DOI 10.31489/2022M2/117-127

UDC 517.95

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An initial boundary value problem for the Boussinesq equation in a Trapezoid

This paper considers an initial boundary value problem for a one-dimensional Boussinesq-type equation in a domain, that is, a trapezoid. Using the methods of the theory of monotone operators, we establish theorems on their unique weak solvability in Sobolev classes.

Keywords: Boussinesq-type equation, boundary value problem, trapezoid, theory of monotone operators.

Introduction

The theory of the Boussinesq equations and its modifications always attracts the attention of both mathematicians and applied scientists. The Boussinesq equation, as well as its modifications, occupies an important place in describing the motion of liquids and gas, including in the theory of unsteady filtration in porous media. Here we note only the works [1–6]. In recent years, boundary problems for these equations have been actively studied, since they model processes in porous media. The processes occurring in porous media acquire special importance for deep understanding in the tasks of exploration and effective development of oil and gas fields.

In this paper, we study the issues of the correct formulation of initial boundary value problems for a one-dimensional Boussinesq-type equation in a domain with a movable boundary. The domain is represented by a trapezoid. Using the method of monotone operators, we prove theorems on the unique weak solvability of the considered boundary value problems.

1 Statement of the initial boundary problem and the main result

Let $\Omega_t = \{0 < x < t\}$, and $\partial\Omega_t$ be the boundary of Ω_t, $0 < t_0 < T < \infty$. In domain $Q_{xt} = \Omega_t \times (t_0, T)$, i.e., a trapezoid, we consider the initial boundary problem for the Boussinesq-type equation

$$\partial_t u - \partial_x \left(|u|\partial_x u\right) = f, \;\; \{x,t\} \in Q_{xt}, \tag{1.1}$$

with boundary

$$u = 0, \;\; \{x,t\} \in \Sigma_{xt} = \partial\Omega_t \times (t_0, T), \tag{1.2}$$

and initial conditions

$$u = u_0, \;\; x \in \Omega_{t_0} = (0, t_0), \tag{1.3}$$

where $f(x,t)$, $u_0(x)$ are given functions.

We have established the following theorems.

Theorem 1.1 (Main result). Let

$$f \in L_{3/2}((t_0,T); W^{-1}_{3/2}(\Omega_t)), \;\; u_0 \in H^{-1}(\Omega_{t_0}).$$

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Then initial boundary problem (1.1)–(1.3) has a unique solution

$$u \in L_3(Q_{xt}).$$

Theorem 1.2 (On smoothness). Let

$$f \in L_{3/2}(Q_{xt}), \ \ u_0 \in L_2(\Omega_{t_0}).$$

Then initial boundary problem (1.1)–(1.3) has a unique solution

$$u \in L_\infty((t_0,T);L_2(\Omega_t)), \quad |u|^{1/2}u \in L_2((t_0,T);H_0^1(\Omega_t)), \quad \partial_t u \in L_{3/2}((t_0,T);W_{3/2}^{-1}(\Omega_t)).$$

2 Auxiliary initial boundary problem in a rectangle

To prove Theorem 1.1, we first consider an auxiliary initial boundary value problem. For this purpose, we pass from variables $\{x,t\}$ to $\{y,t\}$ by formulas $y=\frac{x}{t}$, $t=t$ and transform the trapezoid Q_{xt} into the rectangular domain $Q_{yt}=\Omega\times(t_0,T)$, $0<t_0<T<\infty$, where $y\in\Omega = (0,1)$, $\partial\Omega=\{0\}\cup\{1\}$, $\Sigma_{xt}=\partial\Omega\times(t_0,T)$. This transformation is one-to-one. Introducing the notation $w(y,t)=u(yt,t)=w(\frac{x}{t},t)$, $w_0(y)=u_0(yt_0,t_0)$ and $g(y,t)=f(yt,t)$, we write the auxiliary initial boundary value problem for (1.1)–(1.3) in the following form:

$$\partial_t w - \frac{1}{t^2}\partial_y\left(|w|\partial_y w\right) - \frac{y}{t}\partial_y w = g, \ \ \{y,t\}\in Q_{yt}, \tag{2.1}$$

$$w=0, \ \ \{y,t\}\in\Sigma_{yt}, \tag{2.2}$$

$$w=w_0, \ \ y\in\Omega. \tag{2.3}$$

By virtue of the one-to-one transformation of independent variables $\{x,t\}\to\{y,t\}$ the given functions in problem (2.1)–(2.3) obviously satisfy the conditions:

$$g\in L_{3/2}((t_0,T);W_{3/2}^{-1}(0,1)), \ \ w_0\in H^{-1}(0,1). \tag{2.4}$$

The following theorems are true.

