of the following conjugate boundary value problem

$$\omega_{tt} - A\omega = \lambda\int_0^T K(\tau,t)\omega(\tau,x)d\tau, x\in Q, 0<t<T,$$

$$\omega(T,x) + 2[V_t(T,x) - \xi_2(x)] = 0, \omega_t(T,x) - 2[V(T,x) - \xi_1(x)] = 0, x\in Q,$$

$$G\omega(t,x) \equiv \sum_{i,j}^{n} a_{ij}(x)\omega_{x_j}(t,x)\cos(\nu,x_i) + a(x)\omega(t,x) = 0,$$

$$x\in\gamma,\ 0<t<T.$$

We investigate the maximum of the function $\Pi\,[t,x,V\,(t,x)\,,\omega\,(t,x)\,,\overline{u}]$ with respect the variables $u_1,u_2,...,u_m$, assuming that the set of allowable values of each of them is an open set. Because of the necessary condition of the extremum, we obtain the following relations from (9)

$$\Pi_{u_i}(.) = \omega(t,x)f_{u_i}[t,x,\overline{u}(t,x)] - \beta signu_i(t,x) = 0, \quad i=1,2,3,...,m.$$

Further, the second necessary condition of the extremum is determined [5–7] by the inequalities $\Delta_1 < 0,\ \Delta_2 > 0,...,\ -1^{(k)}\Delta_k > 0,\ k = 1,2,3,...,m$, according to the Sylvester criterion, where $\triangle i$ are the diagonal determinants of the Hess matrix

$$\Gamma(\Pi,\bar{u}) = \begin{pmatrix} \beta\frac{sign(u_1)}{f_{u_1}}f_{u_1u_1} & ... & \beta\frac{sign(u_1)}{f_{u_1}}f_{u_1u_2} \\ ... & ... & ... \\ \beta\frac{sign(u_m)}{f_{u_m}}f_{u_mu_1} & ... & \beta\frac{sign(u_m)}{f_{u_m}}f_{u_mu_m} \end{pmatrix}.$$

Thus, the components of the optimal vector control $\bar{u}^0(t,x)$ should be found from relation (8), taking into account the second necessary optimality condition. We rewrite condition (8) in the form

$$\omega(t,x) = \beta\frac{signu_1(t,x)}{f_{u_1}[t,x,\bar{u}(t,x)]} = \beta\frac{signu_2(t,x)}{f_{u_2}[t,x,\bar{u}(t,x)]} = ... = \beta\frac{signu_m(t,x)}{f_{u_m}[t,x,\bar{u}(t,x)]} = g(t,x).$$

According to the second necessary condition of extremum and the theorem on implicit functions, we have the following relations

$$u_k^0(t,x) = \varphi_k[t,x,g(t,x),\beta]. \tag{10}$$

The unknown function $g(t,x)$ in (9) is defined as a solution to the following equation

$$g(t,x) = \omega[t,x,f(t,x,\varphi_1(t,x,g(t,x),\beta),...,\varphi_m(t,x,g(t,x),\beta))] = W[g(t,x)] \qquad (11)$$

which it is a Fredholm nonlinear integral equation of the 2nd kind with respect to $g(t,x)$, where the integral has double non-linearity with respect to the function $g(t,x)$. The solvability of a nonlinear integral equation (11) is studied by the method of contraction operators. Let $g_0(t,x)$ is a solution of the equation (11). Substituting this function into (10) we find the desired optimal controls

$$u_k^0(t,x) = \varphi_k[t,x,g_0(t,x),\beta], \quad k = 1,...,m.$$

Next substituting the found values of these controls into (5), we obtain the value of the optimal processes

$$V^0\left(t,x\right) = \sum_{n=1}^{\infty} V_n\left(t\right) z_n\left(x\right) = \sum_{n=1}^{\infty}\left(\lambda\int_0^T R_n\left(t,s,\lambda\right) a_n^0\left(s\right) ds + a_n^0\left(t\right)\right) z_n\left(x\right),$$

where

$$a_{\,n}^{\,0}(t) = \psi_{1n}\cos\lambda_n t + \frac{\psi_{2n}}{\lambda_n}\sin\lambda_n t + \frac{1}{\lambda_n}\int_0^t \sin\lambda_n\left(t-\tau\right)\left[f_n\left[\tau, u_{\,1}^{\,0},...,u_{\,m}^{\,0}\right]\right] d\tau.$$

Substituting the found values of the optimal control and optimal processes into the functional (1), we find the minimum value of the functional (1)

$$J\left[u_1^0(t,x),...u_m^0(t,x)\right] = \int_0^T\int_Q \left\{\left[V^0\left(T,x\right) - \xi_1\left(x\right)\right]^2 + \left[V_t^0\left(T,x\right) - \xi_2\left(x\right)\right]^2\right\} dx +$$

$$+\beta\int_0^T\int_Q \sum_{k=1}^{m}\left|u_k^0(t,x)\right| dxdt,$$

The found triple $\left\{\left(\overline{u}^0\left(t,x\right)\right), V^0\left(t,x\right), I\left(\overline{u}^0\left(t,x\right)\right)\right\}$ is determined as the complete solution of the nonlinear optimization problem.

Conclusion

Solving the problem of the minimization of the piece-wise linear functional is a difficult problem. Therefore the results received in the paper have great scientific value. The developed solving procedure of the formulated problem is constructive and useful in applied problems.

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Тербелмелі процестерді шекаралық векторлық басқарумен сызықтыемес оңтайландыру есебінің шешімі туралы

Мақалада Фредгольм интегралдық операторымен интегралды-дифференциалдық дербес теңдеулер арқылы сипатталған тербелмелі процестерді шекаралық векторлық басқарумен сызықтыемес оңтайландыру есебінің шешімділігі зерттелген. Шекаралық векторлық басқарудың құрамдас бөліктері нақты түрдегі сызықтыемес интегралдық теңдеулер жүйесінің шешімі ретінде анықталатыны және бұл жүйенің теңдеулері тең қатынастық қасиетке ие екендігі анықталды. Сызықтыемес оңтайландыру есебінің шешімін құру алгоритмі жасалды.

Кілт сөздер: жалпыланған шешім, сызықтыемес оңтайландыру, шекаралық векторлық бақылау, функционалды, оңтайлылық шарттары, тең қатынастың қасиеті.

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О разрешимости задачи нелинейной оптимизации при граничном векторном управлении колебательными процессами

В статье исследована разрешимость задачи нелинейной оптимизации при граничном векторном управлении колебательными процессами, описываемыми интегро-дифференциальными уравнениями в частных производных с интегральным оператором Фредгольма. Установлено, что компоненты граничного векторного управления определены как решение системы нелинейных интегральных уравнений специфического вида, и уравнения этой системы обладают свойством равных отношений. Разработан алгоритм построения решения задачи нелинейной оптимизации.

Ключевые слова: обобщенное решение, нелинейная оптимизация, граничное векторное управление, функционал, условия оптимальности, свойство равных отношений.

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