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## Partial best approximations and the absolute Cesaro summability of multiple Fourier series

The article is devoted to the problem of absolute Cesaro summability of multiple trigonometric Fourier series. Taking a central place in the theory of Fourier series this problem was developed quite widely in the one-dimensional case and the fundamental results of this theory are set forth in the famous monographs by N.K. Bari, A. Zigmund, R. Edwards, B.S. Kashin and A.A. Saakyan [1–4]. In the case of multiple series, the corresponding theory is not well developed. The multidimensional case has own specifics and the analogy with the one-dimensional case does not always be unambiguous and obvious. In this article, we obtain sufficient conditions for the absolute summability of multiple Fourier series of the function  $f \in L_q(I_s)$  in terms of partial best approximations of this function. Four theorems are proved and four different sufficient conditions for the  $|C; \bar{\beta}|_\lambda$ -summability of the Fourier series of the function  $f$  are obtained. In the first theorem, a sufficient condition for the absolute  $|C; \bar{\beta}|_\lambda$ -summability of the Fourier series of the function  $f$  is obtained in terms of the partial best approximation of this function which consists of  $s$  conditions, in the case when  $\beta_1 = \dots = \beta_s = \frac{1}{q'}$ . Other sufficient conditions are obtained for double Fourier series. Sufficient conditions for the  $|C; \beta_1; \beta_2|_\lambda$ -summability of the Fourier series of the function  $f \in L_q(I_2)$  are obtained in the cases  $\beta_1 = \frac{1}{q'}$ ,  $-1 < \beta_2 < \frac{1}{q'}$  (in the second theorem),  $\frac{1}{q'} < \beta_1 < +\infty$ ,  $\beta_2 = \frac{1}{q'}$  (in the third theorem),  $-1 < \beta_1 < \frac{1}{q'}$ ,  $\frac{1}{q'} < \beta_2 < +\infty$  (in the fourth theorem).

*Keywords:* trigonometric series, Fourier series, Lebesgue space, partial best approximation of a function, absolute summability of the series.

### Introduction

Let  $R^s$  be a  $s$ -dimensional Euclidean space of points  $\bar{x} = (x_1, x_2, \dots, x_s)$  with real coordinates;  $I_s = \{\bar{x} \in R^s : 0 \leq x_j \leq 2\pi, j = 1, 2, \dots, s\}$  is a  $s$ -dimensional cube.

We put  $\gamma_i(nx) = \begin{cases} \cos nx, & i = 1, \\ \sin nx, & i = 2. \end{cases}$

We will consider the following multiple series

$$\sum_{\bar{n} \geq 1} B_{\bar{n}}(\bar{x}) = \sum_{n_1=1}^{\infty} \dots \sum_{n_s=1}^{\infty} B_{n_1, \dots, n_s}(x_1, \dots, x_s), \quad (1)$$

where  $\bar{n} \geq \bar{\alpha} = (\alpha_1, \dots, \alpha_s)$  means  $n_j \geq \alpha_j$  for all  $j = 1, 2, \dots, s$ ;

$$B_{\bar{n}}(\bar{x}) = \sum_{\bar{1} \leq \bar{i} \leq \bar{2}} a_{\bar{n}}^{(\bar{i})} \cdot \prod_{\nu=1}^s \gamma_{i_\nu}(n_\nu x_\nu).$$

Assume that  $A_n^{(\beta)} = \frac{(\beta+1)(\beta+2)\dots(\beta+n)}{n!}$ ,  $\beta \in R$ , where  $n$  is natural number.

The sum

$$\sigma_{\bar{n}}^{(\bar{\beta})}(\bar{x}) = \sum_{\bar{1} \leq \bar{k} \leq \bar{n}} \prod_{j=1}^s A_{n_j - k_j}^{(\beta_j - 1)} \left( A_{n_j}^{(\beta_j)} \right)^{-1} B_{\bar{k}}(\bar{x})$$

is called  $(C; \bar{\beta}) \equiv (C; \beta_1, \dots, \beta_s)$  average of the series (1).

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For a given number  $b_{\bar{n}}$  we define the mixed difference as follows:

$$\Delta b_{\bar{n}} = \sum_{\bar{0} \leq \bar{\varepsilon} \leq \bar{1}} (-1)^{s - \sum_{i=1}^s \varepsilon_i} \cdot b_{\bar{n} - \bar{1} + \bar{\varepsilon}}.$$

Here  $\bar{\varepsilon} = (\varepsilon_1, \dots, \varepsilon_s)$ . The series (1) is called  $|C; \bar{\beta}|_\lambda$ -summable (or absolutely Cesaro summable),  $\lambda \geq 1$ , at the point  $\bar{x} \in I_s$ , if [5]:

$$\sum_{\bar{n} \geq \bar{1}} \left| \Delta \sigma_{\bar{n}}^{(\bar{\beta})}(\bar{x}) \right|^\lambda \cdot \prod_{j=1}^s n_j^{\lambda-1} < +\infty.$$

Further, we put  $\rho_{\bar{k}} = \sqrt{\sum_{\bar{1} \leq \bar{i} \leq \bar{2}} \left| a_{\bar{k}}^{(\bar{i})} \right|^2}$ .