Theorem 2.1 Under conditions (2.4) initial boundary value problem (2.1)–(2.3) is uniquely solvable

$$w\in L_3(Q_{yt}).$$

Theorem 2.2 (On smoothness). Let

$$g\in L_{3/2}(Q_{yt}), \ \ w_0\in L_2(\Omega).$$

Then initial boundary problem (2.1)–(2.3) has a unique solution

$$w \in L_\infty((t_0,T);L_2(\Omega)), \quad |w|^{1/2}w \in L_2((t_0,T);H_0^1(\Omega)), \quad \partial_t w \in L_{3/2}((t_0,T);W_{3/2}^{-1}(\Omega)).$$

3 Auxiliary statements

To prove Theorem 2.1, we first establish a number of auxiliary statements. Denote by A the operator of problem (2.1)–(2.3)

$$A(t,w)=\frac{1}{t^2}A_1(w)+\frac{1}{t}A_{21}(w), \quad \text{where} \ \ A_1(w)=-\partial_y(|w|\partial_y w), \ \ A_2(w)=-y\partial_y w, \tag{3.1}$$

and the operator $A_2(w)$ can be represented as:

$$A_2(w)=A_{21}(w)+A_{22}(w), \ \ \text{where} \ \ A_{21}(w)=w, \ \ A_{22}(w)=-\partial_y(yw). \tag{3.2}$$

Let us show that the operator $A_1(w)+A_{21}(w)$ will have the monotonicity property if we introduce the scalar product in an appropriate way. For this purpose, we take as a scalar product

$$\langle\varphi,\psi\rangle=\int_0^1\varphi\left[\left(-d_y^{\,2}\right)^{-1}\psi\right]dy,\ \ \forall\,\varphi,\psi\in H^{-1}(\Omega), \tag{3.3}$$

where $d_y^{\,2}=\frac{d^2}{dy^2}$, $\tilde{\psi}=\left(-d_y^{\,2}\right)^{-1}\psi:\ -d_y^{\,2}\tilde{\psi}=\psi,\ \tilde{\psi}(0)=\tilde{\psi}(1)=0,\forall\,\psi\in H^{-1}(\Omega)$.

Let us show the validity of the following lemma.

Lemma 3.1. The operator A_1+A_{21} is monotone in the sense of the scalar product (3.3) in the space $H^{-1}(0,1)$, i.e., the following inequality is true:

$$\langle(A_1+A_{21})(w_1)-(A_1+A_{21})(w_2),w_1-w_2\rangle\geq 0,\ \forall\,w_1,w_2\in\mathfrak{D}(\Omega). \tag{3.4}$$

To the proof of Lemma 3.1. It suffices for us to show that the operator A_1 is monotone and condition (3.4) will be satisfied (according to [7], chap. 2, s. 3.1). Indeed, on the one hand, we have

$$\langle A_1(\varphi)-A_1(\psi),\varphi-\psi\rangle=\frac{1}{2}\int_0^1\left(-d_y^2\right)(|\varphi|\varphi-|\psi|\psi)\left(-d_y^2\right)^{-1}(\varphi-\psi)\,d\,y=$$

$$=\frac{1}{2}\int_0^1(|\varphi|\varphi-|\psi|\psi)(\varphi-\psi)\,d\,y,\ \ \forall\,\varphi,\,\psi\in\mathfrak{D}(\Omega).$$

On the other hand, the convexity condition of the functional
$J_1(\varphi)=\frac{1}{3}\int_0^1|\varphi(y)|^3\,d\,y,\ \ \varphi\in\mathfrak{D}(\Omega)$, implies

$$\langle J_1'(\varphi)-J_1'(\psi),\varphi-\psi\rangle\ \geq 0,\ \ \forall\,\varphi,\,\psi\in\mathfrak{D}(\Omega).$$

Thus, we get

$$\int\limits_0^1(|\varphi|\varphi-|\psi|\psi)(\varphi-\psi)\,d\,y\ \geq 0,\ \ \forall\,\varphi,\,\psi\in\mathfrak{D}(\Omega).$$

For the operator A_{21} according to scalar product (3.3) we have:

$$\langle A_{21}(\varphi),\psi\rangle=\int\limits_0^1\varphi\tilde{\psi}\,dy=$$

$$=\int\limits_0^1\varphi\left(-d_y^2\right)^{-1}\psi\,dy=\int\limits_0^1\left(\left(-d_y^2\right)^{-1}\varphi\right)\psi\,dy,\ \ \forall\,\varphi,\,\psi\in\mathfrak{D}(\Omega), \tag{3.5}$$

where $\tilde{\psi}$ is the solution to the following problem: $-d_y^{\,2}\tilde{\psi}=\psi,\ \tilde{\psi}(0)=\tilde{\psi}(1)=0$.

