DOI 10.31489/2021M3/140-155

UDC 517.911

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On a Hilfer Type Fractional Differential Equation with Nonlinear Right-Hand Side

In this article we consider the questions of one-valued solvability and numerical realization of initial value problem for a nonlinear Hilfer type fractional differential equation with maxima. By the aid of uncomplicated integral transformation based on Dirichlet formula, this initial value problem is reduced to the nonlinear Volterra type fractional integral equation. The theorem of existence and uniqueness of the solution of given initial value problem in the segment under consideration is proved. For numerical realization of solution the generalized Jacobi–Galerkin method is applied. Illustrative examples are provided.

Keywords: Ordinary differential equation, equation with maxima, Hilfer operator, one-valued solvability, generalized Jacobi–Galerkin method.

Introduction

Let $(t_0; b) \subset \mathbb{R}^+ \equiv [0; \infty)$ be a finite interval on the set of positive real numbers, and let $\alpha > 0$. The Riemann–Liouville α-order fractional integral of a function $\eta(t)$ is defined as follows:

$$I_{t_0+}^{\alpha}\eta(t) = \frac{1}{\Gamma(\alpha)} \int\limits_{t_0}^{t} (t-s)^{\alpha-1}\eta(s)\,d\,s, \quad \alpha > 0, \quad t \in (t_0; b),$$

where $\Gamma(\alpha)$ is the Gamma function [1; 112].

Let $n-1 < \alpha \le n$, $n \in \mathbb{N}$. The Riemann–Liouville α-order fractional derivative of a function $\eta(t)$ is defined as follows [2, Vol. 1, p. 27]:

$$D_{t_0+}^{\alpha}\eta(t) = \frac{d^n}{dt^n} I_{t_0+}^{n-\alpha}\eta(t), \quad t \in \big(t_0; b\big).$$

The Caputo α-order fractional derivative of a function $\eta(t)$ is defined [2, Vol. 1; 34] by

$${}_*D_{t_0+}^{\alpha}\eta(t) = I_{t_0+}^{n-\alpha}\eta^{(n)}(t) = \frac{1}{\Gamma(n-\alpha)} \int\limits_{t_0}^{t} \frac{\eta^{(n)}(s)\,d\,s}{(t-s)^{\alpha-n+1}}, \quad t \in (t_0; b).$$

Both the derivatives are reduced to the n-th order derivatives for $\alpha = n \in \mathbb{N}$ [2, Vol. 1; 27–34]:

$$D_{t_0+}^{n}\eta(t) =_* D_{t_0+}^{n}\eta(t) = \frac{d^n}{dt^n}\eta(t), \quad t \in (t_0; b).$$

The so-called generalized Riemann–Liouville fractional derivative (referred to as the Hilfer fractional derivative) of order α, $n-1 < \alpha \le n$, $n \in \mathbb{N}$ and type β, $0 \le \beta \le 1$ is defined by the following composition of three operators: [1; 113]:

$$D_{t_0+}^{\alpha,\beta}\eta(t) = I_{t_0+}^{\beta(n-\alpha)} \frac{d^n}{dt^n} I_{t_0+}^{(1-\beta)(n-\alpha)}\eta(t), \quad t \in (t_0; b).$$

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For $\beta = 0$ this operator is reduced to the Riemann–Liouville fractional derivative $D^{\alpha,0}_{t_0+} = D^{\alpha}_{t_0+}$ and the case $\beta = 1$ corresponds to the Caputo fractional derivative $D^{\alpha,1}_{t_0+} =_* D^{\alpha}_{t_0+}$.

Let $\gamma = \alpha + \beta n - \alpha\beta$. It is easy to see that $\alpha \le \gamma \le n$. Then it is convenient to use another designation for the operator $D^{\alpha,\beta}_{t_0+}\eta(t)$:

$$D^{\alpha,\gamma}\eta(t) = D^{\alpha,\beta}_{t_0+}\eta(t).$$

The generalized Riemann–Liouville operator was introduced in [1] by R. Hilfer on the basis of fractional time evolutions that arise during the transition from the microscopic scale to the macroscopic time scale. Using the integral transforms, he investigated the Cauchy problem for the generalized diffusion equation, the solution of which is presented in the form of the Fox H-function. Note [3, 4], the generalized Riemann–Liouville operator was used in studying dielectric relaxation in glass-forming liquids with different chemical compositions. In [5] the properties of the generalized Riemann–Liouville operator were investigated in a special functional space, and an operational method was developed for solving fractional differential equations with this operator. Based on the results of the work [5], the authors of [6] have developed an operational method for solving fractional differential equations containing a finite linear combination of the generalized Riemann–Liouville operators with various parameters.

Fractional calculus plays an important role in the mathematical modelling of many scientific and engineering disciplines (see more detailed information in [7]). In [8] problems of continuum and statistical mechanics are considered. In [9] the mathematical problems of Ebola epidemic model are studied. In [10] and [11] the fractional model for the dynamics of tuberculosis infection and novel coronavirus (COViD-2019), respectively are studied. The construction of various models of theoretical physics by the aid of fractional calculus is described in [2, Vol. 4, 5], [12, 13]. A specific interpretation of the Hilfer fractional derivative, describing the random motion of a particle moving on the real line at Poisson paced times with finite velocity is given in [14]. A detailed review of the application of fractional calculus in solving problems of applied sciences is given in [2, Vol. 6–8], [15]. More detailed information related to the theory of fractional integro-differentiation, including the Hilfer fractional derivative one can find in the monograph [16]. In [17] the unique solvability of boundary value problem for weak nonlinear partial differential equations of mixed type with fractional Hilfer operator is studied by analytical method. In [18] the solvability of nonlocal problem for a mixed type fourth-order differential equation with Hilfer fractional operator is studied. In [19] it is considered an inverse problem for a mixed type integro-differential equation with fractional order Caputo operators (see also [20–22]).

In the modern scientific world information technologies are widely used in various fields of science and engineering [23, 24]. In application of differential equations the numerical methods play an important role. Different methods are used for the numerical solution of differential, integral and integro-differential equations [25–34]. In particular, the book [28] is devoted to Chebyshev and Fourier spectral methods and [30] tells us about polynomial approximations of solving differential equations. The work [35] is devoted to study of nonlinear Volterra integral equations with weakly singular kernels by generalized Jacobi Spectral–Galerkin method.

In the present paper we consider the questions of one-valued solvability and numerical realization for a Hilfer type fractional differential equation with nonlinear right-hand side and maxima. This equation we solve under initial value condition. Differential equations with maxima play an important role in solving control problems of the sale of goods and investment of manufacturing companies in a market economy [36]. In [37] it is justified that the theoretical study of differential equations with maxima is relevant.

