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Upper Estimates of the angle best approximations of generalized Liouville-Weyl derivatives

In this article we consider continuous functions f with period 2π and their approximation by trigonometric polynomials. This article is devoted to the study of estimates of the best angular approximations of generalized Liouville-Weyl derivatives by angular approximation of functions in the three-dimensional case. We consider generalized Liouville-Weyl derivatives instead of the classical mixed Weyl derivative. In choosing the issues to be considered, we followed the general approach that emerged after the work of the second author of this article. Our main goal is to prove analogs of the results of in the three-dimensional case. The concept of general monotonic sequences plays a key role in our study. Several well-known inequalities are indicated for the norms, best approximations of the r-th derivative with respect to the best approximations of the function f. The issues considered in this paper are related to the range of issues studied in the works of Bernstein. Later Stechkin and Konyushkov obtained an inequality for the best approximation $f^{(r)}$. Also, in the works of Potapov, using the angle approximation, some classes of functions are considered. In subsection 1 we give the necessary notation and useful lemmas. Estimates for the norms and best approximations of the generalized Liouville-Weyl derivative in the three-dimensional case are obtained.

Keywords: Lebesgue space, best approximation by three-dimensional angle, trigonometric polynomial, Liouville-Weyl derivative.

Introduction

Let us mention several well-known inequalities for norms and best approximations of the r-th derivative in terms of best approximations of the function f.

The following result was proved by Bernstein for $p = \infty$ (for $1 \le p < \infty$, see [2]) if $f \in L_p$, $1 \le p \le \infty$, and $\sum_{k=0}^{\infty}(k+1)^{r-1}E_k(f)_p < \infty, r \in \mathbb{N}$, then $\|f^{(r)}\|_p \le C(r)\sum_{k=0}^{\infty}(k+1)^{r-1}E_k(f)_p$ [1].

Later on, Stechkin [3] for $p = \infty$ and Konyushkov [4] for $1 < p < \infty$ obtained the following inequality for the best approximations of $f^{(r)}$:

$$E_n(f^{(r)})_p \le C(r,p)\left(n^r E_n(f)_p + \sum_{k=n+1}^{\infty} k^{r-1}E_k(f)_p\right) \quad r,n \in \mathbb{N}.$$

The last inequality was extended by the formula Timan [5] for the case of $1 < p < \infty$ as follows:

$$E_n(f^{(r)})_p \le C(r)\left(n^r E_n(f)_p + (\sum_{k=n+1}^{\infty} k^{\theta r-1}E_k^{\theta}(f)_p)^{\frac{1}{\theta}}\right), \quad \theta = min(2,p) \quad r,n \in \mathbb{N}.$$

Also, A. Jumabayeva and B. Simonov obtained estimates of norms and the angle best approximations of the generalized Liouville-Weyl derivatives by the angle approximation of functions in the two-dimensional case [6, 7].

Let $L_p(T^3)$, $1 < p < \infty$ be the space of measurable functions of three variables that are 2π periodic in each variable and such that

$$\|f\|_p = \left(\int_0^{2\pi}\int_0^{2\pi}\int_0^{2\pi} \mid f(x_1,x_2,x_3)\mid^p dx_1dx_2dx_3\right)^{1/p} < \infty.$$

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L_p^0 is the set of functions $f \in L_p$ such that $\int\limits_0^{2\pi} f(x_1, x_2, x_3)dx_1 = 0$ for almost everyone x_2, x_3, $\int\limits_0^{2\pi} f(x_1, x_2, x_3)dx_2 = 0$ for almost everyone x_1, x_3 and $\int\limits_0^{2\pi} f(x_1, x_2, x_3)dx_3 = 0$ for almost everyone x_1, x_2.

Let $Y_{m_1,m_2,m_3}(f)_p$ be the best approximation by a three-dimensional angle of the function $f \in L_p(\mathbb{T}^3)$, i.e.

$$Y_{m_1,m_2,m_3}(f)_p = \inf_{T_{m_1,\infty,\infty}, T_{\infty,m_2,\infty}, T_{\infty,\infty,m_3}} \|f - T_{m_1,\infty,\infty} - T_{\infty,m_2,\infty} - T_{\infty,\infty,m_3}\|_p,$$

where the function $T_{m_1,\infty,\infty}(x_1, x_2, x_3) \in L_p(\mathbb{T}^3)$ is a trigonometric polynomial of order at most m_1 in x_1, the function $T_{\infty,m_2,\infty}(x_1, x_2, x_3) \in L_p(\mathbb{T}^3)$ is a trigonometric polynomial of order at most m_2 in x_2 and the function $T_{\infty,\infty,m_3}(x_1, x_2, x_3) \in L_p(\mathbb{T}^3)$ is a trigonometric polynomial of order at most m_3 in x_3. In the work of Potapov using the angle approximation, some classes of functions are considered [8, 9].

By $\sigma(f)$ we denote the Fourier series of a function $f \in L_p(T^3)$, that is

$$\sigma(f) := \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} \sum_{k_3=-\infty}^{\infty} c_{k_1,k_2,k_3} e^{i(k_1x_1+k_2x_2+k_3x_3)} = \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \sum_{n_3=0}^{\infty} A_{n_1,n_2,n_3}(x_1, x_2, x_3), \qquad (1)$$

where $c_{k_1,k_2,k_3} = \frac{1}{8\pi^3} \int\limits_0^{2\pi} \int\limits_0^{2\pi} \int\limits_0^{2\pi} f(x_1, x_2, x_3) e^{-i(k_1x_1+k_2x_2+k_3x_3)} dx_1dx_2dx_3$.

The transformed Fourier series of $\sigma(f)$ is given by

$$\sigma(f, \lambda, \beta_1, \beta_2, \beta_3) =$$

$$= \sum_{n_1=-\infty}^{\infty} \sum_{n_2=-\infty}^{\infty} \sum_{n_3=-\infty}^{\infty} \lambda_{n_1,n_2,n_3}[c_{n_1n_2,n_3} e^{i(n_1x_1+\beta_1\frac{\pi}{2})} e^{i(n_2x_2+\beta_2\frac{\pi}{2})} e^{i(n_3x_3+\beta_3\frac{\pi}{2})} \mid n_1 \mid^{\beta_1} \mid n_2 \mid^{\beta_2} \mid n_3 \mid^{\beta_3}],$$

where $\beta_1, \beta_2, \beta_3 \in R$ and $\lambda = \{\lambda_{n_1,n_2,n_3}\}_{n_1n_2n_3 \in N}$ is a sequence of real numbers.

Let $\varphi(x_1x_2x_3) \sim \sigma(f, \lambda, \beta_1, \beta_2, \beta_3)$ is the $(\lambda, \beta_1, \beta_2, \beta_3)$ is the mixed derivative of the function f (or Liouville-–Weyl derivative) and denote it by $f^{(\lambda,\beta_1,\beta_2,\beta_3)}(x_1x_2x_3)$. For example, if $\lambda_{n_1,n_2,n_3} = n_1^{r_1} n_2^{r_2} n_3^{r_3}$, $r_i > 0$, $\beta_i = r_i$ $(i = 1, 2, ...) \Rightarrow f^{(\lambda,\beta_1,\beta_2,\beta_3)} = f^{(r_1,r_2,r_3)}$, where $f^{(r_1r_2r_3)}$-mixed derivative of the function f in the sense of Weyl.

Definition 1.1. [10, 11] A sequence $\lambda := \{\lambda_n\}_{n=1}^{\infty}$ is said to be general monotone, written $\lambda \in GM^3$, if the relations

$$\sum_{k_1=n_1}^{2n_1} \mid \lambda_{k_1,n_2,n_3} - \lambda_{k_1+1,n_2,n_3} \mid \leq C \mid \lambda_{n_1,n_2n_3,} \mid, \quad \sum_{k_2=n_2}^{2n_2} \mid \lambda_{n_1,k_2,n_3} - \lambda_{n_1k_2+1,n_3} \mid \leq C \mid \lambda_{n_1,n_2,n_3} \mid,$$

$$\sum_{k_3=n_3}^{2n_3} \mid \lambda_{n_1,n_2,k_3} - \lambda_{n_1,n_2,k_3+1} \mid \leq C \mid \lambda_{n_1,n_2,n_3,} \mid,$$

$$\sum_{k_1=n_1}^{2n_1} \sum_{k_2=n_2}^{2n_2} \mid \lambda_{k_1,k_2,n_3} - \lambda_{k_1+1,k_2,n_3} - \lambda_{k_1,k_2+1,n_3} + \lambda_{k_1+1,k_2+1,n_3} \mid \leq C \mid \lambda_{n_1,n_2,n_3} \mid,$$

$$\sum_{k_2=n_2}^{2n_2} \sum_{k_3=n_3}^{2n_3} \mid \lambda_{n_1,k_2,k_3} - \lambda_{n_1,k_2+1,k_3} - \lambda_{n_1,k_2,k_3+1} + \lambda_{n_1,k_2+1,k_3} \mid \leq C \mid \lambda_{n_1,n_2,n_3} \mid,$$

$$\sum_{k_1=n_1}^{2n_1} \sum_{k_3=n_3}^{2n_3} \mid \lambda_{k_1,n_2,k_3} - \lambda_{k_1+1,n_2,k_3} - \lambda_{k_1,n_2,k_3+1} + \lambda_{k_1+1,n_2,k_3+1} \mid \leq C \mid \lambda_{n_1,n_2,n_3} \mid,$$

$$\sum_{k_1=n_1}^{2n_1} \sum_{k_2=n_2}^{2n_2} \sum_{k_3=n_3}^{2n_3} \mid \lambda_{k_1,k_2,k_3} - \lambda_{k_1+1,k_2,k_3} - \lambda_{k_1,k_2+1,k_3} - \lambda_{k_1,k_2,k_3+1} +$$

$$+ \lambda_{k_1,k_2+1,k_3+1} + \lambda_{k_1+1,k_2,k_3+1} + \lambda_{k_1+1,k_2+1,k_3} - \lambda_{k_1+1,k_2+1,k_3+1} \mid \leq C \mid \lambda_{n_1,n_2,n_3} \mid$$

hold for all integers n_1,n_2 and n_3, where the constant C is independent of n_1, n_2 and n_3.