Let us introduce the convex functional

$$J_{21}(u)=\frac{1}{2}\int_0^1\left[\left(-d_y^{\,2}\right)^{-\frac{1}{2}}u\right]^2dy. \tag{3.6}$$

For the Gateaux derivative of functional (3.6) we have

$$J_{21}'(u)=\left(-d_y^{\,2}\right)^{-1}u, \tag{3.7}$$

that is, taking into account (3.7), we obtain the following convexity conditions of functional (3.6):

$$\langle J_{21}'(u)-J_{21}'(v),u-v\rangle=\int_0^1\left[\left(-d_y^{\,2}\right)^{-1}(u-v)\right](u-v)\,dy\geq 0\ \ \forall\,u,v\in\mathfrak{D}(\Omega). \tag{3.8}$$

Remark 3.1. On the other hand, inequality (3.8) is a consequence of the positivity of the operator $\left(-d_y^2\right)^{-1}$. Further, based on relations (3.5) and (3.8), we establish that the following monotonicity condition holds for operator A_{21}:

$$\langle A_{21}(t,u) - A_{21}(t,v), u-v\rangle \geq 0 \ \forall\, u,v \in \mathfrak{D}(\Omega), \ \ \forall\, t \in (t_0,T).$$

Thus, we have shown the validity of statement (3.4) of Lemma 3.1.

4 To the proof of Theorem 2.1

Let us preliminarily note that the nonlinear operator $A(t,v) \equiv (0.5\,t^{-2}A_1 + t^{-1}A_{21})v : L_3(\Omega) \to L_{3/2}(\Omega)$ (3.1)–(3.2) of boundary value problem (2.1)–(2.2) has the following properties:

$$A(t,v) : L_3(\Omega) \to L_{3/2}(\Omega) \text{ is a hemicontinuous operator,} \tag{4.1}$$

$$\|A(t_0,v)\|_{L_{3/2}(\Omega)} \leq c\|v\|^2_{L_3(\Omega)}, \ \ c>0, \ \ \forall\, v \in L_3(\Omega), \tag{4.2}$$

$$\langle A(T,v), v\rangle \geq \alpha\|v\|^3_{L_3(\Omega)}, \ \ \alpha>0, \ \ \forall\, v \in L_3(\Omega). \tag{4.3}$$

This follows directly from Lemma 4.1, as well as from ([7], Chap. 2, Proposition 1.1).

Recall the definition of a hemicontinuous operator.

Definition 4.1. Every operator $B:\ V \to V'$, having the following property:

$$\forall\, u,v,w \in V \ \text{ function } \ \lambda \to \langle B(u+\lambda v), w\rangle \ \text{ is continuous as a function from } \mathbb{R} \text{ to } \mathbb{R},$$

is called hemicontinuous.

Now we take as the main space:

$$H = H^{-1}(\Omega), \ \ (u,v)_H = \left(u, (-d_y^2)^{-1}v\right), \tag{4.4}$$

where $(-d_y^2)^{-1}v = \tilde{v}$ is the solution to problem

$$-d_y^2\tilde{v} = v, \ \ \tilde{v}(0) = \tilde{v}(1) = 0, \ \ v \in H^{-1}(\Omega). \tag{4.5}$$

Further, we have

$$V = L_3(\Omega), \ \ V \subset H \subset V', \tag{4.6}$$

where each embedding is dense and continuous. In notation (4.4)–(4.6), we introduce a linear continuous functional

$$L(v) = \langle g, v\rangle = (g, \tilde{v}), \ \text{ i.e. the element } \ g \in L_{3/2}(\Omega) \text{ is defined.}$$

Finally, we introduce

$$a(t,u,v) = \langle A(t,u), v\rangle = \int\limits_0^1 \left[\frac{1}{2\,t^2}|u|uv + \frac{1}{t}\left(-d_y^2\right)^{-1}uv\right]dy, \ \ \forall\, u,v \in L_3(\Omega).$$

We have

$$a(t,u,u) = \langle A(t,u), u\rangle = \frac{1}{2\,t^2}\|u\|^3_{L_3(\Omega)} + \frac{1}{t}\left\|\left(-d_y^2\right)^{-1/2}u\right\|^2_{L_2(\Omega)},$$

and

$$a(t,u,u-v) - a(t,v,u-v) \geq 0 \ \ \forall\, t \in (t_0,T), \tag{4.7}$$

where the form $a(t,u,v)$ corresponds to variational inequalities (3.4) and (3.8). Now, using (4.7), we obtain the following variational formulation for initial boundary problem (2.1)–(2.3):

$$(w'(t),v)_H + a(t,w(t),v) - b(t,w(t),v) = (g(t),v) \ \ \forall\, v \in L_3(\Omega) \subset H^{-1}(\Omega), \tag{4.8}$$

$$w(0) = w_0, \tag{4.9}$$

where $b(t,w(t),v) = t^{-1}\langle A_{22}(w), v\rangle$.

