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On the solvability of a semi-periodic boundary value problem for the nonlinear Goursat equation

In this paper, by means of a change of variables, a nonlinear semi-periodic boundary value problem for the Goursat equation is reduced to a linear gravity problem for hyperbolic equations. Reintroducing a new function, the obtained problem is reduced to a family of boundary value problems for ordinary differential equations and functional relations. When solving a family of boundary value problems for ordinary differential equations, the parameterization method is used. The application of this approach made it possible to establish the coefficients of the unique solvability of the semi-periodic problem for the Goursat equation and to propose constructive algorithms for finding an approximate solution.

Keywords: semi-periodic boundary value problem, second order boundary value problem, Goursat equation, boundary value problem, algorithm, approximate solution.

Introduction

On $\Omega = [0, X] \times [0, T]$ a boundary value problem for the nonlinear Goursat equation is considered

$$\frac{\partial^2 z}{\partial x \partial t} = 2\sqrt{f(x,t)\frac{\partial z}{\partial x}\frac{\partial z}{\partial t}}, \tag{1}$$

$$z(x,0) = 0, \tag{2}$$

$$\frac{\partial z(0,t)}{\partial t} = \psi^2(t), \tag{3}$$

$$z(x,0) = z(x,T), \tag{4}$$

where $f(x,t)$-given function depending on x and t.

Let $C(\Omega, R^n)$ be the spaces of functions $u : \Omega \to R^n$ which are continuous on Ω, with the rate $\|u\|_0 = \max\limits_{(x,t)\in\Omega} |u(x,t)|$.

In this paper, we study a remarkable equation, the importance of which was first noted by Goursat [1–2].

The function $z(x,t) \in C(\Omega, R^n)$ with partial derivatives $\frac{\partial z(x,t)}{\partial x} \in C(\overline{\Omega}, R^n)$, $\frac{\partial z(x,t)}{\partial t} \in C(\overline{\Omega}, R^n)$, $\frac{\partial^2 z(x,t)}{\partial x \partial t} \in C(\Omega, R^n)$ is called the classical solution to the problem (1)-(4), if it satisfies the system (1) with all $(x,t) \in \Omega$ and boundary conditions (2)–(4).

To find the solution of the problem (1)–(4), we make differential substitutions, introduce the functions $u = u(x,t)$, $g = g(x,t)$ by formulas: $u = \sqrt{\frac{\partial z}{\partial t}}$, $g = \sqrt{\frac{\partial z}{\partial x}}$. Differentiating these ratios, respectively, by t and x excluding z using the equation (1), we get the system $\frac{\partial u}{\partial x} = g\sqrt{f(x,t)}$, $\frac{\partial g}{\partial t} = u\sqrt{f(x,t)}$. Excluding g, we arrive at a linear equation for the function $u = u(x,t)$:

$$\frac{\partial^2 u}{\partial x \partial t} = a(x,t)\frac{\partial u}{\partial x} + f(x,t)u, \tag{5}$$

$$u(0,t) = \psi(t), \tag{6}$$

$$u(x,0) = u(x,T), \tag{7}$$

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$$z(x,t)=\int_0^t u^2(x,\eta)d\eta, \tag{8}$$

where $a(x,t)=\frac{1}{2}\frac{\partial}{\partial t}\ln f(x,t)$.

Such problems these were investigated in the works [3–6].

We introduce a new unknown function $w(x,t)=\frac{\partial u(x,t)}{\partial x}$, and the problem (5)–(8) can be written in the form

$$\frac{\partial w}{\partial t}=a(x,t)w+f(x,t)u(x,t), \quad (x,t)\in\Omega, \tag{9}$$

$$w(x,0)=w(x,T), \quad x\in[0,X], \tag{10}$$

$$u(x,t)=\psi(t)+\int_0^x w(\xi,t)d\xi, \quad t\in[0,T], \tag{11}$$

$$z(x,t)=\int_0^t u^2(x,\eta)d\eta. \tag{12}$$

Here, the problem of finding a solution to a semi-periodic boundary value problem for hyperbolic equations (5)–(8) reduced to a family of periodic boundary value problems for a system of ordinary differential equations (9), (10) and functional relationships (11), (12).