Auxiliary results

In order to prove the main result, we formulate auxiliary statements. We denote

$$\Delta_{m_1,m_2,m_3} := \sum_{n_1=2^{m_1-1}}^{2^{m_1}-1} \sum_{n_2=2^{m_2-1}}^{2^{m_2}-1} \sum_{n_3=2^{m_3-1}}^{2^{m_3}-1} A_{n_1,n_2,n_3}(x_1,x_2,x_3), \quad m_1,m_2,m_3 = 1,2,...$$

Lemma 2.1. [11] $\{\lambda_n\} \in GM$ if and only if there exists $C > 0$, such that

$$(i) \mid \lambda_k \mid \leq C \mid \lambda_n \mid \text{ for } n \leq k \leq 2n;\ (ii) \sum_{k=n}^{N} \mid \Delta\lambda_k \mid \leq C(\mid \lambda_n \mid + \sum_{k=n+1}^{N} \frac{\mid \lambda_k \mid}{k}) \text{ for any } n < N.$$

By [11], it follows that if $\{\lambda_{n_1n_2n_3}\} \in GM^3$, then

$$\mid \lambda_{k_1,k_2,k_3} \mid \leq C \mid \lambda_{n_1,n_2,n_3} \mid \quad \text{for } n_1 \leq k_1 \leq 2n_1, n_2 \leq k_2 \leq 2n_2, n_3 \leq k_3 \leq 2n_3.$$

This implies that the condition

$$\sum_{k_1=n_1}^{2n_1} \sum_{k_2=n_2}^{2n_2} \sum_{k_3=n_3}^{2n_3} \mid \lambda_{k_1,k_2,k_3} - \lambda_{k_1+1,k_2,k_3} - \lambda_{k_1,k_2+1,k_3} - \lambda_{k_1,k_2,k_3+1} + \lambda_{k_1,k_2+1,k_3+1} +$$

$$+\lambda_{k_1+1,k_2,k_3+1} + \lambda_{k_1+1,k_2+1,k_3} - \lambda_{k_1+1,k_2+1,k_3+1} \mid \leq C(\mid \lambda_{n_1,n_2,n_3} \mid + \mid \lambda_{2n_1,2n_2,2n_3} \mid)$$

is equivalent to the condition

$$\sum_{k_1=n_1}^{2n_1} \sum_{k_2=n_2}^{2n_2} \sum_{k_3=n_3}^{2n_3} \mid \lambda_{k_1,k_2,k_3} - \lambda_{k_1+1,k_2,k_3} - \lambda_{k_1,k_2+1,k_3} - \lambda_{k_1,k_2,k_3+1} +$$

$$+\lambda_{k_1,k_2+1,k_3+1} + \lambda_{k_1+1,k_2,k_3+1} + \lambda_{k_1+1,k_2+1,k_3} - \lambda_{k_1+1,k_2+1,k_3+1} \mid \leq C \mid \lambda_{n_1,n_2,n_3} \mid .$$

Lemma 2.2. (Minkowskii inequality [12]) Let $1 \leq p < \infty$ and $a_{\nu k} \geq 0$, then

$$(a) \Big(\sum_{k=1}^{\infty} (\sum_{\nu=1}^{k} a_{\nu k})^p\Big)^{\frac{1}{p}} \leq \sum_{\nu=1}^{\infty} \Big(\sum_{k=\nu}^{\infty} a_{\nu k}^p\Big)^{\frac{1}{p}}, (b) \Big(\sum_{k=1}^{\infty} (\sum_{\nu=k}^{\infty} a_{\nu k})^p\Big)^{\frac{1}{p}} \leq \sum_{\nu=1}^{\infty} \Big(\sum_{k=1}^{\nu} a_{\nu k}^p\Big)^{\frac{1}{p}}.$$

Lemma 2.3. [12] For a function $f(u,y)$ defined on measurable set $E = E_1 \times E_2 \subset \mathbb{R}_n$, where $x = (u,y)$, $u = (x_1,...,x_m), y = (x_{m+1},...,x_n)$, the following inequality holds

$$\Big(\int_{E_1} \mid \int_{E_2} f(u,y)dy \mid^p du\Big)^{\frac{1}{p}} \leq \int_{E_2} \Big(\int_{E_1} \mid f(u,y) \mid^p du\Big)^{\frac{1}{p}} dy.$$

Lemma 2.4. [8] Let $f \in L_p(\mathbb{T}^2), 1 < p < \infty, m_i \in \mathbb{N} \cup 0\ \ (i = 1,2)$. Then

$$\|f - S_{m_1,\infty}(f) - S_{\infty,m_2}(f) + S_{m_1,m_2}(f)\|_p \asymp Y_{m_1,m_2}(f)_p,$$

where S_{m_1,m_2} are the partial sums of the Fourier series of a function f.

Lemma 2.5. [8] a) Let $1 < p < \infty$ and (1) be the Fourier series of $f \in L_{p^0}(\mathbb{T}^3)$, then

$$C_1(p)\|f\|_p \leq \Big(\int_0^{2\pi}\int_0^{2\pi}\int_0^{2\pi} (\sum_{m_1=1}^{\infty} \sum_{m_2=1}^{\infty} \sum_{m_3=1}^{\infty} \Delta^2_{m_1m_2m_3})^{\frac{p}{2}} dx_1dx_2dx_3\Big)^{\frac{1}{p}} \leq C_2(p)\|f\|_p.$$

b) Let $1 < p < \infty$. If (1) satisfies the following inequality

$$I_p = \Big(\int_0^{2\pi}\int_0^{2\pi}\int_0^{2\pi} (\sum_{m_1=1}^{\infty} \sum_{m_2=1}^{\infty} \sum_{m_3=1}^{\infty} \Delta^2_{m_1m_2m_3})^{\frac{p}{2}} dx_1dx_2dx_3\Big)^{\frac{1}{p}} < \infty.$$

Then (1) is the Fourier series of a function $f = (x_1,x_2,x_3) \in L_p(\mathbb{T}^3)$ and $\|f\|_p \leq C(p)I_p$.

Main result

The aim of this paper is to prove the following theorem.

Theorem 3.1. Let $1 < p < \infty$, $0 < \theta \le \min(p, 2)$, $\lambda := \{\lambda_{n_1,n_2,n_3}\}_{n_1,n_2,n_3}$ be sequences of positive numbers such that $\lambda \in GM^3$, $\alpha_i \in \mathbb{R}_+$, $r_i \in \mathbb{R}_+ \cup \{0\}$ and $\beta_i \in \mathbb{R} (i = 1, 2)$. If for $f \in L_p^0(\mathbb{T}^3)$ the series

$$\sum_{n_1=1}^{\infty} | \lambda^{\theta}_{n_1+1,1,1} - \lambda^{\theta}_{n_1,1,1} | Y^{\theta}_{n_1,0,0}(f)_p +$$
$$+ \sum_{n_2=1}^{\infty} | \lambda^{\theta}_{1,n_2+1,1} - \lambda^{\theta}_{1,n_2,1} | Y^{\theta}_{0,n_2,0}(f)_p + \sum_{n_3=1}^{\infty} | \lambda^{\theta}_{1,1,n_3+1} - \lambda^{\theta}_{1,1,n_3} | Y^{\theta}_{0,0,n_3}(f)_p +$$
$$+ \sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty} | \lambda^{\theta}_{n_1+1,n_2+1,1} - \lambda^{\theta}_{n_1+1,n_2,1} - \lambda^{\theta}_{n_1,n_2+1,1} + \lambda^{\theta}_{n_1,n_2,1} | Y^{\theta}_{n_1,n_2,0}(f)_p +$$
$$+ \sum_{n_1=1}^{\infty}\sum_{n_3=1}^{\infty} | \lambda^{\theta}_{n_1,1,n_3} - \lambda^{\theta}_{n_1+1,1,n_3} - \lambda^{\theta}_{n_1,1,n_3+1} + \lambda^{\theta}_{n_1+1,1,n_3+1} | Y^{\theta}_{n_1,0,n_3}(f)_p + \qquad (2)$$
$$+ \sum_{n_2=1}^{\infty}\sum_{n_3=1}^{\infty} | \lambda^{\theta}_{1,n_2,n_3} - \lambda^{\theta}_{1,n_2+1,n_3} - \lambda^{\theta}_{1,n_2,n_3+1} + \lambda^{\theta}_{1,n_2+1,n_3+1} | Y^{\theta}_{0,n_2,n_3}(f)_p +$$
$$+ \sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty}\sum_{n_3=1}^{\infty} | \lambda^{\theta}_{n_1,n_2,n_3} - \lambda^{\theta}_{n_1+1,n_2,n_3} - \lambda^{\theta}_{n_1,n_2+1,n_3} - \lambda^{\theta}_{n_1,n_2,n_3+1} +$$
$$+\lambda^{\theta}_{n_1,n_2+1,n_3+1} + \lambda^{\theta}_{n_1+1,n_2,n_3+1} + \lambda^{\theta}_{n_1+1,n_2+1,n_3} - \lambda^{\theta}_{n_1+1,n_2+1,n_3+1} | Y^{\theta}_{n_1,n_2,n_3}(f)_p$$

converges, then there exists a function $\varphi \in L_p^0(\mathbb{T}^3)$, with the Fourier series $\sigma(f, \lambda, \beta_1, \beta_2, \beta_3)$ and

$$\|\varphi\|_p \le \Bigg(\lambda^{\theta}_{1,1,1} \|f\|^{\theta}_p + \sum_{n_1=1}^{\infty} | \lambda^{\theta}_{n_1+1,1,1} - \lambda^{\theta}_{n_1,1,1} | Y^{\theta}_{n_1,0,0}(f)_p +$$
$$+ \sum_{n_2=1}^{\infty} | \lambda^{\theta}_{1,n_2+1,1} - \lambda^{\theta}_{1,n_2,1} | Y^{\theta}_{0,n_2,0}(f)_p + \sum_{n_3=1}^{\infty} | \lambda^{\theta}_{1,1,n_3+1} - \lambda^{\theta}_{1,1,n_3} | Y^{\theta}_{0,0,n_3}(f)_p +$$
$$+ \sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty} | \lambda^{\theta}_{n_1,n_2,1} - \lambda^{\theta}_{n_1+1,n_2,1} - \lambda^{\theta}_{n_1,n_2+1,1} + \lambda^{\theta}_{n_1+1,n_2+1,1} | Y^{\theta}_{n_1,n_2,0}(f)_p + \qquad (3)$$
$$+ \sum_{n_1=1}^{\infty}\sum_{n_3=1}^{\infty} | \lambda^{\theta}_{n_1,1,n_3} - \lambda^{\theta}_{n_1+1,1,n_3} - \lambda^{\theta}_{n_1,1,n_3+1} + \lambda^{\theta}_{n_1+1,1,n_3+1} | Y^{\theta}_{n_1,0,n_3}(f)_p +$$
$$+ \sum_{n_2=1}^{\infty}\sum_{n_3=1}^{\infty} | \lambda^{\theta}_{1,n_2,n_3} - \lambda^{\theta}_{1,n_2+1,n_3} - \lambda^{\theta}_{1,n_2,n_3+1} + \lambda^{\theta}_{1,n_2+1,n_3+1} | Y^{\theta}_{0,n_2,n_3}(f)_p +$$
$$+ \sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty}\sum_{n_3=1}^{\infty} | \lambda^{\theta}_{n_1,n_2,n_3} - \lambda^{\theta}_{n_1+1,n_2,n_3} - \lambda^{\theta}_{n_1,n_2+1,n_3} - \lambda^{\theta}_{n_1,n_2,n_3+1} +$$
$$+\lambda^{\theta}_{n_1,n_2+1,n_3+1} + \lambda^{\theta}_{n_1+1,n_2,n_3+1} + \lambda^{\theta}_{n_1+1,n_2+1,n_3} - \lambda^{\theta}_{n_1+1,n_2+1,n_3+1} | Y^{\theta}_{n_1,n_2,n_3}(f)_p \Bigg)^{\frac{1}{\theta}},$$

