Order of the trigonometric widths of the Nikol’skii-Besov classes with mixed metric in the metric of anisotropic Lorentz spaces

In this paper we estimate the order of the trigonometric width of the Nikol’skii–Besov classes $B_{pq}^{\alpha\tau} (\mathbb{T}^n)$ with mixed metric in the anisotropic Lorentz space $L_{\theta p} (\mathbb{T}^n)$ when $1 < p = (p_1, \ldots, p_n) < 2 < q = (q_1, \ldots, q_n)$. The concept of a trigonometric width in the one-dimensional case was first introduced by R.S. Ismagilov and he established his estimates for certain classes in the space of continuous functions. For a function of several variables exact orders of trigonometric widths of Sobolev class $W_{p}^{r}$, Nikol’skii class $H_{p}^{r}$ in the space $L_{q}$ are established by E.S. Belinsky, V.E. Majorov, Yu. Makovoz, G.G. Magaril-Ilyaev, V.N. Temlyakov. This problem for the Besov class $B_{pq}^{r}$ was investigated by A.S. Romanyuk, D.B. Bazarkhanov. The trigonometric width for the anisotropic Nikol’skii-Besov classes $B_{pq}^{\alpha\tau} (\mathbb{T}^n)$ in the metric of the anisotropic Lorentz spaces $L_{\theta p} (\mathbb{T}^n)$ was found by K.A. Bekmaganbetov and Ye. Toleugazy.

Keywords: trigonometric widths, anisotropic Lorentz space, Nikol’skii–Besov class with mixed metric.

Introduction

Let $V \subset L_1 (\mathbb{T}^n)$ be the normed space and $F \subset V$ be some functional class. The trigonometric width of the class $F$ in the space $V$ is defined as follows (see [1])

$$d_M^F (F, V) = \inf_{t \in F} \sup_{t (\Omega_M; \cdot)} \| f (\cdot) - t (\Omega_M; \cdot) \|_{V},$$

where $t (\Omega_M; x) = \sum_{j=1}^{M} c_j e^{i (k_j, x)}$, $\Omega_M = \{ k_1, \ldots, k_M \}$ is the set of vectors $k_j = (k_{j1}, \ldots, k_{jn})$ from the integer lattice $\mathbb{Z}^n$ and $c_j$ are some numbers ($j = 1, \ldots, M$).

The concept of a trigonometric width in the one-dimensional case was first introduced by R.S. Ismagilov [1] and he established its estimates for certain classes in the space of continuous functions. For a function of several variables exact orders of trigonometric widths of Sobolev class $W_{p}^{r}$, Nikol’skii class $H_{p}^{r}$ in the space $L_{q}$ are established by E.S. Belinsky [2], V.E. Majorov [3], Yu. Makovoz [4], G.G. Magaril-Ilyaev [5], V.N. Temlyakov [6]. This problem for the Besov class $B_{pq}^{r}$ was investigated by A.S. Romanyuk [7], D.B. Bazarkhanov [8]. The trigonometric width for the anisotropic Nikol’skii-Besov classes $B_{pq}^{\alpha\tau} (\mathbb{T}^n)$ in the metric of the anisotropic Lorentz spaces $L_{\theta p} (\mathbb{T}^n)$ was found by K.A. Bekmaganbetov and Ye. Toleugazy [9].

We study the problem of estimating the order of the trigonometric width of the Nikol’skii-Besov classes $B_{pq}^{\alpha\tau} (\mathbb{T}^n)$ with a mixed metric in the metric of anisotropic Lorentz spaces $L_{\theta p} (\mathbb{T}^n)$.