We consider the Hilfer type fractional differential equation on a interval $(t_0; T)$:

$$D^{\alpha,\gamma}x(t) + \omega x(t) = f\Big(t,\, x(t),\, \max\left\{x(\theta)\,|\,\theta \in [q_1 t;\, q_2 t]\right\}\Big) \tag{1}$$

under initial value condition

$$\lim_{t\to t_0} J^{1-\gamma}_{t_0+}x(t) = x_0\,, \quad x(t) = \varphi(t),\ t \notin (t_0, T)\,, \tag{2}$$

where $f(t, u, \vartheta) \in C(\Omega)$, $\varphi(t) \in C\left(\left[0;\, t_0\right] \cup [T;\, \infty]\right)$, $0 < \omega$ is real parameter, $x_0 = \text{const}$, $\Omega \equiv [t_0;\, T] \times \mathbb{X} \times \mathbb{X}$, $0 \le t_0$, $\mathbb{X} \subset \mathbb{R} \equiv (-\infty;\, \infty)$, $\mathbb{X}$ is closed set. Here

$$D^{\alpha,\gamma} = J^{\gamma-\alpha}_{t_0+}\frac{d}{dt}J^{1-\gamma}_{t_0+}, \quad 0 < \alpha \le \gamma \le 1$$

is Hilfer operator and J^{α}_{0+} is the Riemann–Liouville integral operator, which is defined by the formula

$$J^{\alpha}_{t_0+}\eta(t) = \frac{1}{\Gamma(\alpha)} \int\limits_{t_0}^{t} \frac{\eta(s)\, d\, s}{(t-s)^{1-\alpha}}, \quad \alpha > 0.$$

We set $0 < q_1 < q_2 < \infty$ and understand that there are possible cases: 1) $0 < q_1 < q_2 < 1$; 2) $0 < q_1 < 1$, $1 < q_2 < \infty$; 3) $1 < q_1 < q_2 < \infty$.

Fractional integral equation

Lemma. The solution of the differential equation (1) with initial value condition (2) is represented as follows

$$x\,(t) = \Im\,(t;\, x) \equiv x_{\,0}\,(t-t_0)^{\gamma-1} E_{\alpha,\,\gamma}\,(-\omega\,(t-t_0)^{\alpha}) +$$

$$+ \int\limits_{t_0}^{t} (t-s)^{\alpha-1} E_{\alpha,\,\alpha}\,(-\omega\,(t-s)^{\alpha})\, f\,(s,\, x\,(s),\, \max\{x\,(\theta) \,|\, \theta \in [q_1 s;\, q_2 s]\})\; d\, s, \tag{3}$$

where $E_{\alpha,\,\gamma}(z)$ is Mittag–Leffler function and has the form [2, vol. 1, 269–295]

$$E_{\alpha,\,\gamma}(z) = \sum_{k=0}^{\infty} \frac{z^{\,k}}{\Gamma(\alpha\, k+\gamma)}, \quad z,\, \alpha, \gamma \in \mathbb{R} > 0.$$

Proof. We rewrite the differential equation (1) in the form

$$J^{\gamma-\alpha}_{t_0+} D^{\gamma}_{t_0+}\, x\,(t) = -\omega\, x(t) + f\,(t,\, \cdot),$$

where $f\,(t,\, \cdot) = f\,(t,\, x\,(t),\, \max\{x\,(\theta) \,|\, \theta \in [q_1 t;\, q_2 t]\})$.

Applying the operator $J^{\alpha}_{t_0+}$ to both sides of this equation, taking into account the linearity of this operator and the formula [6]

$$J^{\gamma}_{t_0+} D^{\gamma}_{t_0+} x(t) = x(t) - \frac{1}{\Gamma\,(\gamma)}\, J^{1-\gamma}_{t_0+} x(t)|_{\,t=t_0+} (t-t_0)^{\,\gamma-1}$$

we obtain

$$x(t) = \frac{x_{\,0}}{\Gamma\,(\gamma)}\,(t-t_0)^{\,\gamma-1} + J^{\alpha}_{t_0+}\, f\,(t,\cdot) - \omega\, J^{\alpha}_{t_0+}\, x(t). \tag{4}$$

Using the lemma from [38], we represent the solution of equation (4) in the form

$$x(t) = \frac{x_{\,0}}{\Gamma\,(\gamma)}\,(t-t_0)^{\,\gamma-1} + J^{\alpha}_{t_0+}\, f\,(t,\cdot) -$$

$$-\omega \int\limits_{t_0}^{t} (t-s)^{\,\alpha-1}\, E_{\alpha,\,\alpha}\, (-\omega\,(t-s)^{\,\alpha}) \left[\frac{x_{\,0}}{\Gamma\,(\gamma)}\,(s-t_0)^{\,\gamma-1} + J^{\alpha}_{t_0+}\, f\,(s,\cdot)\right] d\, s. \tag{5}$$

We rewrite the representation (5) as the sum of two expressions:

$$I_{\,1}\,(t) = x_{\,0} \left[\frac{(t-t_0)^{\,\gamma-1}}{\Gamma\,(\gamma)} - \frac{\omega}{\Gamma\,(\gamma)} \int\limits_{t_0}^{t} (t-s)^{\,\alpha-1}\, E_{\alpha,\,\alpha}\, (-\omega\,(t-s)^{\,\alpha})\,(s-t_0)^{\,\gamma-1}\; d\, s\right], \tag{6}$$

$$I_{\,2}\,(t) = J^{\alpha}_{t_0+}\, f\,(t,\cdot) - \omega \int\limits_{t_0}^{t} (t-s)^{\,\alpha-1}\, E_{\alpha,\,\alpha}\; (-\omega\,(t-s)^{\,\alpha})\, J^{\alpha}_{t_0+}\, f\,(s,\cdot)\, d\, s. \tag{7}$$

We apply the following representations [2, vol. 1, 269–295]

$$E_{\,\alpha,\,\gamma}\,(z) = \frac{1}{\Gamma\,(\gamma)} + z\, E_{\,\alpha,\,\gamma+\alpha}\,(z), \quad \alpha > 0, \quad \gamma > 0, \tag{8}$$

$$\frac{1}{\Gamma(k)}\int\limits_{t_0}^{z}(z-t)^{k-1}E_{\alpha,\gamma}\left(-\omega t^{\alpha}\right)t^{\gamma-1}dt=z^{\gamma+k-1}E_{\alpha,\gamma+k}\left(-\omega z^{\alpha}\right),\quad k>0,\quad \gamma>0. \tag{9}$$

