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On simple modules with singular highest weights for $\mathfrak{so}_{2l+1}(\mathbb{K})$

In this paper, we study formal characters of simple modules with singular highest weights over classical Lie algebras of type B over an algebraically closed field of characteristic $p \geq h$, where h is the Coxeter number. Assume that the highest weights of these simple modules are restricted. We have given a description of their formal characters. In particular, we have obtained some new examples of simple Weyl modules. In the restricted region, the representation theory of algebraic groups and its Lie algebras are equivalent. Therefore, we can use the tools of the representation theory of semisimple and simply-connected algebraic groups in positive characteristic. To describe the formal characters of simple modules, we construct Jantzen filtrations of Weyl modules of the corresponding highest weights.

Keywords: Lie algebra, simple module, algebraic group, Weyl module, Jantzen filtration.

Introduction

The study of the structure of Weyl modules is one of the central questions of the representation theory of simply-connected and semisimple algebraic groups in positive characteristics. If the structure of the Weyl modules is known, then it is easy to describe the formal characteristics of simple modules associated with them. There is a remarkable Lusztig's conjecture, which facilitates to describe the characters of simple modules. Its validity is proved for sufficiently large characteristics of the ground field. In [1], Fiebig gave the upper bound of the exceptional characteristics for Lusztig's character formula. It depends on the root system and is far from the Coxeter number of the root system. Furthermore, the description of the Kazhdan-Lusztig polynomials for the antidominant elements of the affine Weyl group that appear in the Lusztig's character formula is due to the complicated calculations. In the restricted region, they are known only for small groups, such as $SL_2(\mathbb{K})$, $SL_3(\mathbb{K})$, $Sp_4(\mathbb{K})$, G_2, $SL_4(\mathbb{K})$, $Sp_6(\mathbb{K})$ и $SO_7(\mathbb{K})$. For nonrestricted elements, they are computed in some special cases. Thus, the problem of a description of the formal characters of simple modules using the structure of Weyl modules still remains of current interest.

Let G be a semisimple, simply-connected algebraic group of type B_l over an algebraically closed field of characteristic $p \geq h$, where h is the Coxeter number, and $\mathfrak{g}$ be a Lie algebra of G. In this paper, for all $l > 2$, we give the structure of the Weyl modules with singular highest weights for G with highest weights defined by the dominant elements of the following subsets of the affine Weyl group W_p of G :

$$Y_1 = \{y_{-1} = 1,\ y_i = s_0 s_1 s_2 \cdots s_i \,|\, i = 0,\, 1,\, \cdots, l\};$$

$$Y_2 = \{y_{l+j} = s_0 s_1 s_2 \cdots s_l s_{l-1} \cdots s_{l-j} \,|\, j = 1,\, 2,\, \cdots, l-2\};$$

$$Z_1 = \{z_{-1} = s_0,\ z_0 = 1,\ z_i = y_i s_0 \,|\, i = 1,\, 2,\, \cdots, l\};$$

$$Z_2 = \{z_{l+j} = y_{l+j} s_0 \,|\, j = 1,\, 2,\, \cdots, l-2\}.$$

Here $s_0, s_1, \cdots, s_l$ are the generators of W_p. The affine Weyl group W_p is a Coxeter group of type $\widetilde{B}_l$ with the following defending relations:

$$(s_i s_j)^{m_{ij}} = 1,\ s_0^2 = 1,\ (s_0 s_i)^2 = 1\ (i \neq 1),\ (s_0 s_1)^4 = 1, \tag{1}$$

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where $i,\, j \in \{1,\, 2,\, \cdots,\, l\}$ and

$$m_{ij} = \begin{cases} 1, & \text{if } i = j; \\ 2, & \text{if } |i-j| > 1; \\ 3, & \text{if } |i-j| = 1 \text{ and } (i,j) \notin \{(l-1,l),\, (l,l-1)\}; \\ 4, & \text{if } (i,j) \in \{(l-1,l),\, (l,l-1)\}. \end{cases}$$

Let R be an irreducible root system of type B_l, $\alpha_1, \alpha_2, \cdots, \alpha_l$ be the simple roots, and $\omega_1, \omega_2, \cdots, \omega_l$ be the fundamental weights. Denote by h the Coxeter number of R. Let us shortly discuss the well-known results on the structure of the Weyl modules for simply-connected and semisimple algebraic groups of type B_l. In [2] and [3], Braden and Jantzen described the structure of the Weyl modules with highest restricted weights of the algebraic group of type B_2. The simplicity of the Weyl module $V(s_0 \cdot \nu)$ with the dominant highest weight $s_0 \cdot \nu$, where $\nu \in \overline{C}_1 \setminus C_1$, was proved by Rudakov in [4, Theorem 2] for semisimple Lie algebras over a field of characteristic $p \geq h$. In [5], O'Halloran obtained a set of Weyl modules with a simple radical and described their structure. In small characteristics, the Weyl modules with restricted highest weights were described for $B_4\,(p = 2)$ by Dowd and Sin [6], for $B_3\,(p = 2, 3)$ by Ye and Zhou [7], [8]. In [9], Arslan and Sin studied the nonrestricted case $V(2\omega_l)$ and the Weyl modules with the fundamental weights in characteristic $p = 2$ for B_l whith $l \geq 2$. The structure of the Weyl modules with highest weights in $\{r\omega_1 \,|\, 0 \leq r \leq p-1\}$ was calculated by Cardinali and Pasini [10]. For the group of type B_4 over a field of characteristic $p > 0$, in [11], Wiggins calculated the structure of the Weyl modules with the highest weights in $\{r\omega_4 \,|\, 0 < r \leq p-1\}$. A similar result was obtained by Cavallin for the groups of type B_l over a field of characteristic $p > 2$ and for the highest weights in $\{2\omega_1,\, \omega_1 + 2\omega_l,\, \omega_1 + \omega_j \,|\, 2 \leq j \leq l\}$ [12].

1 Notation and formulation of the main results

Before starting to formulate our results, we introduce some notation and useful facts. Basically, we will use standard notation. Let R be an irreducible root system of type B_l and let G be a simply-connected and semisimple algebraic group with root system R over an algebraically closed field $\mathbb{K}$ of characteristic $p \geq h$, where h is Coxeter number of R. We assume that $R \subset \mathbb{R}^l$, where $\mathbb{R}$ is the field of real numbers. On $\mathbb{R}^l$ there is the usual euclidean inner product $(\cdot,\cdot)$. This leads to the natural pairing $\langle\cdot,\cdot\rangle : \mathbb{R}^l \times \mathbb{R}^l \to \mathbb{R}$ given by $\langle\lambda,\mu\rangle = (\lambda,\mu^\vee)$, where $\mu^\vee = \frac{2}{(\mu,\mu)}\mu$. If $\Delta = \{\alpha_1, \alpha_2, \cdots, \alpha_l\}$ is the set of simple roots and $\{\varepsilon_i \,|\, i = 1,\, 2,\, \cdots,\, l\}$ is the orthonormal basis of $\mathbb{R}^l$ then the positive roots of R can be seen as a set [13]:

$$R^+ = \begin{array}{l} \{\alpha_i + \alpha_{i+1} + \cdots + \alpha_j = \varepsilon_i - \varepsilon_{j+1} \,|\, 1 \leq i \leq j \leq l-1\} \cup \\ \{\alpha_i + \cdots + \alpha_l = \varepsilon_i \,|\, i = 1,\, 2,\, \cdots,\, l\} \cup \\ \{\alpha_i + \cdots + \alpha_j + 2\alpha_{j+1} + \cdots + 2\alpha_l = \varepsilon_i + \varepsilon_{j+1} \,|\, 1 \leq i \leq j < l\}. \end{array} \qquad (2)$$

Let $T \subseteq G$ be a maximal torus, and B be the Borel subgroup corresponding to the negative roots. We denote by U the unipotent radical of B. The set $X(T)$ of additive characters for T can be seen as a subset of $\mathbb{R}^l$ with basis $\omega_1,\, \omega_2,\, \cdots, \omega_l$ satisfying $\langle\omega_i, \alpha_j\rangle = \delta_{ij}$. The set $X(T)$ also has the following property:

$$X(T) = \{\lambda \in \mathbb{R}^l \,|\, \langle\lambda, \alpha\rangle \in \mathbb{Z} \text{ for all } \alpha \in R\}.$$

Any rational G-module can be considered as the direct sum of T-modules:

$$V = \bigoplus_{\lambda \in X(T)} V_\lambda,$$

where $V_\lambda = \{v \in V \,|\, tv = \lambda(t)v,\ \text{for all } t \in T\}$. If $V_\lambda \neq 0$ we say λ is a weight of V, and V_λ is a weight subspace of V. In this case, the vectors of V_λ will be called weight vectors.