By  $L_q(I_s)$  we denote the space of all Lebesgue measurable,  $2\pi$ -periodic in each variable functions  $f(\bar{x})$ , for which

$$\|f\|_q = \left( \int_{I_s} |f(\bar{x})|^q d\bar{x} \right)^{\frac{1}{q}} < +\infty, \quad 1 \leq q < +\infty.$$

Let

$$E_{n_1, \dots, n_s}(f)_q = \inf_{T_{n_1, \dots, n_s}} \|f(\cdot) - T_{n_1, \dots, n_s}(\cdot)\|_q$$

be the full best approximation [6] of the function  $f$  by trigonometric polynomials of order not exceeding  $n_j$  in variables  $x_j (j = 1, \dots, s)$ . Let us also consider the partial best approximation of the function  $f$  which is determined by the formula [6]:

$$E_{n, \infty}^{(j)}(f)_q = \inf_{T_n} \|f(x_1, \dots, x_s) - T_n(x_1, \dots, x_{j-1}, (x_j), x_{j+1}, \dots, x_s)\|_q,$$

where  $T_n(x_1, \dots, x_{j-1}, (x_j), x_{j+1}, \dots, x_s)$  is a trigonometric polynomial in the variable  $x_j$  of order not exceeding  $n$  with coefficients from the space  $L_q(I_{s-1})$ .

It is known that [6; 44]:

$$E_{n_1, \dots, n_s}(f)_q \leq C \cdot \sum_{j=1}^s E_{n_j, \infty}^{(j)}(f)_q, \quad 1 \leq q < \infty.$$

Conditions for the absolute summability of series (1) in the case  $\lambda = 1, s = 2, 0 < \beta_j < \frac{1}{2}, j = 1, 2$  were investigated by I.E. Zhak and M.F. Timan [7], and the questions of  $|C; \bar{\beta}|_\lambda$  summability of the Fourier series of the function  $f \in L_2(I_s)$  were studied by Yu.A. Ponomarenko, M.F. Timan [5], and in the one-dimensional case these questions were studied by I. Szalay [8]. Questions of the absolute summability of multiple trigonometric series were also investigated in [9–18].

### Results

*Theorem 1.* Let  $1 < q \leq 2, 1 \leq \lambda \leq q, \beta_1 = \dots = \beta_s = \frac{1}{q}, \frac{1}{q} + \frac{1}{q'} = 1$ . If  $f \in L_q(I_s)$  and

$$\sum_{n=3}^{\infty} n^{\frac{s\lambda}{q}(2-q)-1} (\ln n)^{\frac{s\lambda}{q}} \left( E_{n, \infty}^{(j)}(f)_q \right)^\lambda < +\infty, \quad j = 1, \dots, s,$$

then the Fourier series of the function  $f \in L_q(I_s)$  will be  $|C; \bar{\beta}|_\lambda$ -summable almost everywhere on  $I_2$ .

*Proof.* We prove the theorem for  $s = 2$  (the methods are similar for the higher dimensions). In [9] it was proved that if  $1 < q \leq 2, 1 \leq \lambda \leq q, \beta_1 = \beta_2 = \frac{1}{q}, \frac{1}{q} + \frac{1}{q'} = 1$  and

$$\sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \left( \sum_{k_1=2^{n_1+1}}^{2^{n_1+1}} \sum_{k_2=2^{n_2+1}}^{2^{n_2+1}} \rho_{k_1 k_2}^q \cdot \ln k_1 \ln k_2 \right)^{\frac{\lambda}{q}} < +\infty,$$

then a double series of the form (1) will be  $|C; \beta_1; \beta_2|_\lambda$ -summable almost everywhere on  $I_2$ .

Let's estimate the last series. For this we use the following inequality

$$\begin{aligned} & \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \left( \sum_{k_1=2^{n_1+1}}^{2^{n_1+1}} \sum_{k_2=2^{n_2+1}}^{2^{n_2+1}} \rho_{k_1 k_2}^q \cdot \ln k_1 \ln k_2 \right)^{\frac{\lambda}{q}} \leq \\ & \leq \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{n_1} \left( \sum_{k_1=2^{n_1+1}}^{2^{n_1+1}} \sum_{k_2=2^{n_2+1}}^{2^{n_2+1}} \rho_{k_1 k_2}^q \cdot \ln k_1 \ln k_2 \right)^{\frac{\lambda}{q}} + \\ & + \sum_{n_2=0}^{\infty} \sum_{n_1=0}^{n_2} \left( \sum_{k_1=2^{n_1+1}}^{2^{n_1+1}} \sum_{k_2=2^{n_2+1}}^{2^{n_2+1}} \rho_{k_1 k_2}^q \cdot \ln k_1 \ln k_2 \right)^{\frac{\lambda}{q}}. \end{aligned} \quad (2)$$

To estimate the first term, we apply Holder's inequalities to the sum over  $n_2$  for  $\theta = \frac{q}{\lambda}, \frac{1}{\theta} + \frac{1}{\theta'} = 1$ . We get

