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Attractors of 2D Navier–Stokes system of equations in a locally periodic porous medium

This article deals with two-dimensional Navier–Stokes system of equations with rapidly oscillating terms in the equations and boundary conditions. Studying the problem in a perforated domain, the authors set homogeneous Dirichlet condition on the outer boundary and the Fourier (Robin) condition on the boundary of the cavities. Under such assumptions it is proved that the trajectory attractors of this system converge in some weak topology to trajectory attractors of the homogenized Navier–Stokes system of equations with an additional potential and nontrivial right hand side in the domain without pores. For this aim, the approaches from the works of A.V. Babin, V.V. Chepyzhov, J.-L. Lions, R. Temam, M.I. Vishik concerning trajectory attractors of evolution equations and homogenization methods appeared at the end of the XX-th century are used. First, we apply the asymptotic methods for formal construction of asymptotics, then, we verify the leading terms of asymptotic series by means of the methods of functional analysis and integral estimates. Defining the appropriate axillary functional spaces with weak topology, we derive the limit (homogenized) system of equations and prove the existence of trajectory attractors for this system. Lastly, we formulate the main theorem and prove it through axillary lemmas.

Keywords: attractors, homogenization, system of Navier–Stokes equations, weak convergence, perforated domains, rapidly oscillating terms, porous medium.

Introduction

In this paper, we study the asymptotic behavior of attractors to initial-boundary-value problems for two-dimensional Navier-Stokes systems of equations in perforated domains as the small parameter ε, characterizing the microinhomogeneous structure of the domain, tends to zero.

One can find some results for homogenization problems in perforated domains and a detailed bibliography in monographs [1–3]. This paper presents the case of the appearance of a potential in the limit (homogenized) equation (cf. similar problem in [4–10]).

We study a weak convergence and limit behavior of attractors to the given system of equations as the small parameter converges to zero. There are recent works (cf. [11–13]) on homogenization of attractors used for this study. Overall results on the theory of attractors and the homogenization of attractors cf., for example, in monographs [14–16], and also see the bibliography in these monographs.

We prove that the trajectory attractors $\mathfrak{A}_\varepsilon$ of the two-dimensional Navier–Stokes system of equations in (cf. also [17–19]) a perforated domain weakly converge as $\varepsilon \to 0$ to the trajectory attractor $\overline{\mathfrak{A}}$ to the homogenized system of equations in the corresponding function space. The small parameter ε characterizes the cavity diameter, as well as the distance between cavities in the perforated medium.

In Section 1, we define main notions and formulate theorems on trajectory attractors of autonomous evolution equations. In Section 2, we describe the geometric structure of a perforated domain, formulate

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the problem under consideration, and introduce some function spaces. Section 3 is devoted to homogenization of attractors to the autonomous two-dimensional system of Navier–Stokes equations with rapidly oscillating terms in a perforated domain.

1 Trajectory Attractors of Evolution Equations

We describe a general scheme of constructing trajectory attractors of autonomous evolution equations. This scheme is used in Section 2 to study trajectory attractors of a two-dimensional system of Navier–Stokes equations in a perforated domain with rapidly oscillating terms in equations and boundary conditions and the corresponding homogenized equation.

We consider the abstract autonomous evolution equations

$$\frac{\partial u}{\partial t} = A(u), \quad t \geq 0, \tag{1}$$

where $A(\cdot) : E_1 \to E_0$ is a given nonlinear operator, E_1 and E_0 are Banach spaces such that $E_1 \subseteq E_0$. For example, $A(u) = \nu\Delta u - (u, \nabla u) + g(\cdot)$ (cf. section 2).

We study a solution $u(s)$ to equation (1) globally, as a function of variable $s \in \mathbb{R}_+$. Here, $s \equiv t$ denotes the time-variable. The set of solutions to equation (1) is called the *trajectory space* of equation (1) and is denoted by $\mathcal{K}^+$. We describe the trajectory space $\mathcal{K}^+$ in detail.

First of all, we consider the solution $u(s)$ to equation (1), defined on a fixed time-segment $[t_1, t_2]$ in $\mathbb{R}$. We study solutions to equation (1) in the Banach space $\mathcal{F}_{t_1,t_2}$, which depends on t_1 and t_2. The space $\mathcal{F}_{t_1,t_2}$ consists of functions, $f(s), s \in [t_1, t_2]$, such that $f(s) \in E$ for almost all $s \in [t_1, t_2]$, where E is a Banach space. It is assumed that $E_1 \subseteq E \subseteq E_0$.

For example, for $\mathcal{F}_{t_1,t_2}$ we can take the space $C([t_1, t_2]; E)$ the space $L_p(t_1, t_2; E)$, or $p \in [1, \infty]$, or the intersection of such spaces (cf. section 2). We assume that $\Pi_{t_1,t_2}\mathcal{F}_{\tau_1,\tau_2} \subseteq \mathcal{F}_{t_1,t_2}$ and

$$\|\Pi_{t_1,t_2} f\|_{\mathcal{F}_{t_1,t_2}} \leq C(t_1, t_2, \tau_1, \tau_2)\|f\|_{\mathcal{F}_{\tau_1,\tau_2}}, \quad \forall f \in \mathcal{F}_{\tau_1,\tau_2}, \tag{2}$$

where $[t_1, t_2] \subseteq [\tau_1, \tau_2]$ and Ψ_{t_1,t_2} is the restriction operator on $[t_1, t_2]$. Constant $C(t_1, t_2, \tau_1, \tau_2)$ is independent of f. Usually, one consider the homogeneous case of the space where $C(t_1, t_2, \tau_1, \tau_2) = C(t_2 - t_1, \tau_2 - \tau_1)$.

Let $S(h)$ for $h \in \mathbb{R}$ denote the translation operator

$$S(h)f(s) = f(h + s).$$

It is obvious that if the variable s of $f(\cdot)$ belongs to $[t_1, t_2]$, then the variable s of $S(h)f(\cdot)$ belongs to $[t_1 - h, t_2 - h]$ for $h \in \mathbb{R}$. We assume that the mapping $S(h)$ is an isomorphism from F_{t_1,t_2} to F_{t_1-h,t_2-h} and

$$\|S(h)f\|_{\mathcal{F}_{t_1-h,t_2-h}} = \|f\|_{\mathcal{F}t_1,t_2}, \quad \forall f \in \mathcal{F}_{t_1,t_2}. \tag{3}$$

This assumption is natural, for example, for the homogeneous space.

We assume that if $f(s) \in \mathcal{F}_{t_1,t_2}$, then $A(f(s)) \in \mathcal{D}_{t_1,t_2}$, where $\mathcal{F}_{t_1,t_2} \subseteq \mathcal{D}_{t_1,t_2}$. The derivative $\frac{\partial f(t)}{\partial t}$ is a generalized function taking the values in E_0, $\frac{\partial f}{\partial t} \in D'((t_1, t_2); E_0)$ We assume that $\mathcal{D}_{t_1,t_2} \subseteq D'((t_1, t_2); E_0)$ for all $(t_1, t_2) \subset \mathbb{R}$. A function $u(s) \in \mathcal{F}_{t_1,t_2}$ is called a *solution* to equation (1) in the space $\mathcal{F}_{t_1,t_2}$ (on the interval (t_1, t_2)) if $\frac{\partial u}{\partial t}(s) = A(u(s))$ if in the sense of distributions in $D'((t_1, t_2); E_0)$.

We also introduce the space

$$\mathcal{F}_+^{loc} = \{f(s),\, s \in \mathbb{R}_+ \mid \Pi_{t_1,t_2} f(s) \in \mathcal{F}_{t_1,t_2}, \quad \forall\, [t_1, t_2] \subset \mathbb{R}_+\}. \tag{4}$$

For example, $\mathcal{F}_{t_1,t_2} = C([t_1, t_2]; E)$ implies $\mathcal{F}_+^{loc} = C(\mathbb{R}_+; E)$, and $\mathcal{F}_{t_1,t_2} = L_p(t_1, t_2; E)$, implies $\mathcal{F}_+^{loc} = L_p^{loc}(\mathbb{R}_+; E)$.

A function $u(s) \in \mathcal{F}_+^{loc}$ is a solution to equation (1) in $\mathcal{F}_+^{loc}$, if $\Pi_{t_1,t_2}u(s) \in \mathcal{F}_{t_1,t_2}$, and the function $\Pi_{t_1,t_2}u(s)$ is a solution to equation (1) for any time-segment $[t_1,t_2] \subset \mathbb{R}_+$.

Let $\mathcal{K}^+$ be a set of solutions to equation (1), on the space $\mathcal{F}_+^{loc}$, but does not necessarily coincides with the set of *all* solutions to equation (1) in $\mathcal{F}_+^{loc}$. Elements of $\mathcal{K}^+$ are called *trajectories*, and $\mathcal{K}^+$ is said to be the *the trajectory space* of equation (1).