Then for the integral (6) we obtain the representation

$$I_1(t)=x_0(t-t_0)^{\gamma-1}E_{\alpha,\gamma}\left(-\omega(t-t_0)^{\alpha}\right). \tag{10}$$

The integral in (7) is easily transformed to the form

$$\int\limits_{t_0}^{t}(t-\xi)^{\alpha-1}E_{\alpha,\alpha}\left(-\omega(t-\xi)^{\alpha}\right)J_{t_0+}^{\alpha}f(\xi,\cdot)\,d\xi=$$

$$=\frac{1}{\Gamma(\alpha)}\int\limits_{t_0}^{t}(t-\xi)^{\alpha-1}E_{\alpha,\alpha}\left(-\omega(t-\xi)^{\alpha}\right)d\xi\int\limits_{t_0}^{\xi}(\xi-s)^{\alpha-1}f(s,\cdot)\,ds=$$

$$=\frac{1}{\Gamma(\alpha)}\int\limits_{t_0}^{t}f(s,\cdot)\,ds\int\limits_{s}^{t}(t-\xi)^{\alpha-1}(\xi-s)^{\alpha-1}E_{\alpha,\alpha}\left(-\omega(t-\xi)^{\alpha}\right)d\xi. \tag{11}$$

Taking (9) into account the second integral in the last equality of (11) can be written as

$$\int\limits_{s}^{t}(t-\xi)^{\alpha-1}(\xi-s)^{\alpha-1}E_{\alpha,\alpha}\left(-\omega(t-\xi)^{\alpha}\right)d\xi=\Gamma(\alpha)(t-\xi)^{2\alpha-1}E_{\alpha,2\alpha}\left(-\omega(t-\xi)^{\alpha}\right).$$

Then, taking into account (8), we represent (7) in the following form

$$I_2(t)=\int\limits_{t_0}^{t}(t-\xi)^{\alpha-1}E_{\alpha,\alpha}\left(-\omega(t-\xi)^{\alpha}\right)f(\xi,\cdot)\,d\xi. \tag{12}$$

Substituting (10) and (12) into the sum $x(t)=I_1(t)+I_2(t)$, we obtain (3). The lemma is proved.

Existence and uniqueness of solution

Theorem. Let the following two conditions be satisfied:
1) $\max\limits_{t_0\le t\le T}|f(t,x,y)|\le M=\text{const}<\infty$;
2) $\left|f\left(t,x_1,y_1\right)-f\left(t,x_2,y_2\right)\right|\le L\left(|x_1-x_2|+|y_1-y_2|\right),\ \ 0<L=\text{const}<\infty$.
Then there exists a unique solution of the initial value problem (1), (2) in the space of continuous functions $C\,(t_0;T)$, which can be found by the method of successive approximations:

$$\begin{cases} x_0(t)=G(t),\\ x_{k+1}(t)=\Im(t;x_k),\quad k=0,1,2,\ldots,\end{cases} \tag{13}$$

where $G(t)=x_0\big(t-t_0\big)^{\gamma-1}E_{\alpha,\gamma}\left(-\omega\left(t-t_0\right)^{\alpha}\right)$.

Proof. Mittag–Leffler function $E_{\alpha,\gamma}(z)$ has the following property [39]: we assume that $0<\alpha<2$, γ is real constant and $\arg z=\pi$. Then there holds

$$|E_{\alpha,\gamma}(z)|\le\frac{A}{1+|z|},$$

where A is positive constant and does not dependent on z. Then it is not difficult to see that from the approximations (13) we obtain that there following estimate holds

$$\left|\left(t-t_0\right)^{1-\gamma}x_0(t)\right|\le|x_0|\cdot\left|E_{\alpha,\gamma}\left(-\omega\left(t-t_0\right)^{\alpha}\right)\right|\le|x_0|\cdot C_0, \tag{14}$$

where $|E_{\alpha,\alpha}(-\omega(t-s)^{\alpha})|\le C_0$.

By virtue of first condition of the theorem and estimate (14), from approximations (13) we obtain

$$|\, x_1(t) - x_0(t)\,| \leq$$

$$\leq \int_{t_0}^{t} \left|(t-s)^{\alpha-1} E_{\alpha,\alpha}\left(-\omega\,(t-s)^{\alpha}\right) f\left(s,\, x_0(s),\, \max\left\{x_0(\theta)\,|\,\theta\in[q_1 s;\, q_2 s]\right\}\right)\right| d\,s \leq$$

$$\leq M\cdot C_0|x_0| \int_{t_0}^{t} (t-s)^{\alpha-1} d\,s \leq \frac{|x_0|}{\alpha} M\cdot C_0\cdot (t-t_0)^{\alpha}. \qquad (15)$$

We continue the Picard iteration process for the integral equation (3) according to the approximations (13). Then, by virtue of conditions of the theorem and taking the estimate (15) into account, we derive

$$|\, x_2(t) - x_1(t)\,| \leq \int_{t_0}^{t} \Big|\,(t-s)^{\alpha-1} E_{\alpha,\alpha}\left(-\omega\,(t-s)^{\alpha}\right)\left[f\left(s,\, x_1(s),\, \max\left\{x_1(\theta)\,|\,\theta\in[q_1 s;\, q_2 s]\right\}\right) -\right.$$

$$\left. -f\left(s,\, x_0(s),\, \max\left\{x_0(\theta)\,|\,\theta\in[q_1 s;\, q_2 s]\right\}\right)\right]|\; d\,s \leq L\int_{t_0}^{t} \left|\,(t-s)^{\alpha-1} E_{\alpha,\alpha}\left(-\omega\,(t-s)^{\alpha}\right)\right| \left[|x_1(s) - x_0(s)| +\right.$$

$$\left. + \Big|\max\left\{x_1(\theta)\,|\,\theta\in[q_1 s;\, q_2 s]\right\} - \max\left\{x_0(\theta)\,|\,\theta\in[q_1 s;\, q_2 s]\right\}\Big|\right] d\,s \leq$$

$$\leq 2C_0 L \int_{t_0}^{t} (t-s)^{\alpha-1} |x_1(s) - x_0(s)| d\,s \leq \frac{2|x_0|}{\alpha} M\cdot C_0^2 L \int_{t_0}^{t} (t-s)^{\alpha-1}(s-t_0)^{\alpha} d\,s.$$