$$Y_{2^{m_1}-1,2^{m_2}-1,2^{m_3}-1}(\varphi)_p \lesssim \Bigg(\lambda^{\theta}_{2^{m_1-1},2^{m_2-1},2^{m_3-1}} Y_{2^{m_1}-1,2^{m_2}-1,2^{m_3}-1}(f)_p +$$
$$+ \sum_{\nu_1=m_1}^{\infty} | \lambda^{\theta}_{2^{\nu_1},2^{m_2-1},2^{m_3-1}} - \lambda^{\theta}_{2^{\nu_1-1},2^{m_2-1},2^{m_3-1}} | \; Y^{\theta}_{2^{\nu_1-1}-1,2^{m_2}-1,2^{m_3}-1}(f)_p +$$

$$+\sum_{\nu_2=m_2}^{\infty} \mid \lambda^{\theta}_{2^{m_1-1},2^{\nu_2},2^{m_3-1}} - \lambda^{\theta}_{2^{m_1-1},2^{\nu_2-1},2^{m_3-1}} \mid \ Y^{\theta}_{2^{m_1}-1,2^{\nu_2-1}-1,2^{m_3}-1}(f)_p+ \tag{4}$$

$$+\sum_{\nu_3=m_3}^{\infty} \mid \lambda^{\theta}_{2^{m_1-1},2^{m_2-1},2^{\nu_3}} - \lambda^{\theta}_{2^{m_1-1},2^{m_2-1},2^{\nu_2-1}} \mid \ Y^{\theta}_{2^{m_1}-1,2^{m_2}-1,2^{\nu_3-1}-1}(f)_p+$$

$$+\sum_{\nu_1=m_1}^{\infty}\sum_{\nu_2=m_2}^{\infty} \mid \lambda^{\theta}_{2^{\nu_1},2^{\nu_2},2^{m_3-1}} - \lambda^{\theta}_{2^{\nu_1-1},2^{\nu_2},2^{m_3-1}} -$$

$$-\lambda^{\theta}_{2^{\nu_1},2^{\nu_2-1},2^{m_3-1}} + \lambda^{\theta}_{2^{\nu_1-1},2^{\nu_2-1},2^{m_3-1}} \mid Y^{\theta}_{2^{\nu_1-1}-1,2^{\nu_2-1}-1,2^{m_3}-1}(f)_p+$$

$$+\sum_{\nu_2=m_2}^{\infty}\sum_{\nu_3=m_3}^{\infty} \mid \lambda^{\theta}_{2^{m_1-1},2^{\nu_2},2^{\nu_3}} - \lambda^{\theta}_{2^{m_1-1},2^{\nu_2-1},2^{\nu_3}} -$$

$$-\lambda^{\theta}_{2^{m_1-1},2^{\nu_2},2^{\nu_3-1}} + \lambda^{\theta}_{2^{m_1-1},2^{\nu_2-1},2^{\nu_3-1}} \mid Y^{\theta}_{2^{m_1}-1,2^{\nu_2-1}-1,2^{\nu_3-1}-1}(f)_p+$$

$$+\sum_{\nu_1=m_1}^{\infty}\sum_{\nu_3=m_3}^{\infty} \mid \lambda^{\theta}_{2^{\nu_1},2^{m_2-1},2^{\nu_3}} - \lambda^{\theta}_{2^{\nu_1-1},2^{m_2-1},2^{\nu_3}} -$$

$$-\lambda^{\theta}_{2^{\nu_1},2^{m_2-1},2^{\nu_3-1}} + \lambda^{\theta}_{2^{\nu_1-1},2^{m_2-1},2^{\nu_3-1}} \mid Y^{\theta}_{2^{\nu_1-1}-1,2^{m_2}-1,2^{\nu_3-1}-1}(f)_p+$$

$$+\sum_{\nu_1=m_1}^{\infty}\sum_{\nu_2=m_2}^{k_2}\sum_{\nu_3=m_3}^{\infty} \mid \lambda^{\theta}_{2^{\nu_1},2^{\nu_2},2^{\nu_3}} - \lambda^{\theta}_{2^{\nu_1-1},2^{\nu_2},2^{\nu_3}} - \lambda^{\theta}_{2^{\nu_1},2^{\nu_2-1},2^{\nu_3}} - \lambda^{\theta}_{2^{\nu_1},2^{\nu_2},2^{\nu_3-1}} +$$

$$\left.+\lambda^{\theta}_{2^{\nu_1},2^{\nu_2-1},2^{\nu_3-1}} + \lambda^{\theta}_{2^{\nu_1-1},2^{\nu_2},2^{\nu_3-1}} + \lambda^{\theta}_{2^{\nu_1-1},2^{\nu_2-1},2^{\nu_3}} - \lambda^{\theta}_{2^{\nu_1-1},2^{\nu_2-1},2^{\nu_3-1}} \mid Y^{\theta}_{2^{\nu_1-1}-1,2^{\nu_2-1}-1,2^{\nu_3-1}-1}(f)_p\right)^{\frac{1}{\theta}}.$$

Proof. Let the series (2) be convergent and $f \in L^0_p(T^3)$. We use the following inequality

$$\lambda^{\theta}_{2^{n_1-1},2^{n_2-1},2^{n_3-1}} \leq \lambda^{\theta}_{111} + \sum_{m_3=2}^{n_3} \mid \lambda^{\theta}_{1,1,2^{m_3-1}} - \lambda^{\theta}_{1,1,2^{m_3-2}} \mid +$$

$$+\sum_{m_2=2}^{n_2} \mid \lambda^{\theta}_{1,2^{m_2-1},1} - \lambda^{\theta}_{1,2^{m_2-2},1} \mid + \sum_{m_1=2}^{n_1} \mid \lambda^{\theta}_{2^{m_1-1},1,1} - \lambda^{\theta}_{2^{m_1-2},1,1} \mid +$$

$$+\sum_{m_1=2}^{n_1}\sum_{m_2=2}^{n_2} \mid \lambda^{\theta}_{2^{m_1-1},2^{m_2-1},1} - \lambda^{\theta}_{2^{m_1-2},2^{m_2-1},1} - \lambda^{\theta}_{2^{m_1-1},2^{m_2-2},1} + \lambda^{\theta}_{2^{m_1-2},2^{m_2-2},1} \mid + \tag{5}$$

$$+\sum_{m_2=2}^{n_2}\sum_{m_3=2}^{n_3} \mid \lambda^{\theta}_{1,2^{m_2-1},2^{m_3-1}} - \lambda^{\theta}_{1,2^{m_2-2},2^{m_3-1}} - \lambda^{\theta}_{1,2^{m_2-1},2^{m_3-2}} + \lambda^{\theta}_{1,2^{m_2-2},2^{m_3-2}} \mid +$$

$$+\sum_{m_1=2}^{n_1}\sum_{m_3=2}^{n_3} \mid \lambda^{\theta}_{2^{m_1-1},1,2^{m_3-1}} - \lambda^{\theta}_{2^{m_1-2},1,2^{m_3-1}} - \lambda^{\theta}_{2^{m_1-1},1,2^{m_3-2}} + \lambda^{\theta}_{2^{m_1-2},1,2^{m_3-2}} \mid +$$

$$+\sum_{m_1=2}^{n_1}\sum_{m_2=2}^{n_2}\sum_{m_3=2}^{n_3} \mid \lambda^{\theta}_{2^{m_1-1},2^{m_2-1},2^{m_3-1}} - \lambda^{\theta}_{2^{m_1-1},2^{m_2-2},2^{m_3-2}} - \lambda^{\theta}_{2^{m_1-2},2^{m_2-1},2^{m_3-2}} -$$

$$-\lambda^{\theta}_{2^{m_1-2},2^{m_2-2},2^{m_3-1}} - \lambda^{\theta}_{2^{m_1-1},2^{m_2-1},2^{m_3-2}} + \lambda^{\theta}_{2^{m_1-2},2^{m_2-1},2^{m_3-1}} - \lambda^{\theta}_{2^{m_1-1},2^{m_2-2},2^{m_3-1}} +$$

$$+\lambda^{\theta}_{2^{m_1-2},2^{m_2-2},2^{m_3-2}} \mid .$$

Let us denote $\Delta_{n_1,n_2,n_3} = \sum_{\nu_1=2^{n_1-1}}^{2^{n_1}-1}\sum_{\nu_2=2^{n_2-1}}^{2^{n_2}-1}\sum_{\nu_3=2^{n_3-1}}^{2^{n_3}-1} A_{\nu_1,\nu_2,\nu_3}(f,x_1,x_2,x_3)(n_1,n_2,n_3 = 1,2...)$. Using (5) and property of GM (Lemma 2.1), we get

$$I_1 = \left\{ \int_0^{2\pi}\int_0^{2\pi}\int_0^{2\pi} \left[\sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty}\sum_{n_3=1}^{\infty} \lambda^2_{2^{n_1-1},2^{n_2-1},2^{n_3-1}} \Delta^2_{n_1,n_2,n_3} \right]^{\frac{p}{2}} dx_1, dx_2, dx_3 \right\}^{\frac{1}{p}} =$$

$$= \Big\| \Big[\sum_{n_1=1}^{\infty} \sum_{n_2=1}^{\infty} \sum_{n_3=1}^{\infty} \lambda^2_{2^{n_1-1},2^{n_2-1},2^{n_3-1}} \Delta^2_{n_1,n_2,n_3} \Big]^{\frac{1}{2}} \Big\|_p =$$

$$= \Big\| \Big[\lambda^2_{1,1,1}\Delta^2_{1,1,1} + \sum_{n_1=2}^{\infty} \lambda^2_{2^{n_1-1},1,1}\Delta^2_{n_1,1,1} + \sum_{n_2=2}^{\infty} \lambda^2_{1,2^{n_2-1},1}\Delta^2_{1,n_2,1} + \sum_{n_3=2}^{\infty} \lambda^2_{1,1,2^{n_3-1}}\Delta^2_{1,1,n^3} +$$

$$+ \sum_{n_1=2}^{\infty}\sum_{n_2=2}^{\infty} \lambda^2_{2^{n_1-1},2^{n_2-1},1}\Delta^2_{n_1,n_2,1} + \sum_{n_1=2}^{\infty}\sum_{n_3=2}^{\infty} \lambda^2_{2^{n_1-1},1,2^{n_3-1}}\Delta^2_{n_1,1,n_3} +$$