We assume that the trajectory space $\mathcal{K}^+$ is *translation invariant* in the following sense: if $u(s) \in \mathcal{K}^+$, then $u(h+s) \in \mathcal{K}^+$ for any $h \geq 0$. This condition is natural for solutions to autonomous equations in homogeneous spaces. We consider the translation operators $S(h)$ in $\mathcal{F}_+^{loc}$:

$$S(h)f(s) = f(s+h), \quad h \geq 0.$$

It is clear that $\{S(h), h \geq 0\}$ is a semigroup in $\mathcal{F}_+^{loc}$: $S(h_1)S(h_2) = S(h_1+h_2)$ for $h_1, h_2 \geq 0$ and $S(0) = I$ is the identity mapping. We replace the variable h with the time-variable t. The semigroup $\{S(t), t \geq 0\}$ is called the *translation semigroup*. By assumption, the translation semigroup maps the trajectory space $\mathcal{K}^+$ onto itself:

$$S(t)\mathcal{K}^+ \subseteq \mathcal{K}^+ \quad \forall\, t \geq 0. \tag{5}$$

In what follows, we study the attraction property of the translation semigroup $\{S(t)\}$, acting on the trajectory space $\mathcal{K}^+ \subset \mathcal{F}_+^{loc}$. We introduce a topology in $\mathcal{F}_+^{loc}$.

Let $\rho_{t_1,t_2}(\cdot,\cdot)$ be a group defined on the space $\mathcal{F}_{t_1,t_2}$ for all segments $[t_1,t_2] \subset \mathbb{R}$. As in (2) and (3) we assume that

$$\rho_{t_1,t_2}\left(\Pi_{t_1,t_2}f, \Pi_{t_1,t_2}g\right) \leq D(t_1,t_2,\tau_1,\tau_2)\rho_{\tau_1,\tau_2}\left(f,g\right), \quad \forall f,g \in \mathcal{F}_{\tau_1,\tau_2},\ [t_1,t_2] \subseteq [\tau_1,\tau_2],$$
$$\rho_{t_1-h,t_2-h}(S(h)f, S(h)g) = \rho_{t_1,t_2}(f,g), \quad \forall f,g \in \mathcal{F}_{t_1,t_2},\ [t_1,t_2] \subset \mathbb{R},\ h \in \mathbb{R}.$$

(For a homogeneous space $D(t_1,t_2,\tau_1,\tau_2) = D(t_2-t_2,\tau_2-\tau_1)$.)

We denote by Θ_{t_1,t_2} the corresponding metric space on $\mathcal{F}_{t_1,t_2}$. For example, ρ_{t_1,t_2} can be the metric generated by the norm $\|\cdot\|_{\mathcal{F}_{t_1,t_2}}$ in the Banach space $\mathcal{F}_{t_1,t_2}$. In applications, it can happen that the metric ρ_{t_1,t_2} generates a weaker topology in Θ_{t_1,t_2} than the strong convergence topology in the Banach space $\mathcal{F}_{t_1,t_2}$.

We denote by Θ_+^{loc} the space $\mathcal{F}_+^{loc}$, equipped with the local convergence topology in Θ_{t_1,t_2} for any $[t_1,t_2] \subset \mathbb{R}_+$. More exactly, by definition, a sequence of functions $\{f_k(s)\} \subset \mathcal{F}_+^{loc}$ converges to a function $f(s) \in \mathcal{F}_+^{loc}$ in $k \to \infty$ as Θ_+^{loc}, if $\rho_{t_1,t_2}(\Pi_{t_1,t_2}f_k, \Pi_{t_1,t_2}f) \to 0$ as $k \to \infty$ for any $[t_1,t_2] \subset \mathbb{R}_+$. It is easy to prove that the topology in Θ_+^{loc} in metrizable by using the Frechet metric

$$\rho_+(f_1,f_2) := \sum_{m\in\mathbb{N}} 2^{-m}\frac{\rho_{0,m}(f_1,f_2)}{1+\rho_{0,m}(f_1,f_2)}. \tag{6}$$

If all metric spaces Θ_{t_1,t_2} are complete, then the metric space Θ_+^{loc} is also complete.

We note that the translation semigroup $\{S(t)\}$ is continuous in the topology of the space Θ_+^{loc}. This fact directly follows from the definition of the topological space Θ_+^{loc}.

We define the Banach space

$$\mathcal{F}_+^b := \{f(s) \in \mathcal{F}_+^{loc} \mid \|f\|_{\mathcal{F}_+^b} < +\infty\}, \tag{7}$$

equipped with the norm

$$\|f\|_{\mathcal{F}_+^b} := \sup_{h\geq 0} \|\Pi_{0,1}f(h+s)\|_{\mathcal{F}_{0,1}}. \tag{8}$$

For example, if $\mathcal{F}_+^{loc} = C(\mathbb{R}_+;E)$, then $\mathcal{F}_+^b = C^b(\mathbb{R}_+;E)$ is equipped with the norm $\|f\|_{\mathcal{F}_+^b} = \sup_{h\geq 0}\|f(h)\|_E$, and if $\mathcal{F}_+^{loc} = L_p^{loc}(\mathbb{R}_+;E)$, then $\mathcal{F}_+^b = L_p^b(\mathbb{R}_+;E)$ is equipped with the norm $\|f\|_{\mathcal{F}_+^b} = \left(\sup_{h\geq 0} \int\limits_h^{h+1} \|f(s)\|_E^p ds\right)^{1/p}$.

We note that $\mathcal{F}_+^b \subseteq \Theta_+^{loc}$. The Banach space $\mathcal{F}_+^b$ is necessary to introduce bounded sets in the trajectory space $\mathcal{K}^+$. To construct a trajectory attractor in $\mathcal{K}^+$, we use the weaker local convergence topology in Θ_+^{loc}, instead of the uniform convergences in the topology of the space $\mathcal{F}_+^b$.

We assume that $\mathcal{K}^+ \subseteq \mathcal{F}_+^b$, i.e., any trajectory $u(s) \in \mathcal{K}^+$ of equation (1) has finite norm (8). We define an attracting set and a trajectory attractor of the translation semigroup $\{S(t)\}$, acting on $\mathcal{K}^+$.

Definition 1.1. A set $\mathcal{P} \subseteq \Theta_+^{loc}$ is called an *attracting set* set of the translation semigroup $\{S(t)\}$, acting on $\mathcal{K}^+$, in the topology of Θ_+^{loc}, if for any bounded set $\mathcal{F}_+^b$ in $\mathcal{B} \subseteq \mathcal{K}^+$ the set $\mathcal{P}$ attracts $S(t)\mathcal{B}$ as $t \to +\infty$ in the topology of Θ_+^{loc},i.e., for any ε-neighborhood $O_\varepsilon(\mathcal{P})$ in Θ_+^{loc} there exists $t_1 \geq 0$ such that $S(t)\mathcal{B} \subseteq O_\varepsilon(\mathcal{P})$ for any $t \geq t_1$. The attraction property of $\mathcal{P}$ can be formulated in the equivalent form: for any bounded set $\mathcal{B} \subseteq \mathcal{K}^+$ in $\mathcal{F}_+^b$ and any $M > 0$

$$\mathrm{dist}_{\Theta_{0,M}}(\Pi_{0,M} S(t)\mathcal{B}, \Pi_{0,M}\mathcal{P}) \to 0 \quad (t \to +\infty),$$

where

$$\mathrm{dist}_{\mathcal{M}}(X, Y) := \sup_{x \in X} \mathrm{dist}_{\mathcal{M}}(x, Y) = \sup_{x \in X} \inf_{y \in Y} \rho_{\mathcal{M}}(x, y)$$

where denotes the Hausdorff semi-distance between sets X and Y in the metric space $\mathcal{M}$.

