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Stabilization of a solution for two-dimensional loaded parabolic equation

In this paper we consider the stabilization problem of the solution of a boundary value problem for the heat equation with a loaded two-dimensional Laplace operator. The loaded terms represent the values of the required function and traces of the first-order partial derivatives of the required function at fixed points. An algorithm for constructing boundary control functions is proposed.

Keywords: problem of boundary stabilization, loaded heat equation, loaded Laplace operator, biorthogonal system, stabilization, algorithm.

Introduction

Along with the direct heat conduction problem - finding the temperature field by solving an equation with known boundary conditions, it is often necessary to solve inverse problems, where the corresponding boundary conditions from a given temperature distribution in space and time need to be determined. Such inverse problems have great practical applications in physics, technology, mechanics, and medicine.

Boundary value problems for loaded heat conduction equations, by themselves, have a large amount of applications; they also constitute a special class of equations with their own specific problems. Such problems arise when studying the unique solvability of semi-periodic (periodic in a time variable) problems in a bounded domain in problems of optimal agroecosystem management, for example, the problem of long-term forecasting and regulation of the level of groundwater and soil moisture.

Recently, among specialists in control problems, interest has significantly increased in the stabilization of solutions to boundary value problems [1]-[4]. First of all, this is due not only to their importance in theoretical terms, but also to the fact that one has to deal with them in many applied problems.

The problem considered in this paper on the stabilizability from the boundary $\partial\Omega$ of a solution of a parabolic equation given in a bounded domain $\Omega \in \mathbb{R}$ consists in choosing a boundary control such that the solution of the boundary value problem tends at $t \to \infty$ to a given stationary solution at a given rate $exp\,(-\sigma_0 t)$. In this case, it is required that the control be with feedback, i.e. so that it responds to unforeseen fluctuations of the system, suppressing the results of their influence on the stabilized solution.

In [5], the stabilization problem for a parabolic equation is reduced to solving an auxiliary boundary value problem in an extended domain of independent variables. This idea was further developed in [6]-[8]. Note that in [5]-[8], stabilization problems for differential equations without load were considered. At the same time, loaded differential equations [9]-[14] are actively used in control problems for nonlocal dynamical systems. In [15]-[20], stabilization problems were studied for a loaded one- and two-dimensional heat equation. In this paper, we consider the stabilization problem on the boundary (forming a parallelepiped) of the solution of the boundary value problem for the heat equation with a loaded two-dimensional Laplace operator, where the loaded terms are the values of the required function and traces of the derivatives of the required function at fixed points.

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1. Problem Formulation

Let $\Omega = \{(x,y) : -\frac{\pi}{2} < x, y < \frac{\pi}{2}\}$ be a domain with boundary $\partial\Omega$. In a parallelepiped $Q = \Omega \times \{t > 0\}$ with a lateral surface $\Sigma = \partial\Omega \times \{t > 0\}$ we consider boundary value problem for a loaded heat equation:

$$\frac{\partial u}{\partial t} - \triangle\, u + \alpha_1\, u\,(x,y,t)|_{x=0} + \alpha_2\, u\,(x,y,t)|_{y=0} + \alpha_3\, \left.\frac{\partial u\,(x,y,t)}{\partial x}\right|_{x=0} + \alpha_4\, \left.\frac{\partial u\,(x,y,t)}{\partial y}\right|_{y=0} = 0, \quad (1)$$

$$u\,(x,y,t)|_{t=0} = u_0\,(x,y)\,,\ \{x,y\} \in \Omega, \quad (2)$$

$$u\,(x,y,t)|_{\Sigma} = p\,(x,y,t) =$$
$$= \left\{ u_1\left(\frac{\pi}{2}, y, t\right); u_2\left(x, \frac{\pi}{2}, t\right); u_3\left(-\frac{\pi}{2}, y, t\right); u_4\left(x, -\frac{\pi}{2}, t\right) \Big| \{x,y,t\} \in \Sigma \right\}, \quad (3)$$

where $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in \mathbb{C}$, $u_0\,(x,y)$ is a known function. Equation (1) is loaded [9], [10]. It is necessary to find such boundary functions $u_1\left(\frac{\pi}{2}, y, t\right)$; $u_2\left(x, \frac{\pi}{2}, t\right)$; $u_3\left(-\frac{\pi}{2}, y, t\right)$; $u_4\left(x, -\frac{\pi}{2}, t\right)$, so that the solution of the boundary value problem (1)-(3) satisfies the inequality:

$$\|u\,(x,y,t)\,\|_{L_2(\Omega)} \le C_0 e^{-\sigma t},\ \ \sigma > 0,\, t > 0, \quad (4)$$

where σ is a given constant, $C_0 \ge \|u_0\,(x,y)\,\|_{L_2(\Omega)}$ is an arbitrary bounded constant.

2. Auxiliary boundary value problem

Let $\Omega_1 = \{(x,y) : -\pi < x, y < \pi\}$ and $Q_1 = \Omega_1 \times \{t > 0\}$.

$$\frac{\partial z}{\partial t} - \triangle\, z + \alpha_1\, z\,(x,y,t)|_{x=0} + \alpha_2\, z\,(x,y,t)|_{y=0} + \alpha_3\, \left.\frac{\partial z\,(x,y,t)}{\partial x}\right|_{x=0} + \alpha_4\, \left.\frac{\partial z\,(x,y,t)}{\partial y}\right|_{y=0} = 0, \quad (5)$$

$$z\,(x,y,t)|_{t=0} = z_0\,(x,y)\,,\ \{x,y\} \in \Omega, \quad (6)$$

$$\left.\begin{array}{l} z\,(-\pi,y,t) = z\,(\pi,y,t)\,; \ \ \dfrac{\partial z\,(-\pi,y,t)}{\partial x} = \dfrac{\partial z\,(\pi,y,t)}{\partial x},\ \ \{y,t\} \in (-\pi;\pi) \times \{t > 0\} \\ z\,(x,-\pi,t) = z\,(x,\pi,t)\,; \ \ \dfrac{\partial z\,(x,-\pi,t)}{\partial y} = \dfrac{\partial z\,(x,\pi,t)}{\partial y},\ \ \{x,t\} \in (-\pi;\pi) \times \{t > 0\} \end{array}\right\} \quad (7)$$

where $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in \mathbb{C}$, $z_0\,(x,y)$ is a known function. It is necessary to find the function z, so that the solution of the auxiliary boundary value problem (5)-(7) satisfies the inequality:

$$\|z\,(x,y,t)\,\|_{L_2(\Omega_1)} \le C_1 e^{-\sigma t},\ \ \sigma > 0,\, t > 0, \quad (8)$$

where σ is a given constant, $C_1 \ge C_0$ is an arbitrary bounded constant.