The problems (5)–(8) and (9)–(12) are equivalent in the sense that if a pair of functions $(u^*(x,t),z^*(x,t))$ are the solution to the problem (5)–(8), then three $(w^*(x,t)=\frac{\partial u^*(x,t)}{\partial x},u^*(x,t),z^*(x,t))$ is the solution of the problem (9)–(12) and vice versa, if three $(w^*(x,t),u^*(x,t),z^*(x,t))$- is the solution of the problem(9)–(12), then a pair of functions $(u^*(x,t),z^*(x,t))$ is the solution of the problem (5)–(8).

Algorithms for finding the solution of a semi-periodic boundary value problem for the nonlinear Goursat equation

To solve problem (9)–(12), we apply the method of a parametrization [7].

For the step $h>0:Nh=T$ we partition $[0,T)=\bigcup\limits_{i=1}^{N}[(i-1)h,ih)$, $N=1,2,...$. In this case, the area Ω is divided into N parts. By $w_i(x,t),u_i(x,t),z_i(x,t)$ we denote, respectively, the restrictions of the function $w(x,t),u(x,t),z(x,t)$ on $\Omega_i=[0,X]\times[(i-1)h,ih),i=\overline{1,N}$. Then problem (9)–(12) be equivalent to the boundary value problem [8–14]

$$\frac{\partial w_i}{\partial t}=a(x,t)w_i+f(x,t)u_i(x,t), \quad (x,t)\in\Omega_i, \tag{13}$$

$$w_1(x,0)-\lim_{t\to T-0}w_N(x,t)=0, \tag{14}$$

$$\lim_{t\to sh-0}w_s(x,t)=w_{s+1}(x,sh), s=\overline{1,N-1}, \tag{15}$$

$$u_i(x,t)=\psi(t)+\int\limits_0^x w_i(\xi,t)d\xi, \tag{16}$$

$$z_i(x,t)=\int\limits_0^t(\psi(\eta)+\int\limits_0^x w_i(\xi,\eta)d\xi)^2d\eta, \tag{17}$$

where (15) is the condition of gluing functions $w(x,t)$ in the internal lines of the partition. By $\lambda_i(x)$ we get the value of the function $w_i(x,t)$ at $t=(i-1)h$, i.e. $\lambda_i(x)=w_i(x,(i-1)h)$ and denote $v_i(x,t)=w_i(x,t)-\lambda_i(x), \quad i=\overline{1,N}$. We obtain an equivalent boundary value problem for the unknown functions $\lambda_i(x)$:

$$\frac{\partial v_i}{\partial t}=a(x,t)v_i+a(x,t)\lambda_i(x)+f(x,t)u_i(x,t), \quad (x,t)\in\Omega_i, \tag{18}$$

$$\lambda_i(x,(i-1)h)=0, x\in[0,X], i=\overline{1,N}, \tag{19}$$

$$\lambda_1(x) - \lambda_N(x) - \lim_{t \to T-0} v_N(x,t) = 0, x \in [0,X], \tag{20}$$

$$\lambda_s(x) + \lim_{t \to sh-0} v_s(x,t) - \lambda_{s+1}(x) = 0, x \in [0,X], s = \overline{1, N-1}, \tag{21}$$

$$u_i(x,t) = \psi(t) + \int_0^x v_i(\xi,t)d\xi + \int_0^x \lambda_i(\xi)d\xi, (x,t) \in \Omega_i, i = \overline{1,N}. \tag{22}$$

$$z_i(x,t) = \int_0^t \left(\psi(\eta) + \int_0^x v_i(\xi,\eta)d\xi + \int_0^x \lambda_i(\xi)d\xi\right)^2 d\eta. \tag{23}$$

