On structures in hypergraphs of models of a theory

Hypergraphs of models of a theory are derived objects allowing to obtain an essential structural information about both given theories and related semantic objects including graph ones. In the present paper we define and study structural properties of hypergraphs of models of a theory including lattice ones. Characterizations for the lattice properties of hypergraphs of models of a theory, as well as for structures on sets of isomorphism types of models of a theory, are given.

Keywords: hypergraph of models, elementary theory, elementarily substructural set, lattice structure.

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Preliminaries

Recall that a hypergraph is a pair of sets $(X, Y)$, where $Y$ is some subset of the Boolean $\mathcal{P}(X)$ of the set $X$.

Let $\mathcal{M}$ be some model of a complete theory $T$. Following [5], we denote by $H(\mathcal{M})$ a family of all subsets $N$ of the universe $M$ of $\mathcal{M}$ that are universes of elementary submodels $\mathcal{N}$ of the model $\mathcal{M}$: $H(\mathcal{M}) = \{N \mid \mathcal{N} \subseteq \mathcal{M}\}$. The pair $(\mathcal{M}, H(\mathcal{M}))$ is called the hypergraph of elementary submodels of the model $\mathcal{M}$ and denoted by $H(\mathcal{M})$.

Definition [8]. Let $\mathcal{M}$ be a model of a theory $T$ with a hypergraph $H = (\mathcal{M}, H(\mathcal{M}))$ of elementary submodels, $A$ be an infinite definable set in $\mathcal{M}$, of arity $n$: $A \subseteq M^n$. The set $A$ is called $\mathcal{H}$-free if for any infinite set $A' \subseteq A$, $A' = A \cap Z^n$ for some $Z \in H(\mathcal{M})$ containing parameters for $A$. Two $\mathcal{H}$-free sets $A$ and $B$ of arities $m$ and $n$ respectively are called $\mathcal{H}$-independent if for any infinite $A' \subseteq A$ and $B' \subseteq B$ there is $Z \in H(\mathcal{M})$ containing parameters for $A$ and $B$ and such that $A' = A \cap Z^n$ and $B' = B \cap Z^n$.

Note the following properties [8].

1. Any two tuples of a $\mathcal{H}$-free set $A$, whose distinct tuples do not have common coordinates, have same type.

Indeed, if there are tuples $a, b \in A$ with $\text{tp}(a) \neq \text{tp}(b)$ then for some formula $\varphi(x)$ the sets of solutions of that formula and of the formula $\neg \varphi(x)$ divide the set $A$ into two nonempty parts $A_1$ and $A_2$, where at least one part, say $A_1$, is infinite. Taking $A_1$ for $A'$ we have $A_1 = A \cap Z^n$ for appropriate $Z \in H(\mathcal{M})$ and $n$. Then by the condition for tuples in $A$ we have $A_2 \cap Z^n = \emptyset$ that is impossible since $Z$ is the universe of an elementary submodel of $\mathcal{M}$.

Thus the formula $\varphi(x)$, defining $A$, implies some complete type in $S^n(\emptyset)$, and if $A$ is $\emptyset$-definable then $\varphi(x)$ is a principal formula.

In particular, if the set $A$ is $\mathcal{H}$-free and $A \subseteq M$, then the formula, defining $A$, implies some complete type in $S^1(\emptyset)$.

2. If $A \subseteq M$ is a $\mathcal{H}$-free set, then $A$ does not have nontrivial definable subsets, with parameters in $A$, i.e., subsets distinct to subsets defined by equalities and inequalities with elements in $A$.

Indeed, if $B \subseteq A$ is a nontrivial definable subset then $B$ is defined by a tuple $\bar{a}$ of parameters in $A$, forming a finite set $A_0 \subseteq A$, and $B$ is distinct to subsets of $A_0$ and to $A \setminus C$, where $C \subseteq A_0$. Then removing from $A$ a set $B \setminus A_0$ or $(A \setminus B) \setminus A_0$, we obtain some $Z \in H(\mathcal{M})$ violating the satisfiability for $B$ or its complement. It contradicts the condition that $Z$ is the universe of an elementary submode of $\mathcal{M}$.

3. If $A$ and $B$ are two $\mathcal{H}$-independent sets, where $A \cup B$ does not have distinct tuples with common coordinates, then $A \cap B = \emptyset$.  

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Indeed, if \( A \cap B \) contains a tuple \( \bar{a} \), then, choosing infinite sets \( A' \subseteq A \) and \( B' \subseteq B \) with \( \bar{a} \in A' \) and \( \bar{a} \notin B' \), we obtain \( \bar{a} \in A' = A \cap Z^n \) for appropriate \( Z \in H(M) \) and \( n \), as so \( \bar{a} \in B \cap Z^n = B' \). This contradiction means that \( A \cap B = \emptyset \).

**Definition** [6]. The complete union of hypergraphs \((X_i, Y_i), i \in I\), is the hypergraph \( \left( \bigcup_{i \in I} X_i, Y \right) \), where \( Y = \bigcup_{i \in I} Z_i \mid Z_i \in Y_i \). If the sets \( X_i \) are disjoint, the complete union is called disjoint too. If the set \( X_i \) form a \( \subseteq \)-chain, then the complete union is called chain.

