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On the calculation of plates by the series representation of the deflection function

In the article calculations of rectangular plates by the series representation of the deflection function are presented. For the considered rectangular plate the investigation was determined by finding the minimum of the potential energy of the plate by Ritz variation method and by direct substitution of series representation of the deflection function in the equilibrium equation by Bubnov-Galerkin method. To illustrate the above variation method, specific examples of the calculation were given for the square hinged bearing plate and for the rectangle plate rigidly clamped along the whole contour and loaded by the uniformly distributed load of the given intensity.

Key words: plate, intensity, deflection, function, biharmonic operator, Ritz method, Bubnov-Galerkin method.

Ritz variational method

Along with the exact methods of finding the plate deflection function by solving the equation (1)

$$D\nabla^2\nabla^2 W = q(x_1, x_2), \tag{1}$$

where q — the intensity of the external distributed load; $\nabla^2\nabla^2 W$ — Biharmonic operator, various methods are applied. They are based on the fact that the problem of integrating the equation (1) can be replaced by the task of finding the minimum of the potential energy of the plate.

$$E = \frac{D}{2}\int\int\left\{\left(\frac{\partial^2 W}{\partial x^2}\right)^2 + \left(\frac{\partial^2 W}{\partial y^2}\right)^2 + 2v\left(\frac{\partial^2 W}{\partial x^2}\cdot\frac{\partial^2 W}{\partial y^2}\right) + \right.$$

$$\left. +2(1-v)\left(\frac{\partial^2 W}{\partial x \partial y}\right)^2\right\} dxdy - \int\int q(x,y)W(x,y)dxdy. \tag{2}$$

The first approximate method of determining the minimum value of the functional (2) was the Ritz method. The essence of it is to provide the desired function, for example deflection function $W(x,y)$ as a series in a system known functions $\varphi_i(x,y)$ with constant coefficients, the system of these functions must be linearly independent and be complete. Thus, we construct a function

$$W_n(x,y) = a_1\varphi_1(x,y) + a_2\varphi_2(x,y) + ... + a_n\varphi_n(x,y). \tag{3}$$

The function $W_n(x,y)$ matched to the expression for the potential energy of the plate, turn it into a function of the unknown coefficients a_n. The coefficients a_n are chosen so that the energy takes a minimum value, and this is achieved with the values of the coefficients, which vanishes the first of its derivatives

$$\frac{\partial E(W_n)}{\partial a_i} = 0, \quad i = 1, 2, ..., n. \tag{4}$$

Let us consider the choice of the functions $\varphi_i(x,y)$ which according to Ritz were called coordinate. Besides the above mentioned conditions of the linear independence and completeness, they must satisfy the geometrical boundary conditions of the problem and preferably, but not necessarily, static boundary conditions.

For example, for a plate with clamped edges coordinate functions must correspond both boundary conditions of deflection and rotation angles equal to zero, and for plates with a hinged edges — they must correspond the condition of equality to zero deflection on the plate circuit.

Another requirement for approximating functions — compliance within the meaning of the problem being solved. For example, if from the physical considerations it is clear that the desired function is even, then the approximating functions must be even too. Conversely, if the desired function is a function of the general form, then the set of approximating functions should consist of even and odd functions [1].

Let's consider the problem of the thin plate bending, in some way supported along the contour and loaded by arbitrary load $q(x,y)$. The expression for the total potential energy in this case is

$$E=\int_0^a\int_0^b\left\{\frac{D}{2}\left[\left(\frac{\partial^2 W}{\partial x^2}+\frac{\partial^2 W}{\partial y^2}\right)^2+2(1-v)\left(\frac{\partial^2 W}{\partial x^2}\cdot\frac{\partial^2 W}{\partial y^2}-\left(\frac{\partial^2 W}{\partial x\partial y}\right)^2\right)\right]-qW\right\}dxdy. \tag{5}$$

In accordance with the idea of the Ritz method we define a desired function of deflection in the form of the expansion (3) with unknown coefficients a_i. Here $\varphi_i(x,y)$ — coordinate functions selected so as to satisfy the geometric boundary conditions on the plate circuit.