Problems (13)–(17) and (18)–(23) are equivalent in the sense that if the triple $\{w_i(x,t), u_i(x,t), z_i(x,t)\}$, $i = \overline{1,N}$ is the solution to the problem (13)–(17), then the system $\{\lambda_i(x) = w_i(x,(i-1)h), v_i(x,t) = w_i(x,t)- -w_i(x,(i-1)h), u_i(x,t), z_i(x,t)\}$, $i = \overline{1,N}$, will be the solution to the problem (18)–(23) and vice versa, if $\{\lambda_i(x), v_i(x,t), u_i(x,t), z_i(x,t)\}$, $i = \overline{1,N}$ - the solution to the problem (18)–(23), then the system $\{\lambda_i(x) + v_i(x,t),\ u_i(x,t), z_i(x,t)\}$, $i = \overline{1,N}$ will be the solution to the problem (7)–(10).

Problem (18)–(19) for fixed $\lambda_i(x)$, $u_i(x,t)$ is a one-parameter family of Cauchy problems for systems of ordinary differential equations, where $x \in [0,X]$ which is equivalent to the integral equation

$$v_i(x,t) = \int_{(i-1)h}^t a(x,\eta)v_i(x,\eta)d\eta + \int_{(i-1)h}^t a(x,\eta)d\eta \cdot \lambda_i(x) + \int_{(i-1)h}^t f(x,\eta)u_i(x,\eta)d\eta. \tag{24}$$

Instead of $v_i(x,t)$ we substitute the corresponding right-hand side (24) and repeating this process l ($l = = 1,2,...$) times we get

$$v_i(x,t) = D_{li}(x,t)\lambda_i(x) + G_{li}(x,t,v_i) + F_{li}(x,t,u_i), i = \overline{1,N}, \tag{25}$$

where

$$D_{l,i}(x,t) = \sum_{j=0}^{l-1} \int_{(i-1)h}^t a(x,\eta_1)\ldots \int_{(i-1)h}^{\eta_j} a(x,\eta_{j+1})d\eta_{j+1}\ldots d\eta_1,$$

$$F_{l,i}(x,t,u_i) = \int_{(i-1)h}^t f(x,\eta_1)u_i(x,\eta_1)d\eta_1 +$$

$$+\sum_{j=1}^{l-1} \int_{(i-1)h}^t a(x,\eta_1)... \int_{(i-1)h}^{\eta_{j-1}} a(x,\eta_j) \int_{(i-1)h}^{\eta_j} f(x,\eta_{j+1})u_i(x,\eta_{j+1})d\eta_{j+1}d\eta_j...d\eta_1,$$

$$G_{l,i}(x,t,v_i) = \int_{(i-1)h}^t a(x,\eta_1)... \int_{(i-1)h}^{\eta_{l-2}} a(x,\eta_{l-1}) \int_{(i-1)h}^{\eta_{l-1}} a(x,\eta_l)v_r(x,\eta_l)d\eta_l d\eta_{l-1}...d\eta_1, \eta_0 = t, i = \overline{1,N}.$$

Passing to the limit as $t \to ih - 0$, in (25) we find $\lim\limits_{t \to ih-0} v_i(x,t), i = \overline{1,N}, x \in [0,X]$, by replacing them into (20)-(21), for unknown functions $\lambda_i(x), i = \overline{1,N}$ we obtain the system of functional equations:

$$Q_l(x,h)\lambda(x) = -F_l(x,h,u) - G_l(x,h,v), \tag{26}$$

where

$$Q_l(x,h) = \begin{pmatrix} I & 0 & \ldots & 0 & -[I + D_{l,N}(x,Nh)] \\ I + D_{l,1}(x,h) & -I & \ldots & 0 & 0 \\ 0 & I + D_{l,2}(x,2h) & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & I + D_{l,N-1}(x,(N-1)h) & -I \end{pmatrix},$$

$$F_l(x,h,u) = (-F_{l,N}(x,Nh,u_N), F_{l,1}(x,h,u_1), ..., F_{l,N-1}(x,(N-1)h,u_{N-1})),$$

$$G_l(x,h,v) = (-G_{l,N}(x,Nh,v_N), G_{l,1}(x,h,v_1), ..., G_{l,N-1}(x,(N-1)h,v_{N-1})),$$

where I - the identity matrix of dimension n.