By Property 3 we have the following theorem on decomposition of restrictions of hypergraphs \( H \), representable by unions of families of \( H \)-independent sets.

**Theorem 1.1** [8]. A restriction of hypergraph \( H = (M, H(M)) \) to a union of a family of \( H \)-free \( H \)-independent sets \( A_i \subseteq M \) is represented as a disjoint complete union of restrictions \( H_i \) of the hypergraph \( H \) to the sets \( A_i \).

Proof. Consider a family of \( H \)-independent sets \( A_i \subseteq M \). By Property 3 these sets are disjoint, and using the definition of \( H \)-independence we immediately obtain that the union of restrictions \( H_i \) of \( H \) to the sets \( A_i \) is complete.

Recall that a subset \( A \) of a linearly ordered structure \( M \) is called convex if for any \( a, b \in A \) and \( c \in M \) whenever \( a < c < b \) we have \( c \in A \). A weakly o-minimal structure is a linearly ordered structure \( M = (M, =, <, \ldots) \) such that any definable (with parameters) subset of the structure \( M \) is a union of finitely many convex sets in \( M \).

In the following definitions \( M \) is a weakly o-minimal structure, \( A, B \subseteq M \), \( M \) be \(|A|^+\)-saturated, \( p, q \in S_1(A) \) be non-algebraic types.

**Definition.** [23]. We say that \( p \) is not weakly orthogonal to \( q \) \((p \not\perp\!\!\!\!\perp q)\) if there exist an \( A \)-definable formula \( H(x, y), \alpha \in p(M) \) and \( \beta_1, \beta_2 \in q(M) \) such that \( \beta_1 \in H(M, \alpha) \) and \( \beta_2 \notin H(M, \alpha) \).

**Definition.** [24]. We say that \( p \) is not quite orthogonal to \( q \) \((p \not\perp q)\) if there exists an \( A \)-definable bijection \( f : p(M) \rightarrow q(M) \). We say that a weakly o-minimal theory is quite o-minimal if the notions of weak and quite orthogonality of 1-types coincide.

In the work [25] the countable spectrum for quite o-minimal theories with non-maximal number of countable models has been described:

**Theorem 1.2.** Let \( T \) be a quite o-minimal theory with non-maximal number of countable models. Then \( T \) has exactly \( 3^k \cdot 6^s \) countable models, where \( k \) and \( s \) are natural numbers. Moreover, for any \( k, s \in \omega \) there exists a quite o-minimal theory \( T \) having exactly \( 3^k \cdot 6^s \) countable models.

Realizations of these theories with a finite number of countable models are natural generalizations of Ehrenfeucht examples obtained by expansions of dense linear orderings by a countable set of constants, and they are called theories of Ehrenfeucht type. Moreover, these realizations are representative examples for hypergraphs of prime models [1, 3, 5]. We consider operators for hypergraphs allowing on one hand to describe the decomposition of hypergraphs of prime models for quite o-minimal theories with few countable models, and on the other hand pointing out constructions leading to the building of required hypergraphs by some simplest ones.

Having nontrivial structures like structures with some orders it is assumed that «complete» decompositions are considered modulo additional conditions guaranteeing the elementarity for substructures with considered universes. So we use the conditional completeness taking unions with the properties of density, linearity etc.

Below we illustrate this conditional completeness for structures with dense linear orders.

Denote by \((M, H_{dlo}(M))\) the hypergraph of (prime) elementary submodels of a countable model \( M \) of the theory of dense linear order without endpoints.

**Remark 1.3.** The class of hypergraphs \((M, H_{dlo}(M))\) is closed under countable chain complete unions, modulo density and having an encompassing dense linear order without endpoints. Thus, any hypergraph \((M, H_{dlo}(M))\) is represented as a countable chain complete, modulo density, union of some its proper subhypergraphs. The notion of weak o-minimality was originally studied by D. Machpherson, D. Marker and C. Steinhorn in [26].

Any countable model of a theory of Ehrenfeucht type is a disjoint union of some intervals, which are ordered both themselves and between them, and of some singletons. Dense subsets of the intervals form universes of elementary substructures. So, in view of Remark 1.3, we have:

**Theorem 1.4** [6]. A hypergraph of prime models of a countable model of a theory of Ehrenfeucht type is represented as a disjoint complete, modulo density, union of some hypergraphs in the form \((M, H_{dlo}(M))\) as well as singleton hypergraphs of the form \((\{c\}, \{\{c\}\})\).
**Remark 1.5.** Taking into consideration links between sets of realizations of 1-types, which are not weakly orthogonal, as well as definable equivalence relations, the construction for the proof of Theorem 1.4 admits a natural generalization for an arbitrary quite o-minimal theory with few countable models. Here conditional complete unions should be additionally coordinated, i.e., considering definable bijections between sets of realizations of 1-types, which are not quite orthogonal.