Preliminaries and auxiliary results

Let $f (x) = f (x_1, \ldots, x_n)$ be a measurable function defined by $\mathbb{T}^n$. Let multiindexes $1 \leq p = (p_1, \ldots, p_n) < \infty$. A Lebesgue space $L_p (\mathbb{T}^n)$ with mixed metric is the set of functions for which the following quantity is finite

$$\| f \|_{L_p (\mathbb{T}^n)} = \left( \int_0^{2\pi} \cdots \left( \int_0^{2\pi} |f (x_1, \ldots, x_n)|^{p_1} \, dx_1 \right)^{p_2 / p_1} \cdots \left( \int_0^{2\pi} |f (x_1, \ldots, x_n)|^{p_n} \, dx_n \right)^{p_{n-1} / p_n} \right)^{1 / p_n}.$$

Here, the expression $\left( \int_0^{2\pi} |f (t)|^p \, dt \right)^{1 / p}$ for $p = \infty$ is understood as $\sup_{0 \leq t < 2\pi} |f (t)|$. 

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For the function \( f \in L_p(\mathbb{T}^n) \) we denote
\[
\Delta_n(f, x) = \sum_{k \in \rho(s)} a_k(f)e^{i(k,x)},
\]
where \( \{a_k(f)\}_{k \in \mathbb{Z}^n} \) are Fourier coefficients of the function \( f \) with respect to the multiple trigonometric system \( \rho(s) = \{k = (k_1, \ldots, k_n) \in \mathbb{Z}^n : 2^{-i} \leq |k_i| < 2^i, i = 1, \ldots, n\} \).

Let \( 0 < \alpha = (\alpha_1, \ldots, \alpha_n) < \infty, 0 < \tau = (\tau_1, \ldots, \tau_n) \leq \infty \) and for any number \( t \) there exists a trigonometric polynomial \( P(\Omega_M, x) \) from Corollary 1, which is approaching the \( f \).

**Lemma 1** [10]. Let \( 2 \leq q < \infty \). Then for any trigonometric polynomial
\[
P(\Omega_M, x) = \sum_{j=1}^{M} e^{i(k_j,x)}
\]
and for any number \( N \leq M \) there exists a trigonometric polynomial \( P(\Omega_N, x) \) containing at most \( N \) harmonics and such that
\[
\|P(\Omega_M, \cdot) - P(\Omega_N, \cdot)\|_{L_q(\mathbb{T}^n)} \leq CMN^{-1/2},
\]
moreover \( \Omega_N \subset \Omega_M \) and all coefficients \( P(\Omega_N, x) \) are the same and do not exceed \( MN^{-1} \).

**Corollary 1** [11]. Let \( 2 < q = (q_1, \ldots, q_n) \leq \infty, 0 < \theta = (\theta_1, \ldots, \theta_n) \leq \infty \). Then for any trigonometric polynomial
\[
P(\Omega_M, x) = \sum_{j=1}^{M} e^{i(k_j,x)}
\]
and for any number \( N \leq M \) there exists a trigonometric polynomial \( P(\Omega_N, x) \) containing at most \( N \) harmonics and such that
\[
\|P(\Omega_M, \cdot) - P(\Omega_N, \cdot)\|_{L_{q\theta}(\mathbb{T}^n)} \leq CMN^{-1/2},
\]
moreover \( \Omega_N \subset \Omega_M \) and all coefficients \( P(\Omega_N, x) \) are the same and do not exceed \( MN^{-1} \).

For any \( s \in \mathbb{Z}^n \) we consider a linear operator
\[
(T_Ns) f(x) = f(x) * \left( \sum_{k \in \rho(s)} e^{i(k,x)} - t(\Omega_N, x) \right),
\]
where \( t(\Omega_N, x) \) is a trigonometric polynomial from Corollary 1, which is approaching the \( \{s\} \).

**Lemma 2.** Let \( 1 < p < 2 \), the multiindex \( q = (q_1, \ldots, q_n) \) be such that \( 2 < q_j < p' \) for all \( j = 1, \ldots, n \) and \( \theta = (\theta_1, \ldots, \theta_n) \leq \infty \). Then the norm operator \( T_Ns \) acting from \( L_p(\mathbb{T}^n) \) to \( L_{q\theta}(\mathbb{T}^n) \) satisfies the following inequality
\[
\|T_Ns\|_{L_p(\mathbb{T}^n) \to L_{q\theta}(\mathbb{T}^n)} \leq C_1 2^{(1,s)} N_s^{-1/2 + 1/p'}.
\]
Proof. Taking into account that the coefficients of the polynomial \( t(\Omega_{N_0}, x) \) are the same and do not exceed \( 2^{(1, s)} N_0^{-1} \) by Parseval’s equality we have
\[
\| T_{N_0} \|_{L_2(T^n)\rightarrow L_2(T^n)} \leq C_1 2^{(1, s)} N_0^{-1}. \tag{1}
\]