By the changing the argument as $s = t_0 + \big(t - t_0\big)\tau$, from the last estimate we obtain

$$|\, x_2(t) - x_1(t)\,| \leq \frac{2|x_0|}{\alpha} M\cdot C_0^2 L \int_{t_0}^{t} \big(t-t_0\big)^{\alpha-1}(1-\tau)^{\alpha-1}\big(t-t_0\big)^{\alpha}\tau^{\alpha}\big(t-t_0\big) d\tau \leq$$

$$\leq \frac{2\Gamma^2(\alpha)}{\Gamma(2\alpha+1)} |x_0| M\cdot L\cdot \left[C_0\cdot (t-t_0)^{\alpha}\right]^2. \qquad (16)$$

Analogously, taking the estimate (16) into account, for the next difference we derive

$$|\, x_3(t) - x_2(t)\,| \leq L\int_{t_0}^{t} \left|\,(t-s)^{\alpha-1} E_{\alpha,\alpha}\left(-\omega\,(t-s)^{\alpha}\right)\right| \big[\,|\, x_2(s) - x_1(s)\,| +$$

$$+ |\max\left\{x_2(\theta)\,|\,\theta\in[q_1 s;\, q_2 s]\right\} - \max\left\{x_1(\theta)\,|\,\theta\in[q_1 s;\, q_2 s]\right\}|\,\big]\, d\,s \leq$$

$$\leq 2C_0 L \int_{t_0}^{t} (t-s)^{\alpha-1} \big|\, x_2(s) - x_1(s)\,\big| d\,s \leq$$

$$\leq \frac{\Gamma^2(\alpha)}{\Gamma(2\alpha+1)} \big|\, x_0\,\big| M\cdot (2L)^2\cdot C_0^3 \int_{t_0}^{t} (t-s)^{\alpha-1}\big(s-t_0\big)^{2\alpha} d\,s \leq$$

$$\leq \frac{\Gamma^3(\alpha)}{\Gamma(3\alpha+1)} \cdot |x_0| \cdot M\cdot (2L)^2 \cdot \left[C_0\cdot (t-t_0)^{\alpha}\right]^3. \qquad (17)$$

Continuing the estimation processes (14)–(17) for arbitrary difference we obtain

$$|\, x_n(t) - x_{n-1}(t)\,| \leq \frac{\Gamma^n(\alpha)}{\Gamma(n\alpha+1)} \cdot \big|x_0\big| \cdot M\cdot (2L)^{n-1} \left[C_0\cdot \big(t-t_0\big)^{\alpha}\right]^n. \qquad (18)$$

For the absolute value of difference $|\,x_n(t)-x_{n-1}(t)\,|$ we show that $\sum\limits_{n=1}^{\infty}|\,x_n(t)-x_{n-1}(t)\,|<\infty$ in the space $C\;(t_0;\,T)$. So, we denote the right-hand side of (18) as

$$a_n=\frac{\Gamma^n(\alpha)}{\Gamma(n\,\alpha+1)}\cdot(2L)^{n-1}\left[C_0\cdot\big(t-t_0\big)^{\alpha}\right]^{\,n}$$

and we put

$$a_{n+1}=\frac{\Gamma^{n+1}(\alpha)}{\Gamma((n+1)\,\alpha+1)}\cdot(2L)^{n}\left[C_0\cdot\big(t-t_0\big)^{\alpha}\right]^{\,n+1}.$$

Then we consider the following limit

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=2L\cdot\Gamma(\alpha)\cdot C_0\cdot\big(t-t_0\big)^{\alpha}\lim_{n\to\infty}\frac{\Gamma(n\,\alpha+1)}{\Gamma((n+1)\,\alpha+1)}. \tag{19}$$

Taking known formula [40]

$$\frac{\Gamma(z+a)}{\Gamma(z+b)}=z^{a-b}\left[1+\frac{(a-b)(a-b-1)}{2z}+O(z^{-2})\right]$$

into account, we obtain

$$\lim_{n\to\infty}\frac{\Gamma(n\,\alpha+1)}{\Gamma((n\,\alpha+\alpha+1)}=\lim_{n\to\infty}(n\,\alpha)^{1-\alpha-1}\left[1+\frac{(1-\alpha-1)(1-\alpha-1-1)}{2n\,\alpha}+O(n\,\alpha)^{-2}\right]=$$

$$=\frac{1}{\alpha^{\alpha}}\lim_{n\to\infty}\frac{1}{n^{\alpha}}\left[1+\frac{\alpha\,(1+\alpha)}{2n\,\alpha}+O(n\,\alpha)^{-2}\right]=0.$$

Consequently, for (19) we have

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=2L\cdot\Gamma(\alpha)\cdot C_0\cdot\big(t-t_0\big)^{\alpha}\cdot\lim_{n\to\infty}\frac{\Gamma(n\,\alpha+1)}{\Gamma((n+1)\,\alpha+1)}=$$

$$=2\Gamma(\alpha)\cdot L\cdot C_0\cdot\big(t-t_0\big)^{\alpha}\cdot\frac{1}{\alpha^{\alpha}}\lim_{n\to\infty}\frac{1}{n^{\alpha}}\left[1+\frac{\alpha\big(1+\alpha\big)}{2n\,\alpha}+O(n\,\alpha)^{-2}\right]=0.$$

Hence, according to d'Alembert's convergence criterion of series, we have

$$\sum_{n=1}^{\infty}|\,x_n(t)-x_{n-1}(t)\,|\le\sum_{n=1}^{\infty}\frac{\Gamma^n(\alpha)}{\Gamma(n\,\alpha+1)}\cdot C_0^n\cdot(2L)^{n-1}\big(t-t_0\big)^{n\,\alpha}<\infty \tag{20}$$

for all $t\ge t_0$. Since we consider the solution of the integral equation (3) in the space of continuous functions $C\;(t_0;\,T)$, it follows from (20) that the sequence of functions $\{x_k(t)\}_{k=1}^{\infty}$ converges absolutely and uniformly to solution of the integral equation (3) with respect to argument t. Hence implies the existence of a solution of the problem (1), (2) on the interval $(t_0;\,T)$. Now we show the uniqueness of this solution. Assuming that the integral equation (3) has two different solutions $x(t)$ and $y(t)$ on the interval $(t_0;\,T)$, we obtain the following integral inequality

$$|\,x(t)-y(t)\,|\le 2L\int\limits_{t_0}^{t}\big|\,(t-s)^{\alpha-1}E_{\alpha,\,\alpha}\,(-\omega\,(t-s)^{\alpha})\,\big|\cdot|\,x(s)-y(s)\,|\;d\,s. \tag{21}$$

Applying Gronwall–Bellman inequality to estimate (21), we obtain that $|\,x\,(t)-y\,(t)\,|\equiv 0$ for all $t\in(t_0;\,T)$. Therefore, the Cauchy problem (1), (2) has a unique solution on the interval $(t_0;\,T)$. The theorem is proved.

