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A new finite difference method for computing approximate solutions of boundary value problems including transition conditions

This article is aimed at computing numerical solutions of new type of boundary value problems (BVPs) for two-linked ordinary differential equations. The problem studied here differs from the classical BVPs such that it contains additional conditions at the point of interaction, so-called transition conditions. Naturally, such type of problems is much more complicated to solve than classical problems. It is not clear how to apply the classical numerical methods to such type of boundary value transition problems (BVTPs). Based on the finite difference method (FDM) we have developed a new numerical algorithm for computing numerical solution of BVTPs for two-linked ordinary differential equations. To demonstrate the reliability and efficiency of the presented algorithm we obtained numerical solution of one BVTP and the results are compared with the corresponding exact solution. The maximum absolute errors (MAEs) are presented in a table.

Keywords: finite difference method, transition condition, boundary value problems, second order differential equation.

Introduction

Sturm-Liouville BVPs arise as mathematical models of many problems in physics and engineering, such as Newton's law of cooling, the population growth of decay, Kirchoff's law in electrical circuits, the steady-state temperature in heated rod, thermodynamics, resistor, and inductor circuits, etc (see, for example, [1–7] and references cited therein). It is obvious that not all BVPs can be solved analytically. Even if a BVP can be solved analytically, the closed-form of the analytical solution may take some complicated form that is unhelpful to use. Therefore we have to apply various numerical methods for determining the approximate solution. There have been developed different numerical methods to solve various type of BVPs. One of them, the so-called FDM, can be applied to a wide class of BVPs, provided that the problem considered has a complete set of continuty and boundary conditions. In this study we will consider a BVP of a new type. The main feature of this problem is the nature of the imposed boundary conditions, which include not only the ends of the interval under consideration but also one inner point of the singularity. Naturally, such type of singular problems is much more difficult to solve than regular problems. We will develop a new modification of classical FDM to solve BVPs involving additional transition conditions at the point of singularity. Such type of singular problems arises in heat and mass transfer problems, in vibrating string problems, and in a varied assortment of physical transfer problems (see, for example, [8–12] and references cited therein).

A new modification of finite difference method

Let us consider a linear BVP for the second order differential equation given by

$$u^{''}(x) + p(x)u^{'}(x) + q(x)u(x) = f(x), \quad x \in [a, c) \cup (c, b], \tag{1}$$

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subject to the boundary conditions (BCs) at end-points $x = a$ and $x = b$ given by

$$u(a) = \alpha, \quad u(b) = \beta \tag{2}$$

together with additional transition conditions at the interior point of singularity $x = c$ given by

$$u(c-0) = \xi u(c+0), \quad u'(c-0) = \psi u'(c+0), \tag{3}$$

where $p(x)$, $q(x)$ and $f(x)$ are continuous on [a,c) and (c,b] with the finite limits $p(c \mp 0)$, $q(c \mp 0)$, $f(c \mp 0)$, α, β, ξ ,ψ are real constants. For convenience, we will use the notations $[a_1, b_1] = [a, c]$, $[a_2, b_2] = [c, b]$. To discretize BVTP (1)-(3) the interval $[a_k, b_k]$, $k = 1, 2$ are divided into finite number f intervals $[x_{k,0}, x_{k,1}]$, $[x_{k,1}, x_{k,2}]$, ..., $[x_{k,N-1}, x_{k,N}]$ with

$$a_k = x_{k,0} < x_{k,1} < ... < x_{k,N} = b_k,$$

where

$$x_{k,i} = a_k + ih_k, \quad h_k = \frac{b_k - a_k}{N}, \quad k = 1, 2 \quad , i = 1, 2, ..., N.$$

