

A.A. Ibrayeva, Sh.Sh. Ibraev*, G.K. Yeshmurat

*Korkyt Ata Kyzylorda University, Kyzylorda, Kazakhstan
(E-mail: anar_ibrayeva@mail.ru, ibrayevsh@mail.ru, gulnuresh@mail.ru)*

Cohomology of simple modules for $\mathfrak{sl}_3(k)$ in characteristic 3

In this paper we calculate cohomology of a classical Lie algebra of type A_2 over an algebraically field k of characteristic $p = 3$ with coefficients in simple modules. To describe their structure, we will consider them as modules over an algebraic group $SL_3(k)$. In the case of characteristic $p = 3$, there are only two peculiar simple modules: a simple that module isomorphic to the quotient module of the adjoint module by the center, and a one-dimensional trivial module. The results on the cohomology of simple nontrivial module are used for calculating the cohomology of the adjoint module. We also calculate cohomology of the simple quotient algebra Lie of A_2 by the center.

Keywords: Lie algebra, simple module, restricted module cohomology, exact sequence.

Introduction

The cohomology theory of modular Lie algebras is one of the interesting questions in the theory of Lie algebras. Many significant results are devoted to the study of the cohomology of classical modular Lie algebras. Their restricted cohomology with coefficients in the dual Weyl modules was studied in [1–3]. Central extensions are described in [4, 5]. In [6, 7] the outer derivations are calculated. As the second cohomology, local deformations are calculated in [8–10].

Among the classical modular Lie algebras, the cohomology of simple modules is completely described only for a three-dimensional Lie algebra of type A_1 [11]. It is known that for other classical modular Lie algebras a complete description of the cohomology of simple modules has not yet been obtained. In this paper we give a complete description of such cohomology for the Lie algebra of type A_2 over an algebraically closed field of characteristic $p = 3$. The first cohomology groups of simple modules for A_2 was computed in [12]. A similar result for the second cohomology groups was obtained in [13]. In all other cases, the computation of the cohomology structure of simple modules for A_2 is close to completion. The results will be published in the next works of the second author.

Let us introduce the basic definitions and notation. Let \mathfrak{g} be a Lie algebra over a field k characteristics of p and M be a \mathfrak{g} -module. We denote the n -th exterior power of the space \mathfrak{g} by $\Lambda^n(\mathfrak{g})$ and let

$$C^n(\mathfrak{g}, M) = \text{Hom}(\Lambda^n, M) = \langle \psi : \mathfrak{g} \times \cdots \times \mathfrak{g} \rightarrow M \rangle_k, \quad n > 0$$

is a space of multilinear skew-symmetric mappings in n arguments with coefficients in M . We put

$$C^n(\mathfrak{g}, M) = 0, \quad n < 0, \quad C^0(\mathfrak{g}, M) = M, \quad C^*(\mathfrak{g}, M) = \bigoplus_{n=-\infty}^{+\infty} C^n(\mathfrak{g}, M).$$

Define the coboundary operator

$$d : C^*(\mathfrak{g}, M) \longrightarrow C^*(\mathfrak{g}, M)$$

as follows:

$$d\psi(l_1, l_2, \dots, l_{n+1}) = \sum_{i < j} (-1)^{i+j} \psi([l_i, l_j], \dots, \hat{l}_i, \dots, \hat{l}_j, \dots, l_{n+1}) + \sum_i (-1)^{i+1} [l_i, \psi(l_1, \dots, \hat{l}_i, \dots, l_{n+1})],$$

where $\psi \in C^n(\mathfrak{g}, M)$. Then $d^2 = 0$, therefore $B^*(\mathfrak{g}, M) \subseteq Z^*(\mathfrak{g}, M)$, where

$$Z^*(\mathfrak{g}, M) = \langle \psi \in C^*(\mathfrak{g}, M) : d\psi = 0 \rangle_k,$$

*Corresponding author.

E-mail: ibrayevsh@mail.ru

$$B^*(\mathfrak{g}, M) = \langle d\psi : \psi \in C^*(\mathfrak{g}, M) \rangle_k.$$

So, we can introduce the factor-space

$$H^*(\mathfrak{g}, M) = Z^*(\mathfrak{g}, M)/B^*(\mathfrak{g}, M).$$

The spaces $C^*(\mathfrak{g}, M)$, $Z^*(\mathfrak{g}, M)$, $B^*(\mathfrak{g}, M)$, $H^*(\mathfrak{g}, M)$ are called *space of cochains*, *space of cocycles*, *spaces of coboundaries*, and *space of cohomologies* of the Lie algebra \mathfrak{g} with coefficients in the \mathfrak{g} -module M respectively.

Similarly, the spaces

$$C^n(\mathfrak{g}, M), Z^n(\mathfrak{g}, M) = Z^*(\mathfrak{g}, M) \cap C^n(\mathfrak{g}, M),$$

$$B^n(\mathfrak{g}, M) = B^*(\mathfrak{g}, M) \cap C^n(\mathfrak{g}, M) \text{ и } H^n(\mathfrak{g}, M) = H^*(\mathfrak{g}, M) \cap C^n(\mathfrak{g}, M)$$

are called *space of n -cochains*, *space of n -cocycles*, *spaces of n -coboundaries*, and *space of n -cohomologies* of the Lie algebra \mathfrak{g} with coefficients in the \mathfrak{g} -module M respectively.

We say that the \mathfrak{g} -module M is *peculiar* if $H^*(\mathfrak{g}, M) \neq 0$. We say that the M is *n -peculiar* module over \mathfrak{g} if $H^n(\mathfrak{g}, M) \neq 0$.