Let

$$X(T)^+ = \{\lambda \in X(T) \,|\, \langle\lambda, \alpha\rangle \geq 0 \text{ for all } \alpha \in R^+\}$$

be the set of dominant weights. If $\rho \in X(T)^+$ is the half-sum of the positive roots, then it is easy to prove that

$$\rho = \omega_1 + \omega_2 + \cdots + \omega_l. \qquad (3)$$

We define the formal characters of V by

$$[V] = \sum_{\lambda \in X(T)} \dim_k V_\lambda e^\lambda \in \mathbb{Z}(X(T)) = \bigoplus_{\lambda \in X(T)} \mathbb{Z}e^\lambda.$$

Let $\mathbb{Z}$ be the set of integers, and $m_1, m_2, \cdots, m_l \in \mathbb{Z}$. If $\lambda = \sum_{i=1}^{l} m_i\omega_i \in X(T)$ then, by (2) and (3), we get

$$\langle \lambda+\rho, \alpha\rangle = \begin{cases} m_i + 1, \text{ if } \alpha = \alpha_i,\ i = 1, 2, \cdots, l; \\ \\ m_i + \cdots + m_j + (j-i+1), \\ \text{if } \alpha = \alpha_i + \cdots + \alpha_j,\ 1 \leq i < j = 1, 2, \cdots, l-1; \\ \\ 2m_i + \cdots + 2m_{l-1} + m_l + 2l - 2i + 1, \\ \text{if } \alpha = \alpha_i + \cdots + \alpha_l,\ i = 1, 2, \cdots, l-1; \\ \\ m_i + \cdots + m_j + 2m_{j+1} + \cdots + 2m_{l-1} + m_l + 2l - i - j, \\ \text{if } \alpha = \alpha_i + \cdots + \alpha_j + 2\alpha_{j+1} + \cdots + 2\alpha_l,\ 1 \leq i \leq j < l. \end{cases} \tag{4}$$

Let $\lambda \in X(T)^+$ and $H^0(\lambda)$ be the vector space over $\mathbb{K}$ of all regular functions $f : G \to \mathbb{K}$ satisfying:

$$f(bg) = \lambda(b^{-1})f(g), \text{ for all } b \in B,\ g \in G.$$

We define on $H^0(\lambda)$ a G-module structure given by

$$gf(h) = f(hg), \text{ where } f \in H^0(\lambda),\ g, h \in G.$$

Also, it is well-known that $H^0(\lambda) = Ind_B^G \mathbb{K}_\lambda$, where $\mathbb{K}_\lambda$ is a one dimensional B-module defined by $\lambda \in X(T)^+$ via the isomorphism $B/U \cong T$. Let $L(\lambda)$ be a maximal semisimple submodule (socle) of $H^0(\lambda)$. Each $L(\lambda)$ is a simple G-module and every simple G-module is isomorphic to $L(\lambda)$ for some $\lambda \in X(T)^+$. The Weyl module $V(\lambda)$ with the highest weight $\lambda \in X(T)^+$ is isomorphic to $H^0(-w_0(\lambda))^*$, where w_0 is the maximal element of the Weyl group W for R. There is the following Weyl character formula:

$$\chi(\lambda) := [V(\lambda)] = [H^0(\lambda)] = \frac{\sum_{w\in W}(-1)^{l(w)}e^{w(\lambda+\rho)}}{\sum_{w\in W}(-1)^{l(w)}e^{w(\rho)}}.$$

Let V be a G-module. We define *a composition coefficient* $[V : L(\lambda)]$ for $\lambda \in X(T)^+$ such that

$$[V] = \sum_{\lambda \in X(T)^+} [V : L(\lambda)][L(\lambda)].$$

If $[V : L(\lambda)] \neq 0$ then we say that $L(\lambda)$ is a *composition factor of* V.

For $\alpha \in R^+$ and $n \in \mathbb{Z}$ let us define the affine reflections $s_{\alpha,n}$ on $X(T)$ by [14, II.6.1]

$$s_{\alpha,n} \cdot \lambda = \lambda - \langle \lambda+\rho, \alpha\rangle + np\alpha \text{ for all } \lambda \in X(T).$$

By W_p denote the *affine Weyl group* generated by all $s_{\alpha,n}$ with $\alpha \in R^+$ and $n \in \mathbb{Z}$. The usual finite Weyl group W of R appears as the subgroup of W_p generated by the reflections $s_{\alpha,0}$ with $\alpha \in R^+$.

Let $\alpha_0 = \omega_1 = \varepsilon_1$ be the unique highest short root of R. We will use the following notation: $s_{\alpha_i,0} := s_i$ for all $i \in \{1, 2, \cdots, l\}$ and $s_0 := s_{\alpha_0,1}$. Then the set of generators of W is $S = \{s_i \,|\, i = 1, 2, \cdots, l\}$ and the set of generators of W_p is $S_p = S \cup \{s_0\}$.

We will also use the affine hyperplanes and the affine alcoves. For $\alpha \in R^+$ and $n \in \mathbb{Z}$ we define

$$H_{\alpha,n} = \{v \in \mathbb{R}^l \,|\, \langle v+\rho, \alpha\rangle = np\}.$$

The set of affine alcoves A is defined as the set of connected components of

$$\mathbb{R}^l \setminus (\bigcup_{\alpha\in R_+,\, n\in\mathbb{Z}} H_{\alpha,n}).$$

The fundamental alcove $C_1 \in A$ is defined by

$$C_1 = \{v \in \mathbb{R}^l \,|\, 0 < \langle v+\rho, \alpha\rangle < p \text{ for all } \alpha \in R^+\}.$$

We denote by $\overline{C}_1$ a closure of C_1.

Let $W_p^+ \subset W_p$ be the set of *dominant elements* defined by

$$W_p^+ = \{w \in W_p \,|\, w\cdot\nu \in X(T)^+ \text{ for any } \nu \in C_1\}.$$

The stabilizer $st(\lambda)$ of $\lambda \in X(T)$ is the set

$$st(\lambda) = \{w \in W_p \,|\, w\cdot\lambda = \lambda\}.$$

If $st(\lambda) \cap S_p = \emptyset$ we say λ is a *regular weight*, otherwise it is called a *singular weight.* Let $\lambda = w\cdot\nu$, where $w \in W_p$ and $\nu \in \overline{C}_1$. It is known that λ is a regular weight if and only if $\nu \in C_1$.

Next we introduce some notation for singular weights. Let $H_0 := H_{\alpha_0,1}$ and $H_i := H_{\alpha_i,0}$ for all $i \in \{1,\, 2,\, \cdots,\, l\}$. By $\nu_{i_1,\, i_2,\, \cdots,\, i_m}$ denote any element of $\overline{C}_1 \setminus C_1$ satisfying the following conditions:

1) $\nu_{i_1,\, i_2,\, \cdots,\, i_m} \in H_{i_1} \cap H_{i_2} \cap \cdots \cap H_{i_m}$
2) $i_1,\, i_2,\, \cdots,\, i_m \in \{0,\, 1,\, \cdots,\, l\}$;
3) $i_1 < i_2 < \cdots < i_m$;
4) $m \in \{1,\, 2,\, \cdots,\, l+1\}$.

Let $\nu \in \overline{C}_1 \setminus C_1$. Denote by $\overline{w}_\nu$ the (left) coset of the stabilizer $st(\nu)$ containing the element $w \in W_p^+$. Then $W_p^+ = \cup_{w\in W_p^+}\overline{w}_\nu$. An action of $\overline{w}_\nu$ on ν defined by $\overline{w}_\nu \cdot \nu = u\cdot\nu$ for any $u \in \overline{w}_\nu$. If $\overline{w}_\nu \cdot \nu \in X(T)^+$ we say $\overline{w}_\nu$ is *dominant for* ν. Then, up to isomorphism, $\overline{w}_\nu$ defines a simple G-module (respectively, a Weyl module) with highest weight $\overline{w}_\nu \cdot \nu$. We will use notation $\overline{w}$ for the coset $\overline{w}_\nu$ when ν is fixed.

Let $W' \subset W_p^+$ and $\nu \in \overline{C}_1 \setminus C_1$. By definition, put

$$W'^+_\nu := \{\overline{w}_\nu \,|\, \overline{w}_\nu\cdot\nu \in X(T)^+ \text{ and } w \in W'\}.$$

We say that W'^+_ν is *the set of dominant elements of* W' *for* ν.