3. Spectral problem for the loaded two-dimensional Laplace operator

Let's find the solution of the problem (5)-(7) by the method of separation of variables

$$z\,(x,y,t) = \sum_{k,n \in \mathbb{Z}} Z_{k,n}\,(t) \cdot \varphi_{k,n}\,(x,y)$$

$$\frac{\partial z}{\partial t} = \sum_{k,n \in \mathbb{Z}} Z'_{k,n}\,(t) \cdot \varphi_{k,n}\,(x,y)\,;$$

$$\triangle z = \sum_{k,n\in\mathbb{Z}} Z_{k,n}(t)\left(\frac{\partial^2\varphi_{k,n}(x,y)}{\partial x^2} + \frac{\partial^2\varphi_{k,n}(x,y)}{\partial y^2}\right) = \sum_{k,n\in\mathbb{Z}} Z_{k,n}(t)\cdot\triangle\varphi_{k,n};$$

$$\alpha_1\, z(x,y,t)|_{x=0} + \alpha_2\, z(x,y,t)|_{y=0} = \sum_{k,n\in\mathbb{Z}} \left(\alpha_1 Z_{k,n}(t)\,\varphi_{k,n}(0,y) + \alpha_2 Z_{k,n}(t)\,\varphi_{k,n}(x,0)\right);$$

$$\alpha_3\, \frac{\partial z(x,y,t)}{\partial x}\bigg|_{x=0} + \alpha_4\, \frac{\partial z(x,y,t)}{\partial y}\bigg|_{y=0} =$$

$$= \sum_{k,n\in\mathbb{Z}} \left(\alpha_3 Z_{k,n}(t)\, \frac{\partial\varphi_{k,n}(x,y)}{\partial x}\bigg|_{x=0} + \alpha_4 Z_{k,n}(t)\, \frac{\partial\varphi_{k,n}(x,y)}{\partial y}\bigg|_{y=0}\right).$$

Now we substitute the obtained expressions into (5):

$$\sum_{k,n\in\mathbb{Z}} \Big(Z'_{k,n}(t)\cdot\varphi_{k,n}(x,y) - Z_{k,n}(t)\cdot\triangle\varphi_{k,n} +$$

$$+Z_{k,n}(t)\left(\alpha_1\varphi_{k,n}(0,y) + \alpha_2\varphi_{k,n}(x,0) + \alpha_3\, \frac{\partial\varphi_{k,n}(x,y)}{\partial x}\bigg|_{x=0} + \alpha_4\, \frac{\partial\varphi_{k,n}(x,y)}{\partial y}\bigg|_{y=0}\right)\Bigg) = 0.$$

Hence, we get:

$$Z'_{k,n}(t)\cdot\varphi_{k,n}(x,y) - Z_{k,n}(t)\cdot\triangle\varphi_{k,n} +$$

$$+ Z_{k,n}(t)\left(\alpha_1\varphi_{k,n}(0,y) + \alpha_2\varphi_{k,n}(x,0) + \alpha_3\, \frac{\partial\varphi_{k,n}(x,y)}{\partial x}\bigg|_{x=0} + \alpha_4\, \frac{\partial\varphi_{k,n}(x,y)}{\partial y}\bigg|_{y=0}\right) = 0.$$

Dividing both sides of the equality by $Z_{k,n}(t)\cdot\varphi_{k,n}(x,y)$, we obtain:

$$\frac{Z'_{k,n}(t)}{Z_{k,n}(t)} = \frac{\triangle\varphi_{k,n} - \alpha_1\varphi_{k,n}(0,y) - \alpha_2\varphi_{k,n}(x,0) - \alpha_3\, \frac{\partial\varphi_{k,n}(x,y)}{\partial x}\Big|_{x=0} - \alpha_4\, \frac{\partial\varphi_{k,n}(x,y)}{\partial y}\Big|_{y=0}}{\varphi_{k,n}(x,y)} = -\lambda_{k,n}.$$

In order to find $\varphi_{k,n}(x,y)$, we consider the following spectral problem.

$$\left.\begin{aligned} \frac{\triangle\varphi_{k,n} - \alpha_1\varphi_{k,n}(0,y) - \alpha_2\varphi_{k,n}(x,0) - \alpha_3\, \frac{\partial\varphi_{k,n}(x,y)}{\partial x}\Big|_{x=0} - \alpha_4\, \frac{\partial\varphi_{k,n}(x,y)}{\partial y}\Big|_{y=0}}{\varphi_{k,n}(x,y)} = -\lambda_{k,n}, \\ \varphi_{k,n}(-\pi,y) = \varphi_{k,n}(\pi,y);\ \ \frac{\partial\varphi_{k,n}(-\pi,y)}{\partial x} = \frac{\partial\varphi_{k,n}(\pi,y)}{\partial x}, \\ \varphi_{k,n}(x,-\pi) = \varphi_{k,n}(x,\pi);\ \ \frac{\partial\varphi_{k,n}(x,-\pi)}{\partial y} = \frac{\partial\varphi_{k,n}(x,\pi)}{\partial y}. \end{aligned}\right\} \tag{9}$$

Let's use the method of separation of variables again. Let $\varphi_{k,n}(x,y) = X_k(x)\cdot Y_n(y)$. Then the problem (9) can be written as follows:

$$\left.\begin{aligned} \frac{X''_k(x) - \alpha_1 X_k(0) - \alpha_3 X'_k(0)}{X_k(x)} = -\lambda_{k,n} - \frac{Y''_n(y) - \alpha_2 Y_n(0) - \alpha_4 Y'_n(0)}{Y_n(y)} = -\mu_k, \\ X_k(-\pi) = X_k(\pi);\ \ X'_k(-\pi) = X'_k(\pi), \\ Y_n(-\pi) = Y_n(\pi);\ \ Y'_n(-\pi) = Y'_n(\pi). \end{aligned}\right\}$$

This problem was reduced to finding solutions of the following two differential equations with periodic conditions:

$$\left.\begin{aligned} X''_k(x) + \mu_k X_k(x) - \big(\alpha_1 X_k(0) + \alpha_3 X'_k(0)\big) = 0, \\ X_k(-\pi) = X_k(\pi);\ \ X'_k(-\pi) = X'_k(\pi). \end{aligned}\right\} \tag{10}$$

$$\left.\begin{aligned} Y_n''(y) - (\mu_k - \lambda_{k,n}) Y_n(y) - \big(\alpha_2 Y_n(0) + \alpha_4 Y_n'(0)\big) = 0, \\ Y_n(-\pi) = Y_n(\pi);\ \ Y_n'(-\pi) = Y_n'(\pi). \end{aligned}\right\} \tag{11}$$

Note that the general solution of loaded ordinary differential equations (10) and (11) is represented as a linear combination of the complete system of periodic functions $\left\{\Phi_m = e^{imt},\ \ m \in \mathbb{Z}\right\}$. Therefore, we will find a solution of problem (10) in the form $X_k(x) = A_k e^{ikx} + C_k$ $(k \in \mathbb{Z})$. Let's consider several cases.

Case (a). Let $\alpha_1 \neq k^2$ $\forall k \in \mathbb{Z} \setminus \{0\}$. Then the solution to the problem (10) will be as follows:

$$X_k(x) = A_k e^{ikx} + C_k \ \Rightarrow\ X_k(0) = A_k + C_k,$$

$$X_k'(x) = ikA_k e^{ikx} \ \Rightarrow\ X_k'(0) = ikA_k,$$

$$X_k''(x) = -k^2 A_k e^{ikx},$$

$$-k^2 A_k e^{ikx} + \mu_k A_k e^{ikx} + \mu_k C_k - (\alpha_1 A_k + \alpha_1 C_k + i\alpha_3 k A_k) = 0.$$

$$\begin{cases} -k^2 A_k + \mu_k A_k = 0, \\ \mu_k C_k - (\alpha_1 A_k + \alpha_1 C_k + i\alpha_3 k A_k) = 0 \end{cases} \Rightarrow \begin{cases} A_k\left(-k^2 + \mu_k\right) = 0, \\ \mu_k C_k - (\alpha_1 A_k + \alpha_1 C_k + i\alpha_3 k A_k) = 0. \end{cases}$$

If $A_k = 0$, then the equation (10) has no nontrivial solutions. Therefore, it follows from the first equation of the last system that A_k can take any nonzero value. For simplicity, we will assume that $A_k = 1$. It should also be noted that from the equality $\mu_k = k^2$ it follows that $k \neq 0$, since for $\mu_k = 0$ the equation (10) also has no nontrivial solutions. Thus:

$$\begin{cases} A_k = 1, \\ \mu_k = k^2,\ k \neq 0 \\ k^2 C_k - \alpha_1 - \alpha_1 C_k - i\alpha_3 k = 0 \end{cases} \Rightarrow \begin{cases} A_k = 1, \\ \mu_k = k^2,\ k \neq 0 \\ C_k = \dfrac{\alpha_1}{k^2 - \alpha_1} + \dfrac{i\alpha_3 k}{k^2 - \alpha_1} \end{cases}$$

Therefore, we obtain a system of eigenfunctions:

$$X_k(x) = e^{ikx} + \frac{\alpha_1}{k^2 - \alpha_1} + \frac{i\alpha_3 k}{k^2 - \alpha_1},\ \ \alpha_1 \neq k^2\ \ \forall k \in \mathbb{Z} \setminus \{0\},$$

which correspond to the eigenvalues $\mu_k = k^2$ $\forall k \in \mathbb{Z} \setminus \{0\}$.