**Elementarily substructural sets**

Let \( \mathcal{M} \) be a model of theory \( T \), \((\mathcal{M}, H(\mathcal{M}))\) be a hypergraph of elementary submodels of \( \mathcal{M} \). The sets \( N \in H(\mathcal{M}) \) are called **elementarily submodel** or **elementarily substructural** in \( \mathcal{M} \).

Elementarily substructural sets in \( \mathcal{M} \) are characterized by the following well-known Tarski–Vaught Theorem, which is called the Tarski–Vaught test.

**Theorem 2.1.** Let \( A \) and \( B \) be structures in a language \( \Sigma \), \( A \subseteq B \). The following are equivalent:

1. \( A \cong B \);
2. for any formula \( \varphi(x_0, x_1, \ldots, x_n) \) in the language \( \Sigma \) and for any elements \( a_1, \ldots, a_n \in A \), if \( B \models \exists x_0 \varphi(x_0, a_1, \ldots, a_n) \) then there is an element \( a_0 \in A \) such that \( B \models \varphi(a_0, a_1, \ldots, a_n) \).

**Corollary 2.2.** A set \( N \subseteq M \) is elementarily substructural in \( M \) if and only if for any formula \( \varphi(x_0, x_1, \ldots, x_n) \) in the language \( \Sigma(\mathcal{M}) \) and for any elements \( a_1, \ldots, a_n \in N \), if \( M \models \exists x_0 \varphi(x_0, a_1, \ldots, a_n) \) then there is an element \( a_0 \in N \) such that \( M \models \varphi(a_0, a_1, \ldots, a_n) \).

**Proposition 2.3.** Let \( A \) be a definable set in an \( \omega_1 \)-saturated model \( M \) of a countable complete theory \( T \). Then exactly one of the following conditions is satisfied:

1. \( A \) is finite and contained in any elementarily substructural set in \( \mathcal{M} \);
2. \( A \) is infinite and has infinitely many distinct intersections with elementarily substructural sets in \( \mathcal{M} \), and all these intersections are infinite; these intersections can be chosen forming an infinite chain/antichain by inclusion.

**Proof.** If \(|A| < \omega\) then \( A \) is contained in \( acl(\emptyset) \), and so it is contained in any elementary submodel of \( \mathcal{M} \).

If \( A = \varphi(M, a) \) is infinite, we construct a countable submodel \( N_0 \prec \mathcal{M} \) containing parameters in \( a \). Since \( A \) is infinite, the set \( A \cap N_0 \) is countable. By compactness, since \( M \) is \( \omega_1 \)-saturated, the set \( A \setminus N_0 \) is finite. Adding to \( N_0 \) new elements of \( A \) we construct a countable model \( N_1 \) such that \( N_0 < N_1 \prec \mathcal{M} \). Continuing the process we build an elementary chain of models \( N_k, k < \omega \), such that \( N_k \prec \mathcal{M} \) and \( A \cap N_k \subset A \cap N_{k+1}, k < \omega \).

Constructing the required antichain of intersections \( A \cap N \) with elementarily substructural sets \( N \), it suffices to use [9, Theorem 2.10] allowing to separate disjoint finite sets, whose elements do not belong to \( acl(\emptyset) \).

The arguments for the proof of Proposition 2.3 stay valid for a countable saturated model \( M \). Thus, we have the following

**Proposition 2.4.** Let \( A \) be a definable set in a countable saturated model \( \mathcal{M} \) of a small theory \( T \). Then exactly one of the following conditions is satisfied:

1. \( A \) is finite and contained in any elementarily substructural set in \( \mathcal{M} \);
2. \( A \) is infinite and has infinitely many distinct intersections with elementarily substructural sets in \( \mathcal{M} \), and all these intersections are infinite; these intersections can be chosen forming an infinite chain/antichain by inclusion.

The following example illustrates that if \( \mathcal{M} \) is not saturated then the conclusions of assertions 2.3 and 2.4 can fail.

**Example 2.5.** Let a set \( A \) is defined by a unary predicate \( P \) and includes infinitely many language constants \( c_i, i \in I \). Then there is, in the language \( \{P\} \cup \{c_i \mid i \in I\} \), a structure \( \mathcal{M} \) having only finite set \( A_0 \) of elements in \( A \), which are not interpreted by constants. Since elementarily substructural sets \( N \) take all constants, there are only finitely many possibilities for intersections \( A \cap N \).

In view of aforesaid arguments it is interesting to describe possible cardinalities both for sets \( H(\mathcal{M}) \) and their restrictions \( H(\mathcal{M}) \upharpoonright A = \{A \cap N \mid N \in H(\mathcal{M})\} \) definable sets \( A \subseteq \mathcal{M} \).

Since in Example 2.5 intersections \( A \cap N \), taking all constants \( c_i \), can include an arbitrary subset of \( A_0 \), then for this example we have \(|H(\mathcal{M}) \upharpoonright A| = 2^{|A_0|}\). The same formula holds for infinite sets \( A_0 \), but in such a case the set \( H(\mathcal{M}) \upharpoonright A \) is transformed from finite one directly to a set with continuum many elements.