Now we formulate the main results of this paper.

Theorem 1. Let G be a simply-connected and semisimple algebraic group of type $B_l\,(l > 2)$ over an algebraically closed field $\mathbb{K}$ of characteristic $p \geq h$, where $h = 2l$ is the Coxeter number. Suppose that $\nu \in \overline{C}_1 \setminus C_1$ and $(Y_1 \cup Y_2 \cup Z_1 \cup Z_2)^+_\nu \neq \emptyset$. If $\overline{w}_\nu \in (Y_1 \cup Y_2 \cup Z_1 \cup Z_2)^+_\nu$ then $\chi(\overline{w}_\nu\cdot\nu) = [L(\overline{w}_\nu\cdot\nu)]$ except in the following cases:

(a) $\chi(\overline{y_i}\cdot\nu_0) = [L(\overline{y_i}\cdot\nu_0)] + [L(\overline{y_{i-1}}\cdot\nu_0)]$, where $i \in \{2,\, 3,\, \cdots,\, 2l-2\}$;
(b) $\chi(\overline{z_2}\cdot\nu_1) = [L(\overline{z_2}\cdot\nu_1)] + [L(\overline{y_0}\cdot\nu_1)]$;
(c) $\chi(\overline{z_i}\cdot\nu_1) = [L(\overline{z_i}\cdot\nu_1)] + [L(\overline{z_{i-1}}\cdot\nu_1)]$, where $i \in \{3,\, 4,\, \cdots,\, 2l-3\}$;
(d) $\chi(\overline{z_{2l-2}}\cdot\nu_1) = [L(\overline{z_{2l-2}}\cdot\nu_1)] + [L(\overline{z_{2l-3}}\cdot\nu_1)] + [L(\overline{y_{2l-2}}\cdot\nu_1)]$;
(e) $\chi(\overline{z_{i-1}}\cdot\nu_i) = [L(\overline{z_{i-1}}\cdot\nu_i)] + [L(\overline{y_{i-1}}\cdot\nu_i)]$, where $i \in \{2,\, 3,\, \cdots,\, l\}$;
(f) $\chi(\overline{z_{2l-i-1}}\cdot\nu_i) = [L(\overline{z_{2l-i-1}}\cdot\nu_i)] + [L(\overline{y_{2l-i-1}}\cdot\nu_i)]$, where $i \in \{2,\, 3,\, \cdots,\, l-1\}$.

A result similar to Theorem 1 was also obtained for simply-connected and semisimple algebraic groups of type D [15] and C [16].

Corollary 1. Let $\mathfrak{g}$ be a simple classical Lie algebra of type $B_l\,(l > 2)$ over an algebraically closed field $\mathbb{K}$ of characteristic $p \geq h$, where h is the Coxeter number. Suppose that $\nu \in \overline{C}_1 \setminus C_1$ and $(Y_1 \cup Y_2 \cup Z_1 \cup Z_2)^+_\nu \neq \emptyset$. If $\overline{w}_\nu \in (Y_1 \cup Y_2 \cup Z_1 \cup Z_2)^+_\nu$ then $[L(\overline{w}_\nu\cdot\nu)] = \chi(\overline{w}_\nu\cdot\nu)$ except in the following cases:

(a) for all $i \in \{2,\, 3,\, \cdots,\, 2l-2\}$,

$$[L(\overline{y_i}\cdot\nu_0)] = \sum_{j=0}^{i}(-1)^{i-j}\chi(\overline{y_j}\cdot\nu_0);$$

(b) $[L(\overline{z_2}\cdot\nu_1)] = -\chi(\overline{y_0}\cdot\nu_1) + \chi(\overline{z_2}\cdot\nu_1))$;
(c) for all $i \in \{3,\, 4,\, \cdots,\, 2l-3\}$,

$$[L(\overline{z_i}\cdot\nu_1)] = (-1)^{i-1}\chi(\overline{y_0}\nu_1) + \sum_{j=2}^{i}(-1)^{i-j}\chi(\overline{z_j}\cdot\nu_1);$$

(d)$[L(\overline{z_{2l-2}}\cdot\nu_1)] = (-1)^{2l-3}\chi(\overline{y_0}\cdot\nu_1) + \sum_{j=2}^{i}(-1)^{2l-2-j}\chi(\overline{z_j}\cdot\nu_1) - \chi(\overline{y_{2l-2}}\cdot\nu_1)$;
(e) $[L(\overline{z_{i-1}}\cdot\nu_i)] = -\chi(\overline{y_{i-1}}\cdot\nu_i) + \chi(\overline{z_{i-1}}\cdot\nu_i)$, where $i\in\{2, 3, \cdots, l\}$;
(f) $[L(\overline{z_{2l-i-1}}\cdot\nu_i)] = -\chi(\overline{y_{2l-i-1}}\cdot\nu_i) + \chi(\overline{z_{2l-i-1}}\cdot\nu_i)$, where $i\in\{2, 3, \cdots, l-1\}$.

2 Preliminary results

Let $V(\lambda)$ be a Weyl module with highest weight $\lambda\in X(T)^+$. Then there is a filtration of submodules

$$V(\lambda) = V(\lambda)^0 \supset V(\lambda)^1 \supset V(\lambda)^2 \supset \cdots \tag{5}$$

such that $V(\lambda)/V(\lambda)^1 \cong L(\lambda)$ and

$$\sum_{j>0}[V(\lambda)^j] = \sum_{\alpha\in R^+}\ \sum_{0<np<\langle\lambda+\rho,\alpha\rangle}\nu_p(np)\chi(s_{\alpha,n}\cdot\lambda), \tag{6}$$

where $\nu_p(m) = max\{i\in\mathbb{N}\,|\,p^i|m\}$ [14, II.8.19]. The filtration (5) is called the *Jantzen filtration* and (6) is called *Jantzen's sum formula.*

Let $\nu = a_1\omega_1 + a_2\omega_2 + \cdots + a_l\omega_l \in X(T)$, where $a_j\in\mathbb{Z}$ for all $j\in\{1, 2, \cdots, l\}$.

Lemma 1. Let $y_i\in Y_1\cup Y_2$ and $\nu = \sum_{i=1}^{l} a_i\omega_i$, where $a_1, \cdots, a_l\in\mathbb{Z}$. Then
(a) $y_0\cdot\nu = (p - a_1 - 2\sum_{i=2}^{l-1} a_i - a_l - 2l + 1)\omega_1 + \sum_{i=2}^{l} a_i\omega_i$;
(b) for all $i\in\{1, \cdots, l-2\}$,

$$y_i\cdot\nu = (p - \sum_{j=1}^{i} a_j - 2\sum_{j=i+1}^{l-1} a_j - a_l - 2l + i)\omega_1 + \\ \sum_{j=2}^{i} a_{j-1}\omega_j + (a_i + a_{i+1} + 1)\omega_{i+1} + \sum_{j=i+2}^{l} a_j\omega_j;$$

(c) $y_{l-1}\cdot\nu = (p - \sum_{j=1}^{l} a_j - l - 1)\omega_1 + \sum_{j=2}^{l-1} a_{j-1}\omega_j + (2a_{l-1} + a_l + 2)\omega_l$;
(d) $y_l\cdot\nu = (p - \sum_{j=1}^{l-1} a_j - l)\omega_1 + \sum_{j=2}^{l-1} a_{j-1}\omega_j + (2a_{l-1} + a_l + 2)\omega_l$;
(e) for all $i\in\{1, \cdots, l-2\}$,

$$y_{l+i}\cdot\nu = (p - \sum_{j=1}^{l-i-1} a_j - l + i)\omega_1 + \\ \sum_{j=2}^{l-i-1} a_{j-1}\omega_j + (a_{l-i-1} + a_{l-i} + 1)\omega_{l-i} + \sum_{j=l-i+1}^{l} a_j\omega_j.$$

Proof. (a) By (4), we have

$$y_0\cdot\nu = \nu - (\langle\nu+\rho,\alpha_0\rangle - p)\alpha_0 = \nu + (p - 2\sum_{i=1}^{l-1} a_i - a_l - 2l + 1)\omega_1 = \\ (p - a_1 - 2\sum_{i=2}^{l-1} a_i - a_l - 2l + 1)\omega_1 + \sum_{i=2}^{l} a_i\omega_i.$$

(b) We use induction on i. By (4),

$$s_1\cdot\nu = (-a_1 - 2)\omega_1 + (a_1 + a_2 + 1)\omega_2 + \sum_{i=3}^{l} a_i\omega_i.$$

Then

$$y_1\cdot\nu = s_1\cdot\nu - (\langle s_1\cdot\nu+\rho,\alpha_0\rangle - p)\alpha_0 = \\ s_1\cdot\nu + (p - 2\sum_{i=2}^{l-1} a_i - a_l - 2l + 3)\omega_1 = \\ (p - a_1 - 2\sum_{i=2}^{l-1} a_i - a_l - 2l + 1)\omega_1 + (a_1 + a_2 + 1)\omega_2 + \sum_{i=3}^{l} a_i\omega_i.$$

Therefore, the statement is true for $i = 1$.