To find a system of four functions $\{\lambda_i(x), v_i(x,t),\ z_i(x,t), u_i(x,t)\}, i = \overline{1,N}$, we have a closed system consisting of the equations (26), (25), (22) и (23). Assuming invertibility of the matrix $Q_l(x,h)$, with all $x \in [0,X]$, from the equation (26), where $v_i(x,t) = 0, u_i(x,t) = \psi(t)$, we find $\lambda^{(0)}(x) = (\lambda_1^{(0)}(x), \lambda_2^{(0)}(x), ..., \lambda_N^{(0)}(x))'$:

$$\lambda^{(0)}(x) = -[Q_l(x,h)]^{-1}\{F_l(x,h,\psi) + G_l(x,h,0)\}.$$

Using the equation (25), at $\lambda_i(x) = \lambda_i^{(0)}(x)$ we find functions $\{v_i^{(0)}(x,t)\}$, $i = \overline{1,N}$, т.е. $v_i^{(0)}(x,t) = D_{l,i}(x,t)\lambda_i^{(0)}(x) + F_{l,i}(x,t,\psi) + G_{l,i}(x,t,0)$.

The functions $u_i^{(0)}(x,t), i = \overline{1,N}$, are defined from the relations

$$u_i^{(0)}(x,t) = \psi(t) + \int_0^x v_i^{(0)}(\xi,t)d\xi + \int_0^x \lambda_i^{(0)}(\xi)d\xi, \quad (x,t) \in \Omega_i,$$

$$z_i^{(0)}(x,t) = \int_0^t (\psi(\eta) + \int_0^x v_i^{(0)}(\xi,\eta)d\xi + \int_0^x \lambda_i^{(0)}(\xi)d\xi)^2 d\eta, (x,t) \in \Omega_i.$$

For the initial approximation of problem (18)-(23) we take the system $(\lambda_i^{(0)}(x), v_i^{(0)}(x,t), u_i^{(0)}(x,t), z_i^{(0)}(x,t))$, $i = \overline{1,N}$ and construct successive approximations on the following algorithm :

Step 1. A) Assuming that $u_i(x,t) = u_i^{(0)}(x,t)$, $i = \overline{1,N}$, , we find the first approximations of $\lambda_i(x)$, $v_i(x,t)$ by finding a solution to the problem (18)–(19). Taking $\lambda^{(1,0)}(x) = \lambda^{(0)}(x)$, $v_i^{(1,0)}(x,t) = v_i^{(0)}(x,t)$, we find the system of couples $\{\lambda_i^1(x), v_i^1(x,t)\}$, $i = \overline{1,N}$, as the limit of the sequence $\lambda_i^{(1,m)}(x), v_i^{(1,m)}(x,t)$, which is defined as follows:

Step 1.1. Assuming the invertibility of the matrix $Q_l(x,h)$, at all $x \in [0,X]$, from equation (26), where $v_i(x,t) = v_i^{(1,0)}(x,t)$, we find $\lambda^{(1,1)}(x) = (\lambda_1^{(1,1)}(x), \lambda_2^{(1,1)}(x), ..., \lambda_N^{(1,1)}(x))'$:

$$\lambda^{(1,1)}(x) = -[Q_l(x,h)]^{-1}\{F_l(x,h,u^{(0)}) + G_l(x,h,v^{(1,0)})\}.$$