$$+ \sum_{n_2=2}^{\infty}\sum_{n_3=2}^{\infty} \lambda^2_{1,2^{n_2-1},2^{n_3-1}}\Delta^2_{1,n_2,n_3} + \sum_{n_1=2}^{\infty}\sum_{n_2=2}^{\infty}\sum_{n_3=2}^{\infty} \lambda^2_{2^{n_1-1},2^{n_2-1},2^{n_3-1}}\Delta^2_{n_1,n_2,n_2} \Big]^{\frac{1}{2}} \Big\|_p \lesssim$$

$$\lesssim \lambda_{1,1,1} \Big\| \Big[\sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty}\sum_{n_3=1}^{\infty} \Delta^2_{n_1,n_2,n_3} \Big]^{\frac{1}{2}} \Big\|_p + \Big\| \Big(\sum_{n_1=2}^{\infty}\sum_{n_2=1}^{\infty}\sum_{n_3=1}^{\infty} \Delta^2_{n_1,n_2,n_3} [\sum_{\nu_1=2}^{n_1} \mid \lambda^\theta_{2^{\nu_1-1},1,1} - \lambda^\theta_{2^{\nu_1-2},1,1} \mid]^{\frac{2}{\theta}} \Big)^{\frac{1}{2}} \Big\|_p +$$

$$+ \Big\| \Big(\sum_{n_1=1}^{\infty}\sum_{n_2=2}^{\infty}\sum_{n_3=1}^{\infty} \Delta^2_{n_1,n_2,n_3} \Big[\sum_{\nu_2=2}^{n_2} \mid \lambda^\theta_{1,2^{\nu_2-1},1} - \lambda^\theta_{1,2^{\nu_2-2},1} \mid \Big]^{\frac{2}{\theta}} \Big)^{\frac{1}{2}} \Big\|_p +$$

$$+ \Big\| \Big(\sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty}\sum_{n_3=2}^{\infty} \Delta^2_{n_1,n_2,n_3} \Big[\sum_{\nu_3=2}^{n_2} \mid \lambda^\theta_{1,1,2^{\nu_3-1}} - \lambda^\theta_{1,1,2^{\nu_3-2}} \mid \Big]^{\frac{2}{\theta}} \Big)^{\frac{1}{2}} \Big\|_p +$$

$$+ \Big\| \Big(\sum_{n_1=2}^{\infty}\sum_{n_2=2}^{\infty}\sum_{n_3=1}^{\infty} \Delta^2_{n_1,n_2,n_3} \Big(\sum_{\nu_1=2}^{n_1}\sum_{\nu_2=2}^{n_2} \mid \lambda^\theta_{2^{\nu_1-1},2^{\nu_2-1},1} - \lambda^\theta_{2^{\nu_1-2},2^{\nu_2-1},1} - \lambda^\theta_{2^{\nu_1-1},2^{\nu_2-2},1} +$$

$$+ \lambda^\theta_{2^{\nu_1-2},2^{\nu_2-2},1} \mid \Big)^{\frac{2}{\theta}} \Big)^{\frac{1}{2}} \Big\|_p + \Big\| \Big(\sum_{n_1=2}^{\infty}\sum_{n_2=1}^{\infty}\sum_{n_3=2}^{\infty} \Delta^2_{n_1,n_2,n_3} \Big(\sum_{\nu_1=2}^{n_1}\sum_{\nu_3=2}^{n_3} \mid \lambda^\theta_{2^{\nu_1-1},1,2^{\nu_3-1}} - \lambda^\theta_{2^{\nu_1-2},1,2^{\nu_3-1}} -$$

$$- \lambda^\theta_{2^{\nu_1-1},1,2^{\nu_3-2}} + \lambda^\theta_{2^{\nu_1-2},1,2^{\nu_3-2}} \mid \Big)^{\frac{2}{\theta}} \Big)^{\frac{1}{2}} \Big\|_p + \Big\| \Big(\sum_{n_1=1}^{\infty}\sum_{n_2=2}^{\infty}\sum_{n_3=2}^{\infty} \Delta^2_{n_1,n_2,n_3} \Big(\sum_{\nu_2=2}^{n_2}\sum_{\nu_3=2}^{n_3} \mid \lambda^\theta_{1,2^{\nu_2-1},2^{\nu_3-1}} -$$

$$- \lambda^\theta_{1,2^{\nu_2-2},2^{\nu_3-1}} - \lambda^\theta_{1,2^{\nu_2-1},2^{\nu_3-2}} + \lambda^\theta_{1,2^{\nu_2-2},2^{\nu_3-2}} \mid \Big)^{\frac{2}{\theta}} \Big)^{\frac{1}{2}} \Big\|_p +$$

$$+ \Big\| \Big(\sum_{n_1=2}^{\infty}\sum_{n_2=2}^{\infty}\sum_{n_3=2}^{\infty} \Delta^2_{n_1,n_2,n_3} \Big[\sum_{\nu_1=2}^{n_1}\sum_{\nu_2=2}^{n_2}\sum_{\nu_3=2}^{n_3} \mid \lambda^\theta_{2^{\nu_1-1},2^{\nu_2-1},2^{\nu_3-1}} - \lambda^\theta_{2^{\nu_1-1},2^{\nu_2-1},2^{\nu_3-2}} -$$

$$- \lambda^\theta_{2^{\nu_1-2},2^{\nu_2-1},2^{\nu_3-1}} - \lambda^\theta_{2^{\nu_1-1},2^{\nu_2-2},2^{\nu_3-1}} + \lambda^\theta_{2^{\nu_1-1},2^{\nu_2-2},2^{\nu_3-2}} + \lambda^\theta_{2^{\nu_1-2},2^{\nu_2-1},2^{\nu_3-2}} +$$

$$+ \lambda^\theta_{2^{\nu_1-2},2^{\nu_2-2},2^{\nu_3-1}} - \lambda^\theta_{2^{\nu_1-2},2^{\nu_2-2},2^{\nu_3-2}} \mid \Big]^{\frac{2}{\theta}} \Big)^{\frac{1}{2}} \Big\|_p =: H_1 + H_2 + H_3 + H_4 + H_5 + H_6 + H_7 + H_8.$$

Let us estimate H_1. Applying Lemma 2.5, we have $H_1 \leq C\lambda_{1,1,1}\|f\|_p < \infty$. Now we estimate H_2 :

$$H_2 = \Big\{ \int_0^{2\pi}\int_0^{2\pi}\int_0^{2\pi} \Big[\sum_{n_1=2}^{\infty}\sum_{n_2=1}^{\infty}\sum_{n_3=1}^{\infty} \Delta^2_{n_1,n_2,n_3} \Big[\sum_{\nu_1=2}^{n_1} \mid \lambda^\theta_{2^{\nu_1-1},1,1} - \lambda^\theta_{2^{\nu_1-2},1,1} \mid \Big]^{\frac{2}{\theta}} \Big]^{\frac{p}{2}} dx_1, dx_2, dx_3 \Big\}^{\frac{1}{p}}.$$

Using Minkowski's inequality and Lemma 2.2 (a) for $\frac{2}{\theta} \geq 1$, we derive

$$\sum_{n_3=1}^{\infty}\sum_{n_2=1}^{\infty}\sum_{n_1=2}^{\infty} \Delta^2_{n_1,n_2,n_3} \Big[\sum_{\nu_1=2}^{n_1} \mid \lambda^\theta_{2^{\nu_1-1},1,1} - \lambda^\theta_{2^{\nu_1-2},1,1} \mid \Big]^{\frac{2}{\theta}} =$$

$$= \sum_{n_1=2}^{\infty} \Big(\Big(\sum_{n_2=1}^{\infty}\sum_{n_3=1}^{\infty} \Big[\sum_{\nu_1=2}^{n_1} \mid \Delta_{n_1,n_2,n_3} \mid^\theta \mid \lambda^\theta_{2^{\nu_1-1},1,1} - \lambda^\theta_{2^{\nu_1-2},1,1} \mid \Big]^{\frac{2}{\theta}} \Big)^{\frac{\theta}{2}} \Big)^{\frac{2}{\theta}} \leq$$

$$\leq \sum_{n_1=2}^{\infty}\left(\sum_{\nu_1=2}^{n_1}\left[\sum_{n_2=1}^{\infty}\sum_{n_3=1}^{\infty}\mid \Delta_{n_1,n_2,n_3}\mid^2\mid \lambda^{\theta}_{2^{\nu_1-1},1,1}-\lambda^{\theta}_{2^{\nu_1-2},1,1}\mid^{\frac{2}{\theta}}\right]^{\frac{\theta}{2}}\right)^{\frac{2}{\theta}}=$$

$$=\left(\left(\sum_{n_1=2}^{\infty}\{\sum_{\nu_1=2}^{n_1}\left[\sum_{n_2=1}^{\infty}\sum_{n_3=1}^{\infty}\mid \Delta_{n_1,n_2,n_3}\mid^2\mid \lambda^{\theta}_{2^{\nu_1-1},1,1}-\lambda^{\theta}_{2^{\nu_1-2},1,1}\mid^{\frac{2}{\theta}}\right]^{\frac{\theta}{2}}\Big\}^{\frac{2}{\theta}}\right)^{\frac{\theta}{2}}\right)^{\frac{2}{\theta}}\leq$$

$$\leq\left(\sum_{\nu_1=2}^{\infty}\left\{\sum_{n_1=\nu_1}^{\infty}\sum_{n_2=1}^{\infty}\sum_{n_3=1}^{\infty}\mid \Delta_{n_1,n_2,n_3}\mid^2\mid \lambda^{\theta}_{2^{\nu_1-1},1,1}-\lambda^{\theta}_{2^{\nu_1-2},1,1}\mid^{\frac{2}{\theta}}\right\}^{\frac{\theta}{2}}\right)^{\frac{2}{\theta}}=$$

$$=\left(\sum_{\nu_1=2}^{\infty}\mid \lambda^{\theta}_{2^{\nu_1-1},1,1}-\lambda^{\theta}_{2^{\nu_1-2},1,1}\mid\left(\sum_{n_1=\nu_1}^{\infty}\sum_{n_2=1}^{\infty}\sum_{n_3=1}^{\infty}\mid \Delta_{n_1,n_2n_3}\mid^2\right)^{\frac{\theta}{2}}\right)^{\frac{2}{\theta}}.$$

Applying this inequality, we obtain

$$H_2\leq\left\{\int_0^{2\pi}\int_0^{2\pi}\int_0^{2\pi}\left[\left\{\sum_{\nu_1=2}^{\infty}\mid \lambda^{\theta}_{2^{\nu_1-1},1,1}-\lambda^{\theta}_{2^{\nu_1-2},1,1}\mid\left(\sum_{n_1=\nu_1}^{\infty}\sum_{n_2=1}^{\infty}\sum_{n_3=1}^{\infty}\mid \Delta_{n_1,n_2n_3}\mid^2\right)^{\frac{\theta}{2}}\right\}^{\frac{2}{\theta}}\right]^{\frac{p}{2}}dx_1,dx_2,dx_3\right\}^{\frac{1}{p}}=$$

$$=\left(\left\{\int_0^{2\pi}\int_0^{2\pi}\int_0^{2\pi}\left(\sum_{\nu_1=2}^{\infty}\mid \lambda^{\theta}_{2^{\nu_1-1},1,1}-\lambda^{\theta}_{2^{\nu_1-2},1,1}\mid\left(\sum_{n_1=\nu_1}^{\infty}\sum_{n_2=1}^{\infty}\sum_{n_3=1}^{\infty}\mid \Delta_{n_1,n_2,n_3}\mid^2\right)^{\frac{\theta}{2}}\right)^{\frac{p}{\theta}}dx_1,dx_2,dx_3\right\}^{\frac{\theta}{p}}\right)^{\frac{1}{\theta}}.$$