Now let \mathfrak{g} be a classical Lie algebra of type A_2 over algebraically closed field k of positive characteristic $p > 0$ and M is a \mathfrak{g} -module. We decompose $C^*(\mathfrak{g}, M)$ into a direct sum of weight subspaces with respect to the maximal torus T of the group $G = SL_3(k)$:

$$C^*(\mathfrak{g}, M) = \bigoplus_{\mu \in X(T)} C^*_\mu(\mathfrak{g}, M),$$

where $X(T)$ is the additive character group of T . Then

$$H^n(\mathfrak{g}, M) = \bigoplus_{\mu \in X(T)} H^n_\mu(\mathfrak{g}, M).$$

Identify the space $C^n(\mathfrak{g}, M)$ with the space $\bigwedge^n \mathfrak{g}^* \otimes M$ and denote by $\prod(V)$ the set of weights of the G -module subspace V of $H^*(\mathfrak{g}, M)$.

Since $\prod(H^n(\mathfrak{g}, M)) \subseteq pX(T) \cap \prod(\bigwedge^n \mathfrak{g}^* \otimes M)$, then we can consider only the elements of the subspace $\overline{C}^n(\mathfrak{g}, M)$ of $C^n(\mathfrak{g}, M)$ with weights contained in the set $pX(T) \cap \prod(\bigwedge^n \mathfrak{g}^* \otimes M)$. The corresponding subspaces of cocycles and cohomologies are denoted by $\overline{Z}^n(\mathfrak{g}, M)$ and $\overline{H}^n(\mathfrak{g}, M)$. Note that

$$H^n(\mathfrak{g}, M) = \overline{H}^n(\mathfrak{g}, M).$$

We will use the following well known formulas:

$$\dim H^n(\mathfrak{g}, M) = \dim \overline{Z}^n(\mathfrak{g}, M) + \dim \overline{Z}^{n-1}(\mathfrak{g}, M) - \dim \overline{C}^{n-1}(\mathfrak{g}, M), \tag{1}$$

$$\dim H^n(\mathfrak{g}, M) = \dim H^{\dim \mathfrak{g} - n}(\mathfrak{g}, M^*). \tag{2}$$

The weight subspaces are invariant under the action of the coboundary operator, therefore the formula (1) is also holds for weight subspaces:

$$\dim H^n_\mu(\mathfrak{g}, M) = \dim \overline{Z}^n_\mu(\mathfrak{g}, M) + \dim \overline{Z}^{n-1}_\mu(\mathfrak{g}, M) - \dim \overline{C}^{n-1}_\mu(\mathfrak{g}, M). \tag{3}$$

Let $L(r, s)$ be a simple \mathfrak{g} -module with the highest weight $r\omega_1 + s\omega_2$, where ω_1, ω_2 are fundamental weights.

It is known that the composition of a representation of $SL_3(k)$ on a vector space L with a d -th power of the Frobenius map defines a new representation, on which the Lie algebra \mathfrak{g} acts trivially. We denote the resulting module by $L^{(d)}$. To each weight μ of the space L there corresponds a weight $p^d\mu$ of the space $L^{(d)}$. The cohomology group $H^n(\mathfrak{g}, M)$, as a $SL_3(k)$ -module, consists of either a twisted module $L^{(d)}$ for some d , or a one-dimensional trivial module k . For the multiplicity of a $SL_3(k)$ -module $L^{(d)}$ in $H^n(\mathfrak{g}, M)$, we use the notation $[H^n(\mathfrak{g}, M) : L^{(d)}]$. Further, for convenience we use the following abbreviations: $H^n(\mathfrak{g}, k) := H^n(\mathfrak{g})$, $\bigoplus_{i=1}^m V := mV$, where V is a $SL_3(k)$ -module.

Let's formulate the main result of this paper:

Theorem 1. Let \mathfrak{g} be a classical Lie algebra of type A_2 over an algebraically closed field k of characteristic $p = 3$ and M be a simple \mathfrak{g} -module. Then there are the following isomorphisms of $SL_3(k)$ -modules:

$$(a) H^0(\mathfrak{g}) \cong H^8(\mathfrak{g}) \cong k, H^2(\mathfrak{g}) \cong H^6(\mathfrak{g}) \cong L(1, 0)^{(1)} \oplus L(0, 1)^{(1)}, H^3(\mathfrak{g}) \cong H^5(\mathfrak{g}) \cong L(1, 0)^{(1)} \oplus L(0, 1)^{(1)} \oplus k;$$

$$(b) H^1(\mathfrak{g}, L(1, 1)) \cong H^7(\mathfrak{g}, L(1, 1)) \cong L(1, 0)^{(1)} \oplus L(0, 1)^{(1)} \oplus k, H^3(\mathfrak{g}, L(1, 1)) \cong H^5(\mathfrak{g}, L(1, 1)) \cong H^0(1, 1)^{(1)}, H^4(\mathfrak{g}, L(1, 1)) \cong 2H^0(1, 1)^{(1)}.$$

In other cases $H^n(\mathfrak{g}, M) = 0$.

Proof of the Theorem 1

As the basis vectors for \mathfrak{g} we choose the special derivations of the algebra of divided powers $O_3(1)$:

$$h_1 = x_1\partial_1 - x_2\partial_2, h_2 = x_2\partial_2 - x_3\partial_3, e_1 = x_1\partial_2, e_2 = x_2\partial_3, e_3 = x_1\partial_3, f_1 = x_2\partial_1, f_2 = x_3\partial_2, f_3 = x_3\partial_1.$$

Over a field of characteristic $p = 3$, the Lie algebra \mathfrak{g} is not simple, it has a one-dimensional center $\langle h_1 - h_2 \rangle_k$. The quotient algebra by the center is a simple Lie algebra; we denote it by $\bar{\mathfrak{g}}$ or $\overline{A_2}$.

It is known that the peculiar modules of the Lie algebra \mathfrak{g} are restricted [11]. According to Lemma 3.1 in [13], only the following two simple restricted modules are peculiar: $L(0, 0) \cong k$ and $L(1, 1) \cong \bar{\mathfrak{g}}$. For $L(1, 1)$ we get the following description:

$$L(1, 1) \cong \langle h_1, h_2, e_1, e_2, e_3, f_1, f_2, f_3 : h_1 - h_2 = 0 \rangle_k.$$

Consider each of these modules separately.

Let $M = L(0, 0) \cong k$.