$$\begin{aligned} & \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{n_1} \left( \sum_{k_1=2^{n_1+1}}^{2^{n_1+1}} \sum_{k_2=2^{n_2+1}}^{2^{n_2+1}} \rho_{k_1 k_2}^q \cdot \ln k_1 \ln k_2 \right)^{\frac{\lambda}{q}} \leq \\ & \leq C \cdot \sum_{n_1=0}^{\infty} 2^{n_1(2-q)\frac{\lambda}{q}} (n_1+1)^{\frac{\lambda}{q}} \cdot \sum_{n_2=0}^{n_1} 2^{n_2(2-q)\frac{\lambda}{q}} (n_2+1)^{\frac{\lambda}{q}} \left( \sum_{k_1=2^{n_1+1}}^{2^{n_1+1}} \sum_{k_2=2^{n_2+1}}^{2^{n_2+1}} (k_1 k_2)^{q-2} \rho_{k_1 k_2}^q \right)^{\frac{\lambda}{q}} \leq \\ & \leq C \cdot \sum_{n_1=0}^{\infty} 2^{n_1(2-q)\frac{\lambda}{q}} (n_1+1)^{\frac{\lambda}{q}} \cdot \left( \sum_{n_2=0}^{n_1} \left( 2^{n_2(2-q)\frac{\lambda}{q}} (n_2+1)^{\frac{\lambda}{q}} \right)^{\theta'} \right)^{\frac{1}{\theta}} \times \\ & \times \left( \sum_{n_2=0}^{n_1} \sum_{k_1=2^{n_1+1}}^{2^{n_1+1}} \sum_{k_2=2^{n_2+1}}^{2^{n_2+1}} (k_1 k_2)^{q-2} \rho_{k_1 k_2}^q \right)^{\frac{\lambda}{q}} \leq \\ & \leq C \cdot \sum_{n_1=0}^{\infty} 2^{n_1(2-q)\frac{\lambda}{q}} (n_1+1)^{\frac{\lambda}{q}} \cdot \left( \sum_{n_2=0}^{n_1} 2^{n_2(2-q)\frac{\lambda}{q-\lambda}} (n_2+1)^{\frac{\lambda}{q-\lambda}} \right)^{\frac{q-\lambda}{q}} \times \\ & \times \left( \sum_{k_1=2^{n_1+1}}^{2^{n_1+1}} \sum_{k_2=2}^{2^{n_1+1}} (k_1 k_2)^{q-2} \rho_{k_1 k_2}^q \right)^{\frac{\lambda}{q}}. \end{aligned} \quad (3)$$

Now let's estimate the following sum:

$$\sum_{n_2=0}^{n_1} 2^{n_2(2-q)\frac{\lambda}{q-\lambda}} (n_2+1)^{\frac{\lambda}{q-\lambda}} \leq (n_1+1)^{\frac{\lambda}{q-\lambda}} \sum_{n_2=0}^{n_1} 2^{n_2(2-q)\frac{\lambda}{q-\lambda}} \leq C(n_1+1)^{\frac{\lambda}{q-\lambda}} 2^{n_1(2-q)\frac{\lambda}{q-\lambda}}.$$

Using this inequality from (3), we obtain

$$\begin{aligned} & \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{n_1} \left( \sum_{k_1=2^{n_1+1}}^{2^{n_1+1}} \sum_{k_2=2^{n_2+1}}^{2^{n_2+1}} \rho_{k_1 k_2}^q \cdot \ln k_1 \ln k_2 \right)^{\frac{\lambda}{q}} \leq \\ & \leq C \cdot \sum_{n_1=0}^{\infty} 2^{2n_1(2-q)\frac{\lambda}{q}} (n_1+1)^{2\frac{\lambda}{q}} \cdot \left( \sum_{k_1=2^{n_1+1}}^{2^{n_1+1}} \sum_{k_2=2}^{2^{n_1+1}} (k_1 k_2)^{q-2} \rho_{k_1 k_2}^q \right)^{\frac{\lambda}{q}}. \end{aligned}$$

It is similarly proved

$$\sum_{n_2=0}^{\infty} \sum_{n_1=0}^{n_2} \left( \sum_{k_1=2^{n_1+1}}^{2^{n_1+1}} \sum_{k_2=2^{n_2+1}}^{2^{n_2+1}} \rho_{k_1 k_2}^q \cdot \ln k_1 \ln k_2 \right)^{\frac{\lambda}{q}} \leq$$

$$\leq C \cdot \sum_{n_2=0}^{\infty} 2^{2n_2(2-q)\frac{\lambda}{q}} (n_2 + 1)^{2\frac{\lambda}{q}} \cdot \left( \sum_{k_1=2^{n_2+1}}^{2^{n_2+1}} \sum_{k_2=2}^{2^{n_2+1}} (k_1 k_2)^{q-2} \rho_{k_1 k_2}^q \right)^{\frac{\lambda}{q}}.$$

Taking into account the last two inequalities from (2), we obtain

$$\begin{aligned} & \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \left( \sum_{k_1=2^{n_1+1}}^{2^{n_1+1}} \sum_{k_2=2^{n_2+1}}^{2^{n_2+1}} \rho_{k_1 k_2}^q \cdot \ln k_1 \ln k_2 \right)^{\frac{\lambda}{q}} \leq \\ & \leq C \cdot \sum_{n=0}^{\infty} 2^{2n(2-q)\frac{\lambda}{q}} (n + 1)^{2\frac{\lambda}{q}} \cdot \left( \sum_{k_1=1}^{2^{n+1}} \sum_{k_2=2^{n+1}}^{2^{n+1}} (k_1 k_2)^{q-2} \rho_{k_1 k_2}^q \right)^{\frac{\lambda}{q}}. \end{aligned} \tag{4}$$

Further, applying the Hardy-Littlewood theorem [19] and using the monotonicity of the logarithmic function and the best approximation from (4), we have

$$\begin{aligned} & \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \left( \sum_{k_1=2^{n_1+1}}^{2^{n_1+1}} \sum_{k_2=2^{n_2+1}}^{2^{n_2+1}} \rho_{k_1 k_2}^q \cdot \ln k_1 \ln k_2 \right)^{\frac{\lambda}{q}} \leq \\ & \leq C \cdot \sum_{n=0}^{\infty} 2^{2n(2-q)\frac{\lambda}{q}} (n + 1)^{2\frac{\lambda}{q}} \cdot E_{2^n, 2^n}^\lambda(f)_q < C \cdot E_{1,1}^\lambda(f)_q + C \cdot E_{2,2}^\lambda(f)_q + \\ & + C \cdot \sum_{n=2}^{\infty} (2^n)^{\frac{2\lambda}{q}(2-q)-1} (n - 1)^{\frac{2\lambda}{q}} \cdot E_{2^n, 2^n}^\lambda(f)_q \cdot \sum_{m=2^{n-1}+1}^{2^n} 1 \leq \\ & \leq C \cdot E_{1,1}^\lambda(f)_q + C \cdot E_{2,2}^\lambda(f)_q + C \sum_{m=3}^{\infty} m^{\frac{2\lambda}{q}(2-q)-1} (\ln m)^{\frac{2\lambda}{q}} \cdot E_{m,m}^\lambda(f)_q. \end{aligned}$$

The theorem is proved.