Definition 1.2.([15]) A set $\mathfrak{A} \subseteq \mathcal{K}^+$ is called a *trajectory attractor* of the translation semigroup $\{S(t)\}$ on $\mathcal{K}^+$ in the topology of Θ_+^{loc} if the following conditions are satisfied: **(i)** $\mathfrak{A}$ is bounded in $\mathcal{F}_+^b$ and compact in Θ_+^{loc}, **(ii)** $\mathfrak{A}$ is strictly invariant under the translation semigroup: $S(t)\mathfrak{A} = \mathfrak{A}$ for all $t \geq 0$, and **(iii)** $\mathfrak{A}$ is an attracting set of the translation semigroup in the topology of $\{S(t)\}$ for $\mathcal{K}^+$ in the topology of Θ_+^{loc}, i.e., for any $M > 0$

$$\mathrm{dist}_{\Theta_{0,M}}(\Pi_{0,M} S(t)\mathcal{B}, \Pi_{0,M}\mathfrak{A}) \to 0 \quad (t \to +\infty).$$

Remark 1.1. Using the terminology of [14], we can say that a trajectory attractor $\mathfrak{A}$ is *global* $(\mathcal{F}_+^b, \Theta_+^{loc})$-*attractor* of the translation semigroup $\{S(t)\}$, acting on $\mathcal{K}^+$, i.e., $\mathfrak{A}$ attracts $S(t)\,\mathcal{B}$ as $t \to +\infty$ in the topology of Θ_+^{loc}, where $\mathcal{B}$ is any bounded (in $\mathcal{F}_+^b$) set in $\mathcal{K}^+$:

$$\mathrm{dist}_{\Theta_+^{loc}}(S(t)\,\mathcal{B}, \mathfrak{A}) \to 0 \quad (t \to +\infty).$$

We formulate the main result concerning the existence and structure of a trajectory attractor of equation (1).

Theorem 1.1.([14, 15, 20]) Let the trajectory space $\mathcal{K}^+$, corresponding to equation (1), be closed in $\mathcal{F}_+^b$ and satisfy the condition (5). Let the translation semigroup $\{S(t)\}$ have an attracting set $\mathcal{P} \subseteq \mathcal{K}^+$, that is bounded in $\mathcal{F}_+^b$ and compact in Θ_+^{loc}. Then the translation semigroup $\{S(t), t \geq 0\}$, acting on $\mathcal{K}^+$, has a trajectory attractor $\mathfrak{A} \subseteq \mathcal{P}$. The set $\mathfrak{A}$ is bounded in $\mathcal{F}_+^b$ and compact in Θ_+^{loc}. We describe the structure of trajectory attractors $\mathfrak{A}$ of equation (1) in terms of complete trajectories of this equation. We consider equation (1) on the whole time-axis

$$\frac{\partial u}{\partial t} = A(u), \ t \in \mathbb{R}. \tag{9}$$

Now, we extend the notion of the trajectory space $\mathcal{K}^+$ of equation (9) introduced on $\mathbb{R}_+$. To the case of the whole axis $\mathbb{R}$. If a function $f(s), \ s \in \mathbb{R}$, is given on the whole time-axis, then the translations $S(h)f(s) = f(s+h)$ are also defined for negative h. A function $u(s), s \in \mathbb{R}$ is called a *complete trajectory* of equation (9), if $\Pi_+ u(s+h) \in \mathcal{K}^+$ for any $h \in \mathbb{R}$. Here, $\Pi_+ = \Pi_{0,\infty}$ denotes the operator of restriction onto the half-axis $\mathbb{R}_+$.

We introduced the spaces $\mathcal{F}_+^{loc}, \mathcal{F}_+^b$ and Θ_+^{loc}. Now, we can introduce the space $\mathcal{F}^{loc}, \mathcal{F}^b$ and Θ^{loc} as follows:

$$\mathcal{F}^{loc} := \{f(s), s \in \mathbb{R} \mid \Pi_{t_1,t_2} f(s) \in \mathcal{F}_{t_1,t_2} \ \forall \ [t_1, t_2] \subseteq \mathbb{R}\};$$
$$\mathcal{F}^b := \{f(s) \in \mathcal{F}^{loc} \mid \|f\|_{\mathcal{F}^b} < +\infty\},$$

where

$$\|f\|_{\mathcal{F}^b} := \sup_{h\in\mathbb{R}} \|\Pi_{0,1} f(h+s)\|_{\mathcal{F}_{0,1}}. \tag{10}$$

The topological space Θ^{loc} coincides (as a set) with $\mathcal{F}^{loc}$, and by definition $f_k(s) \to f(s)\ as(k \to \infty)$ in Θ^{loc}, if $\Pi_{t_1,t_2} f_k(s) \to \Pi_{t_1,t_2} f(s)\ as(k \to \infty)$ in Θ_{t_1,t_2} for any $[t_1,t_2] \subseteq \mathbb{R}$. It is clear that Θ^{loc} is a metric space, as well as Θ^{loc}_+.

Definition 1.3. The *kernel* $\mathcal{K}$ in the space $\mathcal{F}^b$ of equation (9) is the union of all complete trajectories $u(s), s \in \mathbb{R}$, of equation (9), that are bounded in $\mathcal{F}^b$ in the norm (10):

$$\|\Pi_{0,1} u(h+s)\|_{\mathcal{F}_{0,1}} \le C_u, \quad \forall h \in \mathbb{R}.$$

Theorem 1.2. Let the assumptions of Theorem 1.1 hold.Then

$$\mathfrak{A} = \Pi_+ \mathcal{K}.$$

The set $\mathcal{K}$ is compact in Θ^{loc} and bounded in $\mathcal{F}^b$.

The full proof is given in [15, 20]. To prove that some ball in $\mathcal{F}^b_+$ is compact in Θ^{loc}_+ we use the following lemma. Let E_0 and E_1 be the Banach spaces such that $E_1 \subset E_0$. We consider the Banach spaces

$$W_{p_1,p_0}(0,M;E_1,E_0) = \left\{\psi(s), s\in 0,M \mid \psi(\cdot) \in L_{p_1}(0,M;E_1),\ \psi'(\cdot) \in L_{p_0}(0,M;E_0)\right\},$$
$$W_{\infty,p_0}(0,M;E_1,E_0) = \left\{\psi(s), s\in 0,M \mid \psi(\cdot) \in L_{\infty}(0,M;E_1),\ \psi'(\cdot) \in L_{p_0}(0,M;E_0)\right\},$$

(where $p_1 \ge 1$ e $p_0 > 1$) with the norms

$$\|\psi\|_{W_{p_1,p_0}} := \left(\int_0^M \|\psi(s)\|^{p_1}_{E_1} ds\right)^{1/p_1} + \left(\int_0^M \|\psi'(s)\|^{p_0}_{E_0} ds\right)^{1/p_0},$$

$$\|\psi\|_{W_{\infty,p_0}} := \operatorname{ess\,sup}\left\{\|\psi(s)\|_{E_1} \mid s \in [0,M]\right\} + \left(\int_0^M \|\psi'(s)\|^{p_0}_{E_0} ds\right)^{1/p_0}.$$

Lemma 1.1.(Aubin-Lions-Simon, [21]) Let $E_1 \Subset E \subset E_0$. Then the following embeddings are compact:

$$W_{p_1,p_0}(0,T;E_1,E_0) \Subset L_{p_1}(0,T;E), \qquad W_{\infty,p_0}(0,T;E_1,E_0) \Subset C([0,T];E).$$

In the next section, two-dimensional systems of Navier-Stokes equations and their trajectory attractors depending on a small parameter $\varepsilon > 0$ will be studied.

Definition 1.4. We say that trajectory attractors $\mathfrak{A}_\varepsilon$ *converge* to a trajectory attractor $\overline{\mathfrak{A}}$ as $\varepsilon \to 0$ in the topological space Θ^{loc}_+, if for any neighborhood $\mathcal{O}(\mathfrak{A})$ in Θ^{loc}_+ there is $\varepsilon_1 \ge 0$ such that $\mathfrak{A}_\varepsilon \subseteq \mathcal{O}(\overline{\mathfrak{A}})$ for any $\varepsilon < \varepsilon_1$, i.e., for any $M > 0$

$$\operatorname{dist}_{\Theta_{0,M}}(\Pi_{0,M}\mathfrak{A}_\varepsilon, \Pi_{0,M}\overline{\mathfrak{A}}) \to 0\ (\varepsilon \to 0).$$

2 Notation and Setting of the Problem

First, we define a perforated domain. Let Ω be a smooth bounded domain $\mathbb{R}^2$. Denote

$$\Upsilon_\varepsilon = \left\{ j \in \mathbb{Z}^2 : \operatorname{dist}(\varepsilon j, \partial\Omega) \ge \sqrt{2}\varepsilon \right\}, \quad \square \equiv \left\{ \xi : -\frac{1}{2} < \xi_k < \frac{1}{2}, k = 1, 2 \right\}.$$

Given an 1-periodic in ξ smooth function $F(x,\xi)$ such that $F(x,\xi)|_{\xi\in\partial\square} \geq \text{const} > 0$, $F(x,0) = -1$, $\nabla_\xi F \neq 0$ as $\xi \in \square \setminus \{0\}$, we set

$$G_j^\varepsilon = \left\{x \in \varepsilon\,(\square + j) \mid F\left(x, \frac{x}{\varepsilon}\right) \leq 0\right\}, \qquad G_\varepsilon = \bigcup_{j\in\Upsilon_\varepsilon} G_j^\varepsilon$$

and introduce the perforated domain as follows:

$$\Omega^\varepsilon = \Omega \setminus G_\varepsilon.$$

Denote by $G(x)$ the domain $G(x,\xi)$ in a stretched space ξ. Afterwards, we often interprete 1-periodic in ξ functions as functions defined on 2-dimensional torus $\mathbb{T}^2 \equiv \{\xi : \xi \in \mathbb{R}^2/\mathbb{Z}^2\}$. According to the above construction, the boundary $\partial\Omega_\varepsilon$ consists of $\partial\Omega$ and the boundary of the cavities $\partial G_\varepsilon \subset \Omega$.

