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Generalized boundary value problem for a linear ordinary differential equation with a discretely distributed fractional differentiation operator

This paper formulates and solves a generalized boundary value problem for a linear ordinary differential equation with a discretely distributed fractional differentiation operator. The fractional derivative is understood as the Gerasimov–Caputo derivative. The boundary conditions are given in the form of linear functionals, which makes it possible to cover a wide class of linear local and non-local conditions. A representation of the solution is found in terms of special functions. A necessary and sufficient condition for the solvability of the problem under study is obtained, as well as conditions under which the solvability condition is certainly satisfied. The theorem of existence and uniqueness of the solution is proved.

Keywords: fractional differentiation operator, Caputo derivative, boundary value problem, functional, Wright function.

Introduction and statement of the problem

In the interval $0 < x < 1$, let us consider the equation

$$\sum_{j=1}^{m} \beta_j \partial_{0x}^{\alpha_j} u(x) + \lambda u(x) = f(x), \tag{1}$$

where $\alpha_j \in (1,2)$, $\lambda,\ \beta_j \in \mathbb{R}$, $\beta_1 > 0$, $\alpha_1 > \alpha_2 > ... > \alpha_m$, $\partial_{0x}^{\gamma} u(x)$ is the Caputo derivative [1; 11]:

$$\partial_{sx}^{\gamma} u(x) = \operatorname{sign}^n(x-s) D_{sx}^{\gamma-n} u^{(n)}(x), \quad n-1 < \gamma \le n,\, n \in \mathbb{N}, \tag{2}$$

and D_{sx}^{γ} is the Riemann–Liouville fractional integro-differentiation operator of order γ with respect to the variable x [1; 9], which is defined by the formula

$$D_{sx}^{\gamma} u(x) = \frac{\operatorname{sign}(x-s)}{\Gamma(-\gamma)} \int_{s}^{x} \frac{u(t)dt}{|x-t|^{\gamma+1}}, \quad \gamma < 0,$$

$$D_{sx}^{\gamma} u(x) = u(x), \quad \gamma = 0,$$

$$D_{sx}^{\gamma} u(x) = \operatorname{sign}^n(x-s) \frac{d^n}{dx^n} D_{sx}^{\gamma-n} u(x), \quad n-1 < \gamma \le n,\, n \in \mathbb{N}.$$

Operator (2) is also known in the literature as the Gerasimov–Caputo operator [2, 3].

At present, differential equations of fractional order are being extensively studied in connection with practical applications in various areas of physics and mathematical modeling. The theory of fractional differential equations has proven itself well in the study of «classical» viscoelastic models. All this is supported by new applied problems [4–8].

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One of the first works devoted to fractional calculus and its applications is the monograph [9]. The main theoretical results and solution methods are reflected in the works [7], [6], and [10].

Linear fractional ordinary differential equations were studied by many authors; a detailed bibliography on this subject can be found in [1, 5, 6, 11]. A significant contribution to the study of fractional differential equations was made by the authors of [12–15].

Differential equations with discretely distributed differentiation operator

$$\sum_{k=1}^{m} \lambda_k \frac{\partial^{\sigma_k}}{\partial y^{\sigma_k}}$$

can be treated as an operator

$$\int_{\alpha}^{\beta} \left(\lambda(y,t) \frac{\partial^t}{\partial y^t} \right) d\mu(t)$$

with a measure concentrated on a discrete set [16, 17].

Differential equations with discretely distributed differentiation operators and continuously distributed differentiation operators were studied in [18–20], where equations with discretely distributed differentiation operators were used to search for approximate solutions of equations with continuously distributed differentiation operators. We also note the papers [16], [17], [21], [22], where equations with fractional discretely distributed differentiation operators were studied.

In this paper, we investigate a generalized boundary value problem (in the terminology of M.A. Naimark) for equation (1) [22; 16]. An explicit representation of the solution of the problem under study is constructed, a condition for unique solvability is found, and a uniqueness theorem for the solution is proved. We specified boundary conditions in the form of linear functionals, which makes it possible to cover a wide class of linear local and nonlocal conditions. Various boundary value problems for equation (1) were studied in the works [23–25]. Note also that in work [26] a generalized boundary value problem for an ordinary differential equation of fractional order with general conditions was investigated.

A *regular solution* to equation (1) is said to be a function $u = u(x)$ that has an absolutely continuous first-order derivative on the closed interval $[0,1]$ and satisfies equation (1) for all $x \in (0,1)$.