Note that for \( \mathcal{H} \)-free sets \( A \subseteq \mathcal{M} \), modulo \( acl(\emptyset) \) (i.e., for sets \( A \), whose each subset \( B \subseteq A \setminus acl(\emptyset) \) has a representation \( B \cup (acl(\emptyset) \cap A) = A \cap N \) for some \( N \in H(\mathcal{M}) \)), the equality \(|H(\mathcal{M}) \upharpoonright A| = 2^{|A \setminus acl(\emptyset)|}\) holds. Thus, we have the following dichotomy theorem.

**Theorem 2.6.** For any \( \mathcal{H} \)-free, modulo \( acl(\emptyset) \), set \( A \subseteq \mathcal{M} \) its restriction to any elementary submodel \( \mathcal{M}_0 \prec \mathcal{M} \) satisfies either \(|H(\mathcal{M}_0) \upharpoonright A| = 2^n \) for some \( n \in \omega \), or \(|H(\mathcal{M}_0) \upharpoonright A| = 2^\lambda \) form some \( \lambda \geq \omega \).
Similar to Example 2.5, the following example illustrates the dichotomy for hypergraphs of elementary substructures.

**Example 2.7.** Consider the structure $\mathcal{M}$ of rational numbers, $\langle \mathbb{Q}, <, c_q \rangle_{q \in \mathbb{Q}}$, in which every element is interpreted by a constant. This structure does not have proper elementary substructures, therefore $|H(\mathcal{M})| = 1 = 2^0$. Extending $\mathcal{M}$ to a structure $\mathcal{M}_1$ by addition of $n$ elements for pairwise distinct 1-types, defined by cuts, we have $|H(\mathcal{M}_1)| = 2^n$. If $\mathcal{M}$ is extended till a structure $\mathcal{M}_2$ by addition of at least two elements of fixed cut or of infinitely many elements for distinct cuts, then by density the summarized number of added elements occurs infinite and $|H(\mathcal{M}_2)| = 2^k$ holds for some $\lambda \geq \omega$.

At the same time there are examples of hypergraphs of elementary submodels, for which the conclusion of Theorem 2.6 fails. For instance, as shown in [13], there are hypergraphs for the theory of arithmetic of natural numbers such that $|H(\mathcal{M})| = 5$ and the lattice of elementary submodels is isomorphic to the lattice $P_5$.

**Lattice structures associated with hypergraphs of models of a theory**

For given structure $\mathcal{M}$ we define the structure $L(\mathcal{M}) = \langle H(\mathcal{M}); \land, \lor \rangle$ by the following relations for $\mathcal{M}_1, \mathcal{M}_2 < \mathcal{M}$: $\mathcal{M}_1 \land \mathcal{M}_2 = \mathcal{M}_1 \cap \mathcal{M}_2$ and $\mathcal{M}_1 \lor \mathcal{M}_2 = \mathcal{M}(\mathcal{M}_1 \cup \mathcal{M}_2)$.

Consider the following question: when the structure $L(\mathcal{M})$ is a lattice?

Clearly, answering this question we have to characterize the conditions $\mathcal{M}_1 \cap \mathcal{M}_2 < \mathcal{M}$ and $\mathcal{M}(\mathcal{M}_1 \cup \mathcal{M}_2) < \mathcal{M}$. Assuming that $\mathcal{M}$ is infinite, the structures $\mathcal{M}_1 \cap \mathcal{M}_2$ should be infinite too, in particular, $\mathcal{M}_1 \cap \mathcal{M}_2 \neq \emptyset$. By [5, Theorem 3.2], assuming that $\mathcal{M}$ is $\lambda$-saturated, it can not contain separated sets $A$ and $B$ of cardinalities $< \lambda$, such that acl$(A) \cap$ acl$(B) = \emptyset$.

By Theorem 2.1 we have the following theorems characterizing the elementarity of substructures.

**Theorem 3.1.** Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be elementary substructures of structure $\mathcal{M}$ in a language $\Sigma$, $\mathcal{M}_1 \cap \mathcal{M}_2 \neq \emptyset$. The following are equivalent:

1. $(\mathcal{M}_1 \cap \mathcal{M}_2) < \mathcal{M}$;
2. for any formula $\varphi(x_0, x_1, \ldots, x_n)$ of the language $\Sigma$ and for any elements $a_1, \ldots, a_n \in M_1 \cap M_2$ if $\mathcal{M} \models \exists x_0 \varphi(x_0, a_1, \ldots, a_n)$ then there is an element $a_0 \in M_1 \cap M_2$ such that $\mathcal{M}_i \models \varphi(a_0, a_1, \ldots, a_n)$, $i = 1, 2$.

**Theorem 3.2.** Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be elementary substructures of structure $\mathcal{M}$ in a language $\Sigma$. The following are equivalent:

1. $\mathcal{M}(\mathcal{M}_1 \cup \mathcal{M}_2) < \mathcal{M}$;
2. for any formula $\varphi(x_0, x_1, \ldots, x_n)$ of the language $\Sigma$ and for any elements $a_1, \ldots, a_n \in M_1 \cap M_2$ if $\mathcal{M} \models \exists x_0 \varphi(x_0, a_1, \ldots, a_n)$ then there is an element $a_0 \in M(\mathcal{M}_1 \cup \mathcal{M}_2)$ such that $\mathcal{M}(\mathcal{M}_1 \cup \mathcal{M}_2) \models \varphi(a_0, a_1, \ldots, a_n)$.