We introduce the function spaces:

$\mathbf{H} := [L_2(\Omega)]^2$, $\mathbf{H}_\varepsilon := [L_2(\Omega_\varepsilon)]^2$, $\mathbf{V} := [H_0^1(\Omega)]^2$, $\mathbf{V}_\varepsilon := [H^1(\Omega_\varepsilon;\partial\Omega)]^2$ is the set of vector-valued functions in $[H^1(\Omega_\varepsilon)]^2$ with zero trace on $\partial\Omega$. The norms in these spaces are defined by

$$\|v\|^2 := \int_\Omega \sum_{i=1}^2 |v^i(x)|^2 dx,\ \|v\|_\varepsilon^2 := \int_{\Omega_\varepsilon} \sum_{i=1}^2 |v^i(x)|^2 dx,$$
$$\|v\|_1^2 := \int_\Omega \sum_{i=1}^2 |\nabla v^i(x)|^2 dx,\ \|v\|_{1\varepsilon}^2 := \int_{\Omega_\varepsilon} \sum_{i=1}^2 |\nabla v^i(x)|^2 dx.$$

We study the asymptotic behavior of trajectory attractors of the following initial-boundary-value problem for the autonomous two-dimensional system of Navier–Stokes equations:

$$\begin{cases} \dfrac{\partial u_\varepsilon}{\partial t} - \nu\Delta u_\varepsilon + (u_\varepsilon,\nabla)u_\varepsilon = g\big(x,\dfrac{x}{\varepsilon}\big), & x \in \Omega_\varepsilon, \\ (\nabla, u_\varepsilon) = 0, & x \in \Omega_\varepsilon, \\ \nu\dfrac{\partial u_\varepsilon}{\partial n} + B\big(x,\dfrac{x}{\varepsilon}\big)u_\varepsilon = h\big(x,\dfrac{x}{\varepsilon}\big), & x \in \partial G_\varepsilon, t \in (0,+\infty), \\ u_\varepsilon = 0, & x \in \partial\Omega \\ u_\varepsilon = U(x), & x \in \Omega_\varepsilon, t = 0. \end{cases} \tag{11}$$

Here $u_\varepsilon = u_\varepsilon(x,t) = (u_\varepsilon^1, u_\varepsilon^2)$, $g_\varepsilon(x) = g\left(x,\frac{x}{\varepsilon}\right) = (g^1,g^2) \in \mathbf{H}$, $h_\varepsilon(x) = h\left(x,\frac{x}{\varepsilon}\right) = (h^1,h^2) \in \mathbf{H}$, n is the outward normal vector to the boundary, and $\nu > 0$.

Further,

$$B(x,\xi) = \begin{pmatrix} b^1(x,\xi) & 0 \\ 0 & b^2(x,\xi) \end{pmatrix},$$

functions $b^k(x,\xi) \in C(\Omega \times \mathbb{R}^2)$ such that $b^k(x,\xi)$ is 1-periodic by variable ξ functions on $\Omega \times \mathbb{R}^2$ and satisfy the condition

$$\int\limits_{\partial G(x)} b^k(x,\xi)\,d\sigma = 0, \qquad k = 1,2,$$

here, $d\sigma$ is the length element of the curve $\partial G(x)$.

Similarly, vector–function components $h(x,\xi)$ satisfy the conditions: $h^k(x,\xi) \in C(\Omega\times\mathbb{R}^2)$, $h^k(x,\xi)$ is 1-periodic by variable ξ functions on $\Omega \times \mathbb{R}^2$ and

$$\int\limits_{\partial G(x)} h^k(x,\xi)\,d\sigma = 0, \qquad k = 1,2.$$

For $U \in \mathbf{H}$, there exists a weak solution $u(s)$ to the problem (11) in the space $\mathbf{L}_{2,w}^{loc}(\mathbb{R}_+;\mathbf{V}_\varepsilon) \cap \mathbf{L}_{\infty,*w}^{loc}(\mathbb{R}_+;\mathbf{H}_\varepsilon)$, such that $u(0) = U$. Moreover $\dfrac{\partial u_\varepsilon}{\partial t} \in \mathbf{L}_{2,w}^{loc}(\mathbb{R}_+;\mathbf{H}_\varepsilon)$. We consider weak solutions to the problem, i.e., [20], [22].

This satisfies the problem (11) in the sense of distributions, i.e.,

$$u_\varepsilon(x,s) \in \mathbf{L}_{2,w}^{loc}(\mathbb{R}_+;\mathbf{V}_\varepsilon) \cap \mathbf{L}_{\infty,*w}^{loc}(\mathbb{R}_+;\mathbf{H}_\varepsilon) \cap \left\{ v : \frac{\partial u_\varepsilon}{\partial t} \in \mathbf{L}_{2,w}^{loc}(\mathbb{R}_+;\mathbf{H}_\varepsilon) \right\}$$

that satisfy the problem (11) in the sense of distributions, i.e.,

$$\int\limits_{Q_\varepsilon} \frac{\partial u_\varepsilon}{\partial t} \cdot \psi \, dxdt + \nu \int\limits_{Q_\varepsilon} \nabla u_\varepsilon \cdot \nabla \psi \, dxdt + \int\limits_{Q_\varepsilon} (u_\varepsilon, \nabla) u_\varepsilon \cdot \psi \, dxdt +$$

$$+ \sum_{j \in \Upsilon_\varepsilon} \int\limits_0^{+\infty} \int\limits_{\partial G_\varepsilon^j} B\big(x, \frac{x}{\varepsilon}\big) u_\varepsilon \cdot \psi \, d\sigma dt = \int\limits_{Q_\varepsilon} g_\varepsilon(x) \cdot \psi \, dxdt + \sum_{j \in \Upsilon_\varepsilon} \int\limits_0^{+\infty} \int\limits_{\partial G_\varepsilon^j} h_\varepsilon(x) \cdot \psi \, d\sigma dt$$

for any function $\psi \in \mathbf{C}_0^\infty(\mathbb{R}_+;\mathbf{H}_\varepsilon)$. Here $y_1 \cdot y_2$ denotes the inner product vectors $y_1, y_2 \in \mathbb{R}^2$.

To describle the trajectory space $\mathcal{K}_\varepsilon^+$ of the problem (11), we follow the general scheme of and on every segment, introduce the Banach space $[t_1, t_2] \in \mathbb{R}$

$$\mathcal{F}_{t_1,t_2} := \mathbf{L}_{2,w}^{loc}(t_1,t_2;\mathbf{V}_\varepsilon) \cap \mathbf{L}_{\infty,*w}^{loc}(t_1,t_2;\mathbf{H}_\varepsilon) \cap \left\{ v : \frac{\partial u_\varepsilon}{\partial t} \in \mathbf{L}_{2,w}^{loc}(t_1,t_2;\mathbf{H}_\varepsilon) \right\}$$

equipped with the norm

$$\|v\|_{\mathcal{F}_{t_1,t_2}} := \|v\|_{\mathbf{L}_2(t_1,t_2;\mathbf{V})} + \|v\|_{\mathbf{L}_\infty(t_1,t_2;\mathbf{H})} + \left\| \frac{\partial v}{\partial t} \right\|_{\mathbf{L}_2(t_1,t_2;\mathbf{H})}. \tag{12}$$

It is obvious that the condition (2) holds for the norm (12) and the translation semigroup $\{S(h)\}$ satisfies (3).

Setting $\mathcal{D}_{t_1,t_2} = \mathbf{L_2}(t_1,t_2;\mathbf{V})$ we find that $\mathcal{F}_{t_1,t_2} \subseteq \mathcal{D}_{t_1,t_2}$, if $u(s) \in \mathcal{F}_{t_1,t_2}$, then $A(u(s)) \in \mathcal{D}_{t_1,t_2}$. Further, we can consider a weak solution to the problem (11) as a solution to the system of equations in accordance with the general scheme .