Suppose that the statement is true for all $i < t$, where $t \le l - 2$. By (4),

$$s_t\cdot\nu = \sum_{j=1}^{t-2} a_j\omega_j + (a_{t-1} + a_t + 1)\lambda_{t-1} + (-a_t - 2)\omega_t + (a_t + a_{t+1} + 1)\omega_{t+1} + \sum_{j=t+2}^{l} a_j\omega_j.$$

By the induction hypothesis,

$$y_{t-1}\cdot\nu = (p - \sum_{j=1}^{t-1} a_j - 2\sum_{j=t}^{l-1} a_j - 2l + t - 1)\omega_1 + \\ \sum_{j=2}^{t-1} a_{j-1}\omega_j + (a_{t-1} + a_t + 1)\omega_t + \sum_{j=t+1}^{l} a_j\omega_j.$$

Then

$$y_t \cdot \nu = y_{t-1} \cdot (s_t \cdot \nu) = (p - \sum_{j=1}^{t-2} a_j - (a_{t-1} + a_t + 1) - 2(-a_t - 2) -$$
$$2(a_t + a_{t+1} + 1) - 2\sum_{j=t+2}^{l-1} a_j - 2l + t - 1)\omega_1 + \sum_{j=2}^{t-1} a_{j-1}\omega_j +$$
$$(a_{t-1} + a_t + 1 - a_t - 2 + 1)\omega_t + (a_t + a_{t+1} + 1)\omega_{t+1} + \sum_{j=t+2}^{l} a_j\omega_j =$$
$$(p - \sum_{j=1}^{t} a_j - 2\sum_{j=t+1}^{l-1} -2l + t)\omega_1 + \sum_{j=2}^{t} a_{j-1}\omega_j +$$
$$(a_{t-1} + a_t + 1)\omega_{t+1} + \sum_{j=t+2}^{l} a_j\omega_j.$$

(c) By (4) we have

$$s_{l-1} \cdot \nu = \sum_{j=1}^{l-3} a_j\omega_j + (a_{l-2} + a_{l-1} + 1)\lambda_{l-2} + (-m_{l-1} - 2)\omega_{l-1} + (2a_{l-1} + a_l + 2)\omega_l.$$

Then, the statement (b) for $i = l - 2$ gives us

$$y_{l-1} \cdot \nu = y_{l-2} \cdot (s_{l-1} \cdot \nu) = (p - \sum_{j=1}^{l-3} a_j - (a_{l-2} + a_{l-1} + 1) -$$
$$2(-a_{l-1} - 2) - (2a_{l-1} + a_l + 2) - l - 2)\omega_1 +$$
$$\sum_{j=2}^{l-2} a_{j-1}\omega_j + (a_{l-2} + a_{l-1} + 1 - a_{l-1} - 2 + 1)\omega_{l-1} + (2a_{l-1} + a_l + 2)\omega_l +$$
$$\sum_{j=t+2}^{l} a_j\omega_j = (p - \sum_{j=1}^{l} a_j - l - 1)\omega_1 + \sum_{j=2}^{l-1} a_{j-1}\omega_j + (2a_{l-1} + a_l + 2)\omega_l.$$

(d) By (4),

$$s_l \cdot \nu = \sum_{j=1}^{l-2} a_j\omega_j + (a_{l-1} + a_l + 1)\omega_{l-1} + (-a_l - 2)\omega_l.$$

Then using the statement (c), we have

$$y_l \cdot \nu = y_{l-1} \cdot (s_l \cdot \nu) = (p - \sum_{j=1}^{l-2} a_j - (a_{l-1} + a_l + 1) -$$
$$(-a_l - 2) - l - 1)\omega_1 + \sum_{j=2}^{l-1} a_{j-1}\omega_j + (2(a_{l-1} + a_l + 1) + (-a_l - 2) + 2)\omega_l =$$
$$(2a_{l-1} + a_l + 2)\omega_l + \sum_{j=t+2}^{l} a_j\omega_j =$$
$$(p - \sum_{j=1}^{l-1} a_j - l)\omega_1 + \sum_{j=2}^{l-1} a_{j-1}\omega_j + (2a_{l-1} + a_l + 2)\omega_l.$$

(e) By (4),

$$s_{l-1} \cdot \nu = \sum_{j=1}^{l-2} a_j\omega_j + (a_{l-2} + a_{l-1} + 1)\omega_{l-2} + (-a_{l-1} - 2)\omega_{l-1} + (2a_{l-1} + a_l + 2)\omega_l.$$

Then using the statement (d), we have

$$y_{l+1} \cdot \nu = y_l \cdot (s_{l-1} \cdot \nu) = (p - \sum_{j=1}^{l-2} a_j - l + 1)\omega_1 + \sum_{j=2}^{l-2} a_{j-1}\omega_j + (a_{l-2} + a_{l-1} + 1)\omega_{l-1} + a_l\omega_l.$$

Therefore, the statement is true for $i = 1$.

Suppose that the statement is true for all $i < t$, where $t \le l - 2$. By (4)

$$s_{l-t} \cdot \nu = \sum_{j=1}^{l-t-2} a_j\omega_j + (a_{l-t-1} + a_{l-t} + 1)\omega_{l-t-1} +$$
$$(-a_{l-t} - 2)\omega_{l-t} + (a_{l-t} + a_{l-t+1} + 1)\omega_{l-t+1} + \sum_{j=l-t+2}^{l} a_j\omega_j.$$

By the induction hypothesis,

$$y_{l+t-1} \cdot \nu = (p - \sum_{j=1}^{l-t} a_j - l + t - 1)\omega_1 +$$
$$\sum_{j=2}^{l-t} a_{j-1}\omega_j + (a_{l-t} + a_{l-t+1} + 1)\omega_{l-t+1} + \sum_{j=l-t+2}^{l} a_j\omega_j.$$

Then

$$y_{l+t} \cdot \nu = y_{l+t-1} \cdot (s_{l-t} \cdot \nu) = (p - \sum_{j=1}^{l-t-2} a_j - (a_{l-t-1} + a_{l-t} + 1) -$$
$$(-a_{l-t} - 2) - l + t - 1)\omega_1 \sum_{j=2}^{l-t-1} a_{j-1}\omega_j + (a_{l-t-1} + a_{l-t})\omega_{l-t} +$$
$$(-a_{l-t} - 2 + a_{l-t} + a_{l-t+1} + 1 + 1)\omega_{l-t+1} + \sum_{j=l-t+2}^{l} a_j\omega_j =$$
$$(p - \sum_{j=1}^{l-t-1} a_j - l + t)\omega_1 + \sum_{j=2}^{l-t-1} a_{j-1}\omega_j +$$
$$(a_{l-t-1} + a_{l-t})\omega_{l-t} + \sum_{j=l-t+1}^{l} a_j\omega_j.$$

□

Lemma 2. Let $z_i \in Z_1 \cup Z_2$ and $\nu = \sum_{i=1}^{l} a_i\omega_i$, where $a_1,\, a_2,\, \cdots,\, a_l \in \mathbb{Z}$. Then
(a) $z_1 \cdot \nu = a_1\omega_1 + (p - a_1 - a_2 - 2\sum_{i=3}^{l-1} a_i - a_l - 2l + 2)\omega_2 + \sum_{i=3}^{l} a_i\omega_i$;
(b) for all $i \in \{2,\, 3,\, \cdots,\, l-2\}$,

$$z_i \cdot \nu = (\textstyle\sum_{j=1}^{i} a_j + i - 1)\omega_1 + (p - a_1 - 2\sum_{j=2}^{l-1} a_j - a_l - 2l + 1)\omega_2 + \\ \textstyle\sum_{j=3}^{i} a_{j-1}\omega_j + (a_i + a_{i+1} + 1)\omega_{i+1} + \sum_{j=i+2}^{l} a_j\omega_j;$$