Lemma 1. There are the following isomorphisms of $SL_3(k)$ -modules:

- (a) $H^0(\mathfrak{g}) \cong k$;
- (b) $H^2(\mathfrak{g}) \cong L(1, 0)^{(1)} \oplus L(0, 1)^{(1)}$;
- (c) $H^3(\mathfrak{g}) \cong L(1, 0)^{(1)} \oplus L(0, 1)^{(1)} \oplus k$;
- (d) $H^5(\mathfrak{g}) \cong L(1, 0)^{(1)} \oplus L(0, 1)^{(1)} \oplus k$;
- (e) $H^6(\mathfrak{g}) \cong L(1, 0)^{(1)} \oplus L(0, 1)^{(1)}$;
- (f) $H^8(\mathfrak{g}) \cong k$.

In other cases $H^n(\mathfrak{g}) = 0$.

Proof. The statements (a) and (f) are obvious. The triviality of $H^1(\mathfrak{g})$ in characteristic $p = 3$ was proved in [12].

(b) The set $\prod(\overline{C}^2(\mathfrak{g}))$ only consists of the following weights: $0 \pm 3\omega_1, \pm 3(\omega_1 - \omega_2), \pm 3\omega_2$. Therefore, only the trivial one-dimensional module and the twisted simple modules $L(1, 0)^{(1)}, L(0, 1)^{(1)}$, can be as nonzero composition factors of $H^2(\mathfrak{g})$. They are generated by the classes of cocycles with dominant weights $0, 3\omega_1$, and $3\omega_2$ respectively.

The subspace $\overline{C}_0^2(\mathfrak{g})$ is 4-dimensional and spans by the cochains $h_1^* \wedge h_2^*, e_1^* \wedge f_1^*, e_2^* \wedge f_2^*, e_3^* \wedge f_3^*$. If $a_1 h_1^* \wedge h_2^* + a_2 e_1^* \wedge f_1^* + a_3 e_2^* \wedge f_2^* + a_4 e_3^* \wedge f_3^* \in \overline{Z}^2(\mathfrak{g})$ then, by cocycle condition, $a_1 = 0, a_4 = a_2 + a_3$. Therefore $\dim \overline{Z}_0^2(\mathfrak{g}) = 2$. Since $\dim \overline{C}_0^1(\mathfrak{g}) = 2$ and $\dim \overline{Z}_0^1(\mathfrak{g}) = 0$, by (2), $\dim \overline{H}_0^2(\mathfrak{g}) = 2 + 0 - 2 = 0$.

The subspace $\overline{C}_{3\omega_1}^2(\mathfrak{g})$ is one-dimensional and spans by the cochain $f_1^* \wedge f_3^*$. Notice that $a f_1^* \wedge f_3^* \in \overline{Z}^2(\mathfrak{g})$ for all $a \in k$. Therefore $\dim \overline{Z}_{3\omega_1}^2(\mathfrak{g}) = 1$. Since $\dim \overline{C}_{3\omega_1}^1(\mathfrak{g}) = 0$, by (3), $\dim \overline{H}_{3\omega_1}^2(\mathfrak{g}) = 1$. So $[H^2(\mathfrak{g}) : L(1, 0)^{(1)}] = 1$.

Arguing as in the previous case, we obtain $[H^2(\mathfrak{g}) : L(0, 1)^{(1)}] = 1$. Thus $H^2(\mathfrak{g}) \cong L(1, 0)^{(1)} \oplus L(0, 1)^{(1)}$.

(c) The sets of weights $\prod(\overline{C}^3(\mathfrak{g}))$ and $\prod(\overline{C}^2(\mathfrak{g}))$ coincide. Therefore, we consider only the weight subspaces of 3-cochains corresponding to the dominant weights $0, 3\omega_1$, and $3\omega_2$.

The subspace $\overline{C}_0^3(\mathfrak{g})$ is 8-dimensional and spans by the cochains

$$\begin{aligned} &h_1^* \wedge e_1^* \wedge f_1^*, h_2^* \wedge e_1^* \wedge f_1^*, h_1^* \wedge e_2^* \wedge f_2^*, h_2^* \wedge e_2^* \wedge f_2^*, \\ &h_1^* \wedge e_3^* \wedge f_3^*, h_2^* \wedge e_3^* \wedge f_3^*, e_3^* \wedge f_1^* \wedge f_2^*, e_1^* \wedge e_2^* \wedge f_3^*. \end{aligned}$$

Suppose that a linear combination of these vectors with coefficients $b_i, i = 1, \dots, 8$ respectively, is a 3-cocycle. Then the cocycle condition implies that

$$b_1 + b_2 + b_5 + b_7 - b_8 = 0, b_2 + b_3 - b_7 + b_8 = 0, b_3 + b_4 + b_6 + b_7 - b_8 = 0, 2b_4 + 2b_7 - 2b_8 = 0, 2b_5 + 2b_6 + 2b_7 - 2b_8 = 0.$$

Whence it follows that $\dim \overline{Z}_0^3(\mathfrak{g}) = 3$. By (3),

$$\dim H_0^3(\mathfrak{g}) = \dim \overline{Z}_0^3(\mathfrak{g}) + \dim \overline{Z}_0^2(\mathfrak{g}) - \dim \overline{C}_0^2(\mathfrak{g}) = 3 + 2 - 4 = 1.$$

Therefore $[H_0^3(\mathfrak{g}) : k] = 1$.

The weight subspaces $\overline{C}_{3\lambda_1}^3(\mathfrak{g}), \overline{C}_{3\lambda_2}^3(\mathfrak{g})$ are two-dimensional and span respectively with 3-cochains: $h_1^* \wedge f_1^* \wedge f_3^*, h_2^* \wedge f_1^* \wedge f_3^*$, and $h_1^* \wedge f_2^* \wedge f_3^*, h_2^* \wedge f_2^* \wedge f_3^*$. Using the cocycle condition, we get $\dim \overline{Z}_{3\lambda_1}^3(\mathfrak{g}) = \dim \overline{Z}_{3\lambda_2}^3(\mathfrak{g}) = 1$. So, $H^3(\mathfrak{g}) \cong L(1, 0)^{(1)} \oplus L(0, 1)^{(1)} \oplus k$.