Introducing the space (4), we find

$$\mathcal{F}_+^{loc} = \mathbf{L}_2^{loc}(\mathbb{R}_+;\mathbf{V}) \cap \mathbf{L}_\infty^{loc}(\mathbb{R}_+;\mathbf{H}) \cap \left\{ v \;\middle|\; \frac{\partial v}{\partial t} \in \mathbf{L}_2^{loc}(\mathbb{R}_+;\mathbf{H}) \right\},$$

$$\mathcal{F}_{\varepsilon,+}^{loc} = \mathbf{L}_2^{loc}(\mathbb{R}_+;\mathbf{V}_\varepsilon) \cap \mathbf{L}_\infty^{loc}(\mathbb{R}_+;\mathbf{H}_\varepsilon) \cap \left\{ v \;\middle|\; \frac{\partial v}{\partial t} \in \mathbf{L}_2^{loc}(\mathbb{R}_+;\mathbf{H}_\varepsilon) \right\}.$$

We denote by $\mathcal{K}_\varepsilon^+$ a set of all weak solutions to the problem (11). We recall that for any function $U \in \mathbf{H}$ there exists at least one trajectory $u(\cdot) \in \mathcal{K}_\varepsilon^+$ such that $u(0) = U(x)$. Consequently, the trajectory space $\mathcal{K}_\varepsilon^+$ of the problem (11) is not empty.

It is clear that $\mathcal{K}_\varepsilon^+ \subset \mathcal{F}_+^{loc}$ and the trajectory space $\mathcal{K}_\varepsilon^+$ is translation invariant, i.e., if $u(s) \in \mathcal{K}_\varepsilon^+$, then and $u(h+s) \in \mathcal{K}_\varepsilon^+$ for any $h \geq 0$. Consequently,

$$S(h)\mathcal{K}_\varepsilon^+ \subseteq \mathcal{K}_\varepsilon^+, \quad \forall h \geq 0.$$

Further, using the $\mathbf{L}_2(t_1,t_2;\mathbf{V})$–norms, we introduce the metrics $\rho_{t_1,t_2}(\cdot,\cdot)$ in the spaces $\mathcal{F}_{t_1,t_2}$ as follows:

$$\rho_{0,M}(u,v) = \left(\int_0^M \|u(s) - v(s)\|^2 ds \right)^{1/2}, \quad \forall\, u(\cdot), v(\cdot) \in \mathcal{F}_{0,M}.$$

These metrics generate the topology of Θ_+^{loc} in the space $\mathcal{F}_+^{loc}$ (respectively $\Theta_{\varepsilon,+}^{loc}$ in $\mathcal{F}_{\varepsilon,+}^{loc}$). We recall that a sequence $\{v_k\} \subset \mathcal{F}_+^{loc}$ converges to a function $v \in \mathcal{F}_+^{loc}$ as $k \to \infty$ in Θ_+^{loc}, if $\|v_k(\cdot) - v(\cdot)\|_{\mathbf{L}_2(0,M;\mathbf{H})} \to 0$ $(k \to \infty)$ for any $M > 0$. The topology of Θ_+^{loc} is metrizable (nf. (6)) and the corresponding metric space is complete. We consider the topology in the trajectory space $\mathcal{K}_\varepsilon^+$ of the problem (11). The translation semigroup $\{S(t)\}$, acting on $\mathcal{K}_\varepsilon^+$ is continuous in the topology of the space Θ_+^{loc}.

Following the general scheme of 1, we consider the bounded set in $\mathcal{K}_\varepsilon^+$ by using the Banach space $\mathcal{F}_+^b$ (cf. (7)). It is clear that

$$\mathcal{F}_+^b = \mathbf{L}_2^b(\mathbb{R}_+; \mathbf{V}) \cap \mathbf{L}_\infty(\mathbb{R}_+; \mathbf{H}) \cap \left\{ v \;\middle|\; \frac{\partial v}{\partial t} \in \mathbf{L}_2^b(\mathbb{R}_+; \mathbf{H}) \right\}$$

and $\mathcal{F}_+^b$ is a subspace of the space $\mathcal{F}_+^{loc}$.

We consider the translation semigroup $\{S(t)\}$ on $\mathcal{K}_\varepsilon^+$, $S(t) : \mathcal{K}_\varepsilon^+ \to \mathcal{K}_\varepsilon^+$, $t \geq 0$.

Let $\mathcal{K}_\varepsilon$ denote the kernel of the problem (11), consisting of all weak solutions $u(s), s \in \mathbb{R}$ bounded in the space

$$\mathcal{F}^b = \mathbf{L}_2^b(\mathbb{R}; \mathbf{V}) \cap \mathbf{L}_\infty(\mathbb{R}; \mathbf{H}) \cap \left\{ v \;\middle|\; \frac{\partial v}{\partial t} \in \mathbf{L}_2^b(\mathbb{R}; \mathbf{H}) \right\}$$

Proposition 2.1. The problem (11) has trajectory attractors $\mathfrak{A}_\varepsilon$ in the topological space Θ_+^{loc}. The set $\mathfrak{A}_\varepsilon$ is uniformly (with respect to $\varepsilon \in (0,1)$) bounded in $\mathcal{F}_+^b$ and compact in Θ_+^{loc}. Furthermore,

$$\mathfrak{A}_\varepsilon = \Pi_+ \mathcal{K}_\varepsilon,$$

the kernel $\mathcal{K}_\varepsilon$ is nonempty and uniformly (with respect to $\varepsilon \in (0,1)$) bounded in $\mathcal{F}^b$. We recall that the spaces $\mathcal{F}_+^b$ and Θ_+^{loc} depend on ε.

The proof of Proposition 2.1 is similar to the proof in [15] given in a particular case.

3 Homogenization of attractors of initial boundary value problem for the Navier-Stokes system of equations in a perfected domain

3.1 The main assertion

In this subsection, we study the limit behavior of attractors $\mathfrak{A}_\varepsilon$ of the Navier-Stokes equations (11) as $\varepsilon \to 0+$ as and their convergence to a trajectory attractor of the corresponding homogenized equation.

The homogenized (limit) problem has the form:

$$\begin{cases} \dfrac{\partial u_0}{\partial t} - \nu \displaystyle\sum_{i,l=1}^{2} \frac{\partial}{\partial x_i} \left(\widehat{a}_{il}(x) \frac{\partial u_0}{\partial x_l} \right) + (u_0, \nabla) u_0 + V(x) u_0 = \bar{g}(x) + H(x), & x \in \Omega, \\ (\nabla, u_0) = 0, & x \in \Omega, \\ u_0 = 0, & x \in \partial\Omega \\ u_0 = U(x), & x \in \Omega, t = 0, \end{cases} \tag{13}$$

where

$$\widehat{a}_{il}(x) = \int\limits_{Y \setminus G(x)} \left(\frac{\partial N_l(x,\xi)}{\partial \xi_i} + \delta_{il} \right) d\xi, \qquad \bar{g}(x) = \int\limits_{Y \setminus G(x)} g(x,\xi)\, d\xi,$$

$$m_k(x) = -\int\limits_{\partial G(x)} b^k(x,\xi) M^k(x,\xi)\, d\sigma, \qquad V(x) = \begin{pmatrix} m_1(x) & 0 \\ 0 & m_2(x) \end{pmatrix},$$

$$H_k(x) = -\int\limits_{\partial G(x)} h^k(x,\xi) M^k(x,\xi)\, d\sigma, \qquad H(x) = \begin{pmatrix} H_1(x) \\ H_2(x) \end{pmatrix}.$$

Here $M^k(\xi)$ and $N_l(\xi)$ are 1-periodic functions of ξ satisfying the problems

$$\Delta M^k = 0 \text{ in } Y \setminus G(x), \qquad \frac{\partial M^k}{\partial n} = -b^k(x,\xi) \text{ on } \partial G(x),$$

$$\Delta N_l = 0 \text{ in } Y \setminus G(x), \qquad \frac{\partial N_l}{\partial n} = -n_l \text{ on } \partial G(x)$$

and having zero mean over the periodicity cell.

We consider the weak solution to the problem (13), i.e., a function

$$u_0(x,s) \in \mathbf{L}^{loc}_{2,w}(\mathbb{R}_+;\mathbf{V}) \cap \mathbf{L}^{loc}_{\infty,*w}(\mathbb{R}_+;\mathbf{H}) \cap \left\{ v : \frac{\partial u_\varepsilon}{\partial t} \in \mathbf{L}^{loc}_{2,w}(\mathbb{R}_+;\mathbf{H}) \right\}$$

satisfying the problem (11) in the sense of distributions,i.e.,

$$\int\limits_Q \frac{\partial u_0}{\partial t} \cdot \psi \; dxdt + \nu \int\limits_Q \sum_{i,l=1}^{2} \widehat{a}_{il}(x) \frac{\partial u_0}{\partial x_i} \cdot \frac{\partial \psi}{\partial x_l} \; dxdt + \int\limits_Q (u_0, \nabla) u_0 \cdot \psi \; dxdt +$$

$$+ \int\limits_Q V u_0 \cdot \psi \; dxdt = \int\limits_Q \bar{g}(x) \cdot \psi \; dxdt + \int\limits_Q H \cdot \psi \; dxdt$$

for any function $\psi \in \mathbf{C}^\infty_0(\mathbb{R}_+;\mathbf{H})$.