We show that relations (4.8), (4.9) admit unique solvability.

4.1 Existence of the solution

Let us show that variational problem (4.8) and (4.9) has a solution. We will use the Faedo-Galerkin method. Let $v_1,...,v_m,...$ be a "basis" in the space $L_3(\Omega)$. According to relations (4.8) and (4.9), we define an approximate solution $w_m(t)$ of initial boundary value problem (2.1)–(2.3) on a subspace $[v_1,...,v_m]$ spanned by $v_1,...,v_m$:

$$(w_m'(t),v_j) + a(t,w_m(t),v_j) - b(t,w_m(t),v_j) = (g(t),v_j), \ \ 1 \le j \le m, \tag{4.10}$$

$$w_m(0) = w_{0m} \in [v_1,...,v_m], \ \ w_{0m} \to w_0 \ \text{ in } \ H^{-1}(\Omega). \tag{4.11}$$

From equations (4.10)–(4.11), $w_m(t)$ is determined on the interval $[t_0,t_m],\ t_m > t_0$. However, due to the validity of inequality (4.3) $\langle A(t,v),v\rangle \ge \alpha\|v\|^3_{L_3(\Omega)},\ \ \alpha>0$, from (4.10)–(4.11) we obtain

$$\frac{1}{2}\|w_m(t)\|^2_{H^{-1}(0,1)} + \alpha\int\limits_{t_0}^{t}\|w_m(\tau)\|^3_{L_3(\Omega)}\,d\tau \le \frac{C_2}{t_0}\int\limits_{t_0}^{t}\|w_m(\tau)\|^3_{L_{3/2}(\Omega)}\|w_m(\tau)\|_{L_3(\Omega)}\,d\tau +$$

$$+\int\limits_{t_0}^{t}\|g(\tau)\|_{L_{3/2}(\Omega)}\|w_m(\tau)\|_{L_3(\Omega)}\,d\tau + \frac{1}{2}\|w_{0m}\|^2_{H^{-1}(\Omega)}, \tag{4.12}$$

since

$$|b(t,w_m(t),w_m(t))| \le \frac{1}{t_0}\,\|A_{22}w_m(t)\|_{L_{3/2}(\Omega)}\,\|w_m(t)\|_{L_3(\Omega)},$$

$$\|A_{22}w_m(t)\|_{L_{3/2}(\Omega)} \le C_2\|w_m(t)\|_{L_{3/2}(\Omega)},$$

$$\frac{C_2}{t_0}\|w_m(t)\|_{L_{3/2}(\Omega)}\|w_m(t)\|_{L_3(\Omega)} \le \frac{8}{9\sqrt{3\,\alpha}}\left(\frac{C_2}{t_0}\right)^{3/2}\|w_m(t)\|^{3/2}_{L_{3/2}(\Omega)} + \frac{\alpha}{4}\|w_m(t)\|^3_{L_3(\Omega)} \le$$

$$\le \frac{8}{9\sqrt{3\,\alpha}}K^{3/2}\left(\frac{C_2}{t_0}\right)^{3/2}\left[\|w_m(t)\|^2_{H^{-1}(\Omega)}\right]^{3/4} + \frac{\alpha}{4}\|w_m(t)\|^3_{L_3(\Omega)},$$

where K is the embedding constant of $\left(H^{-1}(\Omega)\right)' \hookrightarrow L_{3/2}(\Omega)$, since by assumptions (4.4) and (4.6): $L_3(\Omega) \subset H^{-1}(\Omega) \equiv \left(H^{-1}(\Omega)\right)' \subset L_{3/2}(\Omega) \equiv \left(L_3(\Omega)\right)'$. Here we also use Young's inequality $(p^{-1} + q^{-1} = 1)$:

$$|AB| = \left|\left(d^{1/p}A\right)\left(d^{1/q}\frac{B}{d}\right)\right| \le \frac{d}{p}\,|A|^p + \frac{d}{qd^q}\,|B|^q\,,$$

where

$$A = \frac{C_2}{t_0}\,\|w_m(t)\|_{L_{3/2}(\Omega)}\,,\ \ B = \|w_m(t)\|_{L_3(\Omega)}\,,\ \ d = \frac{2}{\sqrt{4\,\alpha}},\ \ p = 3/2,\ \ q = 3.$$