Now we prove that $H^4(\mathfrak{g}) = 0$. It's obvious that $\prod(\overline{C}^4(\mathfrak{g})) = \prod(\overline{C}^3(\mathfrak{g}))$. Therefore we consider only the weight subspaces

$$\overline{C}_0^4(\mathfrak{g}), \overline{C}_{3\omega_1}^4(\mathfrak{g}), \overline{C}_{3\omega_2}^4(\mathfrak{g}).$$

The subspace $\overline{C}_0^4(\mathfrak{g})$ is 10-dimensional and spans by the cochains

$$\begin{aligned} & h_1^* \wedge h_2^* \wedge e_1^* \wedge f_1^*, h_1^* \wedge h_2^* \wedge e_2^* \wedge f_2^*, h_1^* \wedge h_2^* \wedge e_3^* \wedge f_3^*, h_1^* \wedge e_1^* \wedge e_2^* \wedge f_3^*, \\ & h_2^* \wedge e_1^* \wedge e_2^* \wedge f_3^*, h_1^* \wedge e_3^* \wedge f_1^* \wedge f_2^*, h_2^* \wedge e_3^* \wedge f_1^* \wedge f_2^*, \\ & e_1^* \wedge e_2^* \wedge f_1^* \wedge f_2^*, e_1^* \wedge e_3^* \wedge f_1^* \wedge f_3^*, e_2^* \wedge e_3^* \wedge f_2^* \wedge f_3^*. \end{aligned}$$

Suppose that the linear combination of these vectors coefficients b_i , $i = 1, \dots, 10$ respectively is a 4-cocycle. Then $b_1 = b_2 = b_3 = 0$, $b_4 = b_6$, $b_5 = b_7$. Whence it follows that $\dim \overline{z}_0^4(\mathfrak{g}) = 5$. By (3), $\dim H_0^4(\mathfrak{g}) = 5 + 3 - 8 = 0$. Therefore, $[H^4(\mathfrak{g}) : k] = 0$.

It's obvious that $\overline{Z}_{3\omega_1}^4(\mathfrak{g}, M) = \overline{Z}_{3\omega_1}^4(\mathfrak{g}, M) = 1$. Then by (3), $\overline{H}_{3\omega_1}^4(\mathfrak{g}, M) = \overline{H}_{3\omega_1}^4(\mathfrak{g}, M) = 1 + 1 - 2 = 0$. So, $H^4(\mathfrak{g}) = 0$.

Using (2) and the statements (b), (c), we get the statements (e), (f) respectively. The proof of Lemma 1 is complete.

Now let $M = L(1, 1)$.

Lemma 2. There are the following isomorphisms of $SL_3(k)$ -modules:

- (a) $H^1(\mathfrak{g}, L(1, 1)) \cong L(1, 0)^{(1)} \oplus L(0, 1)^{(1)} \oplus k$;
- (b) $H^3(\mathfrak{g}, L(1, 1)) \cong H^0(1, 1)^{(1)}$;
- (c) $H^4(\mathfrak{g}, L(1, 1)) \cong 2H^0(1, 1)^{(1)}$;
- (d) $H^5(\mathfrak{g}, L(1, 1)) \cong H^0(1, 1)^{(1)}$;
- (e) $H^7(\mathfrak{g}, L(1, 1)) \cong L(1, 0)^{(1)} \oplus L(0, 1)^{(1)} \oplus k$.

In other cases $H^n(\mathfrak{g}, L(1, 1)) = 0$.

Proof. The calculations similar to the previous Lemma 1 yield:

- 1) $\prod(\overline{C}^0(\mathfrak{g}, L(1, 1))) = \prod(\overline{C}^8(\mathfrak{g}, L(1, 1))) = \{0\}$,
- $\prod(\overline{C}^i(\mathfrak{g}, L(1, 1))) = \{0, \pm 3\omega_1, \pm 3(\omega_1 - \omega_2), \pm 3\omega_2\}$ for $i = 1, 2, 6, 7$,
- $\prod(\overline{C}^j(\mathfrak{g}, L(1, 1))) = \prod(\overline{C}^1(\mathfrak{g}, L(1, 1))) \cup \{\pm 3(\omega_1 + \omega_2), \pm 3(2\omega_1 - \omega_2), \pm 3(-\omega_1 + 2\omega_2)\}$ for $j = 3, 4$;
- 2) $\dim \overline{C}_0^0(\mathfrak{g}, L(1, 1)) = \dim \overline{C}_0^8(\mathfrak{g}, L(1, 1)) = 1$, $\dim \overline{C}_0^1(\mathfrak{g}, L(1, 1)) = \dim \overline{C}_0^7(\mathfrak{g}, L(1, 1)) = 8$,
- $\dim \overline{C}_0^2(\mathfrak{g}, L(1, 1)) = \dim \overline{C}_0^6(\mathfrak{g}, L(1, 1)) = 22$, $\dim \overline{C}_0^3(\mathfrak{g}, L(1, 1)) = \dim \overline{C}_0^5(\mathfrak{g}, L(1, 1)) = 38$,
- $\dim \overline{C}_0^4(\mathfrak{g}, L(1, 1)) = 44$;
- 3) $\dim \overline{C}_{3\omega_i}^0(\mathfrak{g}, L(1, 1)) = \dim \overline{C}_{3\omega_i}^8(\mathfrak{g}, L(1, 1)) = 0$, $\dim \overline{C}_{3\omega_i}^1(\mathfrak{g}, L(1, 1)) = \dim \overline{C}_{3\omega_i}^7(\mathfrak{g}, L(1, 1)) = 2$,
- $\dim \overline{C}_{3\omega_i}^2(\mathfrak{g}, L(1, 1)) = \dim \overline{C}_{3\omega_i}^6(\mathfrak{g}, L(1, 1)) = 7$, $\dim \overline{C}_{3\omega_i}^3(\mathfrak{g}, L(1, 1)) = \dim \overline{C}_{3\omega_i}^5(\mathfrak{g}, L(1, 1)) = 14$,
- $\dim \overline{C}_{3\omega_i}^4(\mathfrak{g}, L(1, 1)) = 18$ for $i = 1, 2$;
- 4) $\dim \overline{C}_{3(\omega_1 + \omega_2)}^0(\mathfrak{g}, L(1, 1)) = \dim \overline{C}_{3(\omega_1 + \omega_2)}^8(\mathfrak{g}, L(1, 1)) = 0$,
- $\dim \overline{C}_{3(\omega_1 + \omega_2)}^1(\mathfrak{g}, L(1, 1)) = \dim \overline{C}_{3(\omega_1 + \omega_2)}^7(\mathfrak{g}, L(1, 1)) = 0$,
- $\dim \overline{C}_{3(\omega_1 + \omega_2)}^2(\mathfrak{g}, L(1, 1)) = \dim \overline{C}_{3(\omega_1 + \omega_2)}^6(\mathfrak{g}, L(1, 1)) = 0$,
- $\dim \overline{C}_{3(\omega_1 + \omega_2)}^3(\mathfrak{g}, L(1, 1)) = \dim \overline{C}_{3(\omega_1 + \omega_2)}^5(\mathfrak{g}, L(1, 1)) = 1$,
- $\dim \overline{C}_{3(\omega_1 + \omega_2)}^4(\mathfrak{g}, L(1, 1)) = 2$;
- 5) $\dim \overline{Z}_0^0(\mathfrak{g}, L(1, 1)) = \dim \overline{Z}_0^8(\mathfrak{g}, L(1, 1)) = 0$, $\dim \overline{Z}_0^1(\mathfrak{g}, L(1, 1)) = \dim \overline{Z}_0^7(\mathfrak{g}, L(1, 1)) = 2$,
- $\dim \overline{Z}_0^2(\mathfrak{g}, L(1, 1)) = \dim \overline{Z}_0^6(\mathfrak{g}, L(1, 1)) = 6$, $\dim \overline{Z}_0^3(\mathfrak{g}, L(1, 1)) = \dim \overline{Z}_0^5(\mathfrak{g}, L(1, 1)) = 18$,
- $\dim \overline{Z}_0^4(\mathfrak{g}, L(1, 1)) = 24$;
- 6) $\dim \overline{Z}_{3\omega_i}^0(\mathfrak{g}, L(1, 1)) = \dim \overline{Z}_{3\omega_i}^8(\mathfrak{g}, L(1, 1)) = 0$, $\dim \overline{Z}_{3\omega_i}^1(\mathfrak{g}, L(1, 1)) = \dim \overline{Z}_{3\omega_i}^7(\mathfrak{g}, L(1, 1)) = 1$,
- $\dim \overline{Z}_{3\omega_i}^2(\mathfrak{g}, L(1, 1)) = \dim \overline{Z}_{3\omega_i}^6(\mathfrak{g}, L(1, 1)) = 1$, $\dim \overline{Z}_{3\omega_i}^3(\mathfrak{g}, L(1, 1)) = \dim \overline{Z}_{3\omega_i}^5(\mathfrak{g}, L(1, 1)) = 6$,
- $\dim \overline{Z}_{3\omega_i}^4(\mathfrak{g}, L(1, 1)) = 8$ for $i = 1, 2$;
- 7) $\dim \overline{Z}_{3(\omega_1 + \omega_2)}^0(\mathfrak{g}, L(1, 1)) = \dim \overline{Z}_{3(\omega_1 + \omega_2)}^8(\mathfrak{g}, L(1, 1)) = 0$,
- $\dim \overline{Z}_{3(\omega_1 + \omega_2)}^1(\mathfrak{g}, L(1, 1)) = \dim \overline{Z}_{3(\omega_1 + \omega_2)}^7(\mathfrak{g}, L(1, 1)) = 0$,
- $\dim \overline{Z}_{3(\omega_1 + \omega_2)}^2(\mathfrak{g}, L(1, 1)) = \dim \overline{Z}_{3(\omega_1 + \omega_2)}^6(\mathfrak{g}, L(1, 1)) = 0$,

$$\dim \overline{Z}_{3(\omega_1+\omega_2)}^3(\mathfrak{g}, L(1, 1)) = \dim \overline{Z}_{3(\omega_1+\omega_2)}^5(\mathfrak{g}, L(1, 1)) = 1,$$

$$\dim \overline{Z}_{3(\omega_1+\omega_2)}^4(\mathfrak{g}, L(1, 1)) = 2.$$

Then, by (3), $\dim \overline{H}_\mu^n(\mathfrak{g}, L(1, 1)) = 0$ except in the following cases:

$$\text{i) } \dim \overline{H}_0^1(\mathfrak{g}, L(1, 1)) = \dim \overline{H}_0^7(\mathfrak{g}, L(1, 1)) = 1, \dim \overline{H}_0^3(\mathfrak{g}, L(1, 1)) = \dim \overline{H}_0^5(\mathfrak{g}, L(1, 1)) = 2,$$

$$\dim \overline{H}_0^4(\mathfrak{g}, L(1, 1)) = 4;$$

$$\text{ii) } \dim \overline{H}_{3\omega_i}^1(\mathfrak{g}, L(1, 1)) = \dim \overline{H}_{3\omega_i}^7(\mathfrak{g}, L(1, 1)) = 1 \text{ for } i = 1, 2;$$

$$\text{iii) } \dim \overline{H}_{3(\omega_1+\omega_2)}^3(\mathfrak{g}, L(1, 1)) = \dim \overline{H}_{3(\omega_1+\omega_2)}^5(\mathfrak{g}, L(1, 1)) = 1, \dim \overline{H}_{3(\omega_1+\omega_2)}^4(\mathfrak{g}, L(1, 1)) = 2.$$

Analyzing the dimensions of the weight subspaces of the corresponding cohomology groups, we obtain the required statements of Lemma 2. The proof of Lemma 2 is complete.