Remark 3.1. Denote by $m_k = \sup\limits_{\Omega} m_k(x)$.The coercivity of the limit operator (13), is a delicate problem since the constants m_k are always positive. In particular, the well-posedness of the problem (13), connected with the coercivity of the operator is guaranteed by the inequalities

$$\lambda_0 > \max\{m_1, m_2\}, \tag{14}$$

where λ_0 is the first eigenvalue of the operator $\nu \sum\limits_{i,l=1}^{2} \frac{\partial}{\partial x_i}\left(\widehat{a}_{il}(x) \frac{\partial}{\partial x_l} \right)$ in the space $H^1(\Omega)$. The proof of this assertion can be found in [8].

Under the condition (14) (cf. remark 3.1) the problem (13) has a trajectory attractor $\overline{\mathfrak{A}}$ in the trajectory space $\overline{\mathcal{K}}^+$, of the problem (13); moreover,

$$\overline{\mathfrak{A}} = \Pi_+ \overline{\mathcal{K}}$$

whera $\overline{\mathcal{K}}$ is the kernel of the problem (13) in $\mathcal{F}^b$.

We formulate the main theorem on homogenization of attractors of the system of Navier–Stokes equations.

Theorem 3.1. Let $\lambda_0 > \max\{m_1, m_2\}$, then is topological space Θ^{loc}_+ correctly limited relation

$$\mathfrak{A}_\varepsilon \to \overline{\mathfrak{A}} \quad \text{if} \quad \varepsilon \to 0+. \tag{15}$$

Moreover,

$$\mathcal{K}_\varepsilon \to \overline{\mathcal{K}} \;\text{ if }\; \varepsilon \to 0+ \;\text{ in } \Theta^{loc}. \tag{16}$$

Remark 3.2. We recall that the spaces in theorem 3.1 depend on ε. We assume that all functions under consideration can be extended over the holes with preserving the norms.

3.2 Auxiliaries

We use some results of [8] below.

We consider the auxiliary problem

$$\begin{cases} -\nu\Delta_{xx}u_\varepsilon^k = g^k\big(x,\frac{x}{\varepsilon}\big), & x\in\Omega_\varepsilon,\\ \nu\frac{\partial u_\varepsilon^k}{\partial n} + b^k\big(x,\frac{x}{\varepsilon}\big)u_\varepsilon^k = h^k\big(x,\frac{x}{\varepsilon}\big), & x\in\partial G_\varepsilon, \qquad k=1,2.\\ u_\varepsilon^k = 0, & x\in\partial\Omega. \end{cases} \tag{17}$$

We also require that

$$\int_{\partial G(x)} b^k(x,\xi)d\sigma = 0, \qquad \int_{\partial G(x)} h^k(x,\xi)d\sigma = 0. \tag{18}$$

We look for a solution in the form of a series

$$u_\varepsilon^k = u_0^k(x) + \varepsilon u_1^k(x,\xi) + \varepsilon^2 u_2^k(x,\xi) + \dots, \qquad \xi = \frac{x}{\varepsilon}. \tag{19}$$

Substituting the series (19) into (17) and collecting terms with ε , of the same order in the equation and boundary conditions, we find a recurrent sequence of problems such that the first one has the form

$$\begin{cases} -\nu\Delta_{\xi\xi}u_1^k + \frac{\partial^2 u_0^k}{\partial\xi_1\partial x_1} + \frac{\partial^2 u_0^k}{\partial\xi_2\partial x_2} = 0, & x\in Y\setminus G(x),\\ \nu\frac{\partial u_1^k}{\partial n_\xi} + \nu\frac{\partial u_0^k}{\partial n_x} + b^k\big(x,\xi\big)u_0^k = h^k(x,\xi), & x\in\partial G(x). \end{cases} \tag{20}$$

The integral identity for the problem (20) is a follows:

$$\iint_{Y\setminus G(x)} \left(\frac{\partial u_1^k}{\partial\xi_1}\frac{\partial v}{\partial\xi_1} + \frac{\partial u_1^k}{\partial\xi_2}\frac{\partial v}{\partial\xi_2}\right)d\xi_1 d\xi_2 + \iint_{Y\setminus G(x)} \left(\frac{\partial u_0^k}{\partial x_1}\frac{\partial v}{\partial\xi_1} + \frac{\partial u_0^k}{\partial x_2}\frac{\partial v}{\partial\xi_2}\right)d\xi_1 d\xi_2 + \\ + \int_{\partial G(x)} b^k\big(x,\xi\big)u_0^k v d\sigma = \int_{\partial G(x)} h^k\big(x,\xi\big)v d\sigma, \tag{21}$$

where $v\in H^1_{per}(Y\setminus G(x))$.

From the form of the integral identity we can propose that the functions $u_1^k(x,\xi)$ have the following structure:

$$u_1^k(x,\xi) = L^k(\xi) + M^k(\xi)u_0^k(x) + N_1(\xi)\frac{\partial u_0^k}{\partial x_1} + N_2(\xi)\frac{\partial u_0^k}{\partial x_2}.$$

Substituting the last expression into (21) and collecting the corresponding terms, we obtain the following problem for the functions $N_l(\xi)$ and $M^k(\xi)$:

$$\iint_{Y\setminus G(x)} \left(\frac{\partial N_l}{\partial\xi_1}\frac{\partial v}{\partial\xi_1} + \frac{\partial N_l}{\partial\xi_2}\frac{\partial v}{\partial\xi_2}\right)d\xi_1 d\xi_2 + \iint_{Y\setminus G(x)} \frac{\partial v}{\partial\xi_1}d\xi_1 d\xi_2 = 0, \tag{22}$$

or, in the classical form:

$$\begin{cases} \Delta_{\xi\xi}(N_l+\xi_l) = 0, & x\in Y\setminus G(x),\\ \frac{\partial N_l}{\partial n_\xi} = n_l, & x\in\partial G(x); \end{cases}$$

$$\iint_{Y\setminus G(x)} \left(\frac{\partial M^k}{\partial\xi_1}\frac{\partial v}{\partial\xi_1} + \frac{\partial M^k}{\partial\xi_2}\frac{\partial v}{\partial\xi_2}\right)d\xi_1 d\xi_2 + \int_{\partial G(x)} b^k\big(x,\xi\big)v d\sigma = 0 \tag{23}$$

or

$$\begin{cases} \Delta_{\xi\xi}M^k = 0, & x\in Y\setminus G(x),\\ \frac{\partial M^k}{\partial n_\xi} + b^k\big(x,\xi\big) = 0, & x\in\partial G(x); \end{cases}$$

$$\iint_{Y\setminus G(x)} \left(\frac{\partial L^k}{\partial \xi_1} \frac{\partial v}{\partial \xi_1} + \frac{\partial L^k}{\partial \xi_2} \frac{\partial v}{\partial \xi_2} \right) d\xi_1 d\xi_2 = \int_{\partial G(x)} h^k(x,\xi) v d\sigma \tag{24}$$

or

$$\begin{cases} \Delta_{\xi\xi} L^k = 0, & x \in Y \setminus G(x), \\ \frac{\partial L^k}{\partial n_\xi} = h^k(x,\xi), & x \in \partial G(x). \end{cases}$$

The compatibility condition in the problem (22) can be easily verified by integrating by parts and using (18) in the problems (23) and (24). We note that the functions $L^k(\xi)$, $M^k(\xi)$, and $N_l(\xi)$ are defined up to an additive constant and the natural normalization conditions are the following:

$$\iint_{Y\setminus G(x)} L^k(\xi) d\xi = \iint_{Y\setminus G(x)} M^k(\xi) d\xi = \iint_{Y\setminus G(x)} N_l(\xi) d\xi = 0.$$

In what follows, we assume that these conditions are satisfied.

The next power of εyields the problem for $u_2^k(x,\xi)$:

$$\begin{cases} \Delta_{\xi\xi} u_2^k + 2\left(\frac{\partial^2 u_1^k}{\partial \xi_1 \partial x_1} + \frac{\partial^2 u_1^k}{\partial \xi_2 \partial x_2} \right) + \Delta_{xx} u_0^k = -g^k, & x \in Y \setminus G(x), \\ \frac{\partial u_2^k}{\partial n_\xi} + \frac{\partial u_1^k}{\partial n_x} + b^k(x, \frac{x}{\varepsilon}) u_1^k + h^k(x, \frac{x}{\varepsilon}) u_0^k = 0, & x \in \partial G(x). \end{cases} \tag{25}$$

The following statement is true.