Let $\alpha_1 \neq k^2$, $k = 0$. Then we have

$$\left.\begin{aligned} X_0''(x) + \mu_0 X_0(x) - \big(\alpha_1 X_0(0) + \alpha_3 X_0'(0)\big) = 0, \\ X_0(-\pi) = X_0(\pi);\ \ X_0'(-\pi) = X_0'(\pi). \end{aligned}\right\}$$

$$X_0(x) = A_0 + C_0 \ \Rightarrow\ X_0(0) = A_0 + C_0$$

$$X_0'(x) = X_0''(x) = 0$$

$$\mu_0(A_0 + C_0) - \alpha_1(A_0 + C_0) = 0 \quad \Rightarrow \quad (A_0 + C_0)(\mu_0 - \alpha_1) = 0 \quad \Rightarrow \quad \begin{cases} A_0 + C_0 = 1, \\ \mu_0 = \alpha_1. \end{cases}$$

We get the eigenfunction $X_0(x) = 1$, which corresponds to the eigenvalue $\mu_0 = \alpha_1$.

Hence, the system of eigenfunctions and eigenvalues of the problem (10) for the case (a) has the form:

$$X_k(x) = \begin{cases} e^{ikx} + \dfrac{\alpha_1}{k^2 - \alpha_1} + \dfrac{i\alpha_3 k}{k^2 - \alpha_1},\ \ \mu_k = k^2,\ \ \forall k \in \mathbb{Z} \setminus \{0\} \\ 1,\ \ \mu_0 = \alpha_1,\ \ k = 0. \end{cases}$$

Case (b). Let $\exists a \in \mathbb{Z}: \ \alpha_1 = a^2 \ \forall k \in \mathbb{Z}\backslash\{\pm a\}$. Reasoning similarly to the previous case, we obtain a system of eigenfunctions $X_k(x) = e^{ikx} + \frac{\alpha_1}{k^2-\alpha_1} + \frac{i\alpha_3 k}{k^2-\alpha_1}$, which correspond to the eigenvalues $\mu_k = k^2, \ \forall k \in \mathbb{Z}\backslash\{0; \pm a\}$, and the eigenfunction $X_0(x) = 1$, which corresponds to the eigenvalue $\mu_0 = \alpha_1 = a^2$.

Let $\exists a \in \mathbb{Z}: \ \alpha_1 = a^2, \ k = \pm a$. Then we have

$$\left.\begin{array}{r} X''_{\pm a}(x) + \mu_{\pm a}X_{\pm a}(x) - \left(\alpha_1 X_{\pm a}(0) + \alpha_3 X'_{\pm a}(0)\right) = 0, \\ X_{\pm a}(-\pi) = X_{\pm a}(\pi); \ X'_{\pm a}(-\pi) = X'_{\pm a}(\pi). \end{array}\right\}$$

$$X_{\pm a}(x) = A_{\pm a}e^{\pm iax} + C_{\pm a} \Rightarrow X_{\pm a}(0) = A_{\pm a} + C_{\pm a},$$

$$X'_{\pm a}(x) =_{\pm} iaA_{\pm a}e^{\pm ax} \Rightarrow X'_{\pm a}(0) = \pm iaA_{\pm a},$$

$$X''_{\pm a}(x) = -a^2A_{\pm a}e^{\pm iax},$$

$$-a^2A_{\pm a}e^{\pm iax} + \mu_{\pm a}A_{\pm a}e^{\pm iax} + \mu_{\pm a}C_{\pm a} - \left(a^2A_{\pm a} + a^2C_{\pm a} \pm i\alpha_3 aA_{\pm a}\right) = 0.$$

$$\left\{\begin{array}{r} -a^2A_{\pm a} + \mu_{\pm a}A_{\pm a} = 0, \\ \mu_{\pm a}C_{\pm a} - \left(a^2A_{\pm a} + a^2C_{\pm a} \pm i\alpha_3 aA_{\pm a}\right) = 0 \end{array}\right. \Rightarrow \left\{\begin{array}{r} A_{\pm a} = 1, \\ \mu_{\pm a} = a^2, \\ a^2C_{\pm a} - a^2 - a^2C_{\pm a} \mp i\alpha_3 a = 0 \end{array}\right. \Rightarrow$$

$$\Rightarrow \left\{\begin{array}{r} A_{\pm a} = 1, \\ \mu_{\pm a} = a^2, \\ -a^2 \mp i\alpha_3 a = 0 \end{array}\right. \Rightarrow \left\{\begin{array}{r} A_{\pm a} = 1, \\ \mu_{\pm a} = a^2, \\ a(a \pm i\alpha_3) = 0 \end{array}\right. \Rightarrow \left\{\begin{array}{r} A_{\pm a} = 1, \\ \mu_{\pm a} = a^2, \\ a = 0 \text{ или } \alpha_3 = \pm ia \end{array}\right.$$

Therefore, we get the eigenfunctions $X_{\pm a}(x) = e^{\pm iax}$, which correspond to the eigenvalues $\mu_{\pm a} = \alpha_1 = a^2$.

Hence, the system of eigenfunctions and eigenvalues of the problem (10) for the case (b) has the form:

$$X_k(x) = \left\{\begin{array}{r} e^{ikx} + \dfrac{\alpha_1}{k^2-\alpha_1} + \dfrac{i\alpha_3 k}{k^2-\alpha_1}, \ \mu_k = k^2, \ \forall k \in \mathbb{Z}\backslash\{0; \pm a\} \\ 1, \ \mu_0 = \alpha_1 = a^2, \ k = 0 \\ e^{\pm iax}, \ \mu_{\pm a} = \alpha_1 = a^2, \ k = \pm a \end{array}\right.$$

The solution of the problem (11) is defined similarly.

Case (c). Let $\alpha_2 \neq n \ \forall n \in \mathbb{Z}$. Then the system of eigenfunctions and eigenvalues of the problem (11) has the form:

$$Y_n(y) = \left\{\begin{array}{r} e^{iny} + \dfrac{\alpha_2}{n^2-\alpha_2} + \dfrac{i\alpha_4 n}{n^2-\alpha_2}, \ \lambda_{k,n} = k^2 + n^2, \ \forall k \in \mathbb{Z}, \ \forall n \in \mathbb{Z}\backslash\{0\} \\ 1, \ \lambda_{k,0} = k^2 + \alpha_2, \ \forall k \in \mathbb{Z}, \ n = 0. \end{array}\right.$$

Case (d). Let $\exists b \in \mathbb{Z}: \ \alpha_2 = b^2 \ \forall n \in \mathbb{Z}$. Then the system of eigenfunctions and eigenvalues of the problem (11) has the form:

$$Y_n(y) = \left\{\begin{array}{r} e^{iny} + \dfrac{b^2}{n^2-b^2} + \dfrac{i\alpha_4 n}{n^2-b^2}, \ \lambda_{k,n} = k^2 + n^2, \ \forall k \in \mathbb{Z}, \forall n \in \mathbb{Z}\backslash\{0; \pm b\} \\ 1, \ \lambda_{k,0} = k^2 + \alpha_2 = k^2 + b^2, \ \forall k \in \mathbb{Z}, n = 0 \\ e^{\pm iby}, \ \lambda_{k;\pm b} = k^2 + \alpha_2 = k^2 + b^2, \ \forall k \in \mathbb{Z}, n = \pm b \end{array}\right.$$

Let's write the systems of eigenfunctions and eigenvalues of the problem (9).