Further, using Minkowski's inequality for $\frac{p}{\theta}\geq 1$, Lemmas 2.4 and 2.5, we have

$$H_2\leq\left(\sum_{\nu_1=2}^{\infty}\mid \lambda^{\theta}_{2^{\nu_1-1},1,1}-\lambda^{\theta}_{2^{\nu_1-2},1,1}\mid\left\{\int_0^{2\pi}\int_0^{2\pi}\int_0^{2\pi}\left(\left(\sum_{n_1=\nu_1}^{\infty}\sum_{n_2=1}^{\infty}\sum_{n_3=1}^{\infty}\mid \Delta_{n_1,n_2,n_3}\mid^2\right)^{\frac{\theta}{2}}\right)^{\frac{p}{\theta}}dx_1,dx_2,dx_3\right\}^{\frac{\theta}{p}}\right)^{\frac{1}{\theta}}=$$

$$=\left(\sum_{\nu_1=2}^{\infty}\mid \lambda^{\theta}_{2^{\nu_1-1},1,1}-\lambda^{\theta}_{2^{\nu_1-2},1,1}\mid\left\|\left(\sum_{n_1=\nu_1}^{\infty}\sum_{n_2=1}^{\infty}\sum_{n_3=1}^{\infty}\mid \Delta_{n_1,n_2,n_3}\mid^2\right)^{\frac{1}{2}}\right\|_p^{\theta}\right)^{\frac{1}{\theta}}\lesssim$$

$$\lesssim\left(\sum_{\nu_1=2}^{\infty}\mid \lambda^{\theta}_{2^{\nu_1-1},1,1}-\lambda^{\theta}_{2^{\nu_1-1},1,1}\mid Y^{\theta}_{2^{\nu_1}-1,0,0}(f)_p\right)^{\frac{1}{\theta}}.$$

Thus, we obtain $H_2\lesssim\left(\sum_{\nu_1=2}^{\infty}\mid \lambda^{\theta}_{2^{\nu_1},1,1}-\lambda^{\theta}_{2^{\nu_1-1},1,1}\mid Y^{\theta}_{2^{\nu_1}-1,0,0}(f)_p\right)^{\frac{1}{\theta}}$. From (2) it follows that $H_2<\infty$, H_3,H_4 can be estimated similarly to H_2 and we have

$$H_3\lesssim\left(\sum_{\nu_2=1}^{\infty}\mid \lambda^{\theta}_{1,2^{\nu_2},1}-\lambda^{\theta}_{1,2^{\nu_2-1},1}\mid Y^{\theta}_{0,2^{\nu_2}-1,0}(f)_p\right)^{\frac{1}{\theta}},H_4\lesssim\left(\sum_{\nu_3=1}^{\infty}\mid \lambda^{\theta}_{1,1,2^{\nu_2}}-\lambda^{\theta}_{1,1,2^{\nu_2-1}}\mid Y^{\theta}_{0,0,2^{\nu_3}-1}(f)_p\right)^{\frac{1}{\theta}}.$$

To estimate H_5, we apply the method of estimate for H_4 as in article [9]. First, we obtain the upper estimate of the following sum. Applying Lemmas 2.2 and 2.3 twice for $\frac{2}{\theta}\geq 1$, we get

$$\sum_{n_3=1}^{\infty}\sum_{n_2=2}^{\infty}\sum_{n_1=2}^{\infty}\Delta^2_{n_1,n_2,n_3}\left[\sum_{\nu_1=2}^{n_1}\sum_{\nu_2=2}^{n_2}\mid \lambda^{\theta}_{2^{\nu_1-1},2^{\nu_2-1},1}-\lambda^{\theta}_{2^{\nu_1-2},2^{\nu_2-1},1}-\lambda^{\theta}_{2^{\nu_1-1},2^{\nu_2-2},1}+\lambda^{\theta}_{2^{\nu_1-2},2^{\nu_2-2},1}\mid\right]^{2/\theta}\leq$$

$$\leq\left(\sum_{\nu_2=2}^{\infty}\sum_{\nu_1=2}^{\infty}\mid \lambda^{\theta}_{2^{\nu_1-1},2^{\nu_2-1},1}-\lambda^{\theta}_{2^{\nu_1-2},2^{\nu_2-1},1}-\lambda^{\theta}_{2^{\nu_1-1},2^{\nu_2-2},1}+\lambda^{\theta}_{2^{\nu_1-2},2^{\nu_2-2},1}\mid\left(\sum_{n_1=\nu_1}^{\infty}\sum_{n_2=\nu_2}^{\infty}\sum_{n_3=1}^{\infty}\mid \Delta_{n_1,n_2,n_3}\mid^2\right)^{\frac{\theta}{2}}\right)^{\frac{2}{\theta}}.$$

Hence, Lemma 2.3 with $\frac{p}{\theta}\geq 1$ implies that

$$H_5\leq\Bigg(\Bigg\{\int_0^{2\pi}\int_0^{2\pi}\int_0^{2\pi}\Bigg[\sum_{\nu_2=2}^{\infty}\sum_{\nu_1=2}^{\infty}\mid \lambda^{\theta}_{2^{\nu_1-1},2^{\nu_2-1},1}-\lambda^{\theta}_{2^{\nu_1-2},2^{\nu_2-1},1}-$$

$$-\lambda^{\theta}_{2^{\nu_1-1},2^{\nu_2-2},1}+\lambda^{\theta}_{2^{\nu_1-2},2^{\nu_2-2},1}\mid \left(\sum_{n_1=\nu_1}^{\infty}\sum_{n_2=\nu_2}^{\infty}\sum_{n_3=1}^{\infty}\mid\Delta_{n_1,n_2,n_3}\mid^2\right)^{\frac{\theta}{2}}\Bigg]^{\frac{p}{\theta}}dx_1,dx_2,dx_3\Bigg\}^{\frac{\theta}{p}}\Bigg)^{\frac{1}{\theta}}\leq$$

$$\leq\Bigg(\sum_{\nu_2=2}^{\infty}\sum_{\nu_1=2}^{\infty}\mid\lambda^{\theta}_{2^{\nu_1-1},2^{\nu_2-1},1}-\lambda^{\theta}_{2^{\nu_1-2},2^{\nu_2-1},1}-\lambda^{\theta}_{2^{\nu_1-1},2^{\nu_2-2},1}+$$

$$+\lambda^{\theta}_{2^{\nu_1-2},2^{\nu_2-2},1}\Bigg\{\int\limits_0^{2\pi}\int\limits_0^{2\pi}\int\limits_0^{2\pi}\Bigg[\sum_{n_1=\nu_1}^{\infty}\sum_{n_2=\nu_2}^{\infty}\sum_{n_3=1}^{\infty}\mid\Delta_{n_1,n_2,n_3}\mid^2\Bigg]^{\frac{p}{\theta}}dx_1,dx_2,dx_3\Bigg\}^{\frac{\theta}{p}}\Bigg)^{\frac{1}{\theta}}.$$

By Lemmas 2.4 and 2.5, we obtain

$$H_5\lesssim\Bigg(\sum_{\nu_2=1}^{\infty}\sum_{\nu_1=1}^{\infty}\mid\lambda^{\theta}_{2^{\nu_1},2^{\nu_2},1}-\lambda^{\theta}_{2^{\nu_1-1},2^{\nu_2},1}-\lambda^{\theta}_{2^{\nu_1},2^{\nu_2-1},1}+\lambda^{\theta}_{2^{\nu_1-1},2^{\nu_2-1},1}\mid Y^{\theta}_{2^{\nu_1}-1,2^{\nu_2}-1,0}(f)_p\Bigg)^{\frac{1}{\theta}}.$$

From (2), it follows that $H_5<\infty$. H_6,H_7,H_8 can be estimated similarly to H_5 and we have

$$H_6\lesssim\Bigg(\sum_{\nu_3=1}^{\infty}\sum_{\nu_1=1}^{\infty}\mid\lambda^{\theta}_{2^{\nu_1},1,2^{\nu_3}}-\lambda^{\theta}_{2^{\nu_1-1},1,2^{\nu_3}}-\lambda^{\theta}_{2^{\nu_1},1,2^{\nu_3-1}}+\lambda^{\theta}_{2^{\nu_1-1},1,2^{\nu_3-1}}\mid Y^{\theta}_{2^{\nu_1}-1,0,2^{\nu_3}-1}(f)_p\Bigg)^{\frac{1}{\theta}},$$

$$H_7\lesssim\Bigg(\sum_{\nu_3=1}^{\infty}\sum_{\nu_2=1}^{\infty}\mid\lambda^{\theta}_{1,2^{\nu_2},2^{\nu_3}}-\lambda^{\theta}_{1,2^{\nu_2-1},2^{\nu_3}}-\lambda^{\theta}_{1,2^{\nu_2},2^{\nu_3-1}}+\lambda^{\theta}_{1,2^{\nu_2-1},2^{\nu_3-1}}\mid Y^{\theta}_{0,2^{\nu_2}-1,2^{\nu_3}-1}(f)_p\Bigg)^{\frac{1}{\theta}},$$

$$H_8\lesssim\Bigg(\sum_{\nu_3=1}^{\infty}\sum_{\nu_2=1}^{\infty}\sum_{\nu_1=1}^{\infty}\mid\lambda^{\theta}_{2^{\nu_1},2^{\nu_2},2^{\nu_3}}-\lambda^{\theta}_{2^{\nu_1},2^{\nu_2},2^{\nu_3-1}}-\lambda^{\theta}_{2^{\nu_1-1},2^{\nu_2},2^{\nu_3}}-\lambda^{\theta}_{2^{\nu_1},2^{\nu_2-1},2^{\nu_3}}+$$

$$+\lambda^{\theta}_{2^{\nu_1},2^{\nu_2-1},2^{\nu_3-1}}+\lambda^{\theta}_{2^{\nu_1-1},2^{\nu_2},2^{\nu_3-1}}+\lambda^{\theta}_{2^{\nu_1-1},2^{\nu_2-1},2^{\nu_3}}-\lambda^{\theta}_{2^{\nu_1-1},2^{\nu_2-1},2^{\nu_3-1}}\mid Y^{\theta}_{2^{\nu_1}-1,2^{\nu_2}-1,2^{\nu_3}-1}(f)_p\Bigg)^{\frac{1}{\theta}}.$$