Combining the results of Lemmas 1 and 2, we obtain all the statements of Theorem 1.

Cohomology of the adjoint module

Using Theorem 1, we can easily compute the cohomology of the adjoint module for \mathfrak{g} . There is the following short exact sequence of \mathfrak{g} -modules:

$$0 \rightarrow k \rightarrow \mathfrak{g} \rightarrow L(1, 1) \rightarrow 0.$$

Consider the corresponding long exact cohomological sequence of $SL_3(k)$ -modules

$$\dots \rightarrow H^{n-1}(\mathfrak{g}, L(1, 1)) \rightarrow H^n(\mathfrak{g}) \rightarrow H^n(\mathfrak{g}, \mathfrak{g}) \rightarrow H^n(\mathfrak{g}, L(1, 1)) \rightarrow H^{n+1}(\mathfrak{g}) \rightarrow \dots$$

It is known that $H^2(\mathfrak{g}, \mathfrak{g}) = 0$ [14]. Then, according to Theorem 1, the last long exact cohomological sequence splits into the following five exact sequences:

$$0 \rightarrow H^0(\mathfrak{g}) \rightarrow H^0(\mathfrak{g}, \mathfrak{g}) \rightarrow 0,$$

$$0 \rightarrow H^1(\mathfrak{g}, \mathfrak{g}) \rightarrow H^1(\mathfrak{g}, L(1, 1)) \rightarrow H^2(\mathfrak{g}) \rightarrow 0,$$

$$0 \rightarrow H^3(\mathfrak{g}) \rightarrow H^3(\mathfrak{g}, \mathfrak{g}) \rightarrow H^3(\mathfrak{g}, L(1, 1)) \rightarrow 0,$$

$$0 \rightarrow H^4(\mathfrak{g}, \mathfrak{g}) \rightarrow H^4(\mathfrak{g}, L(1, 1)) \rightarrow H^5(\mathfrak{g}) \rightarrow H^5(\mathfrak{g}, \mathfrak{g}) \rightarrow H^5(\mathfrak{g}, L(1, 1)) \rightarrow H^6(\mathfrak{g}) \rightarrow H^6(\mathfrak{g}, \mathfrak{g}) \rightarrow 0,$$

$$0 \rightarrow H^7(\mathfrak{g}, \mathfrak{g}) \rightarrow H^7(\mathfrak{g}, L(1, 1)) \rightarrow H^8(\mathfrak{g}) \rightarrow H^8(\mathfrak{g}, \mathfrak{g}) \rightarrow 0.$$

The first three short exact sequences yield the following isomorphisms of A -modules respectively:

$$H^0(\mathfrak{g}, \mathfrak{g}) \cong k, H^1(\mathfrak{g}, \mathfrak{g}) \cong k, H^3(\mathfrak{g}, \mathfrak{g}) \cong L(1, 0)^{(1)} \oplus L(0, 1)^{(1)} \oplus H^0(1, 1)^{(1)} \oplus k.$$

Since $3(\omega_1 + \omega_2) \notin \prod(H^i(\mathfrak{g}))$ for $i = 5, 6$, then the fourth exact sequence splits and yields the following isomorphisms:

$$H^4(\mathfrak{g}, \mathfrak{g}) \cong H^4(\mathfrak{g}, L(1, 1)) \cong H^0(1, 1)^{(1)}, H^5(\mathfrak{g}, \mathfrak{g}) \cong L(1, 0)^{(1)} \oplus L(0, 1)^{(1)} \oplus H^0(1, 1)^{(1)} \oplus k,$$

$$H^6(\mathfrak{g}, \mathfrak{g}) \cong L(1, 0)^{(1)} \oplus L(0, 1)^{(1)}.$$

Similarly to the previous case, from the last exact sequence we obtain

$$H^7(\mathfrak{g}, \mathfrak{g}) \cong L(1, 0)^{(1)} \oplus L(0, 1)^{(1)} \oplus k, H^8(\mathfrak{g}, \mathfrak{g}) \cong k.$$

Thus, we get the following

Proposition 1. Let \mathfrak{g} be a classical Lie algebra of type A_2 over an algebraically closed field k of characteristic $p = 3$. Then there are the following isomorphisms of $SL_3(k)$ -modules:

$$\text{(a) } H^0(\mathfrak{g}, \mathfrak{g}) \cong H^1(\mathfrak{g}, \mathfrak{g}) \cong H^8(\mathfrak{g}, \mathfrak{g}) \cong k;$$

$$\text{(b) } H^3(\mathfrak{g}, \mathfrak{g}) \cong H^5(\mathfrak{g}, \mathfrak{g}) \cong L(1, 0)^{(1)} \oplus L(0, 1)^{(1)} \oplus H^0(1, 1)^{(1)} \oplus k;$$

$$\text{(c) } H^4(\mathfrak{g}, \mathfrak{g}) \cong 2H^0(1, 1)^{(1)};$$

$$\text{(d) } H^6(\mathfrak{g}, \mathfrak{g}) \cong L(1, 0)^{(1)} \oplus L(0, 1)^{(1)};$$

$$\text{(e) } H^7(\mathfrak{g}, \mathfrak{g}) \cong L(1, 0)^{(1)} \oplus L(0, 1)^{(1)} \oplus k.$$

In other cases $H^n(\mathfrak{g}, \mathfrak{g}) = 0$.

Cohomology for $\overline{A_2}$

Recall that $\overline{A_2}$ is the quotient algebra of the classical Lie algebra of type A_2 over an algebraically closed field of characteristic $p = 3$ by the center. In this section we compute cohomology of the simple Lie algebra $\overline{A_2}$ with coefficients in the simple modules.

First, we consider an arbitrary Lie algebra \mathfrak{g} with the center $C_{\mathfrak{g}}$ such that the corresponding quotient algebra is a simple algebra. The following result immediately leads to our goal.