Case 1. Let $\alpha_1 \neq k^2$ $\forall k \in \mathbb{Z}$ and $\alpha_2 \neq n^2$ $\forall n \in \mathbb{Z}$. Then the system of eigenfunctions and eigenvalues of the problem (9) has the form:

$$\begin{gathered}
\Big\{\varphi_{k,n}(x,y) = \left(e^{ikx} + \frac{\alpha_1}{k^2-\alpha_1} + \frac{i\alpha_3 k}{k^2-\alpha_1}\right)\left(e^{iny} + \frac{\alpha_2}{n^2-\alpha_2} + \frac{i\alpha_4 n}{n^2-\alpha_2}\right),\\
\lambda_{k,n} = k^2+n^2,\ \forall k,n \in \mathbb{Z}\setminus\{0\};\\
\varphi_{k,0}(x,y) = e^{ikx} + \frac{\alpha_1}{k^2-\alpha_1} + \frac{i\alpha_3 k}{k^2-\alpha_1},\ \ \lambda_{k,0} = k^2+\alpha_2,\ \forall k \in \mathbb{Z}\setminus\{0\};\\
\varphi_{0,n}(x,y) = e^{iny} + \frac{\alpha_2}{n^2-\alpha_2} + \frac{i\alpha_4 n}{n^2-\alpha_2},\ \ \lambda_{0,n} = \alpha_1+n^2,\ \forall n \in \mathbb{Z}\setminus\{0\};\\
\varphi_{0,0}(x,y) = 1,\ \ \lambda_{0,0} = \alpha_1+\alpha_2\Big\}.
\end{gathered} \tag{12}$$

Case 2. Let $\alpha_1 \neq k^2$ $\forall k \in \mathbb{Z}$ and $\exists b \in \mathbb{Z}:\ \alpha_2 = b^2$ $\forall n \in \mathbb{Z}$. Then the system of eigenfunctions and eigenvalues of the problem (9) has the form:

$$\begin{gathered}
\Big\{\varphi_{k,n}(x,y) = \left(e^{ikx} + \frac{\alpha_1}{k^2-\alpha_1} + \frac{i\alpha_3 k}{k^2-\alpha_1}\right)\left(e^{iny} + \frac{\alpha_2}{n^2-\alpha_2} + \frac{i\alpha_4 n}{n^2-\alpha_2}\right),\\
\lambda_{k,n} = k^2+n^2,\ \forall k \in \mathbb{Z}\setminus\{0\},\ n \in \mathbb{Z}\setminus\{0;\pm b\};\\
\varphi_{k,0}(x,y) = e^{ikx} + \frac{\alpha_1}{k^2-\alpha_1} + \frac{i\alpha_3 k}{k^2-\alpha_1},\ \ \lambda_{k,0} = k^2+\alpha_2,\ \forall k \in \mathbb{Z}\setminus\{0\};\\
\varphi_{k,\pm b}(x,y) = \left(e^{ikx} + \frac{\alpha_1}{k^2-\alpha_1} + \frac{i\alpha_3 k}{k^2-\alpha_1}\right)e^{\pm iby},\ \ \lambda_{k,\pm b} = k^2+\alpha_2,\ \forall k \in \mathbb{Z}\setminus\{0\};\\
\varphi_{0,n}(x,y) = e^{iny} + \frac{\alpha_2}{n^2-\alpha_2} + \frac{i\alpha_4 n}{n^2-\alpha_2},\ \ \lambda_{0,n} = \alpha_1+n^2,\ \forall n \in \mathbb{Z}\setminus\{0;\pm b\};\\
\varphi_{0,\pm b}(x,y) = e^{\pm iby},\ \ \lambda_{0,\pm b} = \alpha_1+\alpha_2;\\
\varphi_{0,0}(x,y) = 1,\ \ \lambda_{0,0} = \alpha_1+\alpha_2\Big\}.
\end{gathered} \tag{13}$$

Case 3. Let $\exists a \in \mathbb{Z}:\ \alpha_1 = a^2$ $\forall k \in \mathbb{Z}$ and $\alpha_2 \neq n^2$ $\forall n \in \mathbb{Z}$. Then the system of eigenfunctions and eigenvalues of the problem (9) has the form:

$$\begin{gathered}
\Big\{\varphi_{k,n}(x,y) = \left(e^{ikx} + \frac{\alpha_1}{k^2-\alpha_1} + \frac{i\alpha_3 k}{k^2-\alpha_1}\right)\left(e^{iny} + \frac{\alpha_2}{n^2-\alpha_2} + \frac{i\alpha_4 n}{n^2-\alpha_2}\right),\\
\lambda_{k,n} = k^2+n^2,\ \forall k \in \mathbb{Z}\setminus\{0;\pm a\},\ n \in \mathbb{Z}\setminus\{0\};\\
\varphi_{k,0}(x,y) = e^{ikx} + \frac{\alpha_1}{k^2-\alpha_1} + \frac{i\alpha_3 k}{k^2-\alpha_1},\ \ \lambda_{k,0} = k^2+\alpha_2,\ \forall k \in \mathbb{Z}\setminus\{0;\pm a\};\\
\varphi_{0,n}(x,y) = e^{iny} + \frac{\alpha_2}{n^2-\alpha_2} + \frac{i\alpha_4 n}{n^2-\alpha_2},\ \ \lambda_{0,n} = \alpha_1+n^2,\ \forall n \in \mathbb{Z}\setminus\{0\};\\
\varphi_{\pm a,n}(x,y) = e^{\pm iax}\left(e^{ikx} + \frac{\alpha_1}{k^2-\alpha_1} + \frac{i\alpha_3 k}{k^2-\alpha_1}\right),\ \ \lambda_{\pm a,n} = \alpha_1+n^2,\ \forall n \in \mathbb{Z}\setminus\{0\};\\
\varphi_{\pm a,0}(x,y) = e^{\pm iax},\ \ \lambda_{\pm a,0} = \alpha_1+\alpha_2;\\
\varphi_{0,0}(x,y) = 1,\ \ \lambda_{0,0} = \alpha_1+\alpha_2\Big\}.
\end{gathered} \tag{14}$$