Further, using the generalized Minkowski’s inequalities and Corollary 1 we can write
\[
\| T_{N_0} f \|_{L_{q^*}(T^n)} \leq \| f \|_{L_1(T^n)} \left| \sum_{k\in\rho(s)} e^{t(k\cdot)} - t(\Omega_{N_0}, \cdot) \right|_{L_{q^*}(T^n)} \leq C_2 2^{(1, s)} N_0^{-1/2} \| f \|_{L_1(T^n)}.
\]

Therefore, by definition, \( \| T_{N_0} \|_{L_1(T^n)\rightarrow L_{q^*}(T^n)} \) we find
\[
\| T_{N_0} \|_{L_1(T^n)\rightarrow L_{q^*}(T^n)} \leq C_2 2^{(1, s)} N_0^{-1/2}. \tag{2}
\]

Further, using the Riesz-Thorin interpolation theorem for Lebesgue spaces and anisotropic Lorentz spaces, we obtain
\[
\| T_{N_0} \|_{L_1(T^n)\rightarrow L_{q^*}(T^n)} \leq \| T_{N_0} \|_{L_2(T^n)\rightarrow L_2(T^n)} \| T_{N_0} \|_{L_1(T^n)\rightarrow L_{q^*}(T^n)}, \tag{3}
\]
where \( 0 < \lambda < 1 \) and \( 1/p = (1 - \lambda)/2 + \lambda/1, \ 1/q = (1 - \lambda)/2 + \lambda/q^* \) and \( 1/\theta = (1 - \lambda)/2 + \lambda/\theta^* \).

By substituting (1) and (2) and performing elementary transformations, we receive at the required estimate with the additional condition \( 0 < \theta = (\theta_1, \ldots, \theta_n) < p' = (p', \ldots, p') \). For the remaining values of the parameters \( \theta = (\theta_1, \ldots, \theta_n) \) the validity of the assertion follows from the embedding \( L_{q^*}(T^n) \rightarrow L_{q^*}(T^n) \) for \( 0 < \theta = (\theta_1, \ldots, \theta_n) \leq \theta_2 = (\theta_1, \ldots, \theta_n) \leq \infty \).

Let us formulate a special case of the embedding theorem from E.D. Nursultanov’s paper ([12]) as a Lemma.

Lemma 3 [12]. Let \( 1 \leq p = (p_1, \ldots, p_n) < q = (q_1, \ldots, q_n) < \infty, \ 0 < \tau = (\tau_1, \ldots, \tau_n) \leq \infty \) and \( \alpha = 1/p - 1/q \), then
\[
B_p^\alpha(T^n) \hookrightarrow L_{q^*}(T^n).
\]

Furthermore we need the following sets
\[
Y^n(N, \gamma) = \left\{ s = (s_1, \ldots, s_n) \in \mathbb{Z}_+^n : \sum_{j=1}^n \gamma_j s_j \geq N \right\},
\]
\[
N^n(N, \gamma) = \left\{ s = (s_1, \ldots, s_n) \in \mathbb{Z}_+^n : \sum_{j=1}^n \gamma_j s_j = N \right\}.
\]