The generalized Jakobi–Galerkin method

Now, to the problem (1), (2) we apply the generalized Jacobi–Galerkin method as a numerical realization of solution (3). This solution (3) is nonlinear Volterra type fractional integral equation. On the interval $(-1; 1)$ for the given numbers $\beta_1, \ \beta_2 > -1$ we consider standard Jacobi polynomial $J_n^{(\beta_1,\beta_2)}(\xi)$ of degree n with weight function $\Lambda^{(\beta_1,\beta_2)}(\xi) = (1-\xi)^{\beta_1}(1+\xi)^{\beta_2}$. For the standard Jacobi polynomial the following relation is true

$$\int_{-1}^{1} J_n^{(\beta_1,\beta_2)}(\xi)\, J_m^{(\beta_1,\beta_2)}(\xi)\, \Lambda^{(\beta_1,\beta_2)}(\xi)\, d\xi = \gamma_m^{(\beta_1,\beta_2)} \delta_{m,n}, \tag{22}$$

where $\delta_{m,n}$ is the Kronecker function and

$$\gamma_m^{(\beta_1,\beta_2)}(\xi) = \begin{cases} \frac{2^{\beta_1+\beta_2+1}\,\Gamma(\beta_1+1)\,\Gamma(\beta_2+1)}{\Gamma(\beta_1+\beta_2+2)}, & m=0, \\ \frac{2^{\beta_1+\beta_2+1}\,\Gamma(m+\beta_1+1)\,\Gamma(m+\beta_2+1)}{(2m+\beta_1+\beta_2+1)\, m!\, \Gamma(m+\beta_1+\beta_2+2)}, & m \geq 1. \end{cases}$$

From (22) we note that the set of standard Jacobi polynomial $J_n^{(\beta_1,\beta_2)}(\xi)$ is a complete orthogonal system in the space $L^2_{\Lambda^{(\beta_1,\beta_2)}}(-1; 1)$ with weight function $\Lambda^{(\beta_1,\beta_2)}(\xi)$. In particular, $J_0^{(\beta_1,\beta_2)}(\xi) = 1$.

The shifted Jacobi polynomial of variable t and degree n is defined by the following formula

$$\tilde{J}_n^{(\beta_1,\beta_2)}(t) = J_n^{(\beta_1,\beta_2)}\left(\frac{2\,(t-t_0)}{T-t_0} - 1\right), \quad t \in \big(t_0; T\big). \tag{23}$$

We note that the set of shifted Jacobi polynomial $\tilde{J}_n^{(\beta_1,\beta_2)}(t)$ is a complete orthogonal system with weight function $\Lambda_T^{(\beta_1,\beta_2)}(t) = \big(T - t + t_0\big)^{\beta_1} \big(t - t_0\big)^{\beta_2}$ in the space $L^2_{\Lambda^{(\beta_1,\beta_2)}}\big(t_0; T\big)$ and by the aid of (23) we have the analogue of the (22)

$$\int_{t_0}^{T} \tilde{J}_n^{(\beta_1,\beta_2)}(t)\, \tilde{J}_m^{(\beta_1,\beta_2)}(t)\, \Lambda_T^{(\beta_1,\beta_2)}(t)\, dt = \left(\frac{T+t_0}{2}\right)^{\beta_1+\beta_2+1} \gamma_m^{(\beta_1,\beta_2)}(t)\, \delta_{m,n}. \tag{24}$$

For any integer $N \geq 0$ we denote by $\left\{\xi_j^{(\beta_1,\beta_2)}, \ \eta_j^{(\beta_1,\beta_2)}\right\}_{j=0}^{N}$ the nodes and the corresponding Christoffel numbers of the standard Jacobi–Gauss interpolation on the interval $(-1; 1)$. By the $\tilde{P}_N(t_0; T)$ we denote the set of polynomials of degree at most N on the interval $\big(t_0; T\big)$ and by the $t_j^{(\beta_1,\beta_2)}$ we denote the shifted Jacobi-Gauss quadrature nodes on the interval $(t_0; T)$

$$t_j^{(\beta_1,\beta_2)} = \frac{T-t_0}{2}\left(\xi_j^{(\beta_1,\beta_2)} + 1\right) + t_0, \quad 0 \leq j \leq N.$$

By virtue of the property of the standard Jacobi–Gauss quadrature it's implied that for any $\phi(t) \in \tilde{P}_{2N+1}(t_0; T)$ we have

$$\int_{t_0}^{T} \phi(t)\, \Lambda_T^{(\beta_1,\beta_2)}(t)\, dt = \left(\frac{T+t_0}{2}\right)^{\beta_1+\beta_2+1} \sum_{j=0}^{N} \phi\left(t_j^{(\beta_1,\beta_2)}\right) \eta_j^{(\beta_1,\beta_2)}. \tag{25}$$

By virtue of (25) from (24), we have for any $0 \leq m+n \leq 2N+1$,

$$\sum_{j=0}^{N} \tilde{J}_m^{(\beta_1,\beta_2)}\left(t_j^{(\beta_1,\beta_2)}\right) \tilde{J}_n^{(\beta_1,\beta_2)}\left(t_j^{(\beta_1,\beta_2)}\right) \eta_j^{(\beta_1,\beta_2)} = \gamma_m^{(\beta_1,\beta_2)} \delta_{m,n}.$$

By the aid of shifted Jacobi polynomial $\tilde{J}_n^{(\beta_1,\beta_2)}(t)$ we define the shifted generalized Jacobi function of degree n as (see [41])

$$P_n^{(\beta_1,\beta_2)}(t) = t^{\beta_2} \tilde{J}_n^{(\beta_1,\beta_2)}(t), \quad \beta_1, \ \beta_2 > -1, \quad t \in \big(t_0; T\big). \tag{26}$$

By virtue of (24) and (26), we see that

$$\int_{t_0}^{T} P_n^{(\beta_1,\,\beta_2)}(t)\,P_m^{(\beta_1,\,\beta_2)}(t)\,\Lambda_T^{(\beta_1,\,-\beta_2)}(t)\,d\,t = \left(\frac{T+t_0}{2}\right)^{\beta_1+\beta_2+1}\gamma_m^{(\beta_1,\,\beta_2)}\delta_{m,\,n}.$$

By virtue of (25), for any $\varphi\,(t) = t^{2\,\beta_2}\phi\,(t)$ we have

$$\int_{t_0}^{T}\varphi\,(t)\,\Lambda_T^{(\beta_1,\,-\beta_2)}(t)\,d\,t = \left(\frac{T+t_0}{2}\right)^{\beta_1+\beta_2+1}\sum_{j=0}^{N}\left(t_j^{(\beta_1,\,\beta_2)}\right)^{-2\beta_2}\varphi\left(t_j^{(\beta_1,\,\beta_2)}\right)\eta_j^{(\beta_1,\,\beta_2)}. \tag{27}$$