Below we will use the central finite difference discretization. Namely, we will express the first and second derivatives of the unknown function

$$u(x) = \begin{cases} u_1(x), & for \quad x \in [a_1, b_1), \\ u_2(x), & for \quad x \in (a_2, b_2] \end{cases}$$

as

$$u'_k(x) \approx \frac{u_k(x+h_k) - u_k(x-h_k)}{2h_k}$$

and

$$u''_k(x) \approx \frac{u_k(x+h_k) - 2u_k(x) + u_k(x-h_k)}{{h_k}^2},$$

respectively. Let us denote the value of the unknown function $u(x)$ at the nodal point $x_{k,i}$ by $u_{k,i}$ and substitute in equation (1). We have the following linear system of equations for each $k = 1, 2$

$$\left(1 - \frac{1}{2}h_k p_{k,i}\right) u_{k,i-1} + \left(-2 + {h_k}^2 q_{k,i}\right) u_{k,i} + \left(1 + \frac{1}{2}h_k u_{k,i+1}\right) u_{i+1} = {h_k}^2 f\left(x_{k,i}\right), \tag{4}$$

$$i = 1, 2, 3, ..., N-1,$$

where

$$u(a) = u_{1,0} = \alpha, \quad u(b) = u_{2,N} = \beta.$$

Let us introduce to two new parameters $\beta_1 := u_{1,N}$ and $\alpha_2 := u_{2,0}$ that will be calculated later. For convenience, we will use the notations $\alpha_1 := \alpha$ and $\beta_2 := \beta$.

Note that each of finite difference equation (4) involves solutions $u_{k,i-1}$, $u_{k,i}$, and $u_{k,i+1}$ at the nodal points $x_{k,i-1}$, $x_{k,i}$, and $x_{k,i+1}$, respectively.

This system of linear equations can be written in matrix form

$$A_k U_k = B_k, \quad k = 1, 2, \tag{5}$$

where A_k is the $(N-1)\times(N-1)$ matrix given by

$$A_k = \begin{pmatrix} -2+{h_k}^2 q_{k,1} & 1+\frac{1}{2}h_k p_{k,1} & 0 & \cdots & 0 & 0 \\ 1-\frac{1}{2}h_k p_{k,2} & -2+h^2{}_k q_{k,2} & 1+\frac{1}{2}h_k p_{k,2} & \cdots & 0 & 0 \\ 0 & 1-\frac{1}{2}h_k p_{k,3} & -2+h^2{}_k q_{k,3} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -2+h^2{}_k q_{k,N-3} & 1+\frac{1}{2}h_k p_{k,N-3} & 0 \\ 0 & 0 & \cdots & 1-\frac{1}{2}h_k p_{k,N-2} & -2+h^2{}_k q_{k,N-2} & 1+\frac{1}{2}h_k p_{k,N-2} \\ 0 & 0 & \cdots & 0 & 1-\frac{1}{2}h_k p_{k,N-1} & -2+h^2{}_k q_{k,N-1} \end{pmatrix}$$

and U_k and B_k are column vectors given by

$$U_k = \begin{pmatrix} u_{k,1} \\ u_{k,2} \\ u_{k,3} \\ \vdots \\ u_{k,N-3} \\ u_{k,N-2} \\ u_{k,N-1} \end{pmatrix} \quad and \quad B_k = \begin{pmatrix} b_{k,1} \\ b_{k,2} \\ b_{k,3} \\ \vdots \\ b_{k,N-3} \\ b_{k,N-2} \\ b_{k,N-1} \end{pmatrix},$$

where

$$b_{k,i} = \begin{cases} h^2{}_k f_{k,1} - \left(1-\frac{1}{2}hp_{k,1}\right), & i=1, \\ h^2{}_k f_{k,i}, & i=2,3,...,N-2, \\ h^2{}_k f_{k,N-1} - \left(1+\frac{1}{2}hp_{k,N-1}\right)\beta_k, & i=N-1 \end{cases}$$

and $f_{k,i} = f(x_{k,i})$.

Since the linear system of algebraic equations (5) is tridiagonal, it can be solved by the Crout or Cholesky algorithm (see [2]). To satisfy transition conditions (3) we have the following equations

$$u_{1,N} = \xi u_{2,0},$$

$$\frac{u_{1,N}-u_{1,N-1}}{h_1} = \psi \frac{u_{2,1}-u_{2,0}}{h_2},$$

From which we can easily find the numerical solutions $u_{1,N}$ and $u_{2,0}$. Thus we find all the numerical solutions $u_{k,0}, u_{k,1}, ..., u_{k,N}, k=1,2$.