We have similar calculations for the expression from (4.12):

$$\|g(t)\|_{L_{3/2}(\Omega)}\|w_m(t)\|_{L_3(\Omega)} \le \frac{8}{9\sqrt{3}\,\alpha}K^{3/2}\left[\|g(t)\|^2_{H^{-1}(\Omega)}\right]^{3/4} + \frac{\alpha}{4}\|w_m(t)\|^3_{L_3(\Omega)}.$$

Now, using a variant of Bihari's lemma from ([8], Chapter 1, p.1.3, Example 1.3.1; it is important here that $3/4 < 1$), it follows from (4.12) that $t_m = T$ and that

$$w_m(t) \text{ are bounded in } L_\infty((t_0,T);H^{-1}(\Omega)) \cap L_3(Q_{yt}).$$

Hence, we can extract such a subsequence of $w_\mu(t)$ that

$$w_\mu \to w \quad *\text{-weak in } L_\infty((t_0,T);H^{-1}(\Omega)),$$

$$w_\mu \to w \text{ weak in } L_3(Q_{yt}),$$

$$w_\mu(T) \to \xi \text{ weak in } H^{-1}(\Omega),$$

$$A(t,w_\mu) \to \chi(t) \text{ weak for almost every } t\in(t_0,T) \text{ in } L_{3/2}(Q_{yt}),$$

due to condition (4.2) $\|A(t,v)\|_{L_{3/2}(\Omega)} \le c\|v\|^2_{L_3(\Omega)}$, $c > 0$, and hence $A(t,w_\mu)$ are bounded in $L_{3/2}(Q_{yt})$.

We extend $w_m(t)$, $A(t,w_m(t)),...$ on the real axis with zero outside the interval $[t_0,T]$, and denote the corresponding continuations by $\tilde{w}_m(t)$, $\widetilde{A(t,w_m(t))},...$ It follows from (4.10)–(4.11) that

$$\left(\tilde{w}'_m(t),v_j\right)_H + \langle\widetilde{A(t,w_m(t))},v_j\rangle - t^{-1}\langle\widetilde{A_{22}w_m(t)},v_j\rangle =$$

$$= (\tilde{g}(t),v_j) + (w_{0m},v_j)\delta(t-t_0) - (w_m(T),v_j)\delta(t-T). \tag{4.13}$$

Now we can pass to the limit in (4.13) at $m=\mu$ and fixed j, whence we have

$$\left(\tilde{w}'(t),v_j\right)_H + \langle\tilde{\chi}(t) - t^{-1}\widetilde{A_{22}w(t)},v_j\rangle = (\tilde{g}(t),v_j) + (w_0,v_j)\delta(t-t_0) - (\xi,v_j)\delta(t-T) \;\; \forall j$$

and hence

$$\tilde{w}'(t) + \tilde{\chi}(t) - t^{-1}\widetilde{A_{22}w(t)} = \tilde{g}(t) + w_0\delta(t-0) - \xi\delta(t-T). \tag{4.14}$$

By restricting (4.14) (t_0,T), we get that

$$w'(t) + \chi(t) - t^{-1}A_{22}w(t) = g(t), \tag{4.15}$$

from where $w'(t)\in L_{3/2}(Q_{yt})$, hence $w(t_0)$ and $w(T)$ make sense, and comparing with (4.14), we get that $w(t_0)=w_0$ and $w(T)=\xi$. So, we will prove the existence of a solution if we show that

$$\chi(t) = A(t,w). \tag{4.16}$$

From property (3.4), i.e., (4.7), it follows that

$$X_\mu \equiv \int\limits_{t_0}^{T}\langle A(t,w_\mu(t)) - A(t,v(t)), w_\mu(t)-v(t)\rangle\,dt \ge 0 \;\; \forall v\in L_3(Q_{yt}). \tag{4.17}$$

According to (4.10)–(4.11),

$$\int_{t_0}^{T}\langle A(t,w_\mu),w_\mu\rangle\,dt=\int_{t_0}^{T}t^{-1}\langle A_{22}w_\mu(t),w_\mu(t)\rangle\,dt+\int_{t_0}^{T}(g,w_\mu)\,dt+\frac{1}{2}\|w_{0\mu}\|^2_{H^{-1}(\Omega)}-\frac{1}{2}\|w_\mu(T)\|^2_{H^{-1}(\Omega)} \tag{4.18}$$

and, therefore,

$$X_\mu=\int_{t_0}^{T}(g,w_\mu)\,dt+\frac{1}{2}\|u_{0\mu}\|^2_{H^{-1}(\Omega)}-\frac{1}{2}\|w_\mu(T)\|^2_{H^{-1}(\Omega)}+$$