By the aid of (27) we introduce the inner product in $L^2_{\Lambda_T^{(\beta_1,\,-\beta_2)}}(0;\,T)$ as

$$\langle f,\,g\rangle_{\Lambda_T^{(\beta_1,\,-\beta_2)}} = \left(\frac{T+t_0}{2}\right)^{\beta_1+\beta_2+1}\sum_{j=0}^{N}\left(t_j^{(\beta_1,\,\beta_2)}\right)^{-2\beta_2} f\left(t_j^{(\beta_1,\,\beta_2)}\right)\,g\left(t_j^{(\beta_1,\,\beta_2)}\right)\eta_j^{(\beta_1,\,\beta_2)}.$$

We need also to introduce finite N-dimensional fractional polynomial space [41]

$$\tilde{F}_N^{(\beta_2)}(t_0;\,T) = \left\{t^{\beta_2}\,\psi\,(t):\ \psi\,(t)\in\tilde{P}_N^{(\beta_1,\,\beta_2)}(t_0;\,T)\right\} = \mathrm{span}\left\{P_n^{(\beta_1,\,\beta_2)}(t):\ 0\le n\le N\right\}.$$

Then we note that for any $\phi,\,\psi\in\tilde{F}_N^{(\beta_2)}\big(t_0;\,T\big)$ hold the equalities

$$(\phi,\,\psi)_{\Lambda_T^{(\beta_1,\,-\beta_2)}} = \langle\phi,\,\psi\rangle_{\Lambda_T^{(\beta_1,\,-\beta_2)}}\,.$$

Now in integral equation (3) we make variable transformation $s = \frac{t\,\tau}{T},\ \tau\in\big(t_0;\,T\big)$. Then the we describe integral equation (3) as

$$x\,(t) = \Im\,(t;\,x) \equiv G\,(t) + V\,x\,(t) = G\,(t) + \left(\frac{t}{T}\right)^{\alpha}\int_{t_0}^{t}(T-\tau)^{\alpha-1}E_{\alpha,\,\alpha}\left(-\omega\left(\frac{t}{T}\right)^{\alpha}(T-\tau)^{\alpha}\right)\times$$
$$\times f\left(\frac{t\,\tau}{T},\,x\left(\frac{t\,\tau}{T}\right),\,\max\left\{x\,(\theta)\,|\,\theta\in\left[\frac{q_1 t\,\tau}{T};\ \frac{q_2 t\,\tau}{T}\right]\right\}\right)d\,\tau. \tag{28}$$

For the Hilfer fractional operator's order $0<\alpha<1$ we denote $\alpha-1=-\mu$, where $0<\mu=\mathrm{const}$ Then for $U,\,\varphi\in\tilde{F}_N^{(1-\mu)}(t_0;\,T)$ we apply the generalized Jacobi–Galerkin method to equation (28):

$$(U,\,\varphi)_{\Lambda_T^{(-\mu,\,\mu-1)}} = (G,\,\varphi)_{\Lambda_T^{(-\mu,\,\mu-1)}} + (V\,U,\,\varphi)_{\Lambda_T^{(-\mu,\,\mu-1)}}\,. \tag{29}$$

We set

$$U\,(t) = \sum_{m=0}^{N}x_m(t)\,P_m^{(-\mu,\,1-\mu)}(t),\quad \varphi\,(t) = P_n^{(-\mu,\,1-\mu)}(t),\quad 0\le m,\ n\le N.$$

Then for (29) we have

$$\sum_{m=0}^{N}x_m(t)\,\left(P_m^{(-\mu,\,1-\mu)}(t),\,P_n^{(-\mu,\,1-\mu)}(t)\right)_{\Lambda_T^{(-\mu,\,\mu-1)}} =$$
$$= \left(G\,(t),\,P_n^{(-\mu,\,1-\mu)}(t)\right)_{\Lambda_T^{(-\mu,\,\mu-1)}} + \left(V\,U\,(t),\,P_n^{(-\mu,\,1-\mu)}(t)\right)_{\Lambda_T^{(-\mu,\,\mu-1)}}\,.$$

Hence, we come to nonlinear system

$$\bar{B}\,\bar{x} = \bar{G} + \bar{\vartheta}\,(\bar{x}), \tag{30}$$

after introducing designations:

$$\bar{x} = (x_0,\,x_1,\,\ldots,\,\bar{x}_N)^T,\quad B = (b_{n\,m})_{0\le n\,m\le N}\,,$$
$$b_{n\,m} = \left(P_m^{(-\mu,\,1-\mu)}(t),\,P_n^{(-\mu,\,1-\mu)}(t)\right)_{\Lambda_T^{(-\mu,\,\mu-1)}} = \left(\tfrac{T+t_0}{2}\right)^{2-2\,\mu}\gamma_m^{(-\,\mu,\,1-\,\mu)}\delta_{m,\,n},$$
$$\bar{G} = (G_0,\,G_1,\,\ldots,\,G_N)^T,\ \ G_n(t) = \left(G\,(t),\,P_n^{(-\mu,\,1-\mu)}(t)\right)_{\Lambda_T^{(-\mu,\,\mu-1)}}\,,$$
$$\bar{\vartheta}\,(\bar{x}) = (\vartheta_0,\,\vartheta_1,\,\ldots,\,\vartheta_N)^T,\ \ \vartheta_n(x) = \left(V\,U\,(t),\,P_n^{(-\mu,\,1-\mu)}(t)\right)_{\Lambda_T^{(-\mu,\,\mu-1)}}\,,$$

where by $(u_0,\,u_1,\,\ldots,\,u_N)^T$ we denoted the transposition of the matrix $(u_0,\,u_1,\,\ldots,\,u_N)$.