Lemma 3.1. The functions $M^k(\xi)$ and $N_l(\xi)$ are connected by the integral identity

$$\frac{\partial u_0^k(x)}{\partial x_l} \left(\iint_{Y\setminus G(x)} \frac{\partial M^k}{\partial \xi_l} d\xi_1 d\xi_2 - \int_{\partial G(x)} b^k N_l d\sigma \right) = 0.$$

We also need the integral identity corresponding to the problem (25)

$$\iint_{Y\setminus G(x)} \left(\frac{\partial u_2^k}{\partial \xi_1} \frac{\partial v}{\partial \xi_1} + \frac{\partial u_2^k}{\partial \xi_2} \frac{\partial v}{\partial \xi_2} \right) d\xi_1 d\xi_2 + \iint_{Y\setminus G(x)} \left(\frac{\partial u_1^k}{\partial x_1} \frac{\partial v}{\partial \xi_1} + \frac{\partial u_1^k}{\partial x_2} \frac{\partial v}{\partial \xi_2} \right) d\xi_1 d\xi_2 +$$
$$+ \int_{\partial G(x)} b^k(\xi) u_1^k v d\sigma + u_0^k(x) \int_{\partial G(x)} h^k(x,\xi) v d\sigma - \iint_{Y\setminus G(x)} \frac{\partial M^k}{\partial \xi_1} v d\xi_1 d\xi_2 \cdot \frac{\partial u_0^k}{\partial x_1} -$$
$$- \iint_{Y\setminus G(x)} \frac{\partial M^k}{\partial \xi_2} v d\xi_1 d\xi_2 \cdot \frac{\partial u_0^k}{\partial x_2} - \iint_{Y\setminus G(x)} \left(\frac{\partial N_1}{\partial \xi_1} + 1 \right) v d\xi_1 d\xi_2 \cdot \frac{\partial^2 u_0^k}{\partial x_1^2} -$$
$$- \iint_{Y\setminus G(x)} \left(\frac{\partial N_1}{\partial \xi_2} + \frac{\partial N_2}{\partial \xi_1} \right) v d\xi_1 d\xi_2 \cdot \frac{\partial^2 u_0^k}{\partial x_1 \partial x_2} - \iint_{Y\setminus G(x)} \left(\frac{\partial N_2}{\partial \xi_2} + 1 \right) v d\xi_1 d\xi_2 \cdot \frac{\partial^2 u_0^k}{\partial x_2^2} + \bar{g}^k = 0,$$

where $\bar{g}^k(x) = \iint_{Y\setminus G(x)} g^k(x,\xi) d\xi_1 d\xi_2$.

The solvability condition for the problem (25) leads to the equations for $u_0^k(x)$, which is the required formal homogenized equations. Applying Lemma 3.1, and considering the connection between $b(x,\xi)$ and $h(x,\xi)$ we can write it in the form

$$\nu \sum_{i,l=1}^{2} \frac{\partial}{\partial x_i} \left(\widehat{a}_{il}(x) \frac{\partial u_0^k}{\partial x_l} \right) - \int\limits_{\partial G(x)} b^k(x,\xi) M^k(\xi) d\sigma\, u_0^k(x) = \bar{g}^k(x) + \int\limits_{\partial G(x)} h^k(x,\xi) M^k(\xi) d\sigma,$$

where

$$\widehat{a}_{il}(x) = \iint_{Y\setminus G(x)} \left(\frac{\partial N_l}{\partial \xi_i} + \delta_{il} \right) d\xi_1 d\xi_2, \quad \delta_{il} \text{ is the Kroneker symbol.}$$

Thus, the homogenized problem can be written as

$$\begin{cases} \nu \sum_{i,l=1}^{2} \frac{\partial}{\partial x_i}\Big(\widehat{a}_{il}(x)\frac{\partial u_0^k}{\partial x_l}\Big) - m^k(x)u_0^k(x) = \bar{g}^k(x) + H^k(x), & x \in \Omega, \\ u_0^k(x) = 0, & x \in \partial\Omega, \end{cases} \tag{26}$$

where as $k = 1, 2$ we have $m^k(x) = \int_{\partial G(x)} b^k(x,\xi)M^k(x,\xi)d\sigma$,

$$H^k(x) = \int_{\partial G(x)} h^k(x,\xi)M^k(x,\xi)d\sigma = -\int_{\partial G(x)} b^k(x,\xi)L^k(x,\xi)d\sigma.$$

The following lemma is true (cf. [8]).

Lemma 3.2. If u_ε is a solution to the problem (17), and u_0 is a solution to the problem (26), then there is convergence

$$\nu \int\limits_{Q_\varepsilon} \nabla u_\varepsilon \cdot \nabla\psi \; dxdt + \sum_{j\in\Upsilon_\varepsilon} \int\limits_0^{+\infty} \int\limits_{\partial G_\varepsilon^j} B\big(x, \frac{x}{\varepsilon}\big)u_\varepsilon \cdot \psi \; d\sigma dt - \sum_{j\in\Upsilon_\varepsilon} \int\limits_0^{+\infty} \int\limits_{\partial G_\varepsilon^j} h\big(x, \frac{x}{\varepsilon}\big) \cdot \psi \; d\sigma dt -$$

$$-\int\limits_{Q_\varepsilon} g\big(x, \frac{x}{\varepsilon}\big)u_\varepsilon \cdot \psi \; d\sigma dt \longrightarrow \nu \int\limits_Q \widehat{a}\nabla u_0 \cdot \nabla\psi \; dxdt + \int\limits_Q V u_0 \cdot \psi \; dxdt - \int\limits_Q H \cdot \psi \; dxdt - \int\limits_Q \bar{g} \cdot \psi \; dxdt$$

as $\varepsilon \to 0$.

Following [23] and taking into account Remark 3.2 , we show that

$$(u_\varepsilon, \nabla)u_\varepsilon \longrightarrow (u, \nabla)u \qquad \text{strongly in} \qquad L_2(Q). \tag{27}$$

For this purpose we use the estimate

$$\|(u_\varepsilon, \nabla)u_\varepsilon - (u, \nabla)u\|_{L_2(Q)} \le \|(u_\varepsilon - u, \nabla)u_\varepsilon\|_{L_2(Q)} + \|(u, \nabla)(u_\varepsilon - u)\|_{L_2(Q)} \le$$

$$\le C\Big(\int\limits_Q |u_\varepsilon - u|^2|\nabla u_\varepsilon|^2 dxds\Big)^{\frac{1}{2}} + C\Big(\int\limits_Q |u|^2|\nabla(u_\varepsilon - u)|^2 dxds\Big)^{\frac{1}{2}} \le$$

$$\le C_1\Big(\int\limits_Q |\nabla u_\varepsilon|^3 dxds\Big)^{\frac{1}{3}}\Big(\int\limits_Q |u_\varepsilon - u|^6 dxds\Big)^{\frac{1}{6}} + C_1\Big(\int\limits_Q |u|^6 dxds\Big)^{\frac{1}{6}}\Big(\int\limits_Q |\nabla(u_\varepsilon - u)|^3 dxds\Big)^{\frac{1}{3}}.$$

As proved in [15] the trajectory attractors $\mathfrak{A}_\varepsilon$ and $\overline{\mathfrak{A}}$ of equations (11) and (13) exist in the following space with a stronger topology: $H_w^{(2,2,1)}(Q)$, where

$$H_w^{(2,2,1)}(Q) = L_{2,w}\left(\mathbb{R}_+; \left[W_2^2(\Omega)\right]^2\right) \cap \left\{v : \frac{\partial v}{\partial t} \in L_{2,w}(\mathbb{R}_+; \mathbf{H})\right\}.$$

We set

$$H_{3,w}^{(1,1,0)}(Q) = L_{3,w}\left(\mathbb{R}_+; \left[W_3^1(\Omega)\right]^2\right).$$

Since $H^{(2,2,1)}(Q) \Subset H_3^{(1,1,0)}(Q)$ and $H^{(2,2,1)}(Q) \Subset L_6(Q)$, we find

$$\int\limits_Q |u_\varepsilon - u|^6 dxds \to 0, \qquad \int\limits_Q |\nabla(u_\varepsilon - u)|^3 dxds \to 0$$

as $\varepsilon \to 0$. Here, we used the uniform boundedness of the integral $\int\limits_Q |\nabla u_\varepsilon|^3 dxds \le M$. Thus, we have proved the convergence (27).