*Theorem 2.* Let  $1 < q \leq 2$ ,  $1 \leq \lambda \leq q$ ,  $\beta_1 = \frac{1}{q'}$ ,  $-1 < \beta_2 < \frac{1}{q'}$ ,  $\frac{1}{q} + \frac{1}{q'} = 1$ . If  $f \in L_q(I_2)$  and

$$\sum_{n=3}^{\infty} n^{\frac{\lambda}{q}(3-q(1+\beta_2))-1} (\ln n)^{\frac{\lambda}{q}} \left( E_{n,\infty}^{(j)}(f)_q \right)^\lambda < +\infty, \quad j = 1, 2,$$

then the Fourier series of function  $f \in L_q(I_2)$  will be  $|C; \beta_1; \beta_2|_\lambda$ -summable almost everywhere on  $I_2$ .

*Proof.* Since  $1 < q \leq 2$ ,  $1 \leq \lambda \leq q$ ,  $\beta_1 = \frac{1}{q'}$ ,  $-1 < \beta_2 < \frac{1}{q'}$ ,  $\frac{1}{q} + \frac{1}{q'} = 1$ , then for  $|C; \beta_1; \beta_2|_\lambda$ -summability almost everywhere on  $I_2$  of a double series of the form (1) is sufficient that [12]

$$\sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \left( \sum_{k_1=2^{n_1+1}}^{2^{n_1+1}} \sum_{k_2=2^{n_2+1}}^{2^{n_2+1}} \rho_{k_1 k_2}^q \cdot \ln k_1 \cdot k_2^{q(1-\beta_2)-1} \right)^{\frac{\lambda}{q}} < +\infty.$$

It is known that

$$\begin{aligned} & \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \left( \sum_{k_1=2^{n_1+1}}^{2^{n_1+1}} \sum_{k_2=2^{n_2+1}}^{2^{n_2+1}} \rho_{k_1 k_2}^q \cdot \ln k_1 \cdot k_2^{q(1-\beta_2)-1} \right)^{\frac{\lambda}{q}} \leq \\ & \leq \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{n_1} \left( \sum_{k_1=2^{n_1+1}}^{2^{n_1+1}} \sum_{k_2=2^{n_2+1}}^{2^{n_2+1}} \rho_{k_1 k_2}^q \cdot \ln k_1 \cdot k_2^{q(1-\beta_2)-1} \right)^{\frac{\lambda}{q}} + \\ & + \sum_{n_2=0}^{\infty} \sum_{n_1=0}^{n_2} \left( \sum_{k_1=2^{n_1+1}}^{2^{n_1+1}} \sum_{k_2=2^{n_2+1}}^{2^{n_2+1}} \rho_{k_1 k_2}^q \cdot \ln k_1 \cdot k_2^{q(1-\beta_2)-1} \right)^{\frac{\lambda}{q}}. \end{aligned} \tag{5}$$

Let us estimate the first term. To do this, we apply Holder's inequalities to the sum over  $n_2$  for  $\theta = \frac{q}{\lambda}$ ,  $\frac{1}{\theta} + \frac{1}{\theta'} = 1$ . We get

$$\begin{aligned} & \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{n_1} \left( \sum_{k_1=2^{n_1+1}}^{2^{n_1+1}} \sum_{k_2=2^{n_2+1}}^{2^{n_2+1}} \rho_{k_1 k_2}^q \cdot \ln k_1 \cdot k_2^{q(1-\beta_2)-1} \right)^{\frac{\lambda}{q}} \leq \\ & \leq C \cdot \sum_{n_1=0}^{\infty} 2^{n_1(2-q)\frac{\lambda}{q}} (n_1 + 1)^{\frac{\lambda}{q}} \cdot \sum_{n_2=0}^{n_1} 2^{n_2(1-q\beta_2)\frac{\lambda}{q}} \left( \sum_{k_1=2^{n_1+1}}^{2^{n_1+1}} \sum_{k_2=2^{n_2+1}}^{2^{n_2+1}} (k_1 k_2)^{q-2} \rho_{k_1 k_2}^q \right)^{\frac{\lambda}{q}} \leq \\ & \leq C \cdot \sum_{n_1=0}^{\infty} 2^{n_1(3-q(1+\beta_2))\frac{\lambda}{q}} (n_1 + 1)^{\frac{\lambda}{q}} \left( \sum_{k_1=2^{n_1+1}}^{2^{n_1+1}} \sum_{k_2=2}^{2^{n_1+1}} (k_1 k_2)^{q-2} \rho_{k_1 k_2}^q \right)^{\frac{\lambda}{q}}. \end{aligned}$$

For the second term (5), we similarly obtain

$$\begin{aligned} & \sum_{n_2=0}^{\infty} \sum_{n_1=0}^{n_2} \left( \sum_{k_1=2^{n_1+1}}^{2^{n_1+1}} \sum_{k_2=2^{n_2+1}}^{2^{n_2+1}} \rho_{k_1 k_2}^q \cdot \ln k_1 \cdot k_2^{q(1-\beta_2)-1} \right)^{\frac{\lambda}{q}} \leq \\ & \leq C \cdot \sum_{n_2=0}^{\infty} 2^{n_2(3-q(1+\beta_2))\frac{\lambda}{q}} (n_2 + 1)^{\frac{\lambda}{q}} \cdot \left( \sum_{k_2=2^{n_1+1}}^{2^{n_2+1}} \sum_{k_1=2}^{2^{n_2+1}} (k_1 k_2)^{q-2} \rho_{k_1 k_2}^q \right)^{\frac{\lambda}{q}}. \end{aligned}$$