Numerical illustration

Let us consider the following BVP on the disjoint intervals $[-1,0)$ and $(0,1]$ consisting of linear differential equation

$$u'' = (1+2tan(x)^2)u, \quad x\in[-1,0)\cup(0,1] \tag{6}$$

together with boundary conditions at the end-points $x = -1, 1$ given by

$$u\,(-1) = 2, \quad u\,(1) = -1 \tag{7}$$

and with additional transition conditions at the interior point of singularity $x = 0$ given by

$$u(-0) = 5u(+0), \quad 3u'(-0) = u'(+0). \tag{8}$$

At first we will investigate this problem without transition conditions.We can show that the exact solution of the BVP (6) and BVP (7) is

$$u(x) = \frac{cos(1)}{2} sec(x) + \frac{3cos(1)}{2 + 2sin(x)} \left(sin(x) + xsec(x)\right). \tag{9}$$

Consider the uniform cartesian grid $x_i = -1 + ih$, $i = 1, \quad \ldots, 49$ for $N = 50$, i.e, $h = \frac{x_0 - x_{20}}{N} = \frac{1-(-1)}{50} = 0,04$ where in particular $x_0 = -1, x_{20} = 1$, $u_0 = 2, u_{50} = -1$. By using the central FDM at a typical grid point x_i, we obtain

$$u_{i-1} + (-2 - h^2(1 + 2tan^2(x_i)))u_i + u_{i+1} = 0 \tag{10}$$

for $i = 1, 2, \ldots, 49$. Consequently, the finite difference solution $u_i \approx u(x_i)$ is defined as the solution of the linear algebraic system of equations (10). In a tridiagonal matrix-vector form, this linear algebraic system of equations can be written as

$$Au = B, \tag{11}$$

where

$$A = \begin{pmatrix} -2 - h^2(1 + 2tan^2(x_1)) & 1 & 0 & \cdots & 0 \\ 1 & -2 - h^2(1 + 2tan^2(x_2)) & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & -2 - h^2(1 + 2tan^2(x_{49})) \end{pmatrix},$$

$$u = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_{48} \\ u_{49} \end{pmatrix}, \quad B = \begin{pmatrix} -2 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}.$$

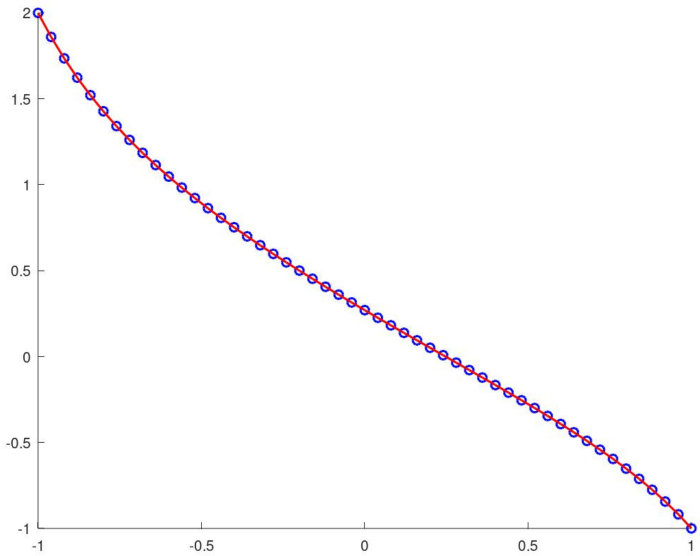

Figure 1. Comparison of FDM-solution and exact solution for the equation (6) and equation (7) for N=50.

Now we shall investigate of BVP (6) and BVP (7) under additional transition conditions (8). We can find the exact solution of this problem in the following form:

$$u = \begin{cases} \frac{25cos(1)}{16}sec(x) - \frac{7cos(1)}{8(2+sin(1))}(sin(x) + xsec(x)) \ , & x \in [-1, 0), \\ \frac{5cos(1)}{16}sec(x) - \frac{21cos(1)}{8(2+sin(2))}(sin(x) + xsec(x)) \ , & x \in [0, 1). \end{cases}$$

Letting $N = 49$ and applying the transition conditions (8) we have two additional algebraic equations

$$u_{24} - 5u_{25} = 0, \quad 3u_{23} - 3u_{24} - u_{25} + u_{26} = 0. \tag{12}$$

The solution of the algebraic system of equations (11) and (12) is obtained by MATLAB/Octave.