Case 4. Let $\exists a \in \mathbb{Z}: \quad \alpha_1 = a^2 \ \forall k \in \mathbb{Z}$ and $\exists b \in \mathbb{Z}: \quad \alpha_2 = b^2 \ \forall n \in \mathbb{Z}$. Then the system of eigenfunctions and eigenvalues of the problem (9) has the form:

$$
\Big\{\varphi_{k,n}(x,y) = \left(e^{ikx} + \frac{\alpha_1}{k^2-\alpha_1} + \frac{i\alpha_3 k}{k^2-\alpha_1}\right)\left(e^{iny} + \frac{\alpha_2}{n^2-\alpha_2} + \frac{i\alpha_4 n}{n^2-\alpha_2}\right),
$$

$$
\lambda_{k,n} = k^2 + n^2, \ \forall k \in \mathbb{Z}\setminus\{0;\pm a\}, n \in \mathbb{Z}\setminus\{0;\pm b\};
$$

$$
\varphi_{k,0}(x,y) = e^{ikx} + \frac{\alpha_1}{k^2-\alpha_1} + \frac{i\alpha_3 k}{k^2-\alpha_1}, \ \lambda_{k,0} = k^2 + \alpha_2, \ \forall k \in \mathbb{Z}\setminus\{0;\pm a\}; \tag{15}
$$

$$
\varphi_{0,n}(x,y) = e^{iny} + \frac{\alpha_2}{n^2-\alpha_2} + \frac{i\alpha_4 n}{n^2-\alpha_2}, \ \lambda_{0,n} = \alpha_1 + n^2, \ \forall n \in \mathbb{Z}\setminus\{0;\pm b\};
$$

$$
\varphi_{\pm a,n}(x,y) = e^{\pm iax}\left(e^{ikx} + \frac{\alpha_1}{k^2-\alpha_1} + \frac{i\alpha_3 k}{k^2-\alpha_1}\right), \ \lambda_{\pm a,n} = \alpha_1 + n^2, \ \forall n \in \mathbb{Z}\setminus\{0;\pm b\};
$$

$$
\varphi_{k,\pm b}(x,y) = \left(e^{ikx} + \frac{\alpha_1}{k^2-\alpha_1} + \frac{i\alpha_3 k}{k^2-\alpha_1}\right)e^{\pm iby}, \ \lambda_{k,\pm b} = k^2 + \alpha_2, \ \forall k \in \mathbb{Z}\setminus\{0;\pm a\};
$$

$$
\varphi_{\pm a,\pm b}(x,y) = e^{\pm i(ax+by)}, \ \lambda_{\pm a,\pm b} = \alpha_1 + \alpha_2;
$$

$$
\varphi_{\pm a,0}(x,y) = e^{\pm iax}, \ \lambda_{\pm a,0} = \alpha_1 + \alpha_2;
$$

$$
\varphi_{0,\pm b}(x,y) = e^{\pm iby}, \ \lambda_{0,\pm b} = \alpha_1 + \alpha_2;
$$

$$
\varphi_{0,0}(x,y) = 1, \ \lambda_{0,0} = \alpha_1 + \alpha_2\Big\}.
$$

Solution of the equation $\frac{Z'_{k,n}(t)}{Z_{k,n}(t)} = -\lambda_{k,n}$ has the form.

$$
Z_{k,n}(t) = C_{k,n} \cdot e^{-\lambda_{k,n} t}, \tag{16}
$$

where $C_{k,n} = z_{0kn}$ are the expansion coefficients of the function $z_0(x,y)$ by system $\{\varphi_{k,n}(x,y), \ k,n \in \mathbb{Z}\}$.

Note that the obtained systems of eigenfunctions (12)-(15) are complete in the space $L_2(\Omega_1)$, forme a basis but is not orthogonal (the completeness of the systems of eigenfunctions (12)-(15) follows from the Paley-Wiener theorem [21], [22]). Therefore, the solution to problem (5)-(7) will be sought in the form

$$
z(x,y,t) = \sum_{k,n} Z_{k,n}(t)\,\psi_{k,n}(x,y), \tag{17}
$$

where $\{\psi_{k,n}(x,y), \ k,n \in \mathbb{Z}\}$ is a biorthogonal basis [23] of the space $L_2(\Omega_1)$ and $\mathbb{Z} = \{0; \pm 1; \pm 2; ...\}$ to the system $\{\varphi_{k,n}(x,y), \ k,n \in \mathbb{Z}\}$.

3. Construction of biorthogonal systems of functions $\{\psi_{k,n}(x,y), \ k,n \in \mathbb{Z}\}$

The biorthogonal systems of functions in $L_2(\Omega_1)$ for (12)-(15) will be constructed as follows:

$$
\{\psi_{k,n}(x,y), \ k,n \in \mathbb{Z}\} = \left\{\frac{1}{4\pi^2}e^{i(kx+ny)}; \ \frac{1}{2\pi}g_0(y)\,e^{ikx}; \ \frac{1}{2\pi}f_0(x)\,e^{iny}; \ k,n \in \mathbb{Z}\setminus\{0\}; \ f_0(x)\cdot g_0(y)\right\}
$$

where $f_0(x)$, $g_0(y)$ are unknown functions. For the (12) $f_0(x)$ we will search in the form:

$$
f_0(x) = C_0 + \sum_{m \in \mathbb{Z}\setminus\{0\}} C_m\left(e^{imx} + \frac{i\alpha_1}{m}\right).
$$

Coefficients C_0 and C_m we determine from the biorthogonality conditions:

$$C_0 = -\frac{1}{2\pi} - \sum_{m\in\mathbb{Z}\setminus\{0\}} C_m B_m^{\alpha_1,\alpha_3}, \;\; C_m = -\frac{1}{2\pi} B_m^{\alpha_1,\alpha_3}, \;\; \text{where } B_m^{\alpha_1,\alpha_3} = \frac{\alpha_1}{m^2-\alpha_1} + \frac{i\alpha_3 m}{m^2-\alpha_1}.$$

Further, applying the found values of C_0 and C_m, we find the required function $f_0\left(x\right)$:

$$f_0\left(x\right) = -\frac{1}{2\pi} \sum_{m\in\mathbb{Z}} B_m^{\alpha_1,\alpha_3} e^{imx}, \;\; \text{where } B_m^{\alpha_1,\alpha_3} = \frac{\alpha_1}{m^2-\alpha_1} + \frac{i\alpha_3 m}{m^2-\alpha_1}.$$

Function $g_0\left(y\right)$ for the (12) is defined similarly:

$$g_0\left(y\right) = -\frac{1}{2\pi} \sum_{m\in\mathbb{Z}} B_m^{\alpha_2,\alpha_4} e^{imy}, \;\; \text{where } B_m^{\alpha_2,\alpha_4} = \frac{\alpha_2}{m^2-\alpha_2} + \frac{i\alpha_4 m}{m^2-\alpha_2}.$$

Thus, biorthogonal system for (12) is:

$$\left\{\psi_{k,n}\left(x,y\right), \; k,n\in\mathbb{Z}\right\} = \left\{ \frac{1}{4\pi^2} e^{i(kx+ny)}; \; -\frac{1}{4\pi^2} \sum_{l\in\mathbb{Z}} B_l^{\alpha_2,\alpha_4} e^{i(ly+kx)}; \; -\frac{1}{4\pi^2} \sum_{m\in\mathbb{Z}} B_m^{\alpha_1,\alpha_3} e^{i(mx+ny)}; \right.$$
$$\left. k,n\in\mathbb{Z}\setminus\{0\}; \; -\frac{1}{4\pi^2} \sum_{m,l\in\mathbb{Z}} B_m^{\alpha_1,\alpha_3} B_l^{\alpha_2,\alpha_4} e^{i(mx+ly)} \right\}. \quad (18)$$

The biorthogonal systems of functions in $L_2\left(\Omega_1\right)$ for (13)-(15) are defined similarly.
Biorthogonal system for (13) is:

$$\left\{\psi_{k,n}\left(x,y\right), \; k,n\in\mathbb{Z}\right\} = \left\{ \frac{1}{4\pi^2} e^{i(kx+ny)}; \; -\frac{1}{4\pi^2} \sum_{l\in\mathbb{Z}\setminus\{\pm b\}} B_l^{\alpha_2,\alpha_4} e^{i(ly+kx)}; \; -\frac{1}{4\pi^2} \sum_{m\in\mathbb{Z}} B_m^{\alpha_1,\alpha_3} e^{i(mx+ny)}; \right.$$
$$\left. k,n\in\mathbb{Z}\setminus\{0\}; \; -\frac{1}{4\pi^2} \sum_{\substack{m\in\mathbb{Z} \\ l\in\mathbb{Z}\setminus\{\pm b\}}} B_m^{\alpha_1,\alpha_3} B_l^{\alpha_2,\alpha_4} e^{i(mx+ly)} \right\}. \quad (19)$$