Lemma 3. Let $\overline{\mathfrak{g}}$ be a simple quotient Lie algebra of a Lie algebra \mathfrak{g} by the center $C_{\mathfrak{g}}$. Then $H^n(\overline{\mathfrak{g}}, \overline{\mathfrak{g}}) \cong H^n(\mathfrak{g}, \overline{\mathfrak{g}})$ for all $n > 0$.

Proof. The space $\overline{\mathfrak{g}}$ can be equipped with the structure of a module over each of the Lie algebras $C_{\mathfrak{g}}$, \mathfrak{g} and $\overline{\mathfrak{g}}$:

$$C_{\mathfrak{g}} \times \overline{\mathfrak{g}} \rightarrow \overline{\mathfrak{g}}, (c, \overline{a}) \mapsto \mu(c)\overline{a}, \text{ where } \mu \text{ is a nonzero linear form on } C_{\mathfrak{g}};$$

$$\mathfrak{g} \times \overline{\mathfrak{g}} \rightarrow \overline{\mathfrak{g}}, (a_1, \overline{a_2}) \mapsto \overline{[a_1, a_2]}, a_1 \in \mathfrak{g}, \overline{a_2} \in \overline{\mathfrak{g}};$$

$$\overline{\mathfrak{g}} \times \overline{\mathfrak{g}} \rightarrow \overline{\mathfrak{g}}, (\overline{a_1}, \overline{a_2}) \mapsto \overline{[a_1, a_2]}, \overline{a_1}, \overline{a_2} \in \overline{\mathfrak{g}}.$$

The short exact sequence of cochain complexes

$$0 \rightarrow (C^*(C_{\mathfrak{g}}, \overline{\mathfrak{g}}), d) \rightarrow (C^*(\mathfrak{g}, \overline{\mathfrak{g}}), d) \rightarrow (C^*(\overline{\mathfrak{g}}, \overline{\mathfrak{g}}), d) \rightarrow 0$$

gives a long exact cohomological sequence

$$\dots \rightarrow H^{n-1}(C_{\mathfrak{g}}, \overline{\mathfrak{g}}) \rightarrow H^n(\overline{\mathfrak{g}}, \overline{\mathfrak{g}}) \rightarrow H^n(\mathfrak{g}, \overline{\mathfrak{g}}) \rightarrow H^n(C_{\mathfrak{g}}, \overline{\mathfrak{g}}) \rightarrow \dots$$

Since $H^n(C_{\mathfrak{g}}, \overline{\mathfrak{g}}) = 0$ for all $n \geq 0$ [15, Lemma 4.2], it follows from the fact that last cohomological sequence is exact that $H^n(\overline{\mathfrak{g}}, \overline{\mathfrak{g}}) \cong H^n(\mathfrak{g}, \overline{\mathfrak{g}})$ for all $n > 0$. The proof of Lemma 3 is complete.

Remark. A special case of Lemma 3 for $n = 1$ was proved in [7]. Using Lemma 3 to Theorem 1, we obtain a complete description of the cohomology of a simple Lie algebra A_2 with coefficients in simple modules.

Acknowledgments

This research was funded by Science Committee grant of the Ministry of Education and Science of the Republic of Kazakhstan (Grant No. AP08855935).

References

- 1 Andersen H.H. Cohomology of induced representations for algebraic groups / H.H. Andersen, J.C. Jantzen // Math. Annal. — 1984. — V. 269. — No 4. — P. 487–524.
- 2 Friedlander E.M. Cohomology of Lie algebras and algebraic groups / E.M. Friedlander, B.J. Parshall // Amer. J. Math. — 1986. — V. 108. — P. 235–253.
- 3 Kumar S. Frobenius splitting of cotangent bundles of flag varieties / S. Kumar, L. Lauritzen, J. Thomsen // Invent. Math. — 1999. — V. 136. — P. 603–621.
- 4 van der Kallen W.L.J. Infinitesimally central extensions of Chevalley groups / W.L.J. van der Kallen. — Berlin, Heidelberg, New York: Springer-Verlag, 1973. — 147 p.
- 5 Ибраев Ш.Ш. О центральных расширениях классических алгебр Ли / Ш.Ш. Ибраев // Сиб. электрон. матем. изв. — 2013. — Т. 10. — С. 450–453.
- 6 Block R. Trace form in Lie algebras / R. Block // Cand. J. Math. — 1962. — V. 14. — P. 553–564.
- 7 Ибраев Ш.Ш. О первой когомологии алгебраической группы и ее алгебры Ли в положительной характеристике / Ш.Ш. Ибраев // Мат. заметки. — 2014. — Вып. 96. — № 4. — С. 512–521.
- 8 Рудаков А.Н. Деформации простых алгебр Ли / А.Н. Рудаков // Изв. АН СССР. Сер. мат. — 1971. — Вып. 35. — С. 113–119.
- 9 Джумадильдаев А.С. К деформации классических простых алгебр Ли / А.С. Джумадильдаев // Успехи мат. наук. — 1976. — Т. 31. — С. 211, 212.
- 10 Кузнецов М.И. Деформации классических алгебр Ли / М.И. Кузнецов, Н.Г. Чебочко // Мат. сб. — 2000. — Т. 191. — № 8. — С. 69–88.
- 11 Джумадильдаев А.С. О когомологиях модулярных алгебр Ли / А.С. Джумадильдаев // Мат. сб. — 1982. — Т. 119 (161). — № 1(9). — С. 132–149.

- 12 Jantzen J.C. First cohomology groups for classical Lie algebras / J.C. Jantzen // Progress in Math. — 1991. — V. 95. — P. 291–315.
- 13 Dzhumadil'daev A.S. Nonsplit extensions of modular Lie algebras of rank 2 / A.S. Dzhumadil'daev, Sh.Sh.Ibraev // Homology, Homotopy and Applications. — 2002. — V. 4. — No 2. — P. 141–163.
- 14 Чебочко Н.Г. Деформации классических алгебр Ли с однородной системой корней в характеристике два. I / Н.Г. Чебочко // Мат. сб. — 2005. — Т. 196. — № 9. — С. 125–156.
- 15 Williams F.L. The cohomology of semisimple Lie algebras with coefficients in a Verma module / F.L. Williams // Trans. Amer. Math. Soc. — 1978. — V. 240. — P. 115–127.