Lemma 4 [13]. Let \( n \in \mathbb{N}, \ n \geq 2, \ 0 < \gamma' = (\gamma'_1, \ldots, \gamma'_n) \leq \gamma = (\gamma_1, \ldots, \gamma_n) < \infty, \ 0 < \delta > 0 \) and \( 0 < \varepsilon = (\varepsilon_1, \ldots, \varepsilon_n) \leq \infty \). Then
\[
\left\| \left\{ 2^{-\delta(s, \gamma)} \right\}_{s \in Y^n(N, \gamma)} \right\|_{L_1(\mathbb{Z}_+^n)} \leq C 2^{-\delta q N} N^{\sum_{j=1}^n (\varepsilon_j - 1)/\varepsilon_j},
\]
where \( \eta = \min \left\{ \gamma_j/\gamma'_j : j = 1, \ldots, n \right\}, \ A = \{ j : \gamma_j/\gamma'_j = n, j = 1, \ldots, n \}, \ j_1 = \min \{ j : j \in A \}. \)

Lemma 5 [13]. Let \( n \in \mathbb{N}, \ n \geq 2, \ 0 < \gamma = (\gamma_1, \ldots, \gamma_n) < \infty, \ 0 < \delta \in \mathbb{R} \) and \( 0 < \varepsilon = (\varepsilon_1, \ldots, \varepsilon_n) \leq \infty \). Then
\[
\left\| \left\{ 2^{-\delta(s, \gamma)} \right\}_{s \in N^n(N, \gamma')} \right\|_{L_1(\mathbb{Z}_+^n)} \leq C 2^{-\delta q N} N^{\sum_{j=1}^n (1)/\varepsilon_j}.
\]

Main result

The main result of this paper includes:

Theorem 1. Let \( 1 < p = (p_1, \ldots, p_n) < 2 < q = (q_1, \ldots, q_n) < p_0' = (p_0', \ldots, p_0'), \ p_0 = \max \{ p_j : j = 1, \ldots, n \}, \ 1 \leq \tau = (\tau_1, \ldots, \tau_n), \ 0 \leq \alpha = (\alpha_1, \ldots, \alpha_n) \) be such that \( \alpha_j > 1 + 1/p_j - 1/p_0 \) for all \( j = 1, \ldots, n \). Let \( \zeta = \min \{ \alpha_j - 1/p_j + 1/q_j : j = 1, \ldots, n \}, \ D = \{ j = 1, \ldots, n : \alpha_j - 1/p_j + 1/q_j = \zeta \}, \ j_1 = \min \{ j : j \in D \}, \ q_j = q_j, \) for all \( j \in D \) and \( q_j \geq q_{j_1} \) for all \( j \notin D \).
Then the following relation holds
\[ d_M^2(B_p^{α,τ}(T^m),L_qθ(T^m)) = M^{-α_n-1/p_{j_1}+1/2}(\log M)^{(|D|-1)(α_j-1/p_{j_1}+1/2)+\sum_{t\in D\setminus\{j_1\}}(1/2-1/τ_j)+} \]
where \(|D|\) is amount of elements of the set \(D\), \(a_+ = \max\{a;0\}\).