The following examples illustrate valuations of the conditions (2) in Theorems 3.1 and 3.2.

**Example 3.3.** Consider a structure $\mathcal{M}$ in a graph language $\{R^{(2)}\}$ with a symmetric irreflexive relation $R$ and elements $a_1, a_2, a_3, a_4$ such that

$$R = \{[a_1, a_2], [a_1, a_3], [a_2, a_3], [a_2, a_4] \}.$$  

The substructures $\mathcal{M}_1 \subset \mathcal{M}$ and $\mathcal{M}_2 \subset \mathcal{M}$ with the universes $\{a_1, a_2, a_3\}$ and $\{a_1, a_2, a_4\}$ respectively satisfy the formula $\varphi(a_1, a_2) = \exists x (R(a_1, x) \land R(a_2, x))$ whereas $\mathcal{M}_1 \cap \mathcal{M}_2$ does not satisfy that formula since appropriate elements for $x$ belong to $\mathcal{M}_1 \cap \mathcal{M}_2$.

**Example 3.4.** Consider a structure $\mathcal{M}$ of graph language $\{R^{(2)}\}$ with symmetric irreflexive relation $R$ and with elements $a_1, a_2, a_3$ such that $R = \{[a_1, a_3], [a_2, a_3]\}$. The substructures $\mathcal{M}_1 \subset \mathcal{M}$ and $\mathcal{M}_2 \subset \mathcal{M}$ with the universes $\{a_1\}$ and $\{a_2\}$ form the substructure $\mathcal{M}(\mathcal{M}_1 \cup \mathcal{M}_2)$ with the universe $\{a_1, a_2\}$ and it does not satisfy the formula $\varphi(a_1, a_2)$ in Example 3.3. At the same time the structure $\mathcal{M}$ satisfies this formula.

Since in some cases elementary substructures of given structure $\mathcal{M}$ form the lattice with respect to the operations $\mathcal{M}_1 \land \mathcal{M}_2 = \mathcal{M}_1 \cap \mathcal{M}_2$ and $\mathcal{M}_1 \lor \mathcal{M}_2 = \mathcal{M}(\mathcal{M}_1 \cup \mathcal{M}_2)$, the study of hypergraphs $H(\mathcal{M})$, for these cases, is reduced to study of the lattices $L(\mathcal{M})$. As Example in [13] shows, the lattices $L(\mathcal{M})$ can be nondistributive unlike the description in Theorem 2.6, where correspondent lattices are distributive, and for finite $H(M)$ even form Boolean algebras.

In the given context hypergraphs/lattices with minimal, i.e. least structures play an important role. These structures can be obtained from an arbitrary structure by addition of constants interpreted by all elements of the structure. Besides, these minimal structures exist for finite sets $H(M)$.

In [27], the following theorem on dichotomy for minimal structures is proved.

**Theorem 3.5.** Let $M_0$ be a minimal structure, $\mathcal{M}$ be its saturated elementary extension and $p \in S_1(M_0)$ be unique non-algebraic 1-type. Then exactly one of the following conditions holds:

$\square$
(I) the structure \( (p(M), \text{Sem}_p) \) is a pregeometry, where \( \text{Sem}_p \) is the relation of semi-isolation on the set of realizations of the type \( p \), i.e. the following conditions are satisfied:

(S1) Monotony: if \( A \subseteq B \) then \( A \subseteq \text{Sem}_p(A) \subseteq \text{Sem}_p(B) \);

(S2) Finite character: \( \text{Sem}_p(A) = \bigcup \{ \text{Sem}_p(A_0) \mid A_0 \text{ is a finite subset of } A \} \);

(S3) Transitivity: \( \text{Sem}_p(A) = \text{Sem}_p(\text{Sem}_p(A)) \);

(S4) Exchange property (Symmetry): if \( a \in \text{Sem}_p(A \cup \{ b \}) \setminus \text{Sem}_p(A) \) then \( b \in \text{Sem}_p(A \cup \{ a \}) \);

(II) for some finite \( A \subseteq M \) there exists an infinite set \( C_0 \subseteq \text{dcl}(A \cup M_0) \) and a definable quasi-order \( \leq \) on \( M \) such that \( C_0 \) orders a type over \( A \\

(D1) for any \( c \in C_0 \) the set \( \{ x \in C_0 \mid c \leq x \} \) is a cofinite subset of \( C_0 \);

(D2) \( C_0 \) is an initial segment of \( M \): if \( c \in C_0 \) and \( m \leq c \), then \( m \in C_0 \).

Basic examples illustrating Theorem 3.5 are represented by ordered structures \( \langle \omega, < \rangle \) and \( \langle \omega + \omega^*, < \rangle \). The conclusion of Theorem 2.6 holds for both structures. Moreover, for \( M_1 \equiv \langle \omega, < \rangle \) and \( M_2 \equiv \langle \omega + \omega^*, < \rangle \) the structures \( L(M_1) \) and \( L(M_2) \) form atomic Boolean algebras, whose atoms are defined by equivalence classes, being closures of singletons, not in \( \omega + \omega^* \), taking all predecessors and successors.

