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# Generalization of the Hardy-Littlewood theorem on Fourier series

In the theory of one-dimensional trigonometric series, the Hardy-Littlewood theorem on Fourier series with monotone Fourier coefficients is of great importance. Multidimensional versions of this theorem have been extensively studied for the Lebesgue space. Significant differences of the multidimensional variants in comparison with the one-dimensional case are revealed and the strengthening of this theorem is obtained. The Hardy-Littlewood theorem is also generalized for various function spaces and various types of monotonicity of the series coefficients. Some of these generalizations can be seen in works of M.F. Timan, M.I. Dyachenko, E.D. Nursultanov, S. Tikhonov. In this paper, a generalization of the Hardy-Littlewood theorem for double Fourier series of a function in the space $L_q\varphi(L_q)(0,2\pi]^2$ is obtained.

*Keywords:* trigonometric series, Fourier series, Lebesgue space, Hardy-Littlewood theorem, Fourier coefficients.

Let $L_q(0,2\pi]^2$, $1 \le q < +\infty$ be the space of all $2\pi$- periodic for each variable, measurable by Lebesgue functions $f(x,y)$, for which

$$\|f\|_q = \left(\int_0^{2\pi}\int_0^{2\pi} |f(x,y)|^q\,dxdy\right)^{\frac{1}{q}} < +\infty.$$

In this article, we study the condition of belonging to the space $L_q\varphi(L_q)(0,2\pi]^2$ for a function of two variables. Let us recall the definition of the space $L_q\varphi(L_q)(0,2\pi]^2$. Let the function $\varphi(t)$ satisfy the following conditions [1]:

a) $\varphi(t)$ is even, non-negative, non-decreasing on $[0,+\infty)$;

b) $\varphi(t^2) \le C\cdot\varphi(t)$, $t\in[0,+\infty)$, $C\ge 1$;

c) $\frac{\varphi(t)}{t^\varepsilon}\downarrow$ on $(0,+\infty)$ with some $\varepsilon>0$.

Measurable $2\pi$- periodic function for each variable $f(x,y)\in L_q\varphi(L_q)(0,2\pi]^2$ if

$$\int_0^{2\pi}\int_0^{2\pi} |f(x,y)|^q\cdot\varphi\left(|f(x,y)|^q\right)dxdy < +\infty.$$

In particular, when $\varphi(t)\equiv 1$ the space $L_q\varphi(L_q)(0,2\pi]^2$ coincides with the Lebesgue space $L_q(0,2\pi]^2$.

We give the following well-known theorem of Hardy-Littlewood.

*Theorem (Hardy-Littlewood).* Let $a_n\downarrow 0$, $n\to+\infty$. For the trigonometric series

$$\sum_{n=1}^{+\infty} a_n\cos nx$$

to be the Fourier series of some functions $f(x)\in L_q$, $1<q<+\infty$, it is necessary and sufficient that

$$\sum_{n=1}^{+\infty} n^{q-2}\cdot a_n^q < +\infty.$$

The Hardy-Littlewood theorem is generalized for various function spaces and various types of monotonicity of the series coefficients. Some of these generalizations can be seen, for example, in works [1–9]. Our main goal is to prove the Hardy-Littlewood theorem for the double Fourier series of a function $f(x,y)\in L_q\varphi(L_q)(0,2\pi]^2$.

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To obtain the main result, we need the following Lemma.

*Lemma A.* Let the function $\Phi(u)$ be even, non-negative, non-decreasing on $[0,+\infty)$. If $\Phi(u^2) \le C \cdot \Phi(u)$ at some constant $C \ge 1$ and $u \in [0,+\infty)$, then for any nonnegative function $\psi(x,y)$ measurable on $[0,1)^2$ and satisfying the condition

$$B \equiv \int_0^1 \int_0^1 \psi(x,y) \cdot \Phi(\psi(x,y))dxdy < +\infty, \tag{1}$$

the following inequality holds

$$A \equiv \int_0^1 \int_0^1 \psi(x,y) \cdot \Phi\left(\frac{1}{xy}\right) dxdy < +\infty. \tag{2}$$

*Proof.* In [10], it was proved that if $\alpha \ge 0$ and $\beta \ge 0$, then

$$\alpha\Phi(\beta) \le C \cdot \alpha \cdot \Phi(\alpha) + \sqrt{\beta} \cdot \Phi(\beta). \tag{3}$$