**Proof.** Let \(f \in B_p^{α,τ}(T^m)\). For any natural number \(M\) there exists the natural number \(m\) such that \(M \approx 2^m m^{|D|-1}\). We will seek an approximating polynomial \(P(Ω_M, x)\) in the following form
\[ P(Ω_M; x) = \sum_{(γ', s)} Δ_γ(f, x) + \sum_{m≤(γ', s) < β_m} t(Ω_{N_s}; x) + Δ_γ(f, x), \]
where
\[ β = \left(\alpha_{j_1} - 1/p_{j_1} + 1/2 - \frac{\log m}{m} \sum_{j∈D\setminus\{j_1\}} \left(1/2 - 1/τ_j - (1/θ_j - 1/τ_j)_+\right)\right)/\left(\alpha_{j_1} - 1/p_{j_1} + 1/q_{j_1}\right). \]
\(γ_j = (α_j - 1/p_{j_1} + 1/q_{j_1})/(α_{j_1} - 1/p_{j_1} + 1/q_{j_1}), j = 1, \ldots, n, γ'_j = γ_j\) for \(j ∈ D\) and \(1 < γ'_j < γ_j\) for \(j \notin D\). The polynomials \(t(Ω_{N_s}, x)\) are chosen for every “block” \(t_s(x) = \sum_{k∈β(s)} e^{i(k,x)}\) according to Corollary 1 and numbers \(N_s = \left[2^{(α_{j_1}-1/p_{j_1}+1/q_0)m}2^{-(α-1/p+1/p_0-1)s}\right]\). Note that according to Lemma 4
\[ \sum_{m≤(γ', s) < β_m} N_s = 2^{(α_{j_1}-1/p_{j_1}+1/q_0)m} \sum_{m≤(γ', s) < β_m} 2^{-(α-1/p+1/p_0-1)s} \leq 2^{(α_{j_1}-1/p_{j_1}+1/q_0)m}\left\| 2^{-(α-1/p+1/p_0-1)s}\right\|_{L_qθ(T^m)} \leq 2^{(α_{j_1}-1/p_{j_1}+1/q_0)m}2^{-(α_{j_1}-1/p_{j_1}+1/q_0)m}m^{|D|-1} = 2^m m^{|D|-1} ≈ M, \]
so that \((α_j - 1/p_{j_1} + 1/p_0 - 1)/(α_{j_1} - 1/p_{j_1} + 1/p_0 - 1) > γ'_j\) at \(j \notin D\). Moreover according to equality (5) and Minkowski’s inequality we have
\[ \|f(·) - P(Ω_M; ·)\|_{L_qθ(T^m)} ≤ C_1 \left( \left\| \sum_{m≤(γ', s) < β_m} \left( Δ_γ(f, ·) - Δ_γ(f, ·) * t(Ω_{N_s}; ·) \right) \right\|_{L_qθ(T^m)} + \left\| \sum_{(γ', s) ≥ β_m} Δ_γ(f, ·) \right\|_{L_qθ(T^m)} \right) \]
\[ = C_1 (I_1(f) + I_2(f)). \]
Firstly we estimate \(I_2(f)\). By Lemma 3 we have
\[ I_2(f) ≤ C_2 \left\| 2^{(1/p-1/q, s)} \|Δ_γ(f, ·)\|_{L_p(T^m)} \right\|_{l_p} \left\| Y^n(β_m, γ') \right\|_{l_p} ≤ I_3(f). \]
According to Hölder’s inequality with parameters \(1/θ = 1/τ + 1/ε\), where \(1/ε = (1/θ - 1/τ)_+\) and Lemma 4, taking into account that \(γ' ≤ γ\) we find
\[ I_3(f) = \left\| 2^{(α, s)} \|Δ_γ(f, ·)\|_{L_p(T^m)} \cdot 2^{-(α_{j_1}-1/p_{j_1}+1/q_{j_1})(γ, s)} \right\|_{s∈ Y^n(β_m, γ')} \leq \left\| 2^{(α, s)} \|Δ_γ(f, ·)\|_{L_p(T^m)} \right\|_{s∈ Y^n(β_m, γ')} \times \left\| 2^{-(α_{j_1}-1/p_{j_1}+1/q_{j_1})(γ, s)} \right\|_{s∈ Y^n(β_m, γ')} \leq \]

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\[ I_2 \leq C_5 M^{-1/2} (\log M)^{D(1/2)} (1/2 - 1/\tau) \]

Now, let us estimate the value \( I_1(f) \). By using the Littlewood-Paley theorem (see [14]), we obtain

\[ I_1(f) \leq C_6 \left\{ \left\| \sum_{m \in \mathbb{N}^n \setminus \{0\}} (\Delta_m f) \ast t(\Omega_{m,s}^+) \right\|_{L^{q}(T^n)} \right\}^{1/2} \]

\[ = C_6 \left\{ \left\| T_{\ast} \Delta_m f \right\|_{L^{q}(T^n)} \right\} \|_{L^{q}(T^n)} \]

(10)

where \( \mathbb{N}^n(m, \beta m, \gamma') = \{ s \in \mathbb{Z}^n_+: m \leq (\gamma', s) < \beta m \} \).