$$+\int_{t_0}^{T}t^{-1}\langle A_{22}w_\mu,w_\mu\rangle\,dt-\int_{t_0}^{T}\langle A(t,w_\mu),v\rangle\,dt-\int_{t_0}^{T}\langle A(t,v),w_\mu-v)\rangle\,dt,$$

whence (since $\liminf\|w_\mu(T)\|^2_{H^{-1}(\Omega)}\geq\|w(T)\|^2_{H^{-1}(\Omega)}$):

$$\limsup X_\mu\leq\int_{t_0}^{T}(g,w)\,dt+\frac{1}{2}\|u_0\|^2_{H^{-1}(\Omega)}-\frac{1}{2}\|w(T)\|^2_{H^{-1}(\Omega)}+$$

$$+\int_{t_0}^{T}t^{-1}\langle A_{22}w,w\rangle\,dt-\int_{t_0}^{T}\langle\chi(t),v\rangle\,dt-\int_{t_0}^{T}\langle A(t,v),w-v)\rangle\,dt. \tag{4.19}$$

From (4.15) we can conclude, since integration by parts is legal, that

$$\int_{t_0}^{T}t^{-1}\langle A_{22}w,w\rangle\,dt+\int_{t_0}^{T}(g,w)\,dt+\frac{1}{2}\|u_0\|^2_{H^{-1}(\Omega)}-\frac{1}{2}\|w(T)\|^2_{H^{-1}(\Omega)}=\int_{t_0}^{T}\langle\chi,w\rangle\,dt.$$

Comparing this equality with (4.17) and (4.19), and also considering (4.18), we get

$$\int_{t_0}^{T}\langle\chi(t)-A(t,v),w-v\rangle\,dt\geq 0. \tag{4.20}$$

Now we use the hemicontinuity property (4.1) of the operator $A(t,w)$ to prove that (4.20) implies (4.16). Let $v=w-\lambda u,\ \lambda>0,\ u\in L_3(Q_{yt})$; then it follows from (4.20) that

$$\lambda\int_{t_0}^{T}\langle\chi(t)-A(t,w-\lambda u),u\rangle\,dt\geq 0,$$

whence

$$\int_{t_0}^{T}\langle\chi(t)-A(t,w-\lambda u),u\rangle\,dt\geq 0; \tag{4.21}$$

when $\lambda\to 0$ in (4.21), then we get that

$$\int_{t_0}^{T}\langle\chi(t)-A(t,w),u\rangle\,dt\geq 0\ \ \forall\,u.$$

Therefore, $\chi(t)=A(t,w)$. The existence of a solution to problem (2.1) and (2.3) is proved.

4.2 Uniqueness of the solution

Let $w_1(t)$ and $w_2(t)$ be two solutions to problem (4.8)-(4.9). Then their difference $w(t) = w_1(t) - w_2(t)$ satisfies the homogeneous problem:

$$w'(t) + A(t, w_1(t)) - A(t, w_2(t)) - t^{-1}A_{22}w(t) = 0, \;\; w(0) = 0,$$

$$\big(w'(t), w(t)\big) + \langle (A(t, w_1(t)) - A(t, w_2(t)), w_1(t) - w_2(t)\rangle - t^{-1}\langle (A_{22}w(t), w(t)\rangle = 0$$

and, due to (4.12) and the monotonicity property of the operator $A(t, w)$, we have:

$$\big(w'(t), w(t)\big) = \frac{d}{2\,dt}\|w(t)\|^2_{H^{-1}(\Omega)} \le \frac{C_2K}{t_0}\|w(t)\|^2_{H^{-1}(\Omega)}, \;\text{ i.e. }\; w(t) \equiv 0,$$

where K is the norm of the operator $\left(-d_y^2\right)^{-1/2} : \; H^{-1}(\Omega) \to \left[H_0^1(\Omega); H^{-1}(\Omega)\right]_{1/2}$, $\left[H_0^1(\Omega); H^{-1}(\Omega)\right]_{1/2}$ is an intermediate space [9].