Substituting (3) into (5), we obtain an expression of the total potential energy of the plate, in the form of the following positive definite quadratic form.

$$E=\frac{1}{2}\delta_{11}a_1^2+\frac{1}{2}\delta_{22}a_2^2+\frac{1}{2}\delta_{33}a_3^2+...+\delta_{12}a_1a_2+\delta_{13}a_1a_3+...+\delta_{23}a_2a_3+...-$$
$$-\triangle_{1p}a_1-\triangle_{2p}a_2-\triangle_{3p}a_3-..., \tag{6}$$

where

$$\delta_{ik}=\delta_{ki}=\int_0^a\int_0^b D\left\{\frac{\partial^2\varphi_i}{\partial x^2}\cdot\frac{\partial^2\varphi_k}{\partial x^2}+\frac{\partial^2\varphi_i}{\partial x^2}\cdot\frac{\partial^2\varphi_k}{\partial y^2}+\frac{\partial^2\varphi_i}{\partial y^2}\cdot\frac{\partial^2\varphi_k}{\partial x^2}+\frac{\partial^2\varphi_i}{\partial y^2}\cdot\frac{\partial^2\varphi_k}{\partial y^2}-\right.$$
$$\left.-(1-v)\left(\frac{\partial^2\varphi_i}{\partial x^2}\cdot\frac{\partial^2\varphi_k}{\partial y^2}+\frac{\partial^2\varphi_i}{\partial y^2}\cdot\frac{\partial^2\varphi_k}{\partial x^2}-2\frac{\partial^2\varphi_i}{\partial x\partial y}\cdot\frac{\partial^2\varphi_k}{\partial x\partial y}\right)\right\}dxdy; \tag{7}$$

$$\triangle_{ip}=\int_0^a\int_0^b q(x,y)\varphi_i(x,y)dxdy.$$

As mentioned above, the expansion coefficients a_i are determined from the condition of the plate potential energy minimum. Consistently performing calculations according to the formula (4), we get the system of linear algebraic equations for the unknown coefficients a_i:

$$\left\{\begin{array}{c}\delta_{11}a_1+\delta_{12}a_2+\delta_{13}a_3+...+\triangle_{1p}=0;\\ \delta_{21}a_1+\delta_{22}a_2+\delta_{23}a_3+...+\triangle_{2p}=0;\\ \delta_{31}a_1+\delta_{32}a_2+\delta_{33}a_3+...+\triangle_{3p}=0;\\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\ \delta_{n1}a_1+\delta_{n2}a_2+\delta_{n3}a_3+...+\triangle_{np}=0.\end{array}\right. \tag{8}$$

The solution of system (8) gives all the coefficients a_i and, therefore, determines the approximate value of deflection function $W_n(x,y)$ by the formula (3).

As a first example, consider a plate with a hinged support on the contour loaded by uniform load q. Let us construct an approximate solution of this problem, keeping in the expanding (3), only one member of the series

$$W_n(x,y) = a_i\varphi_i(x,y).$$

The function $\varphi_1(x,y)$ must satisfy the boundary conditions on the contour of the plate for hinged support

$$x=0, x=a, y=0, y=b; \quad W(x,y)=0;$$

$$x=0, \quad x=a; \quad \frac{\partial^2 W(x,y)}{\partial x^2}=0;$$

$$y=0, \quad y=b; \quad \frac{\partial^2 W(x,y)}{\partial y^2}=0.$$

Let us represent the function $\varphi_1(x,y)$ as a product of two functions

$$\varphi_1(x,y) = \psi_1(x)\chi_1(y).$$

Moreover, the function must satisfy the boundary conditions when $x=0$ and $x=a$, and the function $\psi_1(x)$ — for $y=0$ and $y=b$. Then, according to the formula (7), we have

$$\delta_{11} = D\int\int\{\psi_1^{''2}\chi_1^2 + 2\psi_1^{''}\chi_1\psi_1\chi_1^{''} + \psi_1^2\chi_1^{''2} - \\ -(1-v)(2\psi_1^{''}\chi_1\psi_1\chi_1^{''} - 2\psi_1^{'2}\chi_1^{'2})\}dxdy; \tag{9}$$

$$\triangle_{1p} = \int_0^a\int_0^b q\psi_1\chi_1 dxdy. \tag{10}$$

Functions $\psi_1(x)$ and $\chi_1(y)$ can take the form

$$\psi_1(x) = x^4 - 2ax^3 + a^3x;$$

$$\chi_1(y) = y^4 - 2by^3 + b^3y,$$

where a and b — the dimensions of the plate. Let us calculate the derivatives of these functions

$$\psi_1^{'}(x) = 4x^3 - 6ax^2 + a^3; \quad \psi_1^{''}(x) = 12x^2 - 12ax;$$

$$\chi_1^{'}(y) = 4y^3 - 6by^2 + b^3; \quad \chi_1^{''}(y) = 12y^2 - 12by.$$

It's obvious that

$$\psi_1(0) = \psi_1(a) = \psi_1^{''}(0) = \psi_1^{''}(a); \quad \chi_1(0) = \chi_1(a) = \chi_1^{''}(0) = \chi_1^{''}(a).$$