By replacing the found $\lambda_i^{(1,1)}(x), i = \overline{1,N}$ in (25) we find

$$v_i^{(1,1)}(x,t) = D_{l,i}(x,t)\lambda_i^{(1,1)}(x) + F_{l,i}(x,t,u^{(0)}) + G_{l,i}(x,t,v^{(1,0)}).$$

Step 1.2. From equation (26), where $v_i(x,t) = v_i^{(1,1)}(x,t)$, we define

$$\lambda^{(1,2)}(x) = -[Q_l(x,h)]^{-1}\{F_l(x,h,u^{(0)}) + G_l(x,h,v^{(1,1)})\}.$$

Using as expression (25) again, , we find the functions $\{v_i^{(1,2)}(x,t)\}, i = \overline{1,N}$:

$$v_i^{(1,2)}(x,t) = D_{li}(x,t)\lambda_i^{(1,2)}(x) + F_{li}(x,t,u^{(0)}) + G_{li}(x,t,v^{(1,1)}).$$

On step $(1,m)$ we obtain the system of couples $\{\lambda_i^{(1,m)}(x), v_i^{(1,m)}(x,t)\}, i = \overline{1,N}$. Suppose that the solution of problem (18)-(21) is a sequence of systems of couples $\{\lambda_i^{(1,m)}(x), v_i^{(1,m)}(x,t)\}$ converge as $m \to \infty$ goes to continuous, respectively, on $x \in [0,X], (x,t) \in \Omega_i$ functions $\lambda_i^{(1)}(x)$, $v_i^{(1)}(x,t)$, $i = \overline{1,N}$.

B) Functions $u_i^{(1)}(x,t), i = \overline{1,N}$, are defined from the relations:

$$u_i^{(1)}(x,t) = \psi(t) + \int_0^x v_i^{(1)}(\xi,t)d\xi + \int_0^x \lambda_i^{(1)}(\xi)d\xi, (x,t) \in \Omega_i,$$

$$z_i^{(1)}(x,t)=\int\limits_0^t(\psi(\eta)+\int\limits_0^x v_i^{(1)}(\xi,\eta)d\xi+\int\limits_0^x\lambda_i^{(1)}(\xi)d\xi)^2d\eta,(x,t)\in\Omega_i.$$

Step 2. A) Assuming that $u_i(x,t)=u_i^{(1)}(x,t), i=\overline{1,N}$ we find the second approximations of $\lambda_i(x)$, $v_i(x,t)$ by finding a solution to the problem (18)-(21).

Taking $\lambda^{(2,0)}(x)=\lambda_i^{(1)}(x)$, $v_i^{(2,0)}(x,t)=v_i^{(1)}(x,t)$ we find the system of couples $\{\lambda_i^{(2)}(x),v_i^{(2)}(x,t)\}, i=\overline{1,N}$ as the limit of the sequence $\lambda_i^{(2,m)}(x),v_i^{(2,m)}(x,t)$, which is defined as follows:

Step 2.1. Assuming invertibility of the matrix $Q_l(x,h)$, at all $x\in[0,X]$, from equation (26), where $v_i(x,t)=v_i^{(2,0)}(x,t)$, we find $\lambda^{(2,1)}(x)=(\lambda_1^{(2,1)}(x),\lambda_2^{(2,1)}(x),...,\lambda_N^{(2,1)}(x))'$:

$$\lambda^{(2,1)}(x)=-[Q_l(x,h)]^{-1}\{F_l(x,h,u^{(1)})+G_l(x,h,v^{(2,0)})\}.$$

By replacing the found $\lambda_i^{(2,1)}(x), i=\overline{1,N}$ in (25) we find

$$v_i^{(2,1)}(x,t)=D_{li}(x,t)\lambda_i^{(2,1)}(x)+F_{li}(x,t,u^{(1)})+G_{li}(x,t,v^{(2,0)}).$$