(c) $z_{l-1} \cdot \nu = (\sum_{j=1}^{l-1} a_j + l - 2)\omega_1 + (p - a_1 - 2\sum_{j=2}^{l-1} a_j - a_l - 2l + 1)\omega_2 + \sum_{j=3}^{l-1} a_{j-1}\omega_j + (2a_{l-1} + a_l + 2)\omega_l$;
(d) $z_l \cdot \nu = (\sum_{j=1}^{l} a_j + l - 1)\omega_1 + (p - a_1 - 2\sum_{j=2}^{l-1} a_j - a_l - 2l + 1)\omega_2 + \sum_{j=3}^{l-1} a_{j-1}\omega_j + (2a_{l-1} + a_l + 2)\omega_l$;
(e) for all $i \in \{1,\, \cdots,\, l-2\}$,

$$z_{l+i} \cdot \nu = (\textstyle\sum_{j=1}^{l-i-1} a_j + 2\sum_{j=l-i}^{l-1} a_j + a_l + l + i - 1)\omega_1 + \\ (p - a_1 - 2\textstyle\sum_{j=2}^{l-1} a_j - a_l - 2l + 1)\omega_2 + \\ \textstyle\sum_{j=3}^{l-i-1} a_{j-1}\omega_j + (a_{l-i-1} + a_{l-i} + 1)\omega_{l-i} + \sum_{j=l-i+1}^{l} a_j\omega_j.$$

Proof. By definition, $z_i = y_i s_0$ for all $i \in \{1,\, 2,\, \cdots,\, 2l-2\}$. Then $z_i \cdot \nu = y_i \cdot (s_0 \cdot \nu)$ for all $i \in \{1,\, 2,\, \cdots,\, 2l-2\}$. Since $s_0 = y_0$, using the statement (a), we have

$$s_0 \cdot \nu = (p - a_1 - 2\sum_{i=2}^{l-1} a_i - a_l - 2l + 1)\omega_1 + \sum_{i=2}^{l} a_i\omega_i.$$

Therefore,

$$z_i \cdot \nu = y_i \cdot ((p - a_1 - 2\sum_{i=2}^{l-1} a_i - a_l - 2l + 1)\omega_1 + \sum_{i=2}^{l} a_i\omega_i). \tag{7}$$

Thus, for all $i \in \{1,\, 2,\, \cdots,\, 2l-2\}$, the statement of the lemma for z_i follows from the corresponding statement for y_i of the Lemma 1 and from (7). □

Now we find a system of generators of a stabilizer of the elements $\nu_{i_1,i_2,\cdots,i_m} \in \overline{C}_1 \setminus C_1$.

Lemma 3. Let S_ν be a system of generators of a stabilizer of $\nu \in \overline{C}_1 \setminus C_1$. If $\nu = \nu_{i_1,i_2,\cdots,i_m}$ then $S_\nu = \{s_{i_1},\, s_{i_2},\, \cdots,\, s_{i_m}\}$.

In particular, if $m = 1$ then $S_{\nu_i} = \{s_i\}$ for all $i \in \{0,\, 1,\, 2,\, \cdots,\, l\}$.

Proof. The generators $s_0,\, s_1,\, \cdots,\, s_l$ of W_p act on ν as follows:

$$s \cdot \nu = \begin{cases} \nu - (\langle \nu + \rho, \alpha_0 \rangle - p)\alpha_0 & \text{if } s = s_0 \\ \nu - \langle \nu + \rho, \alpha_i \rangle \alpha_i & \text{if } i \in \{1,\, 2,\, \cdots,\, l\} \end{cases} \tag{8}$$

If $i_1 = 0$ then by definition $\nu_{0,i_2,\cdots,i_m} \in H_0 \cap H_{i_2} \cap \cdots \cap H_{i_m}$. Then

$$\langle \nu_{0,i_2,\cdots,i_m} + \rho, \alpha_0 \rangle = p$$

and

$$\langle \nu_{0,i_2,\cdots,i_m} + \rho, \alpha_i \rangle = 0$$

for all $i \in \{i_2, \cdots, i_m\}$. Therefore, by (8), the condition

$$s \in S_{\nu_{0,i_2,\cdots,i_m}} = \{s \in S_p \,|\, s \cdot \nu_{0,i_2,\cdots,i_m} = \nu_{0,i_2,\cdots,i_m}\}$$

yields $s \in \{s_0,\, s_{i_2},\, \cdots,\, s_{i_m}\} \subset S_p$.

If $i_1 \neq 0$ then by definition $\nu_{i_1,i_2,\cdots,i_m} \in H_{i_1} \cap H_{i_2} \cap \cdots \cap H_{i_m}$. Then

$$\langle \nu_{i_1,i_2,\cdots,i_m} + \rho, \alpha_i \rangle = 0$$

for all $i \in \{i_1, i_2, \cdots, i_m\}$. Therefore, by (8), the condition

$$s \in S_{\nu_{i_1,i_2,\cdots,i_m}} = \{s \in S_p \,|\, s \cdot \nu_{i_1,i_2,\cdots,i_m} = \nu_{i_1,i_2,\cdots,i_m}\}$$

yields $s \in \{s_{i_1},\, s_{i_2},\, \cdots,\, s_{i_m}\} \subset S_p$. □

Lemma 4. Let $w \in Y_1 \cup Y_2$, $\nu \in \overline{C}_1 \setminus C_1$ and $w \cdot \nu \in X(T)^+$.
(a) If $w = 1$ then $\nu \in \{\nu_0\}$.
(b) If $w = y_0$ then $\nu \in \{\nu_1,\, \nu_0\}$.
(c) If $w = y_1$ then $\nu \in \{\nu_1,\, \nu_2,\, \nu_0,\, \nu_{0,2}\}$.
(d) If $w = y_i$, where $i \in \{2,\, 3,\, \cdots,\, l-2\}$, then $\nu \in \{\nu_i,\, \nu_{i+1},\, \nu_0,\, \nu_{0,i},\, \nu_{0,i+1}\}$.
(e) If $w = y_{l-1},\, y_l$ then $\nu \in \{\nu_{l-1},\, \nu_l,\, \nu_0,\, \nu_{0,l-1}\}$.
(f) If $w = y_{l+i}$, where $i \in \{1,\, 2,\, \cdots,\, l-2\}$, then

$$\nu \in \{\nu_{l-i-1},\, \nu_{l-i},\, \nu_0,\ \nu_{0,l-i-1},\ \nu_{0,l-i}\}.$$

Proof. (a) We prove that $\nu = \nu_{i_1,i_2,\cdots,i_m} \in X(T)^+$ if and only if $m = 1$ and $i_1 = 0$. Indeed, if $m = 1$ and $i_1 = 0$ then ν belongs to the upper closure of C_1, since $\langle \nu_0 + \rho, \alpha_i \rangle \neq 0$ for all $i \in \{1,\, 2,\, \cdots,\, l\}$ and $\langle \nu_0 + \rho, \alpha_0 \rangle = p$. Therefore $0 < \langle \nu_0 + \rho, \alpha \rangle \leq p$ for all $\alpha \in \Delta$.

Conversely, if $\nu = \nu_{i_1,i_2,\cdots,i_m} \in X(T)^+$ then $\langle \nu_{i_1,i_2,\cdots,i_m} + \rho, \alpha \rangle \neq 0$ for all $\alpha \in R^+$. In particular, $\langle \nu_{i_1,i_2,\cdots,i_m} + +\rho, \alpha \rangle \neq 0$ for all $\alpha \in \{\alpha_{i_1},\, \alpha_{i_2},\, \cdots,\, \alpha_{i_m}\}$. Then

$$i_1 = i_2 = \cdots = i_m = 0,$$

since

$$\nu_{i_1,i_2,\cdots,i_m} \in \overline{C}_1 \setminus C_1.$$

Therefore, by the conditions 2) and 3) of the definition of $\nu_{i_1,i_2,\cdots,i_m}$, we get $m = 1$ and $i_1 = 0$. This implies $\nu = \nu_0$.

(b) Let $\nu = \nu_{i_1,i_2,\cdots,i_m} = \sum_{i=1}^{l} a_i \omega_i$ and $\nu \notin H_0$. Then $\langle \nu_{i_1,i_2,\cdots,i_m} + \rho, \alpha \rangle \neq 0$ for all $\alpha \in \{\alpha_{i_1},\, \alpha_{i_2},\, \cdots,\, \alpha_{i_m}\}$. This condition yields $a_{i_1} = a_{i_2} = \cdots = a_{i_m} = -1$. By the statement (b) of Lemma 1, $y_0 \cdot \nu \in X(T)^+$ if and only if $i_1 = i_2 = \cdots = i_m = 1$. Then by the conditions 2) and 3) of the definition of $\nu_{i_1,i_2,\cdots,i_m}$, we get $m = 1$ and $i_1 = 1$. This implies that $\nu = \nu_1$.