That is, the selected function satisfies the geometrical and static boundary conditions. Here are integral values in the formula (9)

$$\int_0^a \psi_1''^2(x)dx = \frac{24}{5}a^5; \quad \int_0^a \psi_1''^2(x)\psi_1(x)dx = -\int_0^a \psi_1'^2(x)dx = -\frac{17}{35}a^7;$$

$$\int_0^a \psi_1^2(x)dx = \frac{31}{630}a^9; \quad \int_0^a \chi_1^2(y)dy = \frac{31}{630}b^9; \quad \int_0^a \chi_1''^2(y)dy = \frac{24}{5}b^5;$$

$$\int_0^a \psi_1(x)dx = \frac{1}{5}a^5; \int_0^a \chi_1(y)dy = \frac{1}{5}b^5;$$

$$\int_0^a \chi_1''^2(y)\chi_1(y)dy = -\int_0^a \chi_1'^2(y)dy = -\frac{17}{35}b^7.$$

Now it is easy to get

$$\delta_{11} = D\left(\frac{24}{5}\cdot\frac{31}{630}a^5b^9 + 2\frac{17}{35}\cdot\frac{17}{35}a^7b^7 + \frac{31}{630}\cdot\frac{24}{5}a^9b^5\right) = 0.236a^7b^7\left(\frac{a}{b}+\frac{b}{a}\right)^2 D;$$

$$\triangle_{1p} = \frac{1}{25}qa^5b^5.$$

Since the problem is solved in a first approximation, only one equation is left from the system (8)

$$\delta_{11}a_1 - \triangle_{1p} = 0,$$

from which

$$a_1 = \frac{\triangle_{1p}}{\delta_{11}} = \frac{0.1695}{Da^2b^2(\frac{a}{b}+\frac{b}{a})^2}.$$

If $a = b$, i.e. when the plate is square, we obtain

$$a_1 = 0.0424\frac{q}{a^4D}.$$

Thus, the solution to a first approximation for a given choice of approximating function for a square, hinged supported plate is as follows

$$W(x,y) = (x^4 - 2ax^3 + a^3x)(y^4 - 2ay^3 + a^3y)\cdot\frac{0.0424q}{a^4D}.$$

When we assume in this decision $x = y = 0.5a$, then the deflection at the center of the plate is equal to:

$$W(x,y)\Big|_{\substack{x=0.5a\\y=0.5a}} = 0.00411\frac{qa^4}{D}.$$

Exact solution, obtained by the function expanding into double trigonometric series, gives the following deflection value in the center of the plate:

$$W(x,y)\Big|_{\substack{x=0.5a\\y=0.5a}} = 0.00406\frac{qa^4}{D},$$

i.e. the error does not exceed 1.5%.

In some way worse is the case with the values of the maximum bending moment and a maximum reduced lateral force. Here calculations give the following results

$$M_x^{max}(x,y)\Big|_{\substack{x=0.5a\\ y=0.5a}} = -Da_1(\psi_1^{''}\chi_1 + v\psi_1\chi_1^{''}) = 0.0517qa^2.$$

Exact solution is

$$M_x = 0.0479qa^2,$$

error of 8%.

$$Q_x^*(x,y)\Big|_{\substack{x=0\\ y=0.5a}} = -Da_1(\psi_1^{'''}\chi_1 + (2-v)\psi_1\chi_1^{''}) = 0.375qa.$$

Exact solution is $Q_x^*(x,y) = 0.420qa$, inaccuracy is 10,7%.

We see that even the first approximation allows to get almost exact solution for deflections. For more accurate values of internal forces of the plate in the decomposition (3) a greater number of members should be hold.

Bubnov-Galerkin method

At investigation of the approximate solution by Ritz method we found a minimum potential energy of a plate instead of the solution of the differential equation (1). However, as I.G.Bubnov specified, it is possible to solve directly the differential equation of a bend of a plate (1), substituting in it decomposition of a deflection function in a row on other known coordinate functions with constant coefficients. This method differs from Ritz method in the fact that coefficient of decomposition are found from conditions of orthogonality of coordinate functions. Besides