Therefore

$$\begin{aligned} & \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \left( \sum_{k_1=2^{n_1+1}}^{2^{n_1+1}} \sum_{k_2=2^{n_2+1}}^{2^{n_2+1}} \rho_{k_1 k_2}^q \cdot \ln k_1 \cdot k_2^{q(1-\beta_2)-1} \right)^{\frac{\lambda}{q}} \leq \\ & \leq C \cdot \sum_{n=0}^{\infty} 2^{n(3-q(1+\beta_2))\frac{\lambda}{q}} (n + 1)^{\frac{\lambda}{q}} \cdot \left( \sum_{k_2=2^{n+1}}^{2^{n+1}} \sum_{k_1=2}^{2^{n+1}} (k_1 k_2)^{q-2} \rho_{k_1 k_2}^q \right)^{\frac{\lambda}{q}}. \end{aligned} \tag{6}$$

Now, applying the Hardy-Littlewood theorem [19] and using the monotonicity of the logarithmic function and the best approximation from (6), we obtain

$$\begin{aligned} & \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \left( \sum_{k_1=2^{n_1+1}}^{2^{n_1+1}} \sum_{k_2=2^{n_2+1}}^{2^{n_2+1}} \rho_{k_1 k_2}^q \cdot \ln k_1 \cdot k_2^{q(1-\beta_2)-1} \right)^{\frac{\lambda}{q}} \leq \\ & \leq C \cdot E_{1,1}^{\lambda}(f)_q + C \cdot E_{2,2}^{\lambda}(f)_q + C \sum_{m=3}^{\infty} m^{\frac{\lambda}{q}(3-q(1+\beta_2))-1} (\ln m)^{\frac{\lambda}{q}} \cdot E_{m,m}^{\lambda}(f)_q. \end{aligned}$$

The theorem is proved.

*Theorem 3.* Let  $1 < q \leq 2$ ,  $1 \leq \lambda \leq q$ ,  $\frac{1}{q'} < \beta_1 < +\infty$ ,  $\beta_2 = \frac{1}{q'}$ ,  $\frac{1}{q} + \frac{1}{q'} = 1$ . If  $f \in L_q(I_2)$  and

$$\sum_{n=3}^{\infty} n^{2\frac{\lambda}{q}(2-q)-1} (\ln n)^{\frac{\lambda}{q}} \left( E_{n,\infty}^{(j)}(f)_q \right)^{\lambda} < +\infty, \quad j = 1, 2,$$

then the Fourier series of function  $f \in L_q(I_2)$  will be  $|C; \beta_1; \beta_2|_{\lambda}$ -summable almost everywhere on  $I_2$ .

*Proof.* It was proved in [12] that, if  $1 < q \leq 2$ ,  $1 \leq \lambda \leq q$ ,  $\frac{1}{q'} < \beta_1 < +\infty$ ,  $\beta_2 = \frac{1}{q'}$ ,  $\frac{1}{q} + \frac{1}{q'} = 1$  and

$$\sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \left( \sum_{k_1=2^{n_1+1}}^{2^{n_1+1}} \sum_{k_2=2^{n_2+1}}^{2^{n_2+1}} \rho_{k_1 k_2}^q \cdot \ln k_2 \right)^{\frac{\lambda}{q}} < +\infty,$$

then a double series of the form (1) is  $|C; \beta_1; \beta_2|_{\lambda}$ -summable almost everywhere on  $I_2$ .

Similarly to the previous theorem, we obtain

$$\sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \left( \sum_{k_1=2^{n_1+1}}^{2^{n_1+1}} \sum_{k_2=2^{n_2+1}}^{2^{n_2+1}} \rho_{k_1 k_2}^q \cdot \ln k_2 \right)^{\frac{\lambda}{q}} \leq \\ \leq C \cdot \sum_{n=0}^{\infty} 2^{2n(2-q)\frac{\lambda}{q}} (n+1)^{\frac{\lambda}{q}} \cdot \left( \sum_{k_2=2^{n+1}}^{2^{n+1}} \sum_{k_1=2}^{2^{n+1}} (k_1 k_2)^{q-2} \rho_{k_1 k_2}^q \right)^{\frac{\lambda}{q}}.$$

Hence, applying the Hardy-Littlewood theorem [19] and using the monotonicity of the logarithmic function and the best approximation, we have:

$$\sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \left( \sum_{k_1=2^{n_1+1}}^{2^{n_1+1}} \sum_{k_2=2^{n_2+1}}^{2^{n_2+1}} \rho_{k_1 k_2}^q \cdot \ln k_2 \right)^{\frac{\lambda}{q}} \leq \\ \leq C \cdot E_{1,1}^{\lambda}(f)_q + C \cdot E_{2,2}^{\lambda}(f)_q + C \sum_{m=3}^{\infty} m^{\frac{2\lambda}{q}(2-q)-1} (\ln m)^{\frac{\lambda}{q}} \cdot E_{m,m}^{\lambda}(f)_q.$$

The theorem is proved.