Step 2.2. From equation (26), where $v_i(x,t)=v_i^{(2,1)}(x,t)$ we define

$$\lambda^{(2,2)}(x)=-[Q_l(x,h)]^{-1}\{F_l(x,h,u^{(1)})+G_l(x,h,v^{(2,1)})\}.$$

Using the expression (25), we find functions $\{v_i^{(2,2)}(x,t)\}, i=\overline{1,N}$:

$$v_i^{(2,2)}(x,t)=D_{li}(x,t)\lambda_i^{(2,2)}(x)+F_{li}(x,t,u^{(1)})+G_{li}(x,t,v^{(2,1)}).$$

On the step $(2,m)$ we obtain the system of couples $\{\lambda_i^{(2,m)}(x),v_i^{(2,m)}(x,t)\}, i=\overline{1,N}$. Suppose that the solution of problem (18)-(21) is a sequence of systems of couples $\{\lambda_i^{(1,m)}(x),v_i^{(1,m)}(x,t)\}$ which as $m\to\infty$ converges to $x\in[0,X],(x,t)\in\Omega_i$ functions $\lambda_i^{(2)}(x)$, $v_i^{(2)}(x,t)$, $i=\overline{1,N}$.

B) The functions $u_i^{(2)}(x,t), i=\overline{1,N}$, are defined from the relations:

$$u_i^{(2)}(x,t)=\psi(t)+\int\limits_0^x v_i^{(2)}(\xi,t)d\xi+\int\limits_0^x\lambda_i^{(2)}(\xi)d\xi,(x,t)\in\Omega_i,$$

$$z_i^{(2)}(x,t)=\int\limits_0^t(\psi(\eta)+\int\limits_0^x v_i^{(2)}(\xi,\eta)d\xi+\int\limits_0^x\lambda_i^{(2)}(\xi)d\xi)^2d\eta,(x,t)\in\Omega_i.$$

Sufficient conditions for the convergence of algorithms for finding its solution

The conditions of the following statement ensure the feasibility and convergence of the proposed algorithm, as well as the unique solvability of the problem (18)–(23).

Theorem 1. Let for some $h>0: Nh=T,\ N=1,2,...$ и $l,l=1,2,...,$ $(nN\times nN)$ - the matrix $Q_l(x,h)$ be reversible for all $x\in[0,X]$ let the following inequalities be satisfied:

$$1)\|[Q_l(x,h)]^{-1}\|\le\gamma_l(x,h);$$

$$2)q_l(x,h)=\frac{(a(x)h)^l}{l!}\Big[1+\gamma_l(x,h)\sum_{j=1}^{l}\frac{(a(x)h)^j}{j!}\Big]\le\mu<1,$$

Then there exists a unique solution to problem (18)-(23) and the following estimates are valid

$$a)\max_{i=\overline{1,N}}\|\lambda_i^*(x)-\lambda_i^{(k)}(x)\|+\max_{i=\overline{1,N}}\sup_{t\in[(i-1)h,ih)}\|v_i^*(x,t)-v_i^{(k)}(x,t)\|\le$$

$$\leq z(x)f(x)\sum_{j=k}^{\infty}\frac{1}{j!}(\int\limits_0^x z(\xi)f(\xi)d\xi)^j\int\limits_0^x g(\xi)d\xi\max_{t\in[0,T]}\|\psi(t)\|,$$

$$b)\max_{i=\overline{1,N}}\sup_{t\in[(i-1)h,ih)}\|u_i^*(x,t)-u_i^{(k)}(x,t)\|\leq$$

$$\leq\int\limits_0^x\max_{i=\overline{1,N}}\|\lambda_i^*(\xi)-\lambda_i^{(k)}(\xi)\|+\max_{i=\overline{1,N}}\sup_{t\in[(i-1)h,ih)}\|v_i^*(\xi,t)-v_i^{(k)}(\xi,t)\|d\xi,$$

$$c)\max_{i=\overline{1,N}}\sup_{t\in[(i-1)h,ih)}\|z_i^*(x,t)-z_i^{(k)}(x,t)\|\leq$$