Let $x > 0, \quad y > 0$. Let in (3) $\alpha = \psi(x,y)$ and $\beta = \frac{1}{xy}$. By integrating we get

$$A = \int_0^1 \int_0^1 \psi(x,y) \cdot \Phi\left(\frac{1}{xy}\right) dxdy \le C \cdot \int_0^1 \int_0^1 \psi(x,y) \cdot \Phi(\psi(x,y))dxdy +$$

$$+ \int_0^1 \int_0^1 \frac{1}{\sqrt{xy}} \cdot \Phi\left(\frac{1}{xy}\right) dxdy. \tag{4}$$

However, a non-decreasing function $\Phi(u) \ge 0$ satisfies the condition $\Phi(u^2) \le C \cdot \Phi(u)$ and therefore

$$\Phi(uv) \le \Phi(4) \cdot \left[(\log_2(2+u))^{\log_2 C} + (\log_2(2+v))^{\log_2 C}\right], \tag{5}$$

at $0 \le u < +\infty, \quad 0 \le v < +\infty$.

In fact, at $0 \le uv \le 4$ inequality (5) is obvious. If $uv \ge 4$, then put $k = [\log_2 \log_2(uv)]$, where $[a]$ means the integer part of $a$, and then

$$\Phi(uv) \le C \cdot \Phi((uv)^{\frac{1}{2}}) \le C^2 \cdot \Phi((uv)^{\frac{1}{2^2}}) \le ... \le C^k \cdot \Phi((uv)^{\frac{1}{2^k}}) \le$$

$$\le C^{\log_2 \log_2(uv)} \cdot \Phi\left((uv)^{\frac{2}{\log_2(uv)}}\right) = k(\log_2(uv))^{\log_2 C} \cdot \Phi(4) \le$$

$$\le \Phi(4) \cdot \left\{(\log_2(u+2))^{\log_2 C} + (\log_2(v+2))^{\log_2 C}\right\},$$

that is, inequality (5) is true when $uv > 4$. From condition (1), inequalities (4) and (5) follows, that (2). The Lemma A is proved.

*Theorem.* Let the function $\varphi(t)$ satisfies conditions a) - c) and $f(x,y) \in L_1(0,2\pi]^2$, is an even function with Fourier series

$$\sum_{n_1=1}^{+\infty} \sum_{n_2=1}^{+\infty} a_{n_1,n_2} \cdot \cos n_1 x \cdot \cos n_2 y, \tag{6}$$

where

$$a_{k,l} \ge 0,\ a_{k,l} - a_{k+1,l} - a_{k,l+1} + a_{k+1,l+1} \ge 0,\ a_{k,l} \ge a_{k+1,l},\ a_{k,l} \ge a_{k,l+1},\ k,l \in \mathbb{N}.$$

Then in order for some $q \in (1,+\infty)$ function $f \in L_q\varphi(L_q)(0,2\pi]^2$, it is necessary and sufficient that

$$\sum_{n_1=2}^{+\infty} \sum_{n_2=2}^{+\infty} (n_1 n_2)^{q-2} \varphi(n_1 n_2) \cdot a_{n_1,n_2}^q < +\infty. \tag{7}$$

*Proof.* Sufficiency. Since with every $\varepsilon \in (0,1)$ the inequality is fulfilled

$$\sum_{n_1=1}^{+\infty} \sum_{n_2=1}^{+\infty} (n_1 n_2)^{q-2} \varphi(n_1 n_2) \cdot a_{n_1,n_2}^q =$$

$$= \sum_{n_1=1}^{+\infty} n_1^{q-2} \sum_{n_2=1}^{+\infty} \left\{ a_{n_1,n_2} \cdot n_2^{-\frac{\varepsilon}{q}} \cdot \varphi^{\frac{1}{q}}(n_1 n_2) n_2 \right\}^q \cdot n_2^{-(2-\varepsilon)},$$

then using the inequality (see. [11; 308]):

$$\sum_{n=1}^{+\infty} n^{-c} \left( \sum_{\nu=1}^{n} a_\nu \right)^l \le M(c,l) \cdot \sum_{n=1}^{+\infty} n^{-c} (n \cdot a_n)^l, \quad (c > 1, \quad l > 1, \quad a_\nu \ge 0)$$

and using properties of the function $\varphi(t)$ we get:

$$\sum_{n_1=1}^{+\infty} \sum_{n_2=1}^{+\infty} (n_1 n_2)^{q-2} \varphi(n_1 n_2) \cdot a_{n_1,n_2}^q \ge$$