А.А. Ибраева, Ш.Ш. Ыбраев, Г.К. Ешмурат

Қорқыт Ата атындағы Қызылорда университеті, Қызылорда, Қазақстан

Сипаттамасы 3 $\mathfrak{sl}_3(k)$ үшін жәй модульдердің когомологиялары

Мақалада сипаттамасы $p = 3$ алгебралық тұйық k өрісіне қатысты A_2 түріндегі классикалық Ли алгебрасы коэффициенттерінің жәй модульдердегі когомологиялары есептелген. Құрылымын беру үшін оларды $SL_3(k)$ алгебралық группасының модульдері ретінде қарастырған. Өріс сипаттамасы $p = 3$ болғанда тек екі арнайы жәй модуль бар: кіріктірілген модульдің центр бойынша фактор-модуліне изоморфты модуль және бір өлшемді тривиаль модуль. Жәй тривиаль емес модульдің когомологиялары туралы алынған нәтижелер кіріктірілген модульдің когомологияларын есептеуге қолданылды. Сонымен қатар, A_2 -нің центр бойынша жәй фактор-алгебрасының да когомологиялары есептелді.

Кілт сөздер: Ли алгебрасы, жәй модуль, шектелген модуль, когомология, дәл тізбек.

А.А. Ибраева, Ш.Ш. Ибраев, Г.К. Ешмурат

Қызылординский университет имени Қорқыт Ата, Қызылорда, Казахстан

Когомологии простых модулей для $\mathfrak{sl}_3(k)$ в характеристике 3

В статье вычислены когомологии классической алгебры Ли типа A_2 над алгебраически замкнутым полем k характеристики $p = 3$ с коэффициентами в простых модулях. Для описания структуры авторы рассмотрели их как модули над алгебраической группой $SL_3(k)$. В случае характеристики $p = 3$ существуют только два простых особых модуля: простой модуль, изоморфный фактор-модулю присоединенного модуля по центру, и одномерный тривиальный модуль. Результаты, полученные для когомологии простого нетривиального модуля, применены для вычисления когомологии присоединенного модуля. Кроме того, рассчитаны когомологии простой фактор-алгебры Ли, алгебры A_2 по центру.

Ключевые слова: алгебра Ли, простой модуль, ограниченный модуль, когомология, точная последовательность.

References

- 1 Andersen, H.H., & Jantzen, J.C. (1984). Cohomology of induced representations for algebraic groups. *Math. Ann.*, 269(4), 487–524.
- 2 Friedlander, E.M., & Parshal, B.J. (1986). Cohomology of Lie algebras and algebraic groups. *Amer. J. Math.*, 108, 235–253.
- 3 Kumar, S., Lauritzen, L., & Thomsen, J. (1999). Frobenius splitting of cotangent bundles of flag varieties. *Invent. Math.*, 136, 603–621.

- 4 van der Kallen, W.L.J. (1973). *Infinitesimally central extensions of Chevalley groups*. Berlin, Heidelberg, New York: Springer-Verlag.
- 5 Ibraev, Sh.Sh. (2013). O tsentralnykh rasshireniakh klassicheskikh algebr Li [On central extensions of classical Lie algebras]. *Sibirskie elektronnye matematicheskie izvestiia – Siberian Electronic Mathematical Reports*, 10, 450–453 [in Russian].
- 6 Block, R. (1962). Trace form in Lie algebras. *Cand. J. Math.*, 14, 553–564.
- 7 Ibraev, Sh.Sh. (2014). O pervoi kogomologii algebraicheskoi gruppy i ee algebrы Li v polozhitelnoi kharakteristike [On the first cohomology of an algebraic group and its Lie algebra in positive characteristic]. *Matematicheskie zametki – Mathematical notes*, 96, 4, 512–521 [in Russian].
- 8 Rudakov, A.N. (1971). Deformatsii prostykh algebr Li [Deformations of simple Lie algebras]. *Izvestiia AN SSSR. Seriya Matematika – Izvestiya: Mathematics*, 35, 113–119 [in Russian].
- 9 Dzhumadil'daev, A.S. (1976). K deformatsii klassicheskikh prostykh algebr Li [On the deformations of the classical simple Lie algebras]. *Uspekhi matematicheskikh nauk – Russian Mathematical Surveys*, 31, 211–212 [in Russian].
- 10 Kuznetsov, M.I., & Chebochko, N.G. (2000). Deformatsii klassicheskikh algebr Li [Deformations of classical Lie algebras]. *Matematicheskii sbornik – Sbornic: Mathematics*, 191(8), 69–88 [in Russian].
- 11 Dzhumadil'daev, A.S. (1982). O kogomologiiakh moduliarnykh algebr Li [On cohomology of modular algebras]. *Matematicheskii sbornik – Sbornic: Mathematics*, 119(161), 132–149 [in Russian].
- 12 Jantzen, J.C. (1991). First cohomology groups for classical Lie algebras *Progress in Mathematics*, 95, 291–315.
- 13 Dzhumadil'daev, A.S., & Ibraev, Sh.Sh. (2002). Nonsplit extensions of modular Lie algebras of rank 2. *Homology, Homotopy and Applications*, 4, 141–163.
- 14 Chebochko, N.G. (2005). Deformatsii klassicheskikh algebr Li s odnorodnoi sistemoi kornei v kharakteristike dva. I [Deformations of classical Lie algebras with homogeneous root system in characteristic two. I]. *Matematicheskii sbornik – Sbornic: Mathematics*, 196, 125–156 [in Russian].
- 15 Williams, F.L. (1978). The cohomology of semisimple Lie algebras with coefficients in a Verma module *Trans. Amer. Math. Soc.*, 240, 115–127.