By using Lemma 2 and inequality of different metric for trigonometric polynomials in the Lebesgue spaces with mixed metric (see [14]) for \( 1 < p_j < p_0 \) (\( j = 1, \ldots, n \)), from (10) we have

\[ I_1(f) \leq C_7 \left\{ 2^{1,\mathcal{a}} N_s^{-1/2} \| \Delta_m f \|_{L^{p_0}(T^n)} \right\} \|_{L^{q}(T^n)} \]

\[ \leq C_8 \left\{ N_s^{-1/2} \| \Delta_m f \|_{L^{p_0}(T^n)} \right\} \|_{L^{q}(T^n)} \]

\[ = C_9 \left\{ N_s^{-1/2} \| \Delta_m f \|_{L^{p_0}(T^n)} \right\} \|_{L^{q}(T^n)} \]

(11)

According to Hölder’s inequality with parameters \( 1/2 = 1/\tau + 1/\varepsilon \), where \( 1/\varepsilon = (1/2 - 1/\tau)_+ \) and by (11) we find

\[ I_1(f) \leq C_6 \left\{ 2^{1,\mathcal{a}} \| \Delta_m f \|_{L^{p_0}(T^n)} \right\} \|_{L^{q}(T^n)} \]

\[ \times \left\{ N_s^{-1/2} \| \Delta_m f \|_{L^{p_0}(T^n)} \right\} \|_{L^{q}(T^n)} \]

(11)
\begin{equation}
\leq C_8 \left\| f \right\|_{B_2^\infty(T^n)} \left\| \left\{ N_j^{-\left(1/2+1/p'_0\right)} \alpha_j \right\} \right\|_{L_2} \leq C_8 \left\| \left\{ N_j^{-\left(1/2+1/p'_0\right)} \alpha_j \right\} \right\|_{L_2} \tag{12}
\end{equation}
for any function \( f \in B_2^\infty(T^n) \).

By continuing (12), according to the Lemma 4 we have
\[
I_1(f) \leq C_8 2^{\left(1/2+1/p'_0\right)(\alpha_j-1/p_j+1/p_0)m} \times \left\| \left\{ 2^{\left(1/2+1/p'_0\right)(\alpha_j-1/p_j+1/p_0-1)} \alpha_j \right\} \right\|_{L_2} = C_8 2^{\left(1/2+1/p'_0\right)(\alpha_j-1/p_j+1/p_0)m} \times \left\| \left\{ 2^{\left(1/2+1/p'_0\right)(\alpha_j-1/p_j+1/p_0-1)} \alpha_j \right\} \right\|_{L_2} \leq C_8 2\left(1/2+1/p'_0\right)(\alpha_j-1/p_j+1/p_0)m \left\| \left\{ 2^{\left(1/2+1/p'_0\right)(\alpha_j-1/p_j+1/p_0-1)} \alpha_j \right\} \right\|_{L_2} \leq C_9 2\left(1/2+1/p'_0\right)(\alpha_j-1/p_j+1/p_0)m 2^{\left(1/2+1/p'_0\right)(\alpha_j-1/p_j+1/p_0-1)} m \sum_{j \in D \setminus \{j_1\} \cap \{j_2\}} 1/\epsilon_j = C_9 2\left(\alpha_j-1/p_j+1/2\right)m \sum_{j \in D \setminus \{j_1\} \cap \{j_2\}} (1/2-\tau_j) + \frac{1}{\epsilon_j}
\]
as \((\alpha_j-1/p_j+1/p_0-1)/(\alpha_j-1/p_j+1/p_0-1) > \gamma_j^0\) at \( j \notin D \).

Taking into account that \( M \geq 2^{m(\|D\|+1)} \) we find
\[
I_2(f) \leq C_{10} M^{-\left(\alpha_j-1/p_j+1/2\right)(\log M)(\|D\|+1)} \left(\alpha_j-1/p_j+1/2\right) + \sum_{j \in D \setminus \{j_1\} \cap \{j_2\}} (1/2-\tau_j) + \frac{1}{\epsilon_j}
\tag{13}
\]
Inserting (9) and (13) into (6) we obtain the inequality, which gives the upper estimate in (4).