Remark 4.1. Let us give the interpretation of the solution to problem (4.8)–(4.9) as the solution to problem (2.1)–(2.3). By introducing $\tilde{v}$ in (4.8), we obtain

$$\int\limits_0^1 \partial_t w\, \tilde{v}\, dy + \int\limits_0^1 \left[\frac{1}{2t^2}|w|w + \frac{1}{t}\left(-\partial_y^2\right)^{-1}w + \frac{1}{t}\left(-\partial_y^2\right)^{-\frac{1}{2}}(y\,w)\right]\left(-\partial_y^2\tilde{v}\right)\, dy =$$

$$= \int\limits_0^1 g(t)\tilde{v}\, dy, \;\; \forall\, \tilde{v} \in H_0^1(\Omega).$$

Hence, from here we have

$$\int\limits_0^1 \left(\partial_t w - \partial_y^2\left[\frac{1}{2\,t^2}|w|w + \frac{1}{t}\left(-\partial_y^2\right)^{-1}w + \frac{1}{t}\left(-\partial_y^2\right)^{-\frac{1}{2}}(y\,w)\right]\right)\tilde{v}\, dy = \int\limits_0^1 g(t)\tilde{v}\, dy +$$

$$+ \left[\frac{1}{2\,t^2}|w|w + \frac{1}{t}\left(-\partial_y^2\right)^{-1}w + \frac{1}{t}\left(-\partial_y^2\right)^{-\frac{1}{2}}(y\,w)\right]\partial_y\tilde{v}\Big|_{y=0}^{y=1} \;\forall\, \tilde{v} \in H_0^1(\Omega). \tag{4.22}$$

Or, taking into account equality (3.5), the last identity can be written in the following form

$$\int\limits_0^1 \left(\partial_t w - \frac{1}{t^2}\partial_y\left(|w|\partial_y w\right) - \frac{y}{t}\,\partial_y w - g(t)\right)\tilde{v}\, dy = 0 \;\; \forall\, \tilde{v} \in \mathfrak{D}(\Omega), \tag{4.23}$$

that is, the function $w(y, t)$ satisfies a Boussinesq type equation (2.1). Now, returning to (4.22) and taking into account (4.23), we get

$$\left[\frac{1}{2t^2}|w|w + \frac{1}{t}\left(-\partial_y^2\right)^{-1}w + \frac{1}{t}\left(-\partial_y^2\right)^{-\frac{1}{2}}(y\,w)\right]\partial_y\tilde{v}\big|_{|y=0} = 0 \;\; \forall\, \tilde{v} \in H_0^1(\Omega),$$

$$\left[\frac{1}{2t^2}|w|w + \frac{1}{t}\left(-\partial_y^2\right)^{-1}w + \frac{1}{t}\left(-\partial_y^2\right)^{-\frac{1}{2}}(y\,w)\right]\partial_y\tilde{v}\big|_{|y=1} = 0 \;\; \forall\, \tilde{v} \in H_0^1(\Omega).$$

The last equalities imply the fulfillment of boundary conditions (2.2). Finally, from the continuity of the function $w : [t_0, T] \to H$ we get that initial condition (2.3) makes sense. This completes the proof of Theorem 4.2.

5 To the proof of Theorem

Since the transformation of independent variables $\{x,t\} \to \{y,t\}$ is one-to-one, there is a mutual correspondence of functional classes defining the given functions and solutions of initial boundary value problems. Therefore, from Theorem 2.1 we obtain the validity of the statement of Theorem 1.1 in terms of the existence of a solution to initial boundary value problem (1.1)–(1.3). Let us show the validity of the assertion of Theorem 1.1 in terms of the uniqueness of the solution to problem (1.1)–(1.3).

We show that the operator $A_1(t,u)$ in problem (1.1)–(1.3) will have the monotonicity property if a scalar product is introduced accordingly. For this purpose, we take as the scalar product

$$\langle \varphi, \psi \rangle = \int_0^t \varphi \left[\left(-d_x^{\,2}\right)^{-1} \psi \right] dx, \;\; \forall \varphi, \psi \in H^{-1}(\Omega_t), \;\; \forall t \in (t_0, T), \tag{5.1}$$

where $d_x^{\,2} = \frac{d^2}{dx^2}$, $\tilde{\psi} = \left(-d_x^{\,2}\right)^{-1} \psi : \; -d_x^{\,2} \tilde{\psi} = \psi$, $\tilde{\psi}(0) = \tilde{\psi}(t) = 0, \forall \psi \in H^{-1}(\Omega_t), \;\; \forall t \in (t_0, T)$.

The following lemma is valid.