Biorthogonal system for (14) is:

$$\left\{\psi_{k,n}\left(x,y\right), \; k,n\in\mathbb{Z}\right\} = \left\{ \frac{1}{4\pi^2} e^{i(kx+ny)}; \; -\frac{1}{4\pi^2} \sum_{l\in\mathbb{Z}} B_l^{\alpha_2,\alpha_4} e^{i(ly+kx)}; \; -\frac{1}{4\pi^2} \sum_{m\in\mathbb{Z}\setminus\{\pm a\}} B_m^{\alpha_1,\alpha_3} e^{i(mx+ny)}; \right.$$
$$\left. k,n\in\mathbb{Z}\setminus\{0\}; \; -\frac{1}{4\pi^2} \sum_{\substack{l\in\mathbb{Z} \\ m\in\mathbb{Z}\setminus\{\pm a\}}} B_m^{\alpha_1,\alpha_3} B_l^{\alpha_2,\alpha_4} e^{i(mx+ly)} \right\}. \quad (20)$$

Biorthogonal system for (15) is:

$$\left\{\psi_{k,n}\left(x,y\right), \; k,n\in\mathbb{Z}\right\} =$$

$$= \left\{ \frac{1}{4\pi^2} e^{i(kx+ny)};\ -\frac{1}{4\pi^2} \sum_{l\in\mathbb{Z}\backslash\{\pm b\}} B_l^{\alpha_2,\alpha_4} e^{i(ly+kx)};\ -\frac{1}{4\pi^2} \sum_{m\in\mathbb{Z}\backslash\{\pm a\}} B_m^{\alpha_1,\alpha_3} e^{i(mx+ny)};\right.$$
$$\left. k,n\in\mathbb{Z}\backslash\{0\};\ -\frac{1}{4\pi^2} \sum_{\substack{l\in\mathbb{Z}\backslash\{\pm b\}\\ m\in\mathbb{Z}\backslash\{\pm a\}}} B_m^{\alpha_1,\alpha_3} B_l^{\alpha_2,\alpha_4} e^{i(mx+ly)} \right\}. \quad (21)$$

The constructed biorthogonal systems define biorthogonal basises in $L_2(\Omega_1)$.
Hereinafter, we will assume that in the space $L_2(\Omega_1)$ we have:

- basises $\{\varphi_{k,n}(x,y),\ k,n\in\mathbb{Z}\}$, composed of the systems (12)-(15) of eigenfunctions and eigenvalues;
- the corresponding biorthogonal basises $\{\psi_{k,n}(x,y),\ k,n\in\mathbb{Z}\}$, defined by the relations (18)-(21).

Then solution (16) of the auxiliary boundary value problem (5)-(7) can be written as:
for the Case 1:

$$z(x,y,t) = \sum_{k,n\in\mathbb{Z}\backslash\{0\}} z_{0kn} e^{-(k^2+n^2)t} \psi_{kn}(x,y) + \sum_{k\in\mathbb{Z}\backslash\{0\}} z_{0k0} e^{-(k^2+\alpha_2)t} \psi_{k0}(x,y) +$$
$$+ \sum_{n\in\mathbb{Z}\backslash\{0\}} z_{00n} e^{-(\alpha_1+n^2)t} \psi_{0n}(x,y) + z_{000} e^{-(\alpha_1+\alpha_2)t} \psi_{00}(x,y); \quad (22)$$

for the Case 2:

$$z(x,y,t) = \sum_{\substack{k\in\mathbb{Z}\backslash\{0\}\\ n\in\mathbb{Z}\backslash\{0;\pm b\}}} z_{0kn} e^{-(k^2+n^2)t} \psi_{kn}(x,y) + \sum_{k\in\mathbb{Z}\backslash\{0\}} z_{0k0} e^{-(k^2+\alpha_2)t} \psi_{k0}(x,y) +$$
$$+ \sum_{k\in\mathbb{Z}\backslash\{0\}} z_{0k\pm b} e^{-(k^2+\alpha_2)t} \psi_{k\pm b}(x,y) + \sum_{n\in\mathbb{Z}\backslash\{0;\pm b\}} z_{00n} e^{-(\alpha_1+n^2)t} \psi_{0n}(x,y) +$$
$$+ z_{00\pm b} e^{-(\alpha_1+\alpha_2)t} \psi_{0\pm b}(x,y) + z_{000} e^{-(\alpha_1+\alpha_2)t} \psi_{00}(x,y); \quad (23)$$

for the Case 3:

$$z(x,y,t) = \sum_{\substack{k\in\mathbb{Z}\backslash\{0;\pm a\}\\ n\in\mathbb{Z}\backslash\{0\}}} z_{0kn} e^{-(k^2+n^2)t} \psi_{kn}(x,y) + \sum_{k\in\mathbb{Z}\backslash\{0;\pm a\}} z_{0k0} e^{-(k^2+\alpha_2)t} \psi_{k0}(x,y) +$$
$$+ \sum_{n\in\mathbb{Z}\backslash\{0\}} z_{00n} e^{-(\alpha_1+n^2)t} \psi_{0n}(x,y) + \sum_{n\in\mathbb{Z}\backslash\{0\}} z_{0\pm an} e^{-(\alpha_1+n^2)t} \psi_{\pm an}(x,y) +$$
$$+ z_{0\pm a0} e^{-(\alpha_1+\alpha_2)t} \psi_{\pm a0}(x,y) + z_{000} e^{-(\alpha_1+\alpha_2)t} \psi_{00}(x,y); \quad (24)$$

for the Case 4:

$$z(x,y,t) = \sum_{\substack{k\in\mathbb{Z}\backslash\{0;\pm a\}\\ n\in\mathbb{Z}\backslash\{0;\pm b\}}} z_{0kn} e^{-(k^2+n^2)t} \psi_{kn}(x,y) + \sum_{k\in\mathbb{Z}\backslash\{0;\pm a\}} z_{0k0} e^{-(k^2+\alpha_2)t} \psi_{k0}(x,y) +$$
$$+ \sum_{n\in\mathbb{Z}\backslash\{0;\pm b\}} z_{00n} e^{-(\alpha_1+n^2)t} \psi_{0n}(x,y) + \sum_{n\in\mathbb{Z}\backslash\{0;\pm b\}} z_{0\pm an} e^{-(\alpha_1+n^2)t} \psi_{\pm an}(x,y) +$$
$$+ \sum_{k\in\mathbb{Z}\backslash\{0;\pm a\}} z_{0k\pm b} e^{-(k^2+\alpha_2)t} \psi_{k\pm b}(x,y) + z_{0\pm a\pm b} e^{-(\alpha_1+\alpha_2)t} \psi_{\pm a\pm b}(x,y) +$$
$$+ z_{0\pm a0} e^{-(\alpha_1+\alpha_2)t} \psi_{\pm a0}(x,y) + z_{00\pm b} e^{-(\alpha_1+\alpha_2)t} \psi_{0\pm b}(x,y) + z_{000} e^{-(\alpha_1+\alpha_2)t} \psi_{00}(x,y); \quad (25)$$

where

$$z_{0kn}=\int\limits_{\Omega_1}\overline{\varphi_{kn}\left(x,y\right)}z_0\left(x,y\right)dxdy,\quad k,n\in\mathbb{Z}$$

are the Fourier coefficients of the function $z_0\left(x,y\right)$; and systems $\{\psi_{k,n}\left(x,y\right),\ k,n\in\mathbb{Z}\}$ are defined by (18)-(21).