$$\ge C \sum_{n_1=1}^{+\infty} n_1^{q-2} \cdot \sum_{n_2=1}^{+\infty} n_2^{-(2-\varepsilon)} \left\{ \sum_{\nu_2=1}^{n_2} a_{n_1,\nu_2} \cdot \varphi^{\frac{1}{q}}(n_1 \nu_2) \nu_2^{-\frac{\varepsilon}{q}} \right\}^q =$$

$$= C \cdot \sum_{n_1=1}^{+\infty} n_1^{q-2+\varepsilon} \cdot \sum_{n_2=1}^{+\infty} n_2^{-(2-\varepsilon)} \left\{ \sum_{\nu_2=1}^{n_2} a_{n_1,\nu_2} \cdot \left( \frac{\varphi(n_1 \nu_2)}{n_1^\varepsilon \nu_2^\varepsilon} \right)^{\frac{1}{q}} \right\}^q \ge$$

$$\ge C \cdot \sum_{n_1=1}^{+\infty} n_1^{q-2+\varepsilon} \cdot \sum_{n_2=1}^{+\infty} n_2^{-(2-\varepsilon)} \frac{\varphi(n_1 n_2)}{n_1^\varepsilon n_2^\varepsilon} \left\{ \sum_{\nu_2=1}^{n_2} a_{n_1,\nu_2} \right\}^q =$$

$$= C \cdot \sum_{n_2=1}^{+\infty} n_2^{-2} \cdot \sum_{n_1=1}^{+\infty} n_1^{q-2} \cdot \varphi(n_1 n_2) \left\{ \sum_{\nu_2=1}^{n_2} a_{n_1,\nu_2} \right\}^q =$$

$$= C \cdot \sum_{n_2=1}^{+\infty} n_2^{-2} \cdot \sum_{n_1=1}^{+\infty} \left\{ \left( \sum_{\nu_2=1}^{n_2} a_{n_1,\nu_2} \right) n_1 \cdot \varphi^{\frac{1}{q}}(n_1 n_2) \cdot n_1^{-\frac{\varepsilon}{p}} \right\}^q \cdot n_1^{-(2-\varepsilon)} \ge$$

$$\ge \sum_{n_2=1}^{+\infty} n_2^{-2} \sum_{n_1=1}^{+\infty} n_1^{-(2-\varepsilon)} \cdot \left\{ \sum_{\nu_1=1}^{n_1} \sum_{\nu_2=1}^{n_2} a_{\nu_1,\nu_2} \cdot \varphi^{\frac{1}{q}}(\nu_1 n_2) \cdot \nu_1^{-\frac{\varepsilon}{q}} \right\}^q \ge$$

$$\ge C \cdot \sum_{n_1=1}^{+\infty} \sum_{n_2=1}^{+\infty} (n_1 n_2)^{-2} \cdot \varphi(n_1 n_2) \left\{ \sum_{\nu_1=1}^{n_1} \sum_{\nu_2=1}^{n_2} a_{\nu_1,\nu_2} \right\}^q \equiv S(q,\varphi).$$

Since

$$J(q,\varphi) = \int_0^\pi \int_0^\pi |f(x,y)|^q \cdot \varphi\left[|f(x,y)|^q\right] dxdy =$$

$$= \sum_{n_1=1}^{+\infty} \sum_{n_2=1}^{+\infty} \int_{\frac{\pi}{n_1+1}}^{\frac{\pi}{n_1}} \int_{\frac{\pi}{n_2+1}}^{\frac{\pi}{n_2}} |f(x,y)|^q \cdot \varphi\left[|f(x,y)|^q\right] dxdy,$$

and for $\frac{\pi}{n_1+1} \le x \le \frac{\pi}{n_1}, \quad \frac{\pi}{n_2+1} \le y \le \frac{\pi}{n_2}$

$$|f(x,y)| \le \sum_{\nu_1=1}^{n_1} \sum_{\nu_2=1}^{n_2} a_{\nu_1,\nu_2} + \left| \sum_{\nu_1=1}^{n_1} \sum_{\nu_2=n_2+1}^{+\infty} a_{\nu_1,\nu_2} \cdot \cos \nu_1 x \cdot \cos \nu_2 y \right| +$$

$$+ \left| \sum_{\nu_1=n_1+1}^{+\infty} \sum_{\nu_2=1}^{n_2} a_{\nu_1,\nu_2} \cdot \cos \nu_1 x \cdot \cos \nu_2 y \right| + \left| \sum_{\nu_1=n_1+1}^{+\infty} \sum_{\nu_2=n_2+1}^{+\infty} a_{\nu_1,\nu_2} \cdot \cos \nu_1 x \cdot \cos \nu_2 y \right| \le$$