Problem. Find a regular solution to equation (1) in the interval $(0,1)$, which satisfies the conditions

$$\ell_0[u] = u_0, \tag{3}$$

$$\ell_1[u] = u_1, \tag{4}$$

where u_0, u_1 are given real numbers, ℓ_0, ℓ_1 are linear bounded functionals in $C^1[0,1]$.

Notation and auxiliary statements

We use the following notation (see [27])

$$G_m^{\mu}(x) = G_m^{\mu}(x;\nu_1,...,\nu_m;\gamma_1,...,\gamma_m) \equiv \int_0^{\infty} e^{-t} S_m^{\mu}(x;\nu_1 t,...,\nu_m t;\gamma_1,...,\gamma_m)dt,$$

$$\nu_1 = -\frac{\lambda}{\beta_1}, \quad \nu_j = -\frac{\beta_j}{\beta_1}, \quad \gamma_1 = \alpha_1, \quad \gamma_j = \alpha_1 - \alpha_j, \quad (j = \overline{2,m}),$$

$$S_m^{\mu}(x;z_1,...,z_m;\gamma_1,...,\gamma_m) = (h_1 * h_2 * ... * h_m)(x),$$

by

$$(g*h)(x) = \int_0^x g(x-t)h(t)dt$$

we denote the Laplace convolution of the functions $g(x)$ and $h(x)$,

$$h_j = h_j(x) \equiv x^{\mu_j - 1}\phi(\gamma_j, \mu_j; z_j x^{\gamma_j}),$$

where

$$\phi(\rho, \zeta; z) = \sum_{k=0}^{\infty} \frac{z^k}{k!\Gamma(\rho k + \zeta)}$$

is the Wright function (see [28]).

Further, we assume that the parameters $G_m^{\mu}(x)$ range over

$$x > 0, \quad z_i \in \mathbb{R}, \quad \gamma_j > 0, \quad \mu_j > 0.$$

We note that the function $G_m^{\mu}(x)$ is independent of the distribution of the numbers $\mu_j > 0$, but only depends on their sum $\mu = \sum\limits_{j=1}^{m} \mu_j$.

The following equalities for the function $G_m^{\mu}(x)$ (see [24]) hold:

$$G_m^{\mu}(x) = O(x^{\mu-1}) \quad \text{при} \quad x \to 0,$$

$$D_{0x}^{\nu} G_m^{\mu}(x) = G_m^{\mu-\nu}(x), \quad \text{если} \quad \mu > \nu, \tag{5}$$

$$G_m^{\mu}(x) - \sum_{j=1}^{m} \nu_j D_{0x}^{-\gamma_j} G_m^{\mu}(x) = \frac{x^{\mu-1}}{\Gamma(\mu)}, \quad \mu > 0. \tag{6}$$

In particular, from equalities (5) and (6) we obtain the formula (see [25])

$$\sum_{j=1}^{m} \beta_j G_m^{\mu-\gamma_j}(x) + \lambda G_m^{\mu}(x) = \frac{\beta_1 x^{\mu-\alpha_1-1}}{\Gamma(\mu-\alpha_1)}, \quad \mu > \alpha_1.$$

We also need the following auxiliary statement proved in (see [26]).

It should be noted that hereinafter the ℓ functionality is applied to the function depending on x.

Lemma. Let $K(x,t) \in C([0,1]\times[0,1])$ end $\frac{\partial}{\partial x}K(x,t) \in C([0,1]\times[0,1])$, $\ell-$ linear bounded functional in space $C^1[0,1]$. Then the following relation is true

$$\ell\left[\int_0^1 K(x,t)dt\right] = \int_0^1 \ell[K(x,t)]dt. \tag{7}$$

Main result

Theorem. Let a function $f(x)$ satisfy the conditions

$$x^{1-\mu}f(x) \in C[0,1], \quad f(x) = D_{0x}^{\alpha_1-2}g(x), \quad g(x) \in L[0,1], \quad \mu > 0.$$

and the inequality

$$\det A = \ell_0[\mathcal{W}_2(x)]\ell_1[\mathcal{W}_3(x)] - \ell_0[\mathcal{W}_3(x)]\ell_1[\mathcal{W}_2(x)] \neq 0 \tag{8}$$

be fulfilled. Then a function $u(x)$ defined by the relation

$$u(x)=\int\limits_0^1 f(t)\mathcal{T}(x,t)dt+\overline{\mathcal{W}}(x)\mathcal{H}\begin{pmatrix} u_0 \\ u_1 \end{pmatrix}, \tag{9}$$

$$\mathcal{T}(x,t)=\left[1-\overline{\mathcal{W}}(x)\mathcal{H}\overline{\ell}\right]\mathcal{W}_1(x-t),\quad \overline{\mathcal{W}}(x)=(\mathcal{W}_2(x),\mathcal{W}_3(x)),\quad \overline{\ell}=\begin{pmatrix} \ell_0 \\ \ell_1 \end{pmatrix},$$