Collecting estimates of H_1-H_8 we get $I_1<\infty$. Hence, by Lemma 2.5 (b), there exists a function $g(x_1,x_2,x_3)\in L^0_p$, with the Fourier series

$$\sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty}\sum_{n_3=1}^{\infty}\lambda_{2^{n_1-1},2^{n_2-1},2^{n_3-1}}\Delta_{n_1,n_2,n_3}\tag{6}$$

and

$$\|g\|_p\leq C(p)I_1.\tag{7}$$

We rewrite series (6) in the form of $\sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty}\sum_{n_3=1}^{\infty}\gamma_{n_1,n_2,n_3}A_{n_1,n_2,n_3}(x_1,x_2,x_3)$, where

$$\gamma_{1,1,1}=\lambda_{1,1,1},\gamma_{1,\nu_2,\nu_3}=\lambda_{1,2^{n_2-1},2^{n_3-1}}\ \text{ for }\ 2^{n_2-1}\leq\nu_2\leq2^{n_2}-1,\,2^{n_3-1}\leq\nu_3\leq2^{n_3}-1\quad(n_2=2,3...),$$

$$\gamma_{\nu_1,1,1}=\lambda_{2^{n_1-1},1,1}\ \text{ for }\ 2^{n_1-1}\leq\nu_1\leq2^{n_1}-1\quad(n_2=2,3...),$$

$$\gamma_{\nu_1,1,\nu_3}=\lambda_{2^{n_1-1},1,2^{n_3-1}}\ \text{ for }\ 2^{n_1-1}\leq\nu_1\leq2^{n_1}-1,\ 2^{n_3-1}\leq\nu_3\leq2^{n_3}-1,\quad(n_1,n_3=2,3...),$$

$$\gamma_{\nu_1,\nu_2,1}=\lambda_{2^{n_1-1},2^{n_2-1},1}\ \text{ for }\ 2^{n_1-1}\leq\nu_1\leq2^{n_1}-1,2^{n_2-1}\leq\nu_2\leq2^{n_2}-1,\quad(n_1,n_2=2,3...),$$

$$\gamma_{1,1,\nu_3}=\lambda_{1,1,2^{n_3-1}}\ \text{ for }\ 2^{n_3-1}\leq\nu_1\leq2^{n_3}-1,\quad(n_3=2,3...),$$

$$\gamma_{1,\nu_2,1}=\lambda_{1,2^{n_2-1},1}\ \text{ for }\ 2^{n_2-1}\leq\nu_2\leq2^{n_2}-1,\quad(n_2=2,3...),$$

$$\gamma_{\nu_1,\nu_2,\nu_3}=\lambda_{2^{n_1-1},2^{n_2-1},2^{n_3-1}}\ \text{ for }\ 2^{n_1-1}\leq\nu_1\leq2^{n_1}-1,$$

$$2^{n_2-1}\leq\nu_2\leq2^{n_2}-1,\,2^{n_3-1}\leq\nu_3\leq2^{n_3}-1\quad(n_1,n_2,n_3=2,3...).$$

Now we consider the following series

$$\sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty}\sum_{n_3=1}^{\infty}\lambda_{n_1,n_2,n_3}A_{n_1,n_2,n_3}(x_1,x_2,x_3)=\sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty}\sum_{n_3=1}^{\infty}\gamma_{n_1,n_2,n_3}\Lambda_{n_1,n_2,n_3}A_{n_1,n_2,n_3}(x_1,x_2,x_3)\qquad(8),$$

where

$$\Lambda_{1,1,1}=1, \Lambda_{1,\nu_2,\nu_3}=\frac{\lambda_{1,\nu_2,\nu_3}}{\gamma_{1,\nu_2,\nu_3}}=\frac{\lambda_{1,\nu_2,\nu_3}}{\lambda_{1,2^{n_2-1},2^{n_3-1}}} \text{for } 2^{n_2-1}\le\nu_2\le 2^{n_2}-1,$$

$$2^{n_3-1}\le\nu_3\le 2^{n_3}-1 \quad (n_2,n_3=2,3...),$$

$$\Lambda_{\nu_1,1,1}=\frac{\lambda_{\nu_1,1,1}}{\gamma_{\nu_1,1,1}}=\frac{\lambda_{\nu_1,1,1}}{\lambda_{2^{n_1-1},1,1}} \text{ for } 2^{n_1-1}\le\nu_1\le 2^{n_1}-1, \quad (n_2=2,3...),$$

$$\Lambda_{\nu_1,\nu_2,1}=\frac{\lambda_{\nu_1,\nu_2,1}}{\gamma_{\nu_1,\nu_2,1}}=\frac{\lambda_{\nu_1,\nu_2,1}}{\lambda_{2^{n_1-1},2^{n_2-1},1}} \text{ for } 2^{n_1-1}\le\nu_1\le 2^{n_1}-1,\ 2^{n_2-1}\le\nu_2\le 2^{n_2}-1, \quad (n_1,n_2=2,3...),$$

$$\Lambda_{\nu_1,1,\nu_3}=\frac{\lambda_{\nu_1,1,\nu_3}}{\gamma_{\nu_1,1,\nu_3}}=\frac{\lambda_{\nu_1,1,\nu_3}}{\lambda_{2^{n_1-1},1,2^{n_3-1}}} \text{ for } 2^{n_1-1}\le\nu_1\le 2^{n_1}-1,\ 2^{n_3-1}\le\nu_3\le 2^{n_3}-1, \quad (n_1,n_2=2,3...),$$

$$\Lambda_{1,1,\nu_3}=\frac{\lambda_{1,1,\nu_3}}{\gamma_{1,1,\nu_3}}=\frac{\lambda_{1,1,\nu_3}}{\lambda_{1,1,2^{n_3-1}}} \text{ for } 2^{n_3-1}\le\nu_3\le 2^{n_3}-1,$$

$$\Lambda_{1,\nu_2,1}=\frac{\lambda_{1,\nu_2,1}}{\gamma_{1,\nu_2,1}}=\frac{\lambda_{1,\nu_2,1}}{\lambda_{1,2^{n_2-1},1}} \text{ for } 2^{n_2-1}\le\nu_2\le 2^{n_2}-1,$$

$$\Lambda_{\nu_1,\nu_2,\nu_3}=\frac{\lambda_{\nu_1,\nu_2,\nu_3}}{\gamma_{\nu_1,\nu_2,\nu_3}}=\frac{\lambda_{\nu_1,\nu_2,\nu_3}}{\lambda_{2^{n_1-1},2^{n_2-1},2^{n_3-1}}} \text{ for } 2^{n_1-1}\le\nu_1\le 2^{n_1}-1,\ 2^{n_2-1}\le\nu_2\le 2^{n_2}-1,$$

$$2^{n_3-1}\le\nu_3\le 2^{n_3}-1 \quad (n_2,n_3=2,3...).$$

As shown in [6], the sequence $\{\Lambda_{n_1=1,n_2=1,n_3=1}\}_{n_1=1,n_2=1,n_3=1}^{\infty,\infty,\infty}$ satisfies the conditions of the Marcinkiewicz multiplier theorem [12], then the series (8) is the Fourier series of a function $\varphi(x_1,x_2,x_2)\in L_p$ and $\|\varphi\|_p\le C(\rho,\lambda)\|g\|_p$.

Taking into account (7) and the estimates of H_1-H_8 we get (3).

Let us estimate $Y_{2^{m_1}-1,2^{m_2}-1,2^{m_3}-1}(\varphi)_p$. Using Lemma 2.4, we get

$$Y_{2^{m_1}-1,2^{m_2}-1,2^{m_3}-1}(\varphi)_p\le C\|\varphi-S_{2^{m_1}-1,\infty,\infty}(\varphi)-S_{\infty,2^{m_2}-1,\infty}(\varphi)-$$

$$-S_{\infty,\infty,2^{m_3}-1}(\varphi)+2S_{2^{m_1}-1,2^{m_2}-1,2^{m_3}-1}(\varphi)\|_p.$$

We consider the series (see (8))

$$\sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty}\sum_{n_3=1}^{\infty}\lambda_{n_1,n_2,n_3}A^*_{n_1,n_2,n_3}(x_1,x_2,x_3)=\sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty}\sum_{n_3=1}^{\infty}\gamma_{n_1,n_2,n_3}\Lambda_{n_1,n_2,n_3}A^*_{n_1,n_2,n_3}(x_1,x_2,x_3),$$

where $A^*_{n_1,n_2,n_3}(x_1,x_2,x_3)=0$, if $n_1\le 2^{m_1}-1$ and $n_2\le 2^{m_2}-1$, $n_3\le 2^{m_3}-1$ also $A^*_{n_1,n_2,n_3}(x_1,x_2,x_3)= = A_{n_1,n_2,n_3}(x_1,x_2,x_3)$ otherwise. Since the sequence $\{\Lambda_{n_1=1,n_2=1,n_3=1}\}$ satisfies the conditions of the Marcinkiewicz multiplier theorem, then

$$\Big\|\sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty}\sum_{n_3=1}^{\infty}\lambda_{n_1,n_2,n_3}A^*_{n_1,n_2,n_3}(x_1,x_2,x_3)\Big\|_p\le C\Big\|\sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty}\sum_{n_3=1}^{\infty}\lambda_{2^{n_1-1},2^{n_2-1},2^{n_3-1}}\Delta^*_{n_1,n_2,n_3}\Big\|_p,$$

where $\Delta^*_{n_1,n_2,n_3}=0$, if $n_1\le m_1$ and $n_2\le m_2$, $n_3\le m_3$ $\Delta^*_{n_1,n_2,n_3}=\Delta_{n_1n_2,n_3}$ otherwise.