Lemma 5.1. Operator $A_1(t,u)$ is monotone in the sense of the scalar product (5.1) in the space $H^{-1}(\Omega_t)$, i.e., the following inequalities hold:

$$\langle A_1(t,u_1) - A_1(t,u_2), u_1 - u_2 \rangle \geq 0, \;\; \forall u_1, u_2 \in \mathfrak{D}(\Omega_t), \;\; \forall t \in (t_0, T). \tag{5.2}$$

To the proof of Lemma 5.1. For each $t \in (t_0, T)$ operator A_1 is monotone and condition (5.2) is satisfied (according to [7], chap. 2, p. 3.1). Indeed, on the one hand, we have

$$\langle A_1(t,\varphi) - A_1(t,\psi), \varphi - \psi \rangle = \frac{1}{2} \int_{\Omega_t} \left(-d_x^2\right) \left(|\varphi|\varphi - |\psi|\psi\right) \left(-d_x^2\right)^{-1} (\varphi - \psi)\, d\,x =$$

$$= \frac{1}{2} \int_{\Omega_t} (|\varphi|\varphi - |\psi|\psi)(\varphi - \psi)\, d\,x, \;\; \forall \varphi, \, \psi \in \mathfrak{D}(\Omega_t), \;\; \forall t \in (t_0, T).$$

On the other hand, the convexity condition of the functional $J(t,\varphi) = \frac{1}{3} \int_{\Omega_t} |\varphi(x)|^3 \, d\,x$, $\varphi \in \mathfrak{D}(\Omega_t)$, $\forall t \in (t_0, T)$, implies

$$\langle J'(t,\varphi) - J'(t,\psi), \varphi - \psi \rangle \geq 0, \;\; \forall \varphi, \, \psi \in \mathfrak{D}(\Omega_t), \;\; \forall t \in (t_0, T).$$

Thus, we get

$$\int\limits_{\Omega_t} (|\varphi|\varphi - |\psi|\psi)(\varphi - \psi)\, d\,x \geq 0, \;\; \forall \varphi, \, \psi \in \mathfrak{D}(\Omega_t), \;\; \forall t \in (t_0, T),$$

that is, inequality 5.2 is established. Lemma 5.1 is proved.

Now we are ready to show the uniqueness of the solution to problem (1.1)–(1.3). To do this, using inequality (5.2), we obtain the following variational formulation for initial boundary problem (1.1)–(1.3):

$$(u'(t), v)_{H^{-1}(0,t)} + a_0(t, u(t), v) = (f(t), v)_{H^{-1}(\Omega_t)} \;\; \forall v \in L_3(0,t) \subset H^{-1}(\Omega_t), \;\; \forall t \in (t_0, T), \tag{5.3}$$

$$u(t_0) = u_0, \tag{5.4}$$

where

$$a_0(t,u,v) = \langle A_1(t, u(t)), v \rangle = \int\limits_{\Omega_t} |u(x,t)| \, u(x,t) \, v(x) \, dx, \;\; \forall t \in (t_0, T).$$

Let $u_1(t)$ and $u_2(t)$ be two solutions to problem (5.3)–(5.4). Then their difference $u(t) = u_1(t) - u_2(t)$ satisfies the homogeneous problem:

$$\left(u'(t), u(t)\right)_{H^{-1}(\Omega_t)} + \langle A_1(t, u_1(t)) - A_1(t, u_2(t)), u(t) \rangle = 0, \;\; \forall t \in (t_0, T); \;\; u(0) = 0,$$

and, due to the monotonicity property of the operator $A_1(t,u)$ (5.2), we have:

$$\big(u'(t), u(t)\big)_{H^{-1}(\Omega_t)} = \frac{d}{2\,dt}\|u(t)\|^2_{H^{-1}(\Omega_t)} \leq 0, \text{ i.e. } u(t) \equiv 0.$$

Thus, Theorem (1.1) is completely proved.

Conclusions

The initial boundary value problems for a one-dimensional Boussinesq type equation in a trapezoid domain are studied. Theorems on their unique weak solvability in Sobolev classes are proved by methods of the theory of monotone operators.

Acknowledgments

This research was funded by the Science Committee of the Ministry of Education and Science of the Republic of Kazakhstan (Grants No. AP09258892, 2021-2023).

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Трапециядағы Буссинеск типті теңдеу үшін бастапқы-шекаралық есеп

Мақалада трапеция облысындағы бір өлшемді Буссинеск типті теңдеу үшін бастапқы-шекаралық есеп қарастырылған. Соболев кластарындағы олардың бірегей әлсіз шешілетіндігі туралы теоремалар монотонды операторлар теориясының әдістерімен анықталған.

Кілт сөздер: Буссинеск типті теңдеу, шекаралық есеп, трапеция, монотонды операторлар теориясы.

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Начально-граничная задача для уравнения типа Буссинеска в трапеции

В статье рассмотрена начально-граничная задача для одномерного уравнения типа Буссинеска в области, представляющей собой трапецию. Методами теории монотонных операторов установлены теоремы об их однозначной слабой разрешимости в соболевских классах.

Ключевые слова: уравнение типа Буссинеска, граничная задача, трапеция, теории монотонных операторов.

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