3.3 Proof of Theorem 3.1

Proof. It is clear that (16) implies (15). Therefore, it suffices to prove (16), i.e., for any neighborhood $\mathcal{O}(\overline{\mathcal{K}})$ in Θ^{loc} there is $\varepsilon_1 = \varepsilon_1(\mathcal{O}) > 0$ such that

$$\mathcal{K}_\varepsilon \subset \mathcal{O}(\overline{\mathcal{K}}) \text{ for all } \varepsilon < \varepsilon_1. \tag{28}$$

If (28) fails, then there exists a neighborhood $\mathcal{O}'(\overline{\mathcal{K}})$ in Θ^{loc}, a sequence $\varepsilon_k \to 0+$ $(k \to \infty)$ and a sequence $u_{\varepsilon_k}(\cdot) = u_{\varepsilon_k}(s) \in \mathcal{K}_{\varepsilon_k}$ such that

$$u_{\varepsilon_k} \notin \mathcal{O}'(\overline{\mathcal{K}}) \ \text{ for all } k \in \mathbb{N}. \tag{29}$$

The sequence $\left\{g\left(x, \frac{x}{\varepsilon_n}\right)\right\}$ is bounded in $\mathbf{H}$. Consequently, using the integral identity and the Cauchy-Bunyakovsky inequality, we conclude that the sequence of solutions $\{u_{\varepsilon_n}\}$ is bounded in $\mathcal{F}^b$. Passing to a subsequence, we can assume that

$$u_{\varepsilon_n} \to u_0 \ (n \to \infty) \ \text{ in } \Theta^{loc}.$$

We assert that $u_0 \in \overline{\mathcal{K}}$. The functions $u_{\varepsilon_n}(x,s)$ satisfy the equation

$$\frac{\partial u_{\varepsilon_n}}{\partial t} - \nu\Delta u_{\varepsilon_n} + (u_{\varepsilon_n}, \nabla)u_{\varepsilon_n} = g\big(x, \frac{x}{\varepsilon_n}\big), \quad t \in \mathbb{R}, \tag{30}$$

the condition

$$\nu\frac{\partial u_{\varepsilon_n}}{\partial n} + B\big(x, \frac{x}{\varepsilon_n}\big)u_{\varepsilon_n} = h\big(x, \frac{x}{\varepsilon_n}\big), \quad x \in \partial G_{\varepsilon_n},$$

and the energy identity

$$-\frac{1}{2}\int_{-M}^{M} \|u_{\varepsilon_n}(s)\|^2_{\mathbf{H}}\psi'(s)ds + \nu\int_{-M}^{M} \|u_{\varepsilon_n}(s)\|^2_{\mathbf{V}}\psi(s)ds + \sum_{j\in\Upsilon_\varepsilon}\int_{-M}^{M}\int\limits_{\partial G^j_\varepsilon} B(x,\xi)u^1_{\varepsilon_n}(x,s)\psi(s)d\sigma ds -$$

$$-\sum_{j\in\Upsilon_\varepsilon}\int_{-M}^{M}\int\limits_{\partial G^j_\varepsilon} h(x,\xi)\psi(s)d\sigma ds = \int_{-M}^{M} (g(x,\xi), u_\varepsilon(x,s))_{\mathbf{H}}\,\psi(s)ds \tag{31}$$

for any $M > 0$ and any function $\psi \in C_0^\infty(]-M, M[)$, $\psi \ge 0$. Furthermore, $u_{\varepsilon_n}(s) \rightharpoonup u_0(s)$ $(n \to \infty)$ weakly in $\mathbf{L}_2(-M, M; \mathbf{V})$, $*$-weakly in $\mathbf{L}_\infty(-M, M; \mathbf{H})$ and $\frac{\partial u_{\varepsilon_n}(s)}{\partial t} \rightharpoonup \frac{\partial u_0(s)}{\partial t}$ $(n \to \infty)$ weakly in $\mathbf{L}_2(-M, M; \mathbf{H})$. By the known compactness theorem [22] we can assume that $u_{\varepsilon_n}(s) \to u_0(s)$ $(n \to \infty)$ strongly in $\mathbf{L}_2(-M, M; \mathbf{H})$ and $u_{\varepsilon_n}(x,s) \to u_0(x,s)$ $(n \to \infty)$ for almost all $(x,s) \in D \times (-M, M)$. In particular, $u_{\varepsilon_n}(s) \to u_0(s)$ $(n \to \infty)$ strongly in $\Theta^{loc}_+ = \mathbf{L}^{loc}_2(\mathbb{R}; \mathbf{H})$.

Now, taking into account Lemma 3.2 and the convergence (27), we pass to the limit in (30) and (31) as $\varepsilon \to 0$, based on a standard argument in [22] (see the detailed proof in [15,17,20]). Consequently $u_0 \in \overline{\mathcal{K}}$, i.e., u_0 is a solution to the problem (13), satisfying the corresponding identity (31) with the exterior force $\bar{g}(x)$. At the same time, we have established that $u_{\varepsilon_n}(s) \to u_0(s)$ $(n \to \infty)$ in Θ^{loc}_+ and, consequently, $u_{\varepsilon_n}(s) \in \mathcal{O}'(u_0(s)) \subset \mathcal{O}'\left(\overline{\mathcal{K}}\right)$ for $\varepsilon_n \ll 1$. Thus, we arrive at a contradiction with (29). The theorem is proved.

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Локальды периодты кеуекті ортадағы 2D Навье–Стокс теңдеулер жүйесінің аттракторлары

Мақалада теңдеулерде және шекаралық шарттарда тез тербелмелі мүшелері бар екі өлшемді Навье–Стокс теңдеулер жүйесінің қарастырылды. Тесік облыстағы есепті зерттей отырып, сыртқы шекарадағы Дирихленің біртекті шартын және қуыстардың шекарасындағы Фурье (Робен) шартын анықтаймыз. Осындай болжамдармен осы жүйенің траекториялық аттракторы кейбір әлсіз топологияларда қосымша потенциалымен және тривиалды емес оң жақ бөлігі бар тесігі жоқ облыстағы орташаланған Навье–Стокс теңдеулер жүйесінің траекториялық аттракторына жинақталатыны дәлелденген. Ол үшін А.В. Бабиннің, В.В. Чепыжовтың, Ж.–Л. Лионстың, Р. Темам және М.И. Вишиктің эволюциялық теңдеулердің траекториялық аттракторлары туралы мақалалары мен монографияларының әдістемесі қолданылған. Сондай-ақ, XX ғасырдың соңында пайда болған орташалау әдістері пайдаланылған. Алдымен асимптотикалық әдістерді асимптотиканы формальды құру үшін қолданып содан кейін асимптотикалық қатарлардың негізгі мүшелерін функционалды талдау және интегралды бағалау әдістерін қолдана отырып таңдалған. Сәйкесінше, көмекші әлсіз топологиялы функционалды кеңістікті анықтау арқылы теңдеулердің шекті (орташаланған) жүйесін алынған және осы жүйе үшін траекториялық аттракторлардың бар екені дәлелденген. Содан кейін негізгі теорема тұжырымдалып, ол көмекші леммалардың көмегімен нақтыланған.

Кілт сөздер: аттракторлар, орташалау, Навье–Стокс теңдеулер жүйесі, әлсіз жинақтылық, тесік облыс, тез тербелмелі мүшелер, кеуекті орта.

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Аттракторы 2D системы уравнений Навье-Стокса в локально периодической пористой среде

Рассмотрена двумерная система уравнений Навье–Стокса с быстро осциллирующими членами в уравнениях и граничных условиях. Исследуя задачу в перфорированной области, мы задаем однородное условие Дирихле на внешней границе и условие Фурье (Робена) на границе полостей. При таких предположениях доказываем, что траекторные аттракторы этой системы сходятся в некоторой слабой топологии к траекторным аттракторам усредненной системы уравнений Навье–Стокса с дополнительным потенциалом и нетривиальной правой частью в области без пор. Для этого мы используем подход из статей и монографий А.В. Бабина, В.В. Чепыжова, Ж.-Л. Лионса, Р. Темама и М.И. Вишика о траекторных аттракторах эволюционных уравнений. Кроме того, применяем методы усреднения, появившиеся в конце XX века. Сначала используем асимптотические методы для формального построения асимптотик, далее мы выверяем главные члены асимптотических рядов с помощью методов функционального анализа и интегральных оценок. Определяя соответствующие вспомогательные функциональные пространства со слабой топологией, мы выводим предельную (усредненную) систему уравнений и доказываем существование траекторных аттракторов для этой системы. Затем формулируем основную теорему и доказываем ее с помощью вспомогательных лемм.

Ключевые слова: аттракторы, усреднение, система уравнений Навье-Стокса, слабая сходимость, перфорированная область, быстро осциллирующие члены, пористая среда.