*Theorem 4.* Let  $1 < q \leq 2$ ,  $1 \leq \lambda \leq q$ ,  $-1 < \beta_1 < \frac{1}{q'}$ ,  $\frac{1}{q'} < \beta_2 < +\infty$ ,  $\frac{1}{q} + \frac{1}{q'} = 1$ . If  $f \in L_q(I_2)$  and

$$\sum_{n=2}^{\infty} n^{\frac{\lambda}{q}(3-q(1+\beta_1))-1} \left( E_{n,\infty}^{(j)}(f)_q \right)^{\lambda} < +\infty, \quad j = 1, 2,$$

then the Fourier series of function  $f \in L_q(I_2)$  will be  $|C; \beta_1; \beta_2|_{\lambda}$ -summable almost everywhere on  $I_2$ .

*Proof.* It was proved in [12] that, if  $1 < q \leq 2$ ,  $1 \leq \lambda \leq q$ ,  $-1 < \beta_1 < \frac{1}{q'}$ ,  $\frac{1}{q'} < \beta_2 < +\infty$ ,  $\frac{1}{q} + \frac{1}{q'} = 1$  and

$$\sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \left( \sum_{k_1=2^{n_1+1}}^{2^{n_1+1}} \sum_{k_2=2^{n_2+1}}^{2^{n_2+1}} \rho_{k_1 k_2}^q \cdot k_1^{q(1-\beta_1)-1} \right)^{\frac{\lambda}{q}} < +\infty,$$

then a double series of the form (1) is  $|C; \beta_1; \beta_2|_{\lambda}$ -summable almost everywhere on  $I_2$ .

For the last converging series, similarly to Theorem 2, we obtain

$$\sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \left( \sum_{k_1=2^{n_1+1}}^{2^{n_1+1}} \sum_{k_2=2^{n_2+1}}^{2^{n_2+1}} \rho_{k_1 k_2}^q \cdot k_1^{q(1-\beta_1)-1} \right)^{\frac{\lambda}{q}} \leq \\ \leq C \cdot \sum_{n=0}^{\infty} 2^{n(3-q(1+\beta_1))\frac{\lambda}{q}} \cdot \left( \sum_{k_2=2^{n+1}}^{2^{n+1}} \sum_{k_1=2}^{2^{n+1}} (k_1 k_2)^{q-2} \rho_{k_1 k_2}^q \right)^{\frac{\lambda}{q}}.$$

Hence, applying the Hardy-Littlewood theorem [19] and the monotonicity of the best approximation, we have:

$$\sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \left( \sum_{k_1=2^{n_1+1}}^{2^{n_1+1}} \sum_{k_2=2^{n_2+1}}^{2^{n_2+1}} \rho_{k_1 k_2}^q \cdot k_1^{q(1-\beta_1)-1} \right)^{\frac{\lambda}{q}} \leq \\ \leq C \cdot E_{1,1}^{\lambda}(f)_q + C \sum_{m=2}^{\infty} m^{\frac{\lambda}{q}(3-q(1+\beta_1))-1} \cdot E_{m,m}^{\lambda}(f)_q.$$

The theorem is proved.

## References

- 1 Бари Н.К. Тригонометрические ряды / Н.К. Бари. — М.: Физматгиз, 1961. — 936 с.
- 2 Зигмунд А. Тригонометрические ряды. — Т. 1, 2 / А. Зигмунд. — М.: Мир, 1965.
- 3 Эдвардс Р. Ряды Фурье в современном изложении. — Т. 1, 2 / Р. Эдвардс. — М.: Мир, 1985.
- 4 Кашин Б.С. Ортогональные ряды / Б.С. Кашин, А.А. Саакян. — М.: АФЦ, 1984. — 496 с.
- 5 Тиман М.Ф. Об абсолютной суммируемости кратных рядов Фурье / М.Ф. Тиман, Ю.А. Пономаренко // Укр. мат. журн. — 1971.— 23. — № 3. — С. 346–361.
- 6 Тиман А.Ф. Теория приближения функций действительного переменного / А.Ф. Тиман. — М., 1960. — 624 с.
- 7 Жак И.Е. О суммировании двойных рядов / И.Е. Жак, М.Ф. Тиман // Мат. сб. — 1954. — 35 (77). — № 1. — С. 21–56.
- 8 Салаи И. Об абсолютной суммируемости тригонометрических рядов / И. Салаи // Мат. заметки. — 1981. — 39. — № 6. — С. 823–837.
- 9 Битимханулы С. Условие абсолютной суммируемости кратных тригонометрических рядов / С. Битимханулы // Вестн. Казах. нац. ун-та. Сер. мат., мех., инф. — 2001. — № 1(24). — С. 3–11.
- 10 Битимхан С. Модули гладкости и абсолютная суммируемость кратных тригонометрических рядов / С. Битимхан, Г.А. Акишев // Мат. журн. — 2003. — 3. — № 1(17). — С. 5–14.
- 11 Битимхан С. О достаточных условиях абсолютной суммируемости кратных рядов Фурье / С. Битимхан // Вестн. Караганд. ун-та. Сер. Математика. — 2007. — № 1(45). — С. 50–54.
- 12 Битимхан С. Коэффициентные условия абсолютной суммируемости кратных тригонометрических рядов / С. Битимхан // Материалы V Междунар. науч. конф. Ч. 1. — 2009. — С. 279–281.
- 13 Bitimkhan S. Absolute summability of Fourier series of monotonously decreasing function / S. Bitimkhan // Abstracts of the third congress of the world mathematical society of Turkic countries. — 2009. — V. 1. — P. 95.
- 14 Битимхан С. Об абсолютной суммируемости кратных рядов с монотонными коэффициентами / С. Битимхан, М. Битимхан // Вестн. Караганд. ун-та. Сер. Математика. — 2010. — №1(57). — С. 3–11.
- 15 Akishev G. The conditions of absolute summability of multiple trigonometric series / G. Akishev, S. Bitimkhan // AIP conference proceedings 1676, 020095-1 - 020095-6. — 2015. doi: 10.1063/1.4930521.
- 16 Битимхан С. Абсолютная суммируемость рядов Фурье функции из классов Бесова / С. Битимхан // Вестн. Караганд. ун-та. Сер. Математика. — 2016. — №3(83). — С. 28–32.
- 17 Битимхан С. Об абсолютной чезаровской суммируемости кратных тригонометрических рядов с обобщенно-монотонными коэффициентами / С. Битимхан, Д.Т. Алибиева // Евраз. науч. объедин. — 2018. — №12(46). — С. 8–10.
- 18 Битимхан С. Об условиях абсолютной чезаровской суммируемости кратных тригонометрических рядов Фурье / С. Битимхан // Тр. Ин-та мат. и мех. УрО РАН. — 2019. — 25. — № 2. — С. 42–47.
- 19 Бугров Я.С. Суммируемость преобразований Фурье и абсолютная сходимость кратных рядов Фурье / Я.С. Бугров // Тр. МИАН СССР. — 1989. — 187. — С. 22–30.