$$\mathcal{W}_1(x)=\frac{1}{\beta}G_m^{\alpha_1}(x);\quad \mathcal{W}_2(x)=x+\nu_1 G_m^{\alpha_1+2}(x);\quad \mathcal{W}_3(x)=1+\nu_1 G_m^{\alpha_1+1}(x),$$

$$A=\overline{\ell}\,[\overline{\mathcal{W}}(x)]=\begin{pmatrix} \ell_0[\mathcal{W}_2(x)] & \ell_0[\mathcal{W}_3(x)] \\ \ell_1[\mathcal{W}_2(x)] & \ell_1[\mathcal{W}_3(x)] \end{pmatrix},\quad \mathcal{H}=A^{-1}=\frac{1}{\det A}\begin{pmatrix} \ell_1[\mathcal{W}_3(x)] & -\ell_0[\mathcal{W}_3(x)] \\ -\ell_1[\mathcal{W}_2(x)] & \ell_0[\mathcal{W}_2(x)] \end{pmatrix},$$

is a regular solution to problem (1), (3), (4). The solution to problem (1), (3), (4) is unique if and only if the condition (8) is satisfied.

Proof. Let $u(x)$ be a regular solution to the problem (1), (3), (4). To find a solution to the problem (1), (3), (4) we use the solution of the Cauchy problem for the equation (1), which can be represented as [24, 25]:

$$u(x)=\int\limits_0^1 f(t)\mathcal{W}_1(x-t)dt+\overline{\mathcal{W}}(x)\begin{pmatrix} C_1 \\ C_2 \end{pmatrix}. \tag{10}$$

Further, taking into account the introduced notation and equalities (7), we satisfy (10) the boundary conditions

$$\int\limits_0^1 f(t)\overline{\ell}[\mathcal{W}_1(x-t)]dt+A\begin{pmatrix} C_1 \\ C_2 \end{pmatrix}=\begin{pmatrix} u_0 \\ u_1 \end{pmatrix}.$$

From this we find

$$\begin{pmatrix} C_1 \\ C_2 \end{pmatrix}=\mathcal{H}\begin{pmatrix} u_0 \\ u_1 \end{pmatrix}-\int\limits_0^1 f(t)\mathcal{H}\overline{\ell}[\mathcal{W}_1(x-t)]dt.$$

After elementary transformations, substituting the found value into (10), we obtain a representation of the solution to problem (1), (3), (4) in the form (8). This, in particular, implies the uniqueness of the solution.

Let us now check the fulfillment of the boundary conditions (3), (4).

$$\overline{\ell}\left[\int\limits_0^1 f(t)\mathcal{T}(x,t)dt+\overline{\mathcal{W}}(x)\mathcal{H}\begin{pmatrix} u_0 \\ u_1 \end{pmatrix}\right]=I_1+I_2,$$

where

$$I_1=\overline{\ell}\left[\int\limits_0^1 f(t)\mathcal{T}(x,t)dt\right]\quad \text{end}\quad I_2=\overline{\ell}\left[\overline{\mathcal{W}}(x)\mathcal{H}\begin{pmatrix} u_0 \\ u_1 \end{pmatrix}\right].$$

Taking into account (7), we have

$$I_1=\int\limits_0^1 \overline{\ell}\,[\mathcal{T}(x,t)]\,f(t)dt=\int\limits_0^1\left(\overline{\ell}[\mathcal{W}_1(x,t)]-\overline{\ell}[\overline{\mathcal{W}}(x)]\mathcal{H}\overline{\ell}[\mathcal{W}_1(x,t)]\right)f(t)dt.$$

By virtue of the equality $\overline{\ell}[\overline{\mathcal{W}}(x)] = \mathcal{H}^{-1}$, we obtain that $I_1 = 0$. Similarly, for I_2 we have

$$I_2 = \overline{\ell}[\overline{\mathcal{W}}(x)]\mathcal{H}\begin{pmatrix} u_0 \\ u_1 \end{pmatrix} = \mathcal{H}^{-1}\mathcal{H}\begin{pmatrix} u_0 \\ u_1 \end{pmatrix} = \begin{pmatrix} u_0 \\ u_1 \end{pmatrix}.$$

Let us prove that the function u(x), given by equality (9), is a solution to problem (3), (4) for equation (1).