By Lemma 2.5, we have

$$Y_{2^{m_1}-1,2^{m_2}-1,2^{m_3}-1}(\varphi)_p\lesssim \tag{9}$$

$$\lesssim\Big\{\int_0^{2\pi}\int_0^{2\pi}\int_0^{2\pi}\Big[\sum_{k_1=m_1+1}^{\infty}\sum_{k_2=m_2+1}^{\infty}\sum_{k_3=m_3+1}^{\infty}\lambda^2_{2^{k_1-1},2^{k_2-1},2^{k_3-1}}\Delta^2_{k_1,k_2,k_3}\Big]^{\frac{p}{2}}dx_1,dx_2,dx_3\Big\}^{\frac{1}{p}}.$$

For the sequence $\lambda^{\theta}_{2^{k_1-1},2^{k_2-1},2^{k_3-1}}$, we use inequality (5) where the index of the first element starts with $2^{m_1-1},2^{m_2-1},2^{m_3-1}$, and we take the sum from $2^{m_1-1},2^{m_2-1},2^{m_3-1}$ to $2^{k_1-1},2^{k_2-1},2^{k_3-1}$ respectively. The resulting inequality is substituted into inequality (9).

$$Y_{2^{m_1}-1,2^{m_2}-1,2^{m_3}-1}(\varphi)_p\lesssim$$

$$\lesssim \Bigg(\lambda^{\theta}_{2^{m_1-1},2^{m_2-1},2^{m_3-1}}\{\int_0^{2\pi}\int_0^{2\pi}\int_0^{2\pi}[\sum_{k_1=m_1+1}^{\infty}\sum_{k_2=m_2+1}^{\infty}\sum_{k_3=m_3+1}^{\infty}\Delta^2_{k_1,k_2,k_3}]^{\frac{p}{2}}dx_1,dx_2,dx_3\}^{\frac{1}{p}}+$$

$$+\{\int_0^{2\pi}\int_0^{2\pi}\int_0^{2\pi}[\sum_{k_1=m_1+1}^{\infty}\sum_{k_2=m_2+1}^{\infty}\sum_{k_3=m_3+1}^{\infty}\Delta^2_{k_1,k_2,k_3}\times$$

$$\times\Bigg(\sum_{\nu_1=m_1+1}^{k_1}|\,\lambda^{\theta}_{2^{\nu_1-1},2^{m_2-1},2^{m_3-1}}-\lambda^{\theta}_{2^{\nu_1-2},2^{m_2-1},2^{m_3-1}}\,|\Bigg)^{\frac{2}{\theta}}\Bigg]^{\frac{p}{2}}dx_1,dx_2,dx_3\Bigg\}^{\frac{1}{p}}+$$

$$+\Bigg\{\int_0^{2\pi}\int_0^{2\pi}\int_0^{2\pi}\Bigg[\sum_{k_1=m_1+1}^{\infty}\sum_{k_2=m_2+1}^{\infty}\sum_{k_3=m_3+1}^{\infty}\Delta^2_{k_1,k_2,k_3}\times$$

$$\times(\sum_{\nu_2=m_2+1}^{k_2}|\,\lambda^{\theta}_{2^{m_1-1},2^{\nu_2-1},2^{m_3-1}}-\lambda^{\theta}_{2^{m_1-1},2^{\nu_2-2},2^{m_3-1}}\,|\Bigg)^{\frac{2}{\theta}}\Bigg]^{\frac{p}{2}}dx_1,dx_2,dx_3\Bigg\}^{\frac{1}{p}}+$$

$$+\Bigg\{\int_0^{2\pi}\int_0^{2\pi}\int_0^{2\pi}\Bigg[\sum_{k_1=m_1+1}^{\infty}\sum_{k_2=m_2+1}^{\infty}\sum_{k_3=m_3+1}^{\infty}\Delta^2_{k_1,k_2,k_3}\times$$

$$\times\Bigg(\sum_{\nu_3=m_3+1}^{k_3}|\,\lambda^{\theta}_{2^{m_1-1},2^{m_2-1},2^{\nu_3-1}}-\lambda^{\theta}_{2^{m_1-1},2^{m_2-1},2^{\nu_2-2}}\,|\Bigg)^{\frac{2}{\theta}}\Bigg]^{\frac{p}{2}}dx_1,dx_2,dx_3\Bigg\}^{\frac{1}{p}}+$$

$$+\Bigg\{\int_0^{2\pi}\int_0^{2\pi}\int_0^{2\pi}\Bigg[\sum_{k_1=m_1+1}^{\infty}\sum_{k_2=m_2+1}^{\infty}\sum_{k_3=m_3+1}^{\infty}\Delta^2_{k_1,k_2,k_3}\times$$

$$\times\Bigg(\sum_{\nu_1=m_1+1}^{k_1}\sum_{\nu_2=m_2+1}^{k_2}|\,\lambda^{\theta}_{2^{\nu_1-1},2^{\nu_2-1},2^{m_3-1}}-\lambda^{\theta}_{2^{\nu_1-2},2^{\nu_2-1},2^{m_3-1}}-$$

$$-\lambda^{\theta}_{2^{\nu_1-1},2^{\nu_2-2},2^{m_3-1}}+\lambda^{\theta}_{2^{\nu_1-2},2^{\nu_2-2},2^{m_3-1}}\,|\Bigg)^{\frac{2}{\theta}}\Bigg]^{\frac{p}{2}}dx_1,dx_2,dx_3\Bigg\}^{\frac{1}{p}}+$$

$$+\Bigg\{\int_0^{2\pi}\int_0^{2\pi}\int_0^{2\pi}\Bigg[\sum_{k_1=m_1+1}^{\infty}\sum_{k_2=m_2+1}^{\infty}\sum_{k_3=m_3+1}^{\infty}\Delta^2_{k_1,k_2,k_3}\times$$

$$\times\Bigg(\sum_{\nu_2=m_2+1}^{k_2}\sum_{\nu_3=m_3+1}^{k_3}|\,\lambda^{\theta}_{2^{m_1-1},2^{\nu_2-1},2^{\nu_3-1}}-\lambda^{\theta}_{2^{m_1-1},2^{\nu_2-2},2^{\nu_3-1}}-$$

$$-\lambda^{\theta}_{2^{m_1-1},2^{\nu_2-1},2^{\nu_3-2}}+\lambda^{\theta}_{2^{m_1-1},2^{\nu_2-2},2^{\nu_3-2}}\,|\Bigg)^{\frac{2}{\theta}}\Bigg]^{\frac{p}{2}}dx_1,dx_2,dx_3\Bigg\}^{\frac{1}{p}}+$$

$$+\Bigg\{\int_0^{2\pi}\int_0^{2\pi}\int_0^{2\pi}\Bigg[\sum_{k_1=m_1+1}^{\infty}\sum_{k_2=m_2+1}^{\infty}\sum_{k_3=m_3+1}^{\infty}\Delta^2_{k_1,k_2,k_3}\times$$

$$\times\Bigg(\sum_{\nu_1=m_1+1}^{k_1}\sum_{\nu_3=m_3+1}^{k_3}|\,\lambda^{\theta}_{2^{\nu_1-1},2^{m_2-1},2^{\nu_3-1}}-\lambda^{\theta}_{2^{\nu_1-2},2^{m_2-1},2^{\nu_3-1}}-$$

$$-\lambda^{\theta}_{2^{\nu_1-1},2^{m_2-1},2^{\nu_3-2}}+\lambda^{\theta}_{2^{\nu_1-2},2^{m_2-1},2^{\nu_3-2}}\,|\Bigg)^{\frac{2}{\theta}}\Bigg]^{\frac{p}{2}}dx_1,dx_2,dx_3\Bigg\}^{\frac{1}{p}}+$$

$$+\Bigg\{\int_0^{2\pi}\int_0^{2\pi}\int_0^{2\pi}\Bigg[\sum_{k_1=m_1+1}^{\infty}\sum_{k_2=m_2+1}^{\infty}\sum_{k_3=m_3+1}^{\infty}\Delta^2_{k_1,k_2,k_3}\times$$

$$\times\Bigg(\sum_{\nu_1=m_1+1}^{k_1}\sum_{\nu_2=m_2+1}^{k_2}\sum_{\nu_3=m_3+1}^{k_3}|\,\lambda^{\theta}_{2^{\nu_1-1},2^{\nu_2-1},2^{\nu_3-1}}-\lambda^{\theta}_{2^{\nu_1-2},2^{\nu_2-1},2^{\nu_3-1}}-$$

$$-\lambda^{\theta}_{2^{\nu_1-1},2^{\nu_2-2},2^{\nu_3-1}}-\lambda^{\theta}_{2^{\nu_1-1},2^{\nu_2-1},2^{\nu_3-2}}+\lambda^{\theta}_{2^{\nu_1-1},2^{\nu_2-2},2^{\nu_3-2}}+\lambda^{\theta}_{2^{\nu_1-2},2^{\nu_2-1},2^{\nu_3-2}}+$$

$$+\lambda^{\theta}_{2^{\nu_1-2},2^{\nu_2-2},2^{\nu_3-1}}-\lambda^{\theta}_{2^{\nu_1-2},2^{\nu_2-2},2^{\nu_3-2}}\mid\Big)^{\frac{2}{\theta}}\Big]^{\frac{p}{2}}dx_1,dx_2,dx_3\Big\}^{\frac{1}{p}}\Big)=:L_1+L_2+L_3+L_4+L_5+L_6+L_7+L_8.$$

We estimate L_1 as H_1, to get

$$L_1\le\lambda^{\theta}_{2^{m_1-1},2^{m_2-1},2^{m_3-1}}\Big\{\int_0^{2\pi}\int_0^{2\pi}\int_0^{2\pi}\Big[\sum_{k_1=m_1+1}^{\infty}\sum_{k_2=m_2+1}^{\infty}\sum_{k_3=m_3+1}^{\infty}\Delta^2_{k_1,k_2,k_3}\Big]^{\frac{p}{2}}dx_1,dx_2,dx_3\Big\}^{\frac{1}{p}}\lesssim$$

$$\lesssim\lambda^{\theta}_{2^{m_1-1},2^{m_2-1},2^{m_3-1}}Y_{2^{m_1}-1,2^{m_2}-1\;,2^{m_3}-1}(f)_p.$$

We also have

$$L_2=\Big\{\int_0^{2\pi}\int_0^{2\pi}\int_0^{2\pi}\Big[\sum_{k_1=m_1+1}^{\infty}\sum_{k_2=m_2+1}^{\infty}\sum_{k_3=m_3+1}^{\infty}\Delta^2_{k_1,k_2,k_3}\times$$

$$\times\Big(\sum_{\nu_1=m_1+1}^{k_1}\mid\lambda^{\theta}_{2^{\nu_1-1},2^{m_2-1},2^{m_3-1}}-\lambda^{\theta}_{2^{\nu_1-2},2^{m_2-1},2^{m_3-1}}\mid\Big)^{\frac{2}{\theta}}\Big]^{\frac{p}{2}}dx_1,dx_2,dx_3\Big\}^{\frac{1}{p}}\lesssim$$

$$\lesssim\Big(\sum_{\nu_1=m_1+1}^{\infty}\mid\lambda^{\theta}_{2^{\nu_1-1},2^{m_2-1},2^{m_3-1}}-\lambda^{\theta}_{2^{\nu_1-2},2^{m_2-1},2^{m_3-1}}\mid\times$$

$$\times\Big\{\int_0^{2\pi}\int_0^{2\pi}\int_0^{2\pi}\Big[\sum_{k_1=\nu_1}^{\infty}\sum_{k_2=m_2+1}^{\infty}\sum_{k_3=m_3+1}^{\infty}\Delta^2_{k_1,k_2,k_3}\Big]^{\frac{p}{2}}dx_1,dx_2,dx_3\Big\}^{\frac{\theta}{p}}\Big)^{\frac{1}{\theta}}\lesssim$$

$$\lesssim\Big(\sum_{\nu_1=m_1+1}^{\infty}\mid\lambda^{\theta}_{2^{\nu_1-1},2^{m_2-1},2^{m_3-1}}-\lambda^{\theta}_{2^{\nu_1-2},2^{m_2-1},2^{m_3-1}}\mid\;Y^{\theta}_{2^{\nu_1-1}-1,2^{m_2}-1,2^{m_3}-1}(f)_p\Big)^{\frac{1}{\theta}}.$$