If $\nu \in H_0$ then $i_1 = 0$ and $\langle \nu_{0,i_2,\cdots,i_m} + \rho, \alpha_0 \rangle = p$. Using (5) we get $2\sum_{j=1}^{l-1} a_j + a_l + 2l - 1 = p$. Then by the statement (b) of Lemma 1, $y_0 \cdot \nu \in X(T)^+$ if and only if $i_2 = i_3 = \cdots = i_m = 0$. Then by the conditions 2) and 3) of the definition of $\nu_{0,i_2,\cdots,i_m}$, we get $m = 1$ and $i_1 = 0$. Therefore, $\nu = \nu_0$.

Other statements are easily proved similarly as the previous statement. □

For the elements of $Z_1 \cup Z_2$, using Lemma 2, we have the following

Lemma 5. Let $w \in Z_1 \cup Z_2$, $\nu \in \overline{C}_1 \setminus C_1$ and $w \cdot \nu \in X(T)^+$.
(a) If $w = z_1$ then $\nu \in \{\nu_2,\, \nu_0,\, \nu_{0,2}\}$.
(b) If $w = z_2$ then $\nu \in \{\nu_1,\, \nu_2,\, \nu_3,\, \nu_0,\, \nu_{1,3},\ \nu_{0,2},\, \nu_{0,3}\}$.
(c) If $w = z_i$, where $i \in \{3,\, 4,\, \cdots,\, l-2\}$, then

$$\nu \in \{\nu_1,\, \nu_i,\, \nu_{i+1},\, \nu_{1,i},\, \nu_{1,i+1},\, \nu_0,\, \nu_{0,i},\, \nu_{0,i+1}\}.$$

(d) If $w = z_{l-1},\, z_l$ then $\nu \in \{\nu_1,\, \nu_{l-1},\, \nu_l,\, \nu_{1,l-1},\, \nu_{1,l},\, \nu_0,\, \nu_{0,l-1}\}$.
(e) If $w = z_{l+i}$, where $i \in \{1,\, 2,\, \cdots,\, l-3\}$, then

$$\nu \in \{\nu_1,\, \nu_{l-i-1},\, \nu_{l-i},\, \nu_{1,l-i-1},\, \nu_{1,l-i},\, \nu_0,\, \nu_{0,l-i-1},\, \nu_{0,l-i}\}.$$

(f) If $w = z_{2l-2}$ then $\nu \in \{\nu_1,\, \nu_2,\, \nu_{1,2},\, \nu_0,\, \nu_{0,1},\, \nu_{0,2}\}$.

By Lemmas 4 and 5, if $(Y_1 \cup Y_2 \cup Z_1 \cup Z_2)^+_\nu \neq \emptyset$ then

$$\nu \in \{\nu_0,\, \nu_l,\, \nu_i,\, \nu_{1,i+1},\, \nu_{0,i} \,|\, i = 1,\, 2,\, \cdots,\, l-1\}.$$

We calculate the stabilizers of these elements ν.

Lemma 6. The following statements hold:
(a) for all $i \in \{0, 1, \cdots, l\}$, $st(\nu_i) = \{1,\, s_i\}$;
(b) for all $i \in \{3, 4, \cdots, l\}$, $st(\nu_{1,i}) = \{1,\, s_1,\, s_i,\, s_1 s_i\}$;
(c)for all $i \in \{2, 3, \cdots, l-1\}$, $st(\nu_{0,i}) = \{1,\, s_0,\, s_i,\, s_0 s_i\}$;
(d) $st(\nu_{1,2}) = \{1,\, s_1,\, s_2,\, s_1 s_2,\, s_2 s_1,\, s_1 s_2 s_1\}$;
(e) $st(\nu_{0,1}) = \{1,\, s_0,\, s_1,\, s_1 s_0,\, s_0 s_1,\, s_1 s_0 s_1,\, s_0 s_1 s_0,\, s_1 s_0 s_1 s_0\}$.

Proof. It follows from the defining relations (1) of the affine Weyl group W_p and Lemma 3. □

Using Lemma 6, we can easily describe $(Y_1 \cup Y_2 \cup Z_1 \cup Z_2)^+_\nu$ for all ν listed above. Below, we often omit the index ν of the element $\overline{x}_\nu$ when ν is a fixed element of $\overline{C}_1 \setminus C_1$.

Lemma 7. Consider the elements in $\{\nu_0, \nu_l, \nu_i, \nu_{1,i+1}, \nu_{0,i} \,|\, i = 1, 2, \cdots, l-1\}$. The following hold:

(a) $(Y_1 \cup Y_2 \cup Z_1 \cup Z_2)^+_{\nu_0} = \{\overline{y_i} \,|\, i = 0,\, 1,\, \cdots\, 2l-2\}$, where $\overline{y_0} = \{1,\, y_0\}$ and $\overline{y_i} = \{y_i,\, z_i\}$;

(b) $(Y_1 \cup Y_2 \cup Z_1 \cup Z_2)^+_{\nu_1} = \{\overline{y_0},\, \overline{y_{2l-2}},\, \overline{z_i} \,|\, i = 2,\, 3,\, \cdots\, 2l-2\}$, where $\overline{y_0} = \{y_0,\, y_1\}$, $\overline{y_{2l-2}} = \{y_{2l-2},\, y_{2l-2}s_1\}$ and $\overline{z_i} = \{z_i,\, z_i s_1\}$;

(c) for all $i \in \{2,\, 3,\, \cdots,\, l-1\}$,

$$(Y_1 \cup Y_2 \cup Z_1 \cup Z_2)^+_{\nu_i} = \{\overline{y_{i-1}},\, \overline{z_{i-1}},\, \overline{y_{2l-i-1}},\, \overline{z_{2l-i-1}}\},$$

where $\overline{y_{i-1}} = \{y_{i-1},\, y_i\}$, $\overline{z_{i-1}} = \{z_{i-1},\, z_i\}$, $\overline{y_{2l-i-1}} = \{y_{2l-i-1},\, y_{2l-i}\}$ and $\overline{z_{2l-i-1}} = \{z_{2l-i-1},\, z_{2l-i}\}$;

(d) $(Y_1 \cup Y_2 \cup Z_1 \cup Z_2)^+_{\nu_l} = \{\overline{y_{l-1}},\, \overline{z_{l-1}}\}$, where $\overline{y_{l-1}} = \{y_{l-1},\, y_l\}$ и $\overline{z_{l-1}} = \{z_{l-1},\, z_l\}$;

(e) $(Y_1 \cup Y_2 \cup Z_1 \cup Z_2)^+_{\nu_{1,2}} = \{\overline{z_{2l-3}}\}$, where

$$\overline{z_{2l-3}} = \{z_{2l-3},\, z_{2l-2},\, z_{2l-3}s_1,\, z_{2l-2}s_1,\, z_{2l-3}s_1s_2,\, z_{2l-2}s_1s_2\};$$

(f) for all $i \in \{3,\, 4,\, \cdots,\, l-1\}$, $(Y_1 \cup Y_2 \cup Z_1 \cup Z_2)^+_{\nu_{1,i}} = \{\overline{z_{i-1}},\, \overline{z_{2l-i-1}}\}$, where

$$\overline{z_{i-1}} = \{z_{i-1},\, z_i,\, z_{i-1}s_1,\, z_i s_1\},\quad \overline{z_{2l-i-1}} = \{z_{2l-i-1},\, z_{2l-i},\, z_{2l-i-1}s_1,\, z_{2l-i}s_1\};$$

(g) $(Y_1 \cup Y_2 \cup Z_1 \cup Z_2)^+_{\nu_{1,l}} = \{\overline{z_{l-1}}\}$, where $\overline{z_{l-1}} = \{z_{l-1},\, z_l,\, z_{l-1}s_1,\, z_l s_1\}$.