We use the quadrature formula

$$\langle f,\, g\rangle_{\Lambda_T^{(\beta_1,\,-\beta_2)}} = \left(\frac{T+t_0}{2}\right)^{\beta_1+\beta_2+1} \sum_{j=0}^{N} \left(t_j^{(\beta_1,\,\beta_2)}\right)^{-2\beta_2} f\left(t_j^{(\beta_1,\,\beta_2)}\right) g\left(t_j^{(\beta_1,\,\beta_2)}\right) \eta_j^{(\beta_1,\,\beta_2)}$$

to obtain approximate formulas:

$$G_n(t) \approx \left\langle G(t),\, P_n^{(-\mu,\,1-\mu)}(t)\right\rangle_{\Lambda_T^{(-\mu,\,\mu-1)}} =$$

$$= \left(\frac{T+t_0}{2}\right)^{2-2\mu} \sum_{j=0}^{N} \left(t_j^{(-\mu,\,1-\mu)}\right)^{2\mu-2} G\left(t_j^{(-\mu,\,1-\mu)}\right) P_n^{(-\mu,\,1-\mu)}\left(t_j^{(-\mu,\,1-\mu)}\right) \eta_j^{(-\mu,\,1-\mu)}, \tag{31}$$

$$\bar{\vartheta}(\bar{x}) \approx \frac{(T+t_0)^{2-2\mu}}{2^{2-2\mu}} \sum_{i,\,j=0}^{N} \left(\frac{t_i^{(-\mu,1-\mu)}}{T}\right)^{1-\mu} (T-\tau)^{-\mu} E_{1-\mu,\,1-\mu}\left(-\omega\left(\frac{t_i^{(-\mu,1-\mu)}}{T}\right)^{1-\mu}(T-\tau)^{1-\mu}\right) \times$$

$$\times f\left(t_{ij},\, U(t_{ij}),\, \max\{U(\theta)\,|\,\theta\in[q_1\cdot t_{ij};\, q_2\cdot t_{ij}]\}\right) \times$$

$$\times P_n^{(-\mu,\,1-\mu)}\left(t_i^{(-\mu,1-\mu)}\right) \eta_i^{(-\mu,\,1-\mu)} \eta_j^{(-\mu,\,0)}, \tag{32}$$

where $t_{ij} = \frac{t_i^{(-\mu,\,1-\mu)} t_j^{(-\mu,\,0)}}{T}$.

In approximately solving the system (30) one can use the Newton iterative method.

Illustrarive examples

As an example, we consider the simple equation of the form

$$D_{0+}^{\alpha,\,\beta} u(t) = \lambda\, u(t) + f(t), \;\; t\in(0;\,T)$$

with initial value condition

$$\lim_{t\to+0} J_{0+}^{1-\gamma} u(t) = u_0.$$

The solution of this problem has the form

$$u(t) = u_0\, t^{\gamma-1} E_{\alpha,\,\gamma}(\lambda t^{\alpha}) + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\,\alpha}\left(\lambda(t-s)^{\alpha}\right) f(s)ds, \tag{33}$$

where $\gamma=\alpha+\beta-\alpha\,\beta$.

Example 1. We consider cases $\alpha=\beta=\frac{1}{2}$, $f(t)=t^{\sigma}$, $\sigma>-1$. Since $\gamma=\frac{1}{2}+\frac{1}{2}-\frac{1}{2}\cdot\frac{1}{2}=\frac{3}{4}$, from (33) we have

$$u(t) = u_0\, t^{-\frac{1}{4}} E_{\frac{1}{2},\,\frac{3}{4}}\left(\lambda t^{\frac{1}{2}}\right) + \int_0^t (t-s)^{-\frac{1}{2}} E_{\frac{1}{2},\,\frac{1}{2}}\left(\lambda(t-s)^{\frac{1}{2}}\right) s^{\sigma} ds. \tag{34}$$

Taking into account

$$\frac{1}{\Gamma(\nu)}\int_0^z (z-t)^{\nu-1} E_{\alpha,\,\beta}(\lambda t^{\alpha})\, t^{\beta-1}dt = z^{\beta+\nu-1} E_{\alpha,\,\beta+\nu}(\lambda z^{\alpha}),\;\; \nu>0,\;\; \beta>0,$$

we calculate the integral in (34):

$$\int_0^t (t-s)^{-\frac{1}{2}} E_{\frac{1}{2},\,\frac{1}{2}}\left(\lambda(t-s)^{\frac{1}{2}}\right) s^{\sigma} ds = \Gamma(\sigma+1) t^{\frac{1}{2}+\sigma} E_{\frac{1}{2},\,\frac{3}{2}+\sigma}\left(\lambda t^{\frac{1}{2}}\right). \tag{35}$$

Substituting (35) into (34), we obtain

$$u(t) = \frac{u_0}{\sqrt[4]{t}} E_{\frac{1}{2},\, \frac{3}{4}} \left(\lambda\sqrt{t}\right) + \Gamma(\sigma+1)\, t^{\sigma} \sqrt{t} E_{\frac{1}{2},\, \frac{3}{2}+\sigma} \left(\lambda\sqrt{t}\right). \tag{36}$$

In particular case, when $\sigma = 0$, from (36) yields

$$u(t) = \frac{u_0}{\sqrt[4]{t}} E_{\frac{1}{2},\, \frac{3}{4}} \left(\lambda\sqrt{t}\right) + \sqrt{t} E_{\frac{1}{2},\, \frac{3}{2}} \left(\lambda\sqrt{t}\right). \tag{37}$$

Taking into account

$$E_{\alpha,\, \mu}(z) = \frac{1}{\Gamma(\mu)} + z\, E_{\alpha,\, \alpha+\mu}(z), \quad \alpha > 0, \quad \mu > 0,$$

we obtain

$$\sqrt{t} E_{\frac{1}{2},\, \frac{3}{2}} \left(\lambda\sqrt{t}\right) = \frac{1}{\lambda} E_{\frac{1}{2},\, 1} \left(\lambda\sqrt{t}\right) - \frac{1}{\lambda}.$$

Therefore (37) takes form

$$u(t) = \frac{u_0}{\sqrt[4]{t}} E_{\frac{1}{2},\, \frac{3}{4}} \left(\lambda\sqrt{t}\right) + \frac{1}{\lambda} E_{\frac{1}{2},\, 1} \left(\lambda\sqrt{t}\right) - \frac{1}{\lambda}.$$

Since $E_{\frac{1}{2},\, 1}(z) = \cosh\sqrt{z}$, we present the solution as

$$u(t) = \frac{u_0}{\sqrt[4]{t}} E_{\frac{1}{2},\, \frac{3}{4}} \left(\lambda\sqrt{t}\right) + \frac{1}{\lambda} \left[\cosh\left(\sqrt{\lambda\sqrt{t}}\right) - 1\right].$$

Example 2. The case of Caputo operator: $\alpha = \frac{1}{2}$, $\beta = 1$, $f(t) = t^{\sigma}$, $\sigma > -1$.
Since $\gamma = \frac{1}{2} + 1 - \frac{1}{2} \cdot 1 = 1$, from (33) we have

$$u\,(t) = u_0 E_{\frac{1}{2},\, 1} \left(\lambda t^{\frac{1}{2}}\right) + \int\limits_0^t (t-s)^{-\frac{1}{2}} E_{\frac{1}{2},\, \frac{1}{2}} \left(\lambda (t-s)^{\frac{1}{2}}\right) s^{\sigma} ds. \tag{38}$$