For the proof of the lower estimate we consider the following value
\[
e_M(f)_V = \sup_{f \in F} \inf_{\sum_{j=1}^M b_j e^{i(k_j,x)}} \left\| f - \sum_{j=1}^M b_j e^{i(k_j,x)} \right\|_V,
\]
which is called the best \( M \)-term approximation of the class \( F \) in metric space \( V \).

Moreover, by the definition, the following inequality holds
\[
e_M(f)_V \leq d_M^2(f,V).
\]

By using the condition \( 2 < q_j \) \( (j = 1, \ldots, n) \) we have
\[
e_M(f)_{L_2(T^n)} \leq C_{11} e_M(f)_{L_w^q(T^n)}.
\]

For the proof of the lower estimate we will use double relation, which follows from the general results of S.M. Nikol’skii (see [15]). According to this relation for any function \( f \in L_2(T^n) \) the following equality holds
\[
e_M(f)_{L_2(T^n)} = \inf_{\sum_{j=1}^M b_j e^{i(k_j,x)}} \sup_{f \in F} \left\| f - \sum_{j=1}^M b_j e^{i(k_j,x)} \right\|_{L_2(T^n)}
\tag{14}
\]
where \( L \) is a linear span of a system of functions \( \{ e^{i(k,x)} \}_{k \in \Omega_M} \).

We consider the function
\[
f(x) = m^{-\sum_{j \in D \setminus \{j_1\} \cap \{j_2\}} 1/\tau_j} \sum_{m \leq \gamma_j < s_0 < \gamma_j + n} \prod_{j=1}^n 2^{\left(\alpha_j+1-1/p_j\right)s_j} \sum_{k \in \rho(s_0)} e^{i(k,x)}
\]
where \( D' = \{ j \in D : 2 < \tau_j \} \cup \{j_1\} \), \( s_0 = (s_0^0, \ldots, s_0^n) \), \( s_j^0 = s_j \) at \( j \in D' \) and \( s_j^0 = 0 \) at \( j \notin D' \).
In the paper [16] it was proved that the function $C_{12}f(x)$ belongs to the class $B^0_{r}(\mathbb{T}^n)$.

Let us construct the function $P(x)$ satisfying the condition (14).

Let

$$u(x) = \sum_{(s_i, s_0) \leq m} \sum_{k \in \rho(s_0)} e^{i(k \cdot x)}$$

and $\Omega_M$ be an arbitrary collection of integer vectors $k = (k_1, \ldots, k_n)$.

We denote by

$$v(x) = \sum_{(s_i, s_0) \leq m} \sum_{k \in \rho(s_0) \cap \Omega_M} e^{i(k \cdot x)}$$

the function, containing only those terms of (15), for which $w(x)$ does not contain the harmonics from $\Omega_M$. By Minkowski’s inequality and Parseval’s equality for function $w(x) = u(x) - v(x)$ we have

$$\|w\|_{L_2(\mathbb{T}^n)} \leq C_{13}M^{1/2}.$$  

We consider the function $P(x) = C^{-1}_{13} M^{-1/2} w(x)$, then $\|P\|_{L_2(\mathbb{T}^n)} \leq 1$. Since the function $w(x) = u(x) - v(x)$ does not contain the harmonics from $\Omega_M$, then function $P \in L^1$. Thus, the function $P(x)$ satisfies the conditions from (14).