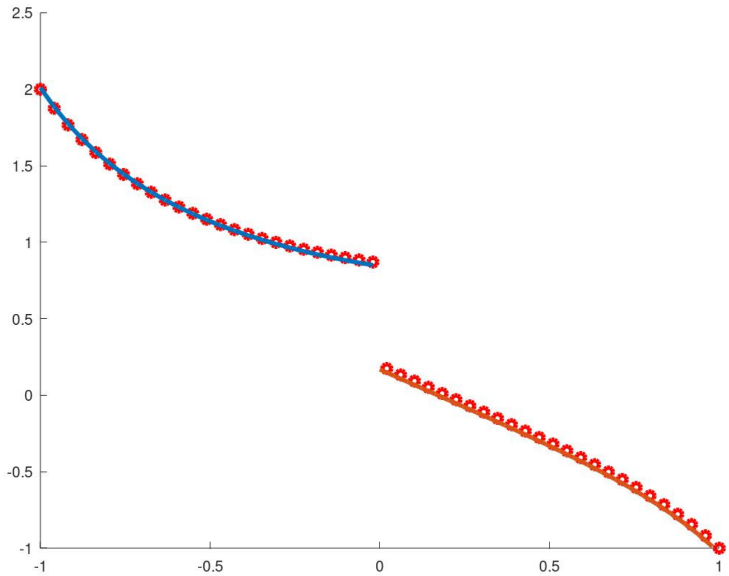

Figure 2. Comparison of FDM-solution and exact solution of BVTP (8)-(10) for N=49.

Conclusion

We have considered BVTP (6)–(8) to test the computational efficiency of the proposed modification of the classical FDM. When solving BVTP (6)–(8) numerically for different values of $N = 20, 50, 100, 1000$ presented in Table 1, we observed that if N increases, h decreases, then maximum absolute error in computed solution decreases.

Table 1

Maximum absolute error

h=2/N	N	MAE
1/10	20	0.0039503
1/25	50	0.00064601
1/50	100	0.00016199
1/500	1000	0.0000016217

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Ауысу шарттарын қамтитын шеттік есептердің жуық шешімдерін есептеу үшін ақырлы айырымдардың жаңа әдісі

Мақала қос байланысымды қарапайым дифференциалдық теңдеулер үшін шеттік есептердің жаңа түрінің сандық шешімдерін есептеуге бағытталған. Мұнда зерттелетін есеп классикалық шеттік есептерден ерекше, онда өзара әрекеттесу нүктесінде ауысу шарттары деп аталатын қосымша шарттар бар. Мұндай есептерді классикалық есептерге қарағанда шешу әлдеқайда қиын. Классикалық сандық әдістерді шеттік ауысу есептердің осы түріне қалай қолдану керектігі түсініксіз. Ақырлы айырымды әдісіне сүйене отырып, қос байланысымды қарапайым дифференциалдық теңдеулер үшін шеттік ауысу есептерді шешудің жаңа сандық алгоритмі құрылды. Ұсынылған алгоритмнің сенімділігі мен тиімділігін көрсету үшін бір шеттік ауысу есептің сандық шешімі табылды және нәтижелер тиісті дәл шешіммен салыстырылды. Максималды абсолютті қателер кестеде келтірілген.

Кілт сөздер: ақырлы айырымдық әдісі, ауысу шарты, классикалық сандық әдістер, шеттік ауысу есептерін шешудің алгоритмі.

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Новый метод конечных разностей для вычисления приближенных решений краевых задач, включающих условия перехода

Статья направлена на вычисление численных решений нового типа краевых задач для двусвязных обыкновенных дифференциальных уравнений. Изучаемая здесь задача отличается от классических краевых задач тем, что она содержит дополнительные условия в точке взаимодействия, так называемые переходные условия. Естественно, что такие задачи гораздо сложнее решать, чем классические. Непонятно, как применить классические численные методы к такому типу краевых переходных задач. На основе метода конечных разностей разработан новый численный алгоритм решения краевых переходных задач для двусвязных обыкновенных дифференциальных уравнений. Для демонстрации надежности и эффективности представленного алгоритма проведено численное решение одной краевой переходной задачи, и результаты сравнивались с соответствующим точным решением. Максимальные абсолютные погрешности представлены в таблице.

Ключевые слова: метод конечных разностей, условие перехода, классические численные методы, алгоритм решения краевых переходных задач.

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