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## **Дербес ең жақсы жуықтаулар және еселі Фурье қатарының абсолютті чезаролық қосындылануы**

Мақала еселі тригонометриялық Фурье қатарының Чезаро бойынша абсолютті қосындылану сұрағына арналған. Фурье қатарлары теориясында ерекше орны бар бұл сұрақ бір өлшемді жағдайда жеткілікті кең зерттелген және бұл теорияның іргелі нәтижелері Н.К. Бари, А. Зигмунд, Р. Эдвардс, Б.С. Кашин және А.А. Саакянның [1–4] белгілі монографияларында келтірілген. Еселі қатарлар

жағдайына сәйкес теория соншалықты күшті жасалмаған. Еселі жағдайдың өз ерекшеліктері бар және бір өлшемді жағдаймен ұқсастық әрқашан бірмәнді емес және айқын бола бермейді. Мақалада  $f \in L_q(I_s)$  функциясының еселі Фурье қатарының абсолютті қосындылануының жеткілікті шарттары осы функцияның дербес ең жақсы жуықтаулары тілінде алынған. Төрт теорема дәлелденіп,  $f \in L_q(I_s)$  функциясының Фурье қатарының  $|C; \bar{\beta}|_\lambda$  қосындылануының әртүрлі төрт жеткілікті шарты нақтыланған. Бірінші теоремада  $f \in L_q(I_s)$  функциясының Фурье қатарының  $|C; \bar{\beta}|_\lambda$  қосындылануы осы функцияның дербес ең жақсы жуықтаулары тіліндегі  $s$  шарттан тұратын жеткілікті шарты  $\beta_1 = \dots = \beta_s = \frac{1}{q^s}$  жағдайында алынған. Басқа жеткілікті шарттар екі еселі Фурье қатарлары үшін алынды.  $f \in L_q(I_2)$  функциясының Фурье қатарының  $|C; \beta_1; \beta_2|_\lambda$  қосындылану шарттары мына жағдайларда алынды:  $\beta_1 = \frac{1}{q^s}$ ,  $-1 < \beta_2 < \frac{1}{q^s}$  (екінші теоремада),  $\frac{1}{q^s} < \beta_1 < +\infty$ ,  $\beta_2 = \frac{1}{q^s}$  (үшінші теоремада),  $-1 < \beta_1 < \frac{1}{q^s}$ ,  $\frac{1}{q^s} < \beta_2 < +\infty$  (төртінші теоремада).

*Кілт сөздер:* тригонометриялық қатар, Фурье қатары, Лебег кеңістігі, функцияның дербес ең жақсы жуықтауы, қатардың абсолютті қосындылануы.

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## Частные наилучшие приближения и абсолютная чезаровская суммируемость кратных рядов Фурье

Статья посвящена вопросу абсолютной суммируемости по Чезаро кратных тригонометрических рядов Фурье. Этот вопрос, занимая центральное место в теории рядов Фурье, в одномерном случае разработан достаточно широко и фундаментальные результаты этой теории изложены в известных монографиях Н.К. Бари, А. Зигмунда, Р. Эдвардса, Б.С. Кашина и А.А. Саакяна [1–4]. В случае кратных рядов соответствующая теория разработана не столь сильно. Многомерный случай имеет свою специфику, и аналогия с одномерным случаем далеко не всегда однозначна и очевидна. В статье получены достаточные условия абсолютной суммируемости кратных рядов Фурье функции  $f \in L_q(I_s)$  в терминах частных наилучших приближений данной функции. Доказаны четыре теоремы и получены четыре разных достаточных условия  $|C; \bar{\beta}|_\lambda$  суммируемости ряда Фурье функции  $f \in L_q(I_s)$ . В первой теореме получено достаточное условие абсолютной  $|C; \bar{\beta}|_\lambda$  суммируемости ряда Фурье функции  $f \in L_q(I_s)$  в терминах частного наилучшего приближения этой функции, которое состоит из  $s$  условий, в случае когда  $\beta_1 = \dots = \beta_s = \frac{1}{q^s}$ . Другие достаточные условия получены для двойных рядов Фурье. Достаточные условия  $|C; \beta_1; \beta_2|_\lambda$  суммируемости ряда Фурье функции  $f \in L_q(I_2)$  получены в случаях  $\beta_1 = \frac{1}{q^s}$ ,  $-1 < \beta_2 < \frac{1}{q^s}$  (во второй теореме),  $\frac{1}{q^s} < \beta_1 < +\infty$ ,  $\beta_2 = \frac{1}{q^s}$  (в третьей теореме),  $-1 < \beta_1 < \frac{1}{q^s}$ ,  $\frac{1}{q^s} < \beta_2 < +\infty$  (в четвертой теореме).