Return to Example 2.7. It is known that the intersection of convex sets is convex, whereas the intersection of dense orders can be not dense. For instance, any interval \( [a, b] \) contains countable dense subsets \( X, Y \) such that \( X \cap Y = \{a, b\} \). It means that for the structure \( M' \equiv (\mathbb{Q}, <, c_0)_{c_0 \in \mathbb{Q}} \) the structure \( L(M') \) forms a lattice, moreover, a Boolean algebra, if and only if each type in \( S_1(\text{Th}(M')) \) has at most one realization in \( M' \). If \( M' \), with the lattice \( L(M') \), realizes \( \lambda \) non-principal 1-types, then \( |L(M')| = 2^\lambda \). Thus, the following proposition holds.

**Proposition 3.6.** For the structure \( L(M') \) the following are equivalent:

1. \( L(M') \) is a lattice;
2. \( L(M') \) forms an atomic Boolean algebra;
3. each type in \( S_1(\text{Th}(M')) \) has at most one realization in \( M' \), and if \( M' \) realizes \( \lambda \) non-principal 1-types, then \( |L(M')| = 2^\lambda \).

Proposition 3.6 admits natural modifications for a series of theories with minimal models, for instance, for models, obtained by replacement of elements in \( M' \) with finite antichains of fixed cardinality marked by unary predicates \( P_q \) instead of constants \( c_q \). Note that admitting replacement of constants \( c_q \) by infinite antichains \( P_q \) the structure \( L(M') \) is not a lattice since \( P_q \) can be divided by some elementary substructures \( M'_1, M'_2 \vartriangleleft M' \) into two disjoint parts, whence \( M'_1 \cap M'_2 \neq M' \).

Clearly, as above, in the general case if there are separable elements in definable sets \( A \subseteq M \) of structure \( M \) then \( L(M) \) is not closed under intersections, i.e., \( L(M) \) is not even a lower semilattice. Thus, the following proposition holds.

**Proposition 3.7.** If \( L(M) \) is a lattice then \( M \) does not have definable sets \( A \subseteq M \) containing elements separable each other, in particular, \( M \) does not contain \( H \)-free sets \( A \subseteq M \).

In view of Proposition 3.7 it is natural, for given structure \( M \), along with \( L(M) \) to consider for sets \( X \subseteq M \) the following relative structures \( L_X(M) \). Denote by \( H_X(M) \) the family of all sets in \( H(M) \) containing the set \( X \).

Then \( L_X(M) := (H_X(M); \wedge, \vee) \) where for structures \( M_1, M_2 \vartriangleleft M \) containing \( X \), \( M_1 \wedge M_2 = M_1 \cap M_2 \) and \( M_1 \vee M_2 = M(M_1 \cup M_2) \).

Note that if \( X \) is a universe of some elementary substructure of structure \( M \) then definable sets \( A \subseteq M \) already do not contain elements separable by sets in \( L_X(M) \). Then, in any case, \( M_1 \cap M_2 \) is a substructure of \( M \) and the elementarity of that substructure is characterized by Theorem 3.1.

The following example illustrates that apart from the density there are other reasons preventing to consider \( L(M) \) as a lattice.

**Example 3.8 [28].** Let \( M = (M; <, P^1, U^2, c_i)_{i \in \omega} \) be a linearly ordered structure such that \( M \) is a disjoint union of interpretations of unary predicates \( P \) and \( \neg P \), where \( \neg P(M) < P(M) \). We identify interpretations of \( P \) and \( \neg P \) with the set \( Q \) of rational numbers with the natural order.

The symbol \( U \) interprets the binary relation defined as follows: for any \( a \in P(M), b \in \neg P(M) \) \( U(a, b) \iff b < a + \sqrt{2} \).

The constants \( c_i \) interpret an infinite strictly increasing sequence on \( P(M) \) as follows: \( c_i = i \in \mathbb{Q} \).

Clearly that \( \text{Th}(M) \) is a weakly o-minimal theory. Let

\[
p(x) := \{ x > c_i \mid i \in \omega \} \cup \{ P(x) \};
\]

\[
g(y) := \{ \forall t(U(c_i, t) \rightarrow t < y) \mid i \in \omega \} \cup \{ \neg P(y) \}.
\]
Obviously, \( p, q \in S_1(\emptyset) \) are nonisolated types and \( p \not\equiv^w q \). Since there are no \( \emptyset \)-definable bijections from \( p(M') \) onto \( q(M') \), where \( M' \) is a model of \( Th(M) \) realizing some of these types then \( Th(M) \) is not quite o-minimal.

As shown in [28], \( Th(M) \) has exactly 4 pairwise non-isomorphic countable models: the prime model \( M \), i.e., with \( p(M) = \emptyset \) and \( q(M) = \emptyset \); the model \( M_1 \) such that \( p(M_1) \) has the ordering type \( [0, 1) \cap \mathbb{Q} \), \( q(M_1) \) has the ordering type \( (0, 1) \cap \mathbb{Q} \); the model \( M_2 \) such that \( p(M_2) \) has the ordering type \( (0, 1) \cap \mathbb{Q} \), \( q(M_2) \) has the ordering type \( [0, 1) \cap \mathbb{Q} \); and the countable saturated model \( M_3 \).