$$\leq t\int\limits_0^x\max_{i=\overline{1,N}}\|\lambda_i^*(\xi)-\lambda_i^{(k)}(\xi)\|d\xi+\int\limits_0^t\int\limits_0^x\max_{i=\overline{1,N}}\sup_{t\in[(i-1)h,ih)}\|v_i^*(\xi,\eta)-v_i^{(k)}(\xi,\eta)\|d\xi d\eta,$$

where $k=1,2,...,\quad a(x)=\max_{t\in[0,T]}\|a(x,t)\|,\quad f(x)=\max_{t\in[0,T]}\|f(x,t)\|,\quad b_1(x)=\gamma_l(x,h)h\sum_{j=0}^{l-1}\frac{(a(x)h)^j}{j!},$

$$b_2(x)=\Big[1+\gamma_l(x,h)\sum_{j=1}^{l}\frac{(a(x)h)^j}{j!}\Big]h\sum_{j=0}^{l-1}\frac{(a(x)h)^j}{j!},\quad b_3(x)=\gamma_l(x,h)\frac{(a(x)h)^l}{l!},$$

$$\theta(x)=\frac{1+b_3(x)}{1-q_l(x,h)}q_l(x,h)+b_3(x),\quad \rho(x)=\frac{1+b_3(x)}{1-q_l(x,h)}b_2(x)+b_1(x),$$

$$d(x)=\rho(x)f(x)\int\limits_0^x[f(\xi)+1][b_1(\xi)+b_2(\xi)]d\xi+\theta(x)b_2(x)[f(x)+1].$$

By virtue of the equivalence of the problems (1.1)–(1.3) и (1.11)–(1.15) from Theorem 1 follows *Theorem 2.* Let the assumptions of Theorem 1 be satisfied. Then problem (1)–(4) as a unique solution $u^*(x,t)$ and the evaluation is performed

$$\max\{\|u^*\|_0,\|\frac{\partial u^*}{\partial x}\|_0\}\leq M_\nu(x,h)\max_{t\in[0,T]}\|\psi(t)\|,$$

where $M_\nu(x,h)=\max\{1+\int_0^x\tilde{\rho}(\xi)d\xi,\ \tilde{\rho}(\xi)\}$,

$$\tilde{\rho}(x)=z(x)\sigma(x)\exp(\int_0^x\rho(\xi)\sigma(\xi)d\xi)\int_0^x g(\xi)d\xi+\theta(x)+[\sigma(x)+1][b_1(x)+b_2(x)].$$

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References

1 Goursat, E. (1897). Sur une equations aux derivees partielles. *Bulletin de la Societe Mathematique de France, 25*, 36–48. DOI: https://doi.org/10.24033/bsmf.549 [in French].

2 Goursat, E. (1900). Sur une transformation de l'equation $s2=4\lambda(x,y)pq$. *Bulletin de la Societe Mathematique de France, 28*, 1–6. DOI: https://doi.org/10.24033/bsmf.616 [in French].

3 Hale, J.K. (1967). Periodic solutions of a class of hyperbolic equations containing a small parameter. *Arch.Rational Meth.Anal., 14*(5), 380–398.

4 Aziz, A.K. (1966). Periodic solutions of hyperbolic partial differential equations. *Proc. Amer. Math. Soc., 17*, 557–566.

5 Kiguradze, T., & Lakshmikantham, V. (2002). On doubly periodic solutions of fourth - order linear hyperbolic equations. *Proc. Nonlinear Analysis., 19*(1), 87–112.

6 Assanova, A.T., Iskakova, N.B., & Orumbayeva, N.T. (2018). Well-posedness of a periodic boundary value problem for the system of hyperbolic equations with delayed argument, *Bulletin of the Karaganda University-Mathematics, 89*(1), 8–14, DOI: 10.31489/2018M1/8-14.