$$\left[\sum_{j=1}^{m}\beta_j\partial_{0x}^{\alpha_j} + \lambda\right] y_i = f(x), \quad i = \overline{1,3},$$

$$y_1 = \int_0^x f(t)\mathcal{W}_1(x-t)dt, \quad y_2 = \int_0^x f(t)\overline{\mathcal{W}}(x)\mathcal{H}\overline{\ell}\mathcal{W}_1(x-t)dt, \quad y_3 = \overline{\mathcal{W}}(x)\mathcal{H}\begin{pmatrix} u_0 \\ u_1 \end{pmatrix}.$$

Taking into account relations (5) – (8) and since the functions y_2 and y_3 is a linear combinations of the functions $\mathcal{W}_i(x), i = \overline{1,3}$ we have

$$\left[\sum_{j=1}^{m}\beta_j\partial_{0x}^{\alpha_j} + \lambda\right] y_i = 0, \quad i = \overline{2,3}.$$

Considering that (see [25])

$$\sum_{j=1}^{m}\beta_j\partial_{0x}^{\alpha_j}\int_0^x f(t)\mathcal{W}_1(x-t)dt = -\frac{\lambda}{\beta_1}\int_0^x f(t)G_m^{\alpha}(x-t)dt + f(x),$$

we get

$$\left[\sum_{j=1}^{m}\beta_j\partial_{0x}^{\alpha_j} + \lambda\right] y_1 = f(x).$$

It means that the solution satisfies equation (1).

Let us show that if condition (8) is satisfied, that is,

$$\ell_0[\mathcal{W}_2(x)]\ell_1[\mathcal{W}_3(x)] - \ell_0[\mathcal{W}_3(x)]\ell_1[\mathcal{W}_2(x)] = 0, \tag{11}$$

then the solution to the homogeneous problem is not unique.
Consider the function

$$\widetilde{u}(x) = (k_1(x), k_2(x))\begin{pmatrix} C_1 \\ C_2 \end{pmatrix},$$

where C_1 and C_2 are arbitrary constants,

$$k_1(x) = \overline{\mathcal{W}}(x)\begin{pmatrix} \ell_0[\mathcal{W}_2(x)] \\ -\ell_0[\mathcal{W}_3(x)] \end{pmatrix}, \quad k_2(x) = \overline{\mathcal{W}}(x)\begin{pmatrix} -\ell_1[W_2(x)] \\ \ell_1[W_3(x)] \end{pmatrix}.$$

Then it follows from (11) that the function $\widetilde{u}(x)$ is a solution to a homogeneous problem

$$\sum_{j=1}^{m}\beta_j\partial_{0x}^{\alpha_j}\widetilde{u}(x) + \lambda\widetilde{u}(x) = 0, \quad \ell_0[\widetilde{u}] = 0, \quad \ell_1[\widetilde{u}] = 0.$$

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Бөлшек дискретті үлестірілген дифференциалдау операторы бар сызықтық қарапайым дифференциалдық теңдеу үшін жалпыланған шеттік есеп

Мақалада бөлшек дискретті үлестірілген дифференциалдау операторы бар сызықтық қарапайым дифференциалдық теңдеу үшін жалпыланған шеттік есеп құрастырылып және шешілген. Бөлшек туынды Герасимов-Капуто туындысы мағынасында түсініледі. Шеттік шарттар сызықтық функционалдар түрінде берілген, бұл сызықтық жергілікті және жергілікті емес жағдайлардың жеткілікті кең класын қамтуға мүмкіндік береді. Шешімнің мәні арнайы функциялар арқылы табылды. Зерттелетін есептің шешілу мүмкіндігінің қажетті және жеткілікті шарты, сондай-ақ шешілу шарты сөзсіз орындалатын шарттар алынды. Шешімнің бар болуы және бірегейлігі теоремасы дәлелденді.

Кілт сөздер: бөлшек дифференциалдау операторы, Капуто туындысы, шеттік есеп, функционал, Райт функциясы.

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Обобщенная краевая задача для линейного обыкновенного дифференциального уравнения с оператором дробного дискретно распределенного дифференцирования

В статье сформулирована и решена обобщенная краевая задача для линейного обыкновенного дифференциального уравнения с оператором дробного дискретно распределенного дифференцирования. Дробная производная понимается в смысле производной Герасимова–Капуто. Краевые условия задаются в форме линейных функционалов, это позволяет охватить достаточно широкий класс линейных локальных и нелокальных условий. В терминах специальных функций найдено представление решения. Получено необходимое и достаточное условие разрешимости исследуемой задачи, а также условия, при которых условие разрешимости заведомо выполняется. Доказана теорема существования и единственности решения.

Ключевые слова: оператор дробного дифференцирования, производная Капуто, краевая задача, функционал, функция Райта.

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