From (16) and (22)-(25) it follows immediately that if

for the Case 1:

$$\begin{aligned} z_{0kn}&=0\ \text{ for }\ k^2+n^2<\sigma,\\ z_{0k0}&=0\ \text{ for }\ k^2+Re\left(\alpha_2\right)<\sigma,\\ z_{00n}&=0\ \text{ for }\ Re\left(\alpha_1\right)+n^2<\sigma,\\ z_{00\pm b}\neq 0\,\text{and}\,z_{000}&\neq 0\ \text{ for }\ Re\left(\alpha_1\right)+Re\left(\alpha_2\right)\geq\sigma,\\ z_{00\pm b}=0\,\text{and}\,z_{000}&=0\ \text{ for }\ Re\left(\alpha_1\right)+Re\left(\alpha_2\right)<\sigma, \end{aligned}$$

then solution (22) of the problem (5)-(7) will satisfy the inequality (8);

for the Case 2:

$$\begin{aligned} z_{0kn}&=0\ \text{ for }\ k^2+n^2<\sigma,\\ z_{0k0}=0\,\text{and}\,z_{0k\pm b}&=0\ \text{ при }\ k^2+Re\left(\alpha_2\right)<\sigma,\\ z_{00n}&=0\ \text{ for }\ Re\left(\alpha_1\right)+n^2<\sigma,\\ z_{00\pm b}\neq 0\,\text{and}\,z_{000}&\neq 0\ \text{ for }\ Re\left(\alpha_1\right)+Re\left(\alpha_2\right)\geq\sigma,\\ z_{00\pm b}=0\,\text{and}\,z_{000}&=0\ \text{ for }\ Re\left(\alpha_1\right)+Re\left(\alpha_2\right)<\sigma, \end{aligned}$$

then solution (23) of the problem (5)-(7) will satisfy the inequality (8);

for the Case 3:

$$\begin{aligned} z_{0kn}&=0\ \text{ for }\ k^2+n^2<\sigma,\\ z_{0k0}&=0\ \text{ for }\ k^2+Re\left(\alpha_2\right)<\sigma,\\ z_{00n}=0\,\text{and}\,z_{0\pm an}&=0\ \text{ for }\ Re\left(\alpha_1\right)+n^2<\sigma,\\ z_{00\pm b}\neq 0\,\text{and}\,z_{000}&\neq 0\ \text{ for }\ Re\left(\alpha_1\right)+Re\left(\alpha_2\right)\geq\sigma,\\ z_{00\pm b}=0\,\text{and}\,z_{000}&=0\ \text{ for }\ Re\left(\alpha_1\right)+Re\left(\alpha_2\right)<\sigma, \end{aligned}$$

then solution (24) of the problem (5)-(7) will satisfy the inequality (8);

for the Case 4:

$$\begin{aligned} z_{0kn}&=0\ \text{ for }\ k^2+n^2<\sigma,\\ z_{0k0}=0\,\text{and}\,z_{0k\pm b}&=0\ \text{ for }\ k^2+Re\left(\alpha_2\right)<\sigma,\\ z_{00n}=0\,\text{and}\,z_{0\pm an}&=0\ \text{ for }\ Re\left(\alpha_1\right)+n^2<\sigma,\\ z_{000}\neq 0,\ z_{00\pm b}\neq 0,\ z_{0\pm a0}\neq 0\,\text{and}\,z_{0\pm a\pm b}&\neq 0\ \text{ for }\ Re\left(\alpha_1\right)+Re\left(\alpha_2\right)\geq\sigma,\\ z_{000}=0,\ z_{00\pm b}=0,\ z_{0\pm a0}=0\,\text{and}\,z_{0\pm a\pm b}&=0\ \text{ for }\ Re\left(\alpha_1\right)+Re\left(\alpha_2\right)<\sigma, \end{aligned}$$

then solution (25) of the problem (5)-(7) will satisfy the inequality (8);

We introduce the following notation for the sets of pairs of indices $(k,n)\quad k,n\in\mathbb{Z}$:

$$I_1=\left\{(k,n)\,|k^2+n^2\geq\sigma\right\},\ \ \overline{I_1}=\left\{(k,n)\,|k^2+n^2<\sigma\right\}$$

$$I_2=\left\{(k,0)\,,(k,\pm b)\,|k^2+Re\left(\alpha_2\right)\geq\sigma\right\},\ \ \overline{I_2}=\left\{(k,0)\,,(k,\pm b)\,|k^2+Re\left(\alpha_2\right)<\sigma\right\}$$

$$I_3=\left\{(0,n)\,,(\pm a,n)\,|Re\left(\alpha_1\right)+n^2\geq\sigma\right\},\ \ \overline{I_3}=\left\{(0,n)\,,(\pm a,n)\,|\left(\alpha_1\right)+n^2<\sigma\right\}$$

$$I_4=\{(0,0)\,,(0,\pm b)\,,(\pm a,0)\,,(\pm a,\pm b)\,|Re\left(\alpha_1\right)+Re\left(\alpha_2\right)\geq\sigma\}$$

$$\overline{I_4} = \{(0,0), (0,\pm b), (\pm a, 0), (\pm a, \pm b) \,|Re(\alpha_1) + Re(\alpha_2) < \sigma\}$$
$$\overline{I} = \overline{I_1} \cup \overline{I_2} \cup \overline{I_3} \cup \overline{I_4}.$$

Let condition

$$z_{0kn} = 0 \text{ при } (k,n) \in \overline{I}$$

satisfies for (22)-(25), then stabilized solution $z_{stab}(x,y,t)$ of the problem (5)-(7), satisfying the inequality (8), can be written as:

for the Case 1:

$$z(x,y,t) = \sum_{(k,n)\in I_1} z_{0kn} e^{-\left(k^2+n^2\right)t} \psi_{kn}(x,y) + \sum_{(k,0)\in I_2} z_{0k0} e^{-\left(k^2+\alpha_2\right)t} \psi_{k0}(x,y) +$$
$$+ \sum_{(0,n)\in I_3} z_{00n} e^{-\left(\alpha_1+n^2\right)t} \psi_{0n}(x,y) + A(\alpha_1,\alpha_2) e^{-(\alpha_1+\alpha_2)t} \psi_{00}(x,y),$$

where

$$A(\alpha_1,\alpha_2) = \begin{cases} z_{000}, & I_4 \neq \oslash \\ 0, & I_4 = \oslash \end{cases}$$

for the Case 2:

$$z(x,y,t) = \sum_{(k,n)\in I_1} z_{0kn} e^{-\left(k^2+n^2\right)t} \psi_{kn}(x,y) + \sum_{(k,0)\in I_2} z_{0k0} e^{-\left(k^2+\alpha_2\right)t} \psi_{k0}(x,y) +$$
$$+ \sum_{(k,\pm b)\in I_2} z_{0k\pm b} e^{-\left(k^2+\alpha_2\right)t} \psi_{k\pm b}(x,y) + \sum_{(0,n)\in I_3} z_{00n} e^{-\left(\alpha_1+n^2\right)t} \psi_{0n}(x,y) +$$
$$+ A_1(\alpha_1,\alpha_2) e^{-(\alpha_1+\alpha_2)t} \psi_{0\pm b}(x,y) + A_2(\alpha_1,\alpha_2) e^{-(\alpha_1+\alpha_2)t} \psi_{00}(x,y),$$

where

$$A_1(\alpha_1,\alpha_2) = \begin{cases} z_{00\pm b}, & I_4 \neq \oslash \\ 0, & I_4 = \oslash \end{cases}; \quad A_2(\alpha_1,\alpha_2) = \begin{cases} z_{000}, & I_4 \neq \oslash \\ 0, & I_4 = \oslash \end{cases}$$