According to (14) and by Lemma 5 we obtain

$$e_M(f)_{L_2(\mathbb{T}^n)} \geq C_{14} M^{-1/2} \left| \int_{\mathbb{T}^n} f(x) \omega(x) dx \right| \geq$$

$$\geq C_{14} M^{-1/2} m^{-\sum_{j \in \mathbb{N} \backslash \{1\}} 1/T_j} \sum_{(s_i, s_0) = m} \prod_{j=1}^{n} 2^{-(\alpha_j + 1 - 1/p_j) s_0} \sum_{k \in \rho(s_0)} 1 =$$

$$\geq C_{14} M^{-1/2} m^{-\sum_{j \in \mathbb{N} \backslash \{1\}} 1/T_j} \sum_{(s_i, s_0) = m} \prod_{j=1}^{n} 2^{-(\alpha_j - 1/p_j) s_0} =$$

$$= C_{14} M^{-1/2} m^{-\sum_{j \in \mathbb{N} \backslash \{1\}} 1/T_j} \left\| \sum_{j \in \mathbb{N} \backslash \{1\}} \left\{ 2^{-(\alpha_j - 1/p_j) s_0} \right\}_{\mathbb{N} \backslash \{1\}} \right\|_{L_1} \geq$$

$$\geq M^{-1/2} m^{-\sum_{j \in \mathbb{N} \backslash \{1\}} 1/T_j} 2^{-(\alpha_j - 1/p_j) m \|\mathbb{N} \| \|D\|^{-1}}.$$  

(16)

where $s = (s_{j_1}, \ldots, s_{j_1(D)})$.

Taking into account that $M \approx 2^m m^{\|D\|^{-1}}$ from (16) we have

$$e_M(f)_{L_2(\mathbb{T}^n)} \geq C_{15} 2^{-(\alpha_j - 1/p_j) m} \sum_{j \in \mathbb{N} \backslash \{1\}} (1/2 - 1/T_j) \geq$$

$$= C_{16} M^{-(\alpha_j - 1/p_j) m} (\log M)^{\|D\|^{-1}} (\alpha_j - 1/p_j) m + \sum_{j \in \mathbb{N} \backslash \{1\}} (1/2 - 1/T_j).$$  

(17)

By (17) lower estimate in (4) follows.

Remark. Note, that for $p = (p, \ldots, p)$, $\tau = (\tau, \ldots, \tau)$ and $q = \theta = (q, \ldots, q)$ the statement of the theorem coincides with the corresponding result of A.S. Romanyuk [7].

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References


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Анизотропты Лоренц кеңістігінде метрикасындағы аралас метрикалы Никольский-Бесов класындағы тригонометриялық көлденеңін реті

Тригонометриялық көлденең ұзындығы бірлешеді және жағдайда алғаш рет Р.С. Исымгилов енгізді және ұзілісінің функциялар кеңістікіндегі біркіткен кластер ұзындығы оларға бағалаудар бәлгіледі. Кен әйірмеалы функциялар ұзындық $L_q$ кеңістікінде Соболевтік $W^r_p$ Никольскийдің $B^{r,p}_{p,q}$ кластерлердің тригонометриялық көлденеңдерінің нормалары менді бұл бағалаударды Қ.А.Бекмаганбетов және Е. Толегазының бірнеше табілді.

Құрам сөздер: тригонометриялық көлденең, анизотропты Лоренц кеңістіктері, аралас метрикалы Никольский-Бесов класы.
Порядок тригонометрического поперечника класса Никольского-Бесова со смешанной метрикой в метрике анизотропного пространства Лоренца

Понятие тригонометрического поперечника в одномерном случае впервые введено Р.С. Исмагиловым и им были установлены оценки для некоторых классов в пространстве непрерывных функций. Для функций многих переменных точные порядки тригонометрических поперечников класса Соболева $W^r_p$ Никольского $H^r_p$ в пространстве $L_q$ установлены Э.С.Белинским, В.Е. Майоровым, Ю. Маковозом, Г.Г. Магарил-Ильйевым, В.Н. Темляковым. Эта задача для класса Бесова $B^r_{pq}$ исследована А.С. Романюком и Д.В. Базархановым. Тригонометрический поперечник для анизотропного класса Никольского-Бесова $B^{r}_{pq}(\mathbb{T}^n)$ в метрике анизотропных пространств Лоренца $L_{aq}(\mathbb{T}^n)$ был найден К.А. Бекмаганбетовым и Е. Толеугазы.

Ключевые слова: тригонометрический поперечник, анизотропные пространства Лоренца, класс Никольского-Бесова со смешанной метрикой.

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