Therefore \( M_1 \cap M_2 \not\preceq M_3 \). By this reason as well as by the possibility of violation of density in intersections, the structure \( L(M_3) \) does not form a lower semilattice.

Remark 3.9. Along with Example if we consider the known Ehrenfeucht’s example with three models: a prime model \( M_0 \), a weakly saturated model \( M_1 \), and a countable saturated model \( M_2 \), then the structure \( L(M_2) \) is not a lattice in view of presence of dense definable intervals but includes the three-element linearly ordered lattice consisting of the universes \( M_0, M_1, M_2 \).

Lattice structures on sets of isomorphism types of models of a theory

Following Example 3.8 and Remark 3.9 we consider a question on existence of natural lattices associated with hypergraphs \((M, H(M))\) which a distinct to \( L(M) \). Related lattices are lattices represented by Rudin–Keisler preorders \( RK(T) \) [1] for isomorphism types of prime models of a theory \( T \), over finite sets, or their lattice fragments.

The description [29] of structures \( RK(T) \) for Ehrenfeucht quite o-minimal theories \( T \) implies that these structures, for the considered theories, form finite lattices \( LRK(T) \) consisting of \( 2^k \cdot 3^s \) elements and, in view of the main result of the paper [25], the number \( I(T, \omega) \) of pairwise non-isomorphic countable models of \( T \) equals \( 3^k \cdot 6^s \), \( k, s \in \omega \).

The Hasse diagrams illustrating these lattices \( LRK(T) \) are represented in Figures 1–9 for the following values \( k \) and \( s \):

1) \( k = 1, s = 0 \);
2) \( k = 0, s = 1 \);
3) \( k = 2, s = 0 \);
4) \( k = 3, s = 0 \);
5) \( k = 0, s = 2 \);
6) \( k = 0, s = 3 \);
7) \( k = 1, s = 1 \);
8) \( k = 2, s = 1 \);
9) \( k = 1, s = 2 \).

Figure 1. \( k = 1, s = 0 \)  
Figure 2. \( k = 0, s = 1 \)  
Figure 3. \( k = 2, s = 0 \)  
Figure 4. \( k = 3, s = 0 \)
Theorem 4.1. Let $T$ be an Ehrenfeucht quite $\omega$-minimal theory, $I(T, \omega) = 3^k \cdot 6^s$, $k, s \in \omega$. Then:

1. $\text{LRK}(T)$ is a lattice;
2. $\text{LRK}(T)$ is a Boolean algebra $\iff k \geq 1$ and $s = 0$; in such a case the Boolean lattice $\text{LRK}(T)$ has a cardinality $2^k$;
3. $\text{LRK}(T)$ is linearly ordered $\iff k + s \leq 1$.

Proof. Let $\Gamma = \Gamma_1 \cup \Gamma_2$ be a maximal independent set of nonisolated types in $S_1(T)$, where realizations of each type in $\Gamma_1$ generate three models, with prime one, and realizations of each type in $\Gamma_2$ generate six models, with prime one, $|\Gamma_1| = k$, $|\Gamma_2| = s$.

(1) We argue to show that $\text{LRK}(T)$ is a lattice. Indeed, for isomorphism types $\overline{M}_1$ and $\overline{M}_2$ of prime model $M_1$ and $M_2$ over some finite sets $A$ and $B$, respectively, we define sets $X, Y \subseteq \Gamma \times \{0, 1\}$ defining these isomorphism types such that $X = \{(p, 0) \mid M_1 \models p(a) \text{ for some } a \in A, \text{ and } |p(M_1)| = 1 \text{ or } p \in \Gamma_1\} \cup \{(p, 1) \mid M_1 \models p(a) \text{ for some } a \in A, |p(M_1)| \geq \omega, p \in \Gamma_2\}$ and $Y = \{(q, 0) \mid M_2 \models q(b) \text{ for some } b \in B, \text{ and } |q(M_2)| = 1 \text{ or } q \in \Gamma_1\} \cup \{(q, 1) \mid M_2 \models q(b) \text{ for some } b \in B, |q(M_2)| \geq \omega, q \in \Gamma_2\}$. Then the isomorphism type for $M_1 \wedge M_2$ corresponds to the set $U \subseteq \Gamma \times \{0, 1\}$ consisting of all common pairs of $X$ and $Y$, as well as all possible pairs $(p, 0)$, if $(p, 0) \in X$ and $(p, 1) \in Y$, or $(p, 1) \in X$ and $(p, 0) \in Y$. And the isomorphism type for $M_1 \vee M_2$ corresponds to the set $V \subseteq \Gamma \times \{0, 1\}$ consisting of the following pairs:

i) all common pairs of $X$ and $Y$;
ii) all pairs $(p, i) \in X$ such that $Y \cap \{(p, 0), (p, 1)\} \emptyset$,
iii) all pairs $(p, i) \in Y$ such that $X \cap \{(p, 0), (p, 1)\} \emptyset$,
iv) all pairs $(p, 1)$ such that $(p, 0) \in X$ and $(p, 1) \in Y$, or $(p, 1) \in X$ and $(p, 0) \in Y$.