$$\le 9 \cdot \sum_{\nu_1=1}^{n_1} \sum_{\nu_2=1}^{n_2} a_{\nu_1,\nu_2},$$

then

$$J(q,\varphi) \le 9^q\pi^2 \cdot \sum_{n_1=1}^{+\infty}\sum_{n_2=1}^{+\infty}(n_1n_2)^{-2}\cdot\left\{\cdot\sum_{\nu_1=1}^{n_1}\sum_{\nu_2=1}^{n_2}a_{\nu_1,\nu_2}\right\}^q\varphi\left(\left\{9\cdot\sum_{\nu_1=1}^{n_1}\sum_{\nu_2=1}^{n_2}a_{\nu_1,\nu_2}\right\}^q\right)\le$$

$$\le 9^q\pi^2 \cdot \sum_{n_1=1}^{+\infty}\sum_{n_2=1}^{+\infty}(n_1n_2)^{-2}\cdot\left\{\cdot\sum_{\nu_1=1}^{n_1}\sum_{\nu_2=1}^{n_2}a_{\nu_1,\nu_2}\right\}^q\varphi\left(9^q\cdot a_{1,1}^q\cdot(n_1n_2)\right).$$

From the condition (7) it follows the finiteness of value $S(p,\varphi)$, and by virtue of the properties of the function $\varphi(t)$ it follows the finiteness of value $J(p,\varphi)$, that is (6).

Now, let us prove the necessity. First, we prove that if condition (6) is satisfied, then there exists a number $q_0$ such that $1 < q_0 < q$ and $f(x,y)\in L_{q_0}[0,\pi]^2$.

Indeed, applying Holder's inequality at $\theta = \frac{q}{q_0}$, we obtain:

$$\int_0^\pi\int_0^\pi |f(x,y)|^{q_0}dxdy = \int_0^\pi\int_0^\pi |f(x,y)|^{q_0}\cdot\varphi^{\frac{q_0}{q}}\left(\frac{1}{xy}\right)\cdot\varphi^{-\frac{q_0}{q}}\left(\frac{1}{xy}\right)dxdy \le$$

$$\le\left\{\int_0^\pi\int_0^\pi |f(x,y)|^{q}\cdot\varphi\left(\frac{1}{xy}\right)dxdy\right\}^{\frac{q_0}{q}}\cdot\left\{\int_0^\pi\int_0^\pi \varphi^{-\frac{q_0}{q}\cdot\theta'}\left(\frac{1}{xy}\right)dxdy\right\}^{\frac{1}{\theta'}},$$

where $\frac{1}{\theta}+\frac{1}{\theta'}=1$. Hence, by virtue of Lemma A and the monotony of the function $\varphi(t)$ we obtain $f(x,y)\in L_{p_0}$.

Now, we show that the following series converges

$$\sum_{n_1=1}^{+\infty}\sum_{n_2=1}^{+\infty}\frac{a_{n_1,n_2}}{n_1n_2}.$$

Indeed, applying the Hausdorff-Young's theorem, we get:

$$\sum_{n_1=1}^{+\infty}\sum_{n_2=1}^{+\infty}\frac{a_{n_1,n_2}}{n_1n_2}\le$$

$$\le\left(\sum_{n_1,n_2=1}^{+\infty}(a_{n_1,n_2})^{\max(p_0,2)}\right)^{\frac{1}{\max(p_0,2)}}\cdot\left(\sum_{n_1,n_2=1}^{+\infty}(n_1n_2)^{-\min(q_0,2)}\right)^{\frac{1}{\min(q_0,2)}}\le+\infty,$$

where $\frac{1}{q_0}+\frac{1}{p_0}=1$.

Then by Lemma 1 of work [2]

$$a_{n_1,n_2}\le\int_0^{\frac{\pi}{n_1}}\int_0^{\frac{\pi}{n_2}}|f(t,\tau)|dtd\tau.$$