*Ключевые слова:* тригонометрический ряд, ряд Фурье, пространство Лебега, частное наилучшее приближение функции, абсолютная суммируемость ряда.

### References

- 1 Bari, N.K. (1961). *Trigonometricheskie riady [Trigonometric series]*. Moscow: Fizmatgiz [in Russian].
- 2 Zigmund, A. (1965). *Trigonometricheskie riady [Trigonometric series]*. Vol. 1, 2. Moscow: Mir [in Russian].
- 3 Edwards, R. (1985). *Riady Fure v sovremennom izlozhenii [Fourier series in modern presentation]*. Vol. 1, 2. Moscow: Mir [in Russian].
- 4 Kashin, B.S., & Saakyan, A.A. (1984). *Ortogonalnye riady [Orthogonal series]*. Moscow: AFTs [in Russian].
- 5 Timan, M.F., & Ponomarenko, Yu.A. (1971). Ob absolutnoi summiruемости kratnykh riadov Fure [Absolute summability of multiple Fourier series]. *Ukrainskii matematicheskii zhurnal — Ukrainian mathematical journal*, 23(3), 346–361 [in Russian].
- 6 Timan, A.F. (1960). *Teoriia priblizheniia funktsii deistvitelnogo peremennogo [The theory of approximation of functions of a real variable]*. Moscow [in Russian].



- 7 Zhak, I.Ye., & Timan, M.F. (1954). O summirovanii dvoynykh riadov [On the summation of double series]. *Matematicheskii sbornik — Mathematical collection*, 35(77), 1, 21–56 [in Russian].
- 8 Salai, I. (1981). Ob absolutnoi summiruemosti trigonometricheskikh riadov [Absolute summability of trigonometric series]. *Matematicheskie zametki — Mathematical notes*, 39(6), 823–837 [in Russian].
- 9 Bitimkhanuly, S. (2001). Uslovia absolutnoi summiruemosti kratnykh trigonometricheskikh riadov [Conditions for the absolute summability of multiple trigonometric series]. *Vestnik Kazakhskogo gosudarstvennogo universiteta. Seriya matematika, mekhanika, informatika — KazSU Bulletin. series: mathematics, mechanics, computer science*, 1(24), 3–11 [in Russian].
- 10 Bitimkhan, S., & Akishev, G.A. (2003). Moduli gladkosti i absolutnaia summiruemost kratnykh trigonometricheskikh riadov [Modules of smoothness and absolute summability of multiple trigonometric series]. *Matematicheskii zhurnal — Mathematical journal*, 3, 1(17), 5–14 [in Russian].
- 11 Bitimkhan, S. (2007). O dostatochnykh usloviakh absolutnoi summiruemosti kratnykh riadov Fure [Sufficient conditions for the absolute summability of multiple Fourier series]. *Vestnik Karagandinskogo universiteta. Seriya Matematika — Bulletin of the Karaganda University, Mathematics series*, 1(45), 50–54 [in Russian].
- 12 Bitimkhan, S. (2009). Koeffitsientnye uslovia absolutnoi summiruemosti kratnykh trigonometricheskikh riadov [Coefficient conditions for the absolute summability of multiple trigonometric series]. *Materialy v Mezhdunarodnoi nauchnoi konferentsii — Materials of the 5th int. scientific. conferences, part 1*, 279–281 [in Russian].
- 13 Bitimkhan, S. (2009). Absolute summability of Fourier series of monotonously decreasing function. *Abstracts of the third congress of the world mathematical society of Turkic countries*, 1, P. 95.
- 14 Bitimkhan, S., & Bitimkhan, M. (2010). Ob absolutnoi summiruemosti kratnykh riadov s monotonnymi koeffitsientami [Absolute summability of multiple series with monotone coefficients]. *Vestnik Karagandinskogo universiteta. Seriya Matematika — Bulletin of the Karaganda University, Mathematics series*, 1(57), 3–11 [in Russian].
- 15 Akishev, G., & Bitimkhan, S. (2015). The conditions of absolute summability of multiple trigonometric series. *AIP conference proceedings 1676*. doi: 10.1063/1.4930521, 020095-1 - 020095-6.
- 16 Bitimkhan, S. (2016). Absolutnaia summiruemost riadov Furie funktsii iz klassov Besova [Absolute summability of the Fourier series of a function from the Besov classes]. *Vestnik Karagandinskogo universiteta. Seriya Matematika — Bulletin of the Karaganda University, Mathematics series*, 3(83), 28–32 [in Russian].
- 17 Bitimkhan, S., & Alibiyeva, D.T. (2018). Ob absolutnoi chezarovskoi summiruemosti kratnykh trigonometricheskikh riadov s obobshchenno-monotonnymi koeffitsientami [Absolute Cesaro summability of multiple trigonometric series with generalized monotone coefficients]. *Yevraziskoe nauchnoe obedinenie — Eurasian Scientific Association*, 12(46), 8–10 [in Russian].
- 18 Bitimkhan, S. (2019). Ob usloviakh absolutnoi chezarovskoi summiruemosti kratnykh trigonometricheskikh riadov Fure [Conditions for the absolute Cesaro summability of multiple trigonometric Fourier series]. *Trudy Instituta matematiki i mekhaniki UrO RAN — Proceedings of the Institute of Mathematics and Mechanics, Ural Branch of the Russian Academy of Sciences*, 25(2), 42–47 [in Russian].
- 19 Bugrov, Ya.S. (1989). Summiruemost preobrazovani Fure i absolutnaia skhodimost kratnykh riadov Fure [Summability of Fourier transforms and absolute convergence of multiple Fourier series]. *Trudy MIAN SSSR — Proceedings of the Steklov Institute of the USSR*, 187, 22–30 [in Russian].