for the Case 3:

$$z(x,y,t) = \sum_{(k,n)\in I_1} z_{0kn} e^{-\left(k^2+n^2\right)t} \psi_{kn}(x,y) + \sum_{(k,0)\in I_2} z_{0k0} e^{-\left(k^2+\alpha_2\right)t} \psi_{k0}(x,y) +$$
$$+ \sum_{(0,n)\in I_3} z_{00n} e^{-\left(\alpha_1+n^2\right)t} \psi_{0n}(x,y) + \sum_{(\pm a,n)\in I_3} z_{0\pm an} e^{-\left(\alpha_1+n^2\right)t} \psi_{\pm an}(x,y) +$$
$$+ A_1(\alpha_1,\alpha_2) e^{-(\alpha_1+\alpha_2)t} \psi_{\pm a0}(x,y) + A_2(\alpha_1,\alpha_2) e^{-(\alpha_1+\alpha_2)t} \psi_{00}(x,y),$$

where

$$A_1(\alpha_1,\alpha_2) = \begin{cases} z_{0\pm a0}, & I_4 \neq \oslash \\ 0, & I_4 = \oslash \end{cases}; \quad A_2(\alpha_1,\alpha_2) = \begin{cases} z_{000}, & I_4 \neq \oslash \\ 0, & I_4 = \oslash \end{cases}$$

for the Case 4:

$$z(x,y,t) = \sum_{(k,n)\in I_1} z_{0kn} e^{-\left(k^2+n^2\right)t} \psi_{kn}(x,y) + \sum_{(k,0)\in I_2} z_{0k0} e^{-\left(k^2+\alpha_2\right)t} \psi_{k0}(x,y) +$$
$$+ \sum_{(0,n)\in I_3} z_{00n} e^{-\left(\alpha_1+n^2\right)t} \psi_{0n}(x,y) + \sum_{(\pm a,n)\in I_3} z_{0\pm an} e^{-\left(\alpha_1+n^2\right)t} \psi_{\pm an}(x,y) +$$
$$+ \sum_{(k,\pm b)\in I_2} z_{0k\pm b} e^{-\left(k^2+\alpha_2\right)t} \psi_{k\pm b}(x,y) + A_1(\alpha_1,\alpha_2) e^{-(\alpha_1+\alpha_2)t} \psi_{\pm a\pm b}(x,y) +$$

$$+A_2(\alpha_1,\alpha_2)\,e^{-(\alpha_1+\alpha_2)t}\psi_{\pm a0}(x,y)+$$
$$+A_3(\alpha_1,\alpha_2)\,e^{-(\alpha_1+\alpha_2)t}\psi_{0\pm b}(x,y)+A_4(\alpha_1,\alpha_2)\,e_{-(\alpha_1+\alpha_2)t}\psi_{00}(x,y),$$

where

$$A_1(\alpha_1,\alpha_2)=\begin{cases} z_{0\pm a\pm b}, & I_4\neq\oslash \\ 0, & I_4=\oslash \end{cases};\quad A_2(\alpha_1,\alpha_2)=\begin{cases} z_{0\pm a0}, & I_4\neq\oslash \\ 0, & I_4=\oslash \end{cases}$$

$$A_3(\alpha_1,\alpha_2)=\begin{cases} z_{00\pm b}, & I_4\neq\oslash \\ 0, & I_4=\oslash \end{cases};\quad A_4(\alpha_1,\alpha_2)=\begin{cases} z_{000}, & I_4\neq\oslash \\ 0, & I_4=\oslash \end{cases}$$

Algorithm for solving the stabilization problem

The results of the previous sections make it possible to implement the following algorithm for the approximate construction of boundary control functions (and even in the form of synthesis that work out random perturbations) that provide a monotonic (no slower than a given exponent) decrease on time according to the formula (4) of the $L_2(\Omega)$-norm solution.

Step 1. To the original boundary value problem (1)–(3) on a parallelepiped the base of which is a square with side π, with the nonhomogeneous Dirichlet boundary conditions and an initial condition on the square Ω determined by the given function $u_0(x,y)$ is posed an auxiliary boundary value problem (5)–(7) on an extended parallelepiped, the base of which is a square with side 2π, with periodicity conditions (instead of the Dirichlet conditions) and an initial function $z_0(x,y)$ on the bottom base of the extended parallelepiped Ω_1. The function $z_0(x,y)$ will be defined as a continuation of the given function $u_0(x,y)$.

Thus, in the auxiliary boundary value problem (5)–(7) it is necessary to redefine the function $z_0(x,y)$ on the square Ω_1, so that for the solution $z(x,y,t)$ of the problem (5)–(7) the requirement (8) will be satisfied. In this case, the condition (4) will be also satisfied for its restriction $u(x,y,t)$ and the required boundary control $p(x,y,t)\ \ \{x,y,t\}\in\Sigma$ will be defined as the trace of the function $z(x,y,t)$ при $\{x,y,t\}\in\Sigma$.

Step 2. Construction of complete biorthogonal systems of functions on the square Ω_1 by solving the corresponding spectral problems.

Step 3. Find the expansion coefficients of the required function $z_0(x,y)$ on the square Ω_1 according to the complete biorthogonal system constructed in the previous step, so that condition (8) is satisfied. Note that condition (8) ensures requirement (4) for the solution of the boundary value problem (1)–(3).

Step 4. Using the found solution $z(x,y,t)$ of the auxiliary boundary value problem (5)–(7), as its restriction on the parallelepiped Q we find the solution $u(x,y,t)$ of the original boundary value problem (1)–(3), satisfying the required condition (4). Boundary control $p(x,y,t)\ \ \{x,y,t\}\in\Sigma$ we find as a trace of the solution $z_{stab}(x,y,t)$, i.e.

$$p(x,y,t)=z_{stab}(x,y,t)|_{\{x,y,t\}\in\Sigma}\,.$$

Conclusion

The paper proposes a problem formulation of boundary stabilization (forming a parallelepiped) of the solution of the boundary value problem for the heat equation with a loaded two-dimensional Laplace operator, where the loaded terms are the values of the required function and traces of the derivatives of the required function at fixed points, and an algorithm for the approximate construction of boundary controls.

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М.Т. Дженалиев, М.И. Рамазанов, А.Х. Аттаев, Н.К. Гульманов

Екіөлшемді жүктелген параболалық теңдеуінің шешімін тұрақтандыру

Мақалада екіөлшемді жүктелген Лаплас операторымен жылуөткізгіштік теңдеуі үшін шекаралық есептің шешімін тұрақтандыру есебі қарастырылды. Жүктелген қосылғыштар белгіленген нүктелердегі ізделінді функцияның мәндері мен оның бірінші ретті дербес туындыларының іздері болып табылады. Сондай-ақ есепте шекаралық басқару функцияларын құру алгоритмі ұсынылған.

Кілт сөздер: шекара бойынша тұрақтандыру есебі, жылуөткізгіштіктің жүктелген теңдеуі, жүктелген Лаплас операторы, биортогоналды жүйе, тұрақтандыру, алгоритм.

М.Т. Дженалиев, М.И. Рамазанов, А.Х. Аттаев, Н.К. Гульманов

Стабилизация решения для двумерного нагруженного параболического уравнения

В статье рассмотрена задача стабилизации решения граничной задачи для уравнения теплопроводности с нагруженным двумерным оператором Лапласа. Нагруженные слагаемые представляют собой значения искомой функции и следы ее частных производных первого порядка в фиксированных точках. Предложен алгоритм построения граничных управляющих функций.

Ключевые слова: задача стабилизации по границе, нагруженное уравнение теплопроводности, нагруженный оператор Лапласа, биортогональная система, стабилизация, алгоритм.

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