7 Dzhumabaev, D.S. (2018). Well-posedness of nonlocal boundary value problem for a system of loaded hyperbolic equations and an algorithm for finding its solution, *Journal of Mathematical analysis and Applications, 461*(1), 817–836. DOI: 10.1016/j.jmaa.2017.12.005.

8 Orumbayeva, N.T., & Sabitbekova, G. (2017). A boundary value problem for nonlinear differential equation with arbitrary functions. *Bulletin of the Karaganda University-Mathematics, 85*(1), 71–76. DOI: 10.31489/2017M1/71-76.

9 Orumbayeva, N.T., & Keldibekova, A.B. (2019). On the solvability of the duo-periodic problem for the hyperbolic equation system with a mixed derivative. *Bulletin of the Karaganda University-Mathematics, 93*(1), 59–71. DOI: 10.31489/2019M1/59-71.

10 Assanova, A., Iskakova, N., & Orumbayeva, N. (2020). On the well-posedness of periodic problems for the system of hyperbolic equations with finite time delay. *Mathematical Methods in the Applied Sciences., 43*(2), 881–902. DOI:10.1002/mma.5970.

11 Orumbayeva, N.T., & Sabitbekova, G. (2017). On a Solution of a Nonlinear Semi-periodic Boundary Value Problem for a Differential Equation with Arbitrary Functions. *Springer Proceedings in Mathematics and Statistics., 216*, 158–163.

12 Orumbayeva, N.T. (2016). On solvability of non-linear semi-periodic boundary-value problem for system of hyperbolic equations. *Russian Mathematics (Iz. VUZ).*(9), 26–41.

13 Orumbayeva, N.T., & Shayakhmetova, B. (2016). On a method of finding a solution of semi-periodic boundary value problem for hyperbolic equations . *AIP Conference Proceedings, 1759*, 120-121.

14 Orumbayeva, N.T., & Keldibekova, A.B. (2020). On One Solution of a Periodic Boundary-Value Problem for a Third-Order Pseudoparabolic Equation. *Lobachevskii Journal of Mathematics, 41*(9) 1857–1865. DOI: 10.1134/S1995080220090218.

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Сызықтық емес Гурс теңдеуі үшін жартылайпериодтық шеттік есептің шешімділігі туралы

Мақалада айнымалыларды ауыстыру арқылы Гурс теңдеуі үшін сызықты емес жартылайпериодты шеттік есебі гиперболалық теңдеулер үшін сызықтық гравитация есебіне келтірілген. Жаңа функцияны қайта енгізу арқылы алынған есеп қарапайым дифференциалдық теңдеулер мен функционалдық қатынастарға арналған шекаралық есептер тобына түседі. Қарапайым дифференциалдық теңдеулер үшін шекаралық есептер тобын шешкен кезде параметрлеу әдісі қолданылды. Бұл тәсілді пайдалану Гурс теңдеуі үшін периодтық есептің біржақты шешілу коэффициенттерін анықтауға және жуық шешімді іздеудің конструктивті алгоритмдерін ұсынуға мүмкіндік береді.

Кілт сөздер: жартылайпериодты шеттік есеп, екінші ретті шеттік есеп, Гурс теңдеуі, шеттік есеп, алгоритм, жуық шешім.

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О разрешимости полупериодической краевой задачи для нелинейного уравнения Гурса

В статье с помощью замены переменных нелинейная полупериодическая краевая задача для уравнения Гурса сведена к линейной задаче гравитации для гиперболических уравнений. Повторно введена новая функция, а полученная задача сведена к семейству краевых задач для обыкновенных дифференциальных уравнений и функциональных соотношений. При решении семейства краевых задач для обыкновенных дифференциальных уравнений использован метод параметризации. Применение данного подхода позволило установить коэффициенты однозначной разрешимости полуперидической задачи для уравнения Гурса и предложить конструктивные алгоритмы поиска приближенного решения.

Ключевые слова: полупериодическая краевая задача, краевая задача второго порядка, уравнение Гурса, краевая задача, алгоритм, приближенное решение.