Taking (35) into account, from (38) we obtain

$$u\,(t) = u_0 E_{\frac{1}{2},\, 1} \left(\lambda\sqrt{t}\right) + \Gamma(\sigma+1) t^{\sigma} \sqrt{t} E_{\frac{1}{2},\, \frac{3}{2}+\sigma} \left(\lambda\sqrt{t}\right). \tag{39}$$

We are looking for real solutions. Since $E_{\frac{1}{2},\, 1}(z) = \cosh\sqrt{z}$, then for $\lambda \geq 0$ we present the solution (39) as

$$u\,(t) = u_0 \cosh\left(\sqrt{\lambda\sqrt{t}}\right) + \Gamma(\sigma+1)\, t^{\sigma} \sqrt{t} E_{\frac{1}{2},\, \frac{3}{2}+\sigma} \left(\lambda\sqrt{t}\right).$$

For the cases $\sigma = 0$ and $\lambda > 0$ we have

$$u\,(t) = u_0\, \cosh\left(\sqrt{\lambda\sqrt{t}}\right) + \sqrt{t} E_{\frac{1}{2},\, \frac{3}{2}} \left(\lambda\sqrt{t}\right). \tag{40}$$

Taking

$$E_{\alpha,\, \mu}(z) = \frac{1}{\Gamma(\mu)} + z\, E_{\alpha,\, \alpha+\mu}(z), \quad \alpha > 0, \quad \mu > 0$$

into account, the last summand easily presents as

$$\sqrt{t} E_{\frac{1}{2},\, \frac{3}{2}} \left(\lambda\sqrt{t}\right) = \frac{1}{\lambda} E_{\frac{1}{2},\, 1} \left(\lambda\sqrt{t}\right) - \frac{1}{\lambda}.$$

So, taking $E_{\frac{1}{2},\, 1}(z) = \cosh\sqrt{z}$ into account, from representation (40) we obtain the simple form of solution

$$u\,(t) = \frac{1}{\lambda} \left[(\lambda\, u_0 + 1)\, \cosh\left(\sqrt{\lambda\sqrt{t}}\right) - 1\right].$$

Now we consider an example of a nonlinear differential equation.

Example 3. The equation

$$ {}_C D_{0t}^{\alpha} y(t) = \frac{1}{5}\Gamma(\alpha+1)t^{-2\alpha}\left(y^2(t) + 4\cdot\max\left\{y^2(\theta)\,|\,\theta\in\left[\frac{1}{3}\,t;\,t\right]\right\}\right),\ \ \alpha > \frac{1}{2} \tag{41} $$

on the interval (0; 1) has a solution

$$ y(t) = t^{\alpha}. \tag{42} $$

Indeed,

$$ \frac{1}{5}\Gamma(\alpha+1)t^{-2\alpha}\left(y^2(t) + 4\cdot\max\left\{y^2(\theta)\,|\,\theta\in\left[\frac{1}{3}\,t;\,t\right]\right\}\right) = \frac{1}{5}\,\Gamma(\alpha+1)t^{-2\alpha}\left(5\,t^{2\alpha}\right) = \Gamma(\alpha+1) \tag{43} $$

and

$$ {}_C D_{0t}^{\alpha} y(t) = {}_C\ D_{0t}^{\alpha}\left(t^{\alpha}\right) = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+1-\alpha)}t^{\alpha-\alpha} = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+1-\alpha)} = \Gamma(\alpha+1). \tag{44} $$

From (43) and (44) we come to the conclusion that function (42) is a solution of the Caputo fractional differential equation (41) on the interval (0; 1).

Remark. The function (42) is not a solution of fractional differential equation (41) on the semiaxis $(1;\infty)$. If we consider the solvability of the differential equation (41) on the entire positive semiaxis $\mathbb{R}^+\equiv(0;\infty)$, then this equation suffers a discontinuity of the first kind at the point $t=1$.

Conclusion

In this paper we consider the questions of unique solvability of initial value problem for a nonlinear fractional differential equation (1) with maxima on the given segment $(t_0;\,T)$. We reduce this initial value problem to the fractional order nonlinear integral equation of Volterra type. Then we used the method of successive approximation and proved the theorem on existence and uniqueness of solution of the problem under consideration. We apply the generalized Jacobi–Galerkin method as a numerical realization of solution of the fractional order nonlinear integral equation (3). We make a variable transformation in integral equation (3): $s=\frac{t\,\tau}{T}$, $\quad \tau\in\left(t_0;\,T\right)$. Applying the generalized Jacobi–Galerkin method to equation (28), we come to the system (30). By using the quadrature formula we obtain the necessary approximation formulas (31) and (32).

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Сызықтық емес оң жағы бар Хильфер типтес бөлшек дифференциалдық теңдеу туралы

Мақалада максималды сызықтық емес бөлшек дифференциалдық теңдеу үшін бастапқы есепті біркелкі шешу және сандық іске асыру мәселелері қарастырылды. Дирихле формуласына негізделген қарапайым интегралдық түрлендіруді қолдана отырып, қарастырылып отырған бастапқы міндет Вольтерр типіндегі сызықты емес бөлшек интегралдық теңдеуге дейін азаяды. Қарастырылған сегментте берілген бастапқы есепті шешудің бар болуы мен бірегейлігі теоремасы дәлелденді. Шешімді сандық түрде жүзеге асыру үшін Галеркин Якобидің жалпыланған спектрлік әдісі қолданылған. Көрнекі мысалдар келтірілген.

Кілт сөздер: қарапайым дифференциалдық теңдеу, максимумдармен теңдеу, Хильфер операторы, бір мәнді шешімділік, Галеркин Якобидің жалпыланған спектрлік әдісі.

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Об одном дробном дифференциальном уравнении типа Хильфера с нелинейной правой частью

В статье рассмотрены вопросы однозначной разрешимости и численной реализации начальной задачи для нелинейного дробного дифференциального уравнения типа Хильфера с максимумами. С помощью несложного интегрального преобразования, основанного на формуле Дирихле, рассматриваемая начальная задача сведена к нелинейному дробно-интегральному уравнению типа Вольтерра. Доказана теорема существования и единственности решения заданной начальной задачи на рассматриваемом отрезке. Для численной реализации решения применен обобщенный спектральный метод Галеркина-Якоби. Приведены наглядные примеры.

Ключевые слова: обыкновенное дифференциальное уравнение, уравнение с максимумами, оператор Хильфера, однозначная разрешимость, обобщенный спектральный метод Галеркина-Якоби.

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