(h) $(Y_1 \cup Y_2 \cup Z_1 \cup Z_2)^+_{\nu_{0,1}} = \{\overline{y_{2l-2}}\}$, where

$$\overline{y_{2l-2}} = \{y_{2l-2},\, z_{2l-2},\, y_{2l-2}s_1,\, z_{2l-2}s_1,\, y_{2l-2}s_1s_0,\, z_{2l-2}s_1s_0,\, y_{2l-2}s_1s_0s,\, z_{2l-2}s_1s_0s_1\};$$

(l) for all $i \in \{2,\, 3,\, \cdots,\, l-1\}$,

$$(Y_1 \cup Y_2 \cup Z_1 \cup Z_2)^+_{\nu_{0,i}} = \{\overline{y_{i-1}},\, \overline{y_{2l-i-1}}\},$$

where $\overline{y_{i-1}} = \{y_{i-1},\, y_i,\, z_{i-1},\, z_i\}$ and $\overline{y_{2l-i-1}} = \{y_{2l-i-1},\, y_{2l-i},\, z_{2l-i-1},\, z_{2l-i}\}$.

Proof. Let $w \in Y_1 \cup Y_2 \cup Z_1 \cup Z_2$. By definition,

$$\overline{w}_\nu = \{wx \,|\, x \in st(\nu)\}. \tag{9}$$

Then using (9) and Lemmas 4 – 6, we obtain the required statements. □

3 Proof of the Theorem 1

Using Lemma 7 and the sum formula (6), we can easily prove Theorem 1.

By sum formula (6), for all cases listed in the statements (d)–(l) of Lemma 7,

$$\sum_{j>0} [V(\overline{w} \cdot \nu)^j] = 0.$$

In the following cases $\sum_{j>0} [V(\overline{w} \cdot \nu)^j]$ is also trivial:

1) $\overline{w} \in \{\overline{y_0},\, \overline{y_1}\} \subset (Y_1 \cup Y_2 \cup Z_1 \cup Z_2)^+_{\nu_0}$;

2) $\overline{w} \in \{\overline{y_0},\, \overline{z_{2l-2}}\} \subset (Y_1 \cup Y_2 \cup Z_1 \cup Z_2)^+_{\nu_1}$;

3) $\overline{w} \in \{\overline{y_{i-1}},\, \overline{y_{2l-i-1}}\} \subset (Y_1 \cup Y_2 \cup Z_1 \cup Z_2)^+_{\nu_i}$, where $i \in \{2,\, 3,\, \cdots,\, l-1\}$;

4) $\overline{w} = \overline{y_{l-1}} \in (Y_1 \cup Y_2 \cup Z_1 \cup Z_2)^+_{\nu_l}$.

Therefore, in all these cases $\chi(\overline{w} \cdot \nu) = [L(\overline{w} \cdot \nu)]$.

Thus, it remains to prove only the statements (a)–(f).

(a) By the statement (a) of Lemma 7, $\overline{y_0}_{\nu_0} = \{1,\, y_0\}$ and $\overline{y_i}_{\nu_0} = \{y_i,\, z_i\}$ for all $i \in \{1,\, 2,\, \cdots,\, 2l-2\}$. Therefore, $\chi(\nu_0) = \chi(y_0 \cdot \nu_0)$ and $\chi(y_i \cdot \nu_0) = \chi(z_i)$ for all $i \in \{1,\, 2,\, \cdots,\, 2l-2\}$. Then using the sum formula (6), we have

$$\sum_{j>0} [V(\overline{y_i} \cdot \nu_0)^j] = \sum_{k=2}^{i} (-1)^{i-k} \chi(\overline{y_{k-1}} \cdot \nu_0) \tag{10}$$

for all $i \in \{2, 3, \cdots, 2l-2\}$. If $i = 2$ then by (10),

$$\sum_{j>0}[V(\overline{y}_2 \cdot \nu_0)^j] = \chi(\overline{y_1} \cdot \nu_0) = [L(\overline{y_1} \cdot \nu_0)].$$

This implies that $\chi(\overline{y_2} \cdot \nu_0) = [L(\overline{y_2} \cdot \nu_0)] + [L(\overline{y_1} \cdot \nu_0)]$.

Now suppose that the statement (a) is true for all $i < t$, where $t \le 2l-2$. Then by (10),

$$\sum_{j>0}[V(\overline{y_t} \cdot \nu_0)^j] = \sum_{k=2}^{i}(-1)^{t-k}([L(\overline{y_{k-1}} \cdot \nu_0)] + [L(\overline{y_{k-2}} \cdot \nu_0)]) = [L(\overline{y_{t-1}} \cdot \nu_0)].$$

This yields $\chi(\overline{y_t} \cdot \nu_0) = [L(\overline{y_t} \cdot \nu_0)] + [L(\overline{y_{t-1}} \cdot \nu_0)]$. So, the statement (a) is true for all $i \in \{2, 3, \cdots 2l-2\}$.

(b). By the statement (b) of Lemma 7, $\chi(\nu_1) = \chi(z_1 \cdot \nu_1) = 0$. Then by (6),

$$\begin{gathered}\textstyle\sum_{j>0}[V(\overline{z_i} \cdot \nu_1)^j] = (-1)^i\chi(\overline{y_0} \cdot \nu_1) + \\ \textstyle\sum_{k=2}^{i-1}(-1)^{i-k-1}\chi(\overline{z_k} \cdot \nu_1) + \delta(i = 2l-2)\chi(\overline{y_{2l-2}} \cdot \nu_1)\end{gathered} \tag{11}$$

for all $i \in \{2, 3, \cdots, 2l-2\}$. If $i = 2$ then using (11), we get

$$\sum_{j>0}[V(\overline{z_2} \cdot \nu_1)^j] = \chi(\overline{y_0} \cdot \nu_1) = [L(\overline{y_0} \cdot \nu_1)].$$

This yields the statement (b).

(c). We use (11) and the induction on i. If $i = 3$ then by (11) and by the statement (b) of this Theorem 1, we have

$$\sum_{j>0}[V(\overline{z_3} \cdot \nu_1)^j] = -\chi(\overline{y_0} \cdot \nu_1) + \chi(\overline{z_2} \cdot \nu_1) = [L(\overline{z_2} \cdot \nu_1)].$$

This yields $\chi(\overline{z_3} \cdot \nu_1) = [L(\overline{z_3} \cdot \nu_1)] + [L(\overline{z_2} \cdot \nu_1)]$.

Now suppose that the statement (c) is true for all $i < t$, where $t \le 2l-3$. Then by (11),

$$\begin{gathered}\textstyle\sum_{j>0}[V(\overline{z_t} \cdot \nu_0)^j] = (-1)^t\chi(\overline{y_0} \cdot \nu_1) + \sum_{k=2}^{t-1}(-1)^{t-k-1}\chi(\overline{z_k} \cdot \nu_1) = \\ (-1)^t[L(\overline{y_0} \cdot \nu_1)] + (-1)^{t-3}([L(\overline{z_2} \cdot \nu_1)] + [L(\overline{y_0} \cdot \nu_1)]) + \\ \textstyle\sum_{k=3}^{t-1}(-1)^{t-k-1}([L(\overline{z_k} \cdot \nu_1)] + [L(\overline{z_{k-1}} \cdot \nu_1)]) = [L(\overline{z_{t-1}} \cdot \nu_1)].\end{gathered}$$

It follows that $\chi(\overline{z_t} \cdot \nu_0) = [L(\overline{z_t} \cdot \nu_0)] + [L(\overline{z_{t-1}} \cdot \nu_0)]$. Therefore, the statement (c) is true for all $i \in \{3, 4, \cdots 2l-3\}$.

(d). By (11),

$$\sum_{j>0}[V(\overline{z_{2l-2}} \cdot \nu_0)^j] = \chi(\overline{y_0} \cdot \nu_1) + \sum_{k=2}^{2l-3}(-1)^{2l-k-3}\chi(\overline{z_k} \cdot \nu_1) + \chi(\overline{y_{2l-2}}).$$

Using the previous statements (b) and (c) of this Theorem 1, we obtain

$$\begin{gathered}\textstyle\sum_{j>0}[V(\overline{z_{2l-2}} \cdot \nu_0)^j] = [L(\overline{y_0} \cdot \nu_1)] + (-1)^{2l-5}([L(\overline{z_2} \cdot \nu_1)] + [L(\overline{y_0} \cdot \nu_1)]) + \\ \textstyle\sum_{k=3}^{2l-3}(-1)^{2l-k-3}([L(\overline{z_k} \cdot \nu_1)] + [L(\overline{z_{k-1}} \cdot \nu_1)]) + [L(\overline{y_{2l-2}} \cdot \nu_1)] = \\ [L(\overline{z_{2l-3}} \cdot \nu_1)] + [L(\overline{y_{2l-2}} \cdot \nu_1)].\end{gathered}$$

It follows that $\chi(\overline{z_{2l-2}} \cdot \nu_1) = [L(\overline{z_{2l-2}} \cdot \nu_1)] + [L(\overline{z_{2l-3}} \cdot \nu_1)] + [L(\overline{y_{2l-2}} \cdot \nu_1)]$.