Now, let us estimate the sum:

$$C(f,\varphi,q)=\sum_{n_1=2}^{+\infty}\sum_{n_2=2}^{+\infty}(n_1n_2)^{q-2}\varphi(n_1n_2)\cdot a_{n_1,n_2}^q\le$$

$$\le\sum_{n_1=2}^{+\infty}\sum_{n_2=2}^{+\infty}(n_1n_2)^{q-2}\varphi(n_1n_2)\cdot\left\{\int_0^{\frac{\pi}{n_1}}\int_0^{\frac{\pi}{n_2}}|f(t,\tau)|dtd\tau\right\}^q\le$$

$$\le C\cdot\sum_{n_1=2}^{+\infty}\sum_{n_2=2}^{+\infty}\int_{\frac{\pi}{n_1}}^{\frac{\pi}{n_1-1}}\int_{\frac{\pi}{n_2}}^{\frac{\pi}{n_2-1}}\left\{[\varphi(n_1n_2)]^{\frac{1}{q}}\cdot n_1n_2\cdot\int_0^{\frac{\pi}{n_1}}\int_0^{\frac{\pi}{n_2}}|f(t,\tau)|dtd\tau\right\}^q dxdy\le$$

$$\leq C \cdot \sum_{n_1=2}^{+\infty} \sum_{n_2=2}^{+\infty} \int\limits_{\frac{\pi}{n_1}}^{\frac{\pi}{n_1-1}} \int\limits_{\frac{\pi}{n_2}}^{\frac{\pi}{n_2-1}} \left\{ \frac{1}{xy} \cdot \int\limits_0^x \int\limits_0^y |f(t,\tau)| \cdot \left[ \varphi\left(\frac{1}{t\tau}\right)\right]^{\frac{1}{q}} dt d\tau \right\}^q dxdy =$$

$$= C \cdot \int\limits_0^{\pi} \int\limits_0^{\pi} \left\{ \frac{1}{xy} \cdot \int\limits_0^x \int\limits_0^y |f(t,\tau)| \cdot \left[ \varphi\left(\frac{1}{t\tau}\right)\right]^{\frac{1}{q}} dt d\tau \right\}^q dxdy.$$

Hence, by virtue of Lemma 2 of [2] and Lemma A we obtain:

$$C(f,\varphi,q) \leq C \cdot \int\limits_0^{\pi} \int\limits_0^{\pi} |f(t,\tau)|^q \cdot \varphi\left(\frac{1}{t\tau}\right) dt d\tau < +\infty.$$

The theorem is proved.

*Remark*. This theorem for a function of one variable is proved by M. F. Timan [1].

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## Фурье қатары туралы Харди-Литтлвуд теоремасының жалпыламасы

Бір өлшемді тригонометриялық қатарлар теориясында Фурье коэффициенті бірсарынды болатын Фурье қатары туралы Харди-Литтлвуд теоремасы маңызды орын иеленеді. Бұл теореманың көп өлшемді нұсқалары Лебег кеңістігі үшін кеңінен зерттелген. Көп өлшемді нұсқаларының бір өлшемді жағдайдан айтарлықтай айырмашылықтары бар екені анықталған және бұл теореманың күшейтілулері алынған болатын. Харди-Литтлвуд теоремасы сонымен қатар әртүрлі функциялық кеңістіктер үшін және қатар коэффициентінің біркелкілігі түрлері үшін жалпыланған болатын. Осы жалпылаулардың кейбірін М.Ф. Тиман, М.И. Дьяченко, Е.Д. Нурсултанов, С. Тихонов жұмыстарынан көруге болады. Осы жұмыста Харди-Литтлвуд теоремасының $L_q\varphi(L_q)(0,2\pi]^2$ кеңістігінен алынған функцияның екі еселі Фурье қатары үшін жалпыламасы алынған.

*Кілт сөздер:* тригонометриялық қатар, Фурье қатары, Лебег кеңістігі, Харди-Литтлвуд теоремасы, Фурье коэффициенттері.

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## Обобщение теоремы Харди-Литтлвуда о рядах Фурье

В теории одномерных тригонометрических рядов важное значение имеет теорема Харди-Литтлвуда о рядах Фурье с монотонными коэффициентами Фурье. Многомерные варианты этой теоремы широко исследованы для пространства Лебега. Выявлены существенные отличия многомерных вариантов по сравнению с одномерным случаем и получено усиление этой теоремы. Теорема Харди-Литтлвуда также обобщена для различных функциональных пространств и видов монотонности коэффициентов ряда. Некоторые из этих обобщений встречается в работах М.Ф. Тимана, М.И. Дьяченко, Е.Д. Нурсултанова и С. Тихонова. В настоящей работе получено обобщение теоремы Харди-Литтлвуда для двойных рядов Фурье функции из пространства $L_q\varphi(L_q)(0,2\pi]^2$.

*Ключевые слова:* тригонометрический ряд, ряд Фурье, пространство Лебега, теорема Харди-Литтлвуда, коэффициенты Фурье.

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