Similarly, we obtain the estimates for L_3, L_4

$$L_3\lesssim\Big(\sum_{\nu_2=m_2+1}^{\infty}\mid\lambda^{\theta}_{2^{m_1-1},2^{\nu_2-1},2^{m_3-1}}-\lambda^{\theta}_{2^{m_1-1},2^{\nu_2-2},2^{m_3-1}}\mid\;Y^{\theta}_{2^{m_1}-1,2^{\nu_2-1}-1,2^{m_3}-1}(f)_p\Big)^{\frac{1}{\theta}},$$

$$L_4\lesssim\Big(\sum_{\nu_3=m_3+1}^{\infty}\mid\lambda^{\theta}_{2^{m_1-1},2^{m_2-1},2^{\nu_3-1}}-\lambda^{\theta}_{2^{m_1-1},2^{m_2-1},2^{\nu_2-2}}\mid\;Y^{\theta}_{2^{m_1}-1,2^{m_2}-1,2^{\nu_3-1}-1}(f)_p\Big)^{\frac{1}{\theta}}.$$

We estimate L_5 as follows:

$$L_5=\Big\{\int_0^{2\pi}\int_0^{2\pi}\int_0^{2\pi}\Big[\sum_{k_1=m_1+1}^{\infty}\sum_{k_2=m_2+1}^{\infty}\sum_{k_3=m_3+1}^{\infty}\Delta^2_{k_1,k_2,k_3}\times\Big(\sum_{\nu_1=m_1+1}^{k_1}\sum_{\nu_2=m_2+1}^{k_2}\mid\lambda^{\theta}_{2^{\nu_1-1},2^{\nu_2-1},2^{m_3-1}}-$$

$$-\lambda^{\theta}_{2^{\nu_1-2},2^{\nu_2-1},2^{m_3-1}}-\lambda^{\theta}_{2^{\nu_1-1},2^{\nu_2-2},2^{m_3-1}}+\lambda^{\theta}_{2^{\nu_1-2},2^{\nu_2-2},2^{m_3-1}}\mid\Big)^{\frac{2}{\theta}}\Big]^{\frac{p}{2}}dx_1,dx_2,dx_3\Big\}^{\frac{1}{p}}\lesssim$$

$$\lesssim\Big(\sum_{\nu_1=m_1+1}^{\infty}\sum_{\nu_2=m_2+1}^{\infty}\mid\lambda^{\theta}_{2^{\nu_1-1},2^{\nu_2-1},2^{m_3-1}}-\lambda^{\theta}_{2^{\nu_1-2},2^{\nu_2-1},2^{m_3-1}}-\lambda^{\theta}_{2^{\nu_1-1},2^{\nu_2-2},2^{m_3-1}}+$$

$$+\lambda^{\theta}_{2^{\nu_1-2},2^{\nu_2-2},2^{m_3-1}}\mid\times\int_0^{2\pi}\int_0^{2\pi}\int_0^{2\pi}\Big[\sum_{k_1=\nu_1}^{\infty}\sum_{k_2=\nu_2}^{\infty}\sum_{k_3=m_3+1}^{\infty}\Delta^2_{k_1,k_2,k_3}\Big]^{\frac{p}{2}}dx_1,dx_2,dx_3\Big\}^{\frac{\theta}{p}}\Big)^{\frac{1}{\theta}}\lesssim$$

$$\lesssim\Big(\sum_{\nu_1=m_1+1}^{\infty}\sum_{\nu_2=m_2+1}^{\infty}\mid\lambda^{\theta}_{2^{\nu_1-1},2^{\nu_2-1},2^{m_3-1}}-\lambda^{\theta}_{2^{\nu_1-2},2^{\nu_2-1},2^{m_3-1}}-$$

$$-\lambda^{\theta}_{2^{\nu_1-1},2^{\nu_2-2},2^{m_3-1}}+\lambda^{\theta}_{2^{\nu_1-2},2^{\nu_2-2},2^{m_3-1}}\mid Y^{\theta}_{2^{\nu_1-1}-1,2^{\nu_2-1}-1,2^{m_3}-1,}(f)_p\Big)^{\frac{1}{\theta}}.$$

Similarly, we obtain the estimates for L_6, L_7 :

$$L_6 \lesssim \Bigg(\sum_{\nu_2=m_2+1}^{\infty} \sum_{\nu_3=m_3+1}^{\infty} |\, \lambda^{\theta}_{2^{m_1-1},2^{\nu_2-1},2^{\nu_3-1}} - \lambda^{\theta}_{2^{m_1-1},2^{\nu_2-2},2^{\nu_3-1}} -$$

$$-\lambda^{\theta}_{2^{m_1-1},2^{\nu_2-1},2^{\nu_3-2}} + \lambda^{\theta}_{2^{m_1-1},2^{\nu_2-2},2^{\nu_3-2}} \,|\, Y^{\theta}_{2^{m_1}-1,2^{\nu_2-1}-1,2^{\nu_3-1}-1}(f)_p \Bigg)^{\frac{1}{\theta}}.$$

$$L_7 \lesssim \Bigg(\sum_{\nu_1=m_1+1}^{\infty} \sum_{\nu_3=m_3+1}^{\infty} |\, \lambda^{\theta}_{2^{\nu_1-1},2^{m_2-1},2^{\nu_3-1}} - \lambda^{\theta}_{2^{\nu_1-2},2^{m_2-1},2^{\nu_3-1}} -$$

$$-\lambda^{\theta}_{2^{\nu_1-1},2^{m_2-1},2^{\nu_3-2}} + \lambda^{\theta}_{2^{\nu_1-2},2^{m_2-1},2^{\nu_3-2}} \,|\, Y^{\theta}_{2^{\nu_1-1}-1,2^{m_2}-1,2^{\nu_3-1}-1}(f)_p \Bigg)^{\frac{1}{\theta}}.$$

Finally, we estimate L_8 as follows:

$$L_8 \lesssim \Bigg(\sum_{\nu_1=m_1+1}^{\infty} \sum_{\nu_2=m_2+1}^{k_2} \sum_{\nu_3=m_3+1}^{\infty} |\, \lambda^{\theta}_{2^{\nu_1-1},2^{\nu_2-1},2^{\nu_3-1}} - \lambda^{\theta}_{2^{\nu_1-2},2^{\nu_2-1},2^{\nu_3-1}} -$$

$$-\lambda^{\theta}_{2^{\nu_1-1},2^{\nu_2-2},2^{\nu_3-1}} - \lambda^{\theta}_{2^{\nu_1-1},2^{\nu_2-1},2^{\nu_3-2}} + \lambda^{\theta}_{2^{\nu_1-1},2^{\nu_2-2},2^{\nu_3-2}} + \lambda^{\theta}_{2^{\nu_1-2},2^{\nu_2-1},2^{\nu_3-2}} +$$

$$+\lambda^{\theta}_{2^{\nu_1-2},2^{\nu_2-2},2^{\nu_3-1}} - \lambda^{\theta}_{2^{\nu_1-2},2^{\nu_2-2},2^{\nu_3-2}} \,|\, Y^{\theta}_{2^{\nu_1-1}-1,2^{\nu_2-1}-1,2^{\nu_3-1}-1}(f)_p \Bigg)^{\frac{1}{\theta}}.$$

Taking into account the estimates for $L_1 - L_8$, we obtain (4). The theorem is proved.

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Жалпыланған Лиувилл-Вейл туындыларының бұрыштық ең жақын жуықтауларының жоғарғы бағалаулары

Мақалада 2π периодты f үзіліссіз функциялар және оларды тригонометриялық көпмүшеліктермен жуықтауқа жалпыланған Лиувиль-Вейл туындылары арқылы берілген үш өлшемді функциялардың бұрыштық ең жақын жуықтауын бастапқы берілген функциялардың бұрыштық ең жақын жуықтауы арқылы бағалауы қарастырылған. Авторлар классикалық Вейл аралас туындыларының орнына жалпыланған Лиувиль-Вейл туындыларын зерттеген. Қарастырылатын мәселелерді таңдағанда осы мақаладағы екінші автордың жұмысынан кейін қалыптасқан жалпы тәсілді ұстанған. Басты мақсат жұмыстың нәтижелерін үш өлшемді жағдайда дәлелдеу. Жалпы монотонды тізбектер туралы түсінік осы зерттеуде басты рөл атқарады. Функцияның ең жақын жуықтауларына қатысты r-туындысының ең жақын жуықтаулары, норма үшін бірнеше белгілі теңсіздіктер көрсетілген. Мақалада қарастырылған мәселелер Бернштейннің зерттелген еңбектерінің мәселелеріне жатады. Кейінірек Стечкин және Конюшков $f^{(r)}$ ең жақын жуықтау үшін теңсіздік алынды. Сонымен қатар Потаповтың еңбектерінде бұрыштарды жақындату арқылы функциялардың кейбір кластары қарастырылған. Бірінші бөлімде қажетті түсініктермен және пайдалы леммалар берілген. Үш өлшемді жағдайда жалпыланған Лиувилль-Вейл туындысы арқылы берілген функциялардың нормасының және бұрыштық ең жақын жуықтауының бағалауы алынды.

Кілт сөздер: Лебег кеңістігі, үш өлшемді бұрышпен ең жақын жуықтау, тригонометриялық көпмүшелік, Лиувилл-Вейл туындысы.

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Верхние оценки угловых наилучших приближений обобщенных производных Лиувилля-Вейля

В статье рассмотрены непрерывные функции f с периодом 2π и их приближения тригонометрическими полиномами. Изучены оценки наилучших угловых приближений обобщены производных Лиувилля-Вейля угловым приближением функций в трехмерном случае. Авторами обобщенные производные Лиувилля-Вейля вместо классического смешанного производного Вейля. При выборе рассматриваемых вопросов они следовали общему подходу, сформировавшемуся после работы второго автора настоящей статьи. Главная цель — доказать аналоги результатов работы в трехмерном случае. Понятие общих монотонных последовательностей играет ключевую роль в исследовании. Указаны несколько известных неравенств для норм, наилучших приближений r-го производного по наилучшим приближениям функции f. Вопросы, рассмотренные в настоящей работе, относятся к кругу проблем, изученных в работах Бернштейном. Позднее Стечкин и Конюшков получили неравенство для наилучшего приближения $f^{(r)}$. Также в работах Потапова при помощи приближения углом изучены некоторые классы функций. В подразделе 1 авторами даны необходимые обозначения и полезные леммы. Получены оценки норм и наилучшие приближения обобщенного производного Лиувилля-Вейля в трехмерном случае.

Ключевые слова: пространство Лебега, наилучшее приближение трехмерным углом, тригонометрический полином, производная Лиувилля-Вейля.

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