(e). Let $i \in \{2, 3, \cdots, l\}$. By the statement (c) of Lemma 7,

$$\chi(\nu_i) = \chi(y_j \cdot \nu_i) = \chi(z_j \cdot \nu_i) = 0$$

for all $j \in \{0, 1, \cdots, i-2\}$. Then by (6),

$$\sum_{j>0}[V(\overline{z_{i-1}} \cdot \nu_i)^j] = \chi(\overline{y_{i-1}} \cdot \nu_i)$$

for all $i \in \{2, 3, \cdots, l\}$. So, for all $i \in \{2, 3, \cdots, l\}$,

$$\chi(\overline{z_{i-1}} \cdot \nu_i) = [L(\overline{z_{i-1}} \cdot \nu_i)] + [L(\overline{y_{i-1}} \cdot \nu_i)].$$

(f). Let $i \in \{2, 3, \cdots l-1\}$. By the statement (c) of Lemma 7,

$$\chi(\nu_i) = \chi(y_j \cdot \nu_i) = \chi(z_j \cdot \nu_i) = 0$$

for all $j \in \{0, 1, \cdots, i-2\} \cup \{i+1, i+2, \cdots, 2l-i-1\}$ and $\chi(\overline{z_{i-1}}) = \chi(\overline{z_i})$. Then by (6),

$$\sum_{j>0} [V(\overline{z_{2l-i-1}} \cdot \nu_i)^j] = \chi(\overline{y_{2l-i-1}} \cdot \nu_i)$$

for all $i \in \{2, 3, \cdots, l-1\}$. So, for all $i \in \{2, 3, \cdots l-1\}$,

$$\chi(\overline{z_{2l-i-1}} \cdot \nu_i) = [L(\overline{z_{2l-i-1}} \cdot \nu_i)] + [L(\overline{y_{2l-i-1}} \cdot \nu_i)].$$

The proof of Theorem 1 is complete. □

Remark 1. Let $(Y_1 \cup Y_2 \cup Z_1 \cup Z_2)^+_\nu \neq \emptyset$. If ν lies in the intersection of two hyperplanes then, by Theorem 1, all Weyl modules with highest weights $\overline{w} \cdot \nu$ with $\overline{w} \in (Y_1 \cup Y_2 \cup Z_1 \cup Z_2)^+_\nu$ are simple.

From the proof of Theorem 1 we immediately obtain the following

Corollary 2. Let G be a simply-connected and semisimple algebraic group of type $B_l\,(l > 2)$ over an algebraically closed field $\mathbb{K}$ of characteristic $p \geq h$, where h is the Coxeter number. Then the Weyl modules with the following highest weights are simple:

(a) ν_0, $\overline{y_1} \cdot \nu_0$;
(b) $\overline{y_0} \cdot \nu_1$, $\overline{z_2} \cdot \nu_1$, $\overline{z_{2l-2}} \cdot \nu_1$;
(c) $\overline{y_{i-1}} \cdot \nu_i$, where $i \in \{2, 3, \cdots, l\}$;
(d) $\overline{y_{2l-i-1}} \cdot \nu_i$, where $i \in \{2, 3, \cdots, l-1\}$;
(e) $\overline{z_{2l-3}} \cdot \nu_{1,2}$,
(f) $\overline{z_{i-1}} \cdot \nu_{1,i}$, $\overline{z_{2l-i-1}} \cdot \nu_{1,i}$, where $i \in \{3, 4, \cdots, l-1\}$;
(g) $\overline{z_{l-1}} \cdot \nu_{1,l}$;
(h) $\overline{y_{2l-2}} \cdot \nu_{0,1}$;
(l) $\overline{y_{i-1}} \cdot \nu_{0,i}$, $\overline{y_{2l-i-1}} \cdot \nu_{0,i}$, where $i \in \{2, 3, \cdots, l-1\}$.

Remark 2. It is known that, in the restricted region, the differential of each simple G-module is a simple $\mathfrak{g}$-module, where $\mathfrak{g}$ is the Lie algebra of G. Therefore, Corollary 2 generalizes the Rudakov simplicity criterion [4, Theorem 2] for semisimple Lie algebras of type B_l. The Weyl module $V(\overline{y_0} \cdot \nu_1)$ satisfies the Rudakov simplicity criterion. In all other cases the highest weights obtained in Corollary 2 do not satisfy the Rudakov simplicity criterion.

Using Lemmas 1 and 2, one can easily describe highest weights of the simple Weyl modules listed in Corollary 2. For example, by definition, $\overline{y_0} \cdot \nu_1 = y_0 \cdot \nu_1$, and ν_1 satisfies the condition $\langle \nu_1 + \rho, \alpha_1 \rangle = 0$, since $\nu_1 \in H_1$. If we write $\nu_1 = \sum_{j=1}^{l} a_j \omega_j$ then the above condition yields $a_1 = -1$. Then by the statement (a) of Lemma 1,

$$\overline{y_0} \cdot \nu_1 = y_0 \cdot \nu_1 = (p - 2\sum_{j=2}^{l-1} a_j - a_l - 2l + 2)\omega_1 + \sum_{j=2}^{l} a_j \omega_j,$$

where $2\sum_{j=2}^{l-1} a_j + a_l + 2l - 3 < p$.

Remark 3. The simplicity of the following Weyl modules for the algebraic group of type B_4 was proved in [11, Theorem 1, (d)]:

- $V((p-4)\omega_4) = V(\overline{z_2} \cdot \nu_{1,3})$, where $\nu_{1,3} = -\omega_1 - \omega_3 + (p-4)\omega_4 \in H_1 \cap H_3$;
- $V((p-5)\omega_4) = V(\overline{y_1} \cdot \nu_{0,2})$, where $\nu_{0,2} = -\omega_2 + (p-5)\omega_4 \in H_0 \cap H_2$;
- $V((p-6)\omega_4) = V(\overline{y_0} \cdot \nu_1)$, where $\nu_1 = -\omega_1 + (p-6)\omega_4 \in H_1$;
- $V((p-7)\omega_4) = V(\nu_0)$, where $\nu_0 = (p-7)\omega_4 \in H_0$.

Thus, Corollary 2 gives several new examples of simple Weyl modules for the algebraic groups of type B_l.

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$\mathfrak{so}_{2l+1}(\mathbb{K})$ үшін сингуляр үлкен салмақты жәй модульдер туралы

Мақалада сипаттамасы $p \geq h$, мұндағы h – Кокстер саны болатын алгебралық тұйық өрістегі B түріндегі классикалық Ли алгебрасының сингуляр үлкен салмақты жәй модульдерінің формалды характерлері зерттелді. Бұл жәй модульдердің үлкен салмақтары шектелген деп есептелінеді. Олардың формалды характерлерінің сипаттамасы берілді. Дербес жағдайда, жәй Вейль модульдерінің жаңа мысалдары алынды. Шектелген салмақтар жағдайында алгебралық группалар мен олардың Ли алгебраларының көріністер теориясы эквивалентті. Сондықтан, оң сипаттамалы өрістердегі жартылай жәй бірбайланысқан алгебралық группалардың көріністер теориясының әдістері қолданылған. Жәй модульдердің формалды характерлерін сипаттау үшін үлкен салмағы сәйкес келетін Вейль модульдерінің Янцен фильтрациясын құру пайдаланылады.

Кілт сөздер: Ли алгебрасы, жәй модуль, алгебралық группа, Вейль модулі, Янцен фильтрациясы.

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О простых модулях со старшим сингулярным весом для $\mathfrak{so}_{2l+1}(\mathbb{K})$

В статье изучены формальные характеры простых модулей со старшим сингулярным весом классической алгебры Ли типа B над алгебраически замкнутым полем характеристики $p \geq h$, где h – число Кокстера. Предположено, что старшие веса этих простых модулей ограничены. Авторами описаны их формальные характеры. В частности, получены новые примеры простых модулей Вейля. В области ограниченных весов теории представлений алгебраических групп и их алгебр Ли эквивалентны. По этой причине можно применять инструменты теории представлений полупростых односвязных алгебраических групп в положительной характеристике. Для описания формальных характеров простых модулей использована конструкция фильтрации Янцена модулей Вейля соответствующих старших весов.

Ключевые слова: алгебра Ли, простой модуль, алгебраическая группа, модуль Вейля, фильтрация Янцена.

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