The defined correspondence witnesses that $\text{LRK}(T)$ is a lattice.

(2) If $s \neq 0$ then $\text{LRK}(T)$ is not a Boolean algebra by Stone Theorem, since the cardinality of each finite Boolean algebra equals $2^n$ for some $n \in \omega$ whereas $|\text{LRK}(T)| = 2^k \cdot 3^s$. If $s = 0$ then $\text{LRK}(T)$ is a Boolean algebra of a cardinality $2^k$ such that for isomorphism types $\overline{M}_1$ and $\overline{M}_2$ of prime models $M_1$ and $M_2$ over some finite sets $A$ and $B$, respectively, and for sets $X, Y \subseteq \Gamma$ such that $X = \{p(x) \in \Gamma \mid M_1 \models p(a) \text{ for some } a \in A\}$ and $Y = \{q(x) \in \Gamma \mid M_2 \models q(b) \text{ for some } b \in B\}$, the isomorphism type $\overline{M}_1 \wedge \overline{M}_2$ corresponds to the set $X \cap Y$, and the isomorphism type $\overline{M}_1 \vee \overline{M}_2$ corresponds to the set $X \cup Y$.

(3) If $k + s \leq 1$ then $\text{LRK}(T)$ is linearly ordered as shown in Figures 1 and 2. If $k + s > 1$ and for distinct types $p, q \in \Gamma$ the isomorphism types of models $M_p$ and $M_q$ are incomparable in $\text{LRK}(T)$.

The description for distributions of disjoint unions of Ehrenfeucht theories and the arguments for the proof of Theorem 4.1 allow to formulate the following theorem modifying Theorem 4.1.

Theorem 4.2. Let $T$ be a disjoint union of theories $T_1$ and $T_2$ in disjoint languages and having finite numbers $I(T_1, \omega)$ and $I(T_2, \omega)$ of countable models. Then:

1. $\text{LRK}(T)$ is a (Boolean) lattice $\iff$ $\text{LRK}(T_1)$ and $\text{LRK}(T_2)$ are (Boolean) lattices;
2. $\text{LRK}(T)$ is linearly ordered $\iff$ $\text{LRK}(T_1)$ and $\text{LRK}(T_2)$ are linearly ordered, and

$$
\min\{I(T_1, \omega), I(T_2, \omega)\} = 1.
$$

Proof. (1) If $\text{LRK}(T)$ is a (Boolean) lattice, then $\text{LRK}(T_1)$ and $\text{LRK}(T_2)$ are (Boolean) lattices, since $\text{LRK}(T_1)$ and $\text{LRK}(T_2)$ are isomorphic to sublattices $L_1$ and $L_2$ of the lattice $\text{LRK}(T)$, and elements/complements
of elements in $\text{LRK}(T)$ are represented as pairs of elements/complements of elements in $L_1$ and $L_2$. If $\text{LRK}(T_1)$ and $\text{LRK}(T_2)$ are (Boolean) lattices, then $\text{LRK}(T)$ is a (Boolean) lattice again in view of aforesaid representation.

![Figure 7. $k = 1, s = 1$](image1)

![Figure 8. $k = 2, s = 1$](image2)

![Figure 9. $k = 1, s = 2$](image3)

![Figure 10. 6-Element diagram](image4)

![Figure 11. 9-Element diagram](image5)

(2) If $\text{LRK}(T)$ is linearly ordered then $\text{LRK}(T_1)$ and $\text{LRK}(T_2)$ are linearly ordered, being isomorphic to substructures of $\text{LRK}(T)$. Here $T_1$ or $T_2$ should be $\omega$-categorical, since otherwise prime models over pairs $(p_1,q_1)$ and $(p_2,q_2)$ occur $\text{LRK}(T)$-incomparable, where $p_1, p_2 \in S_1(T_1)$, $q_1, q_2 \in S_1(T_2)$, $p_1, q_1$ are isolated, $p_2, q_2$ are nonisolated.

If structures $\text{LRK}(T_1)$ and $\text{LRK}(T_2)$ linearly ordered, and $\min\{I(T_1,\omega), I(T_2,\omega)\} = 1$, then $\text{LRK}(T)$ is linearly ordered, since $\text{LRK}(T) \simeq \text{LRK}(T_1)$ for $I(T_2,\omega) = 1$, and $\text{LRK}(T) \simeq \text{LRK}(T_2)$ for $I(T_1,\omega) = 1$.

In Figures 10 and 11 we illustrate Theorem by structures $\text{LRK}(T)$ in [30], for disjoint unions of theories, which are not lattices.

Acknowledgements

This research was partially supported by Committee of Science in Education and Science Ministry of the Republic of Kazakhstan (Grant No. AP05132546) and Russian Foundation for Basic Researches (Project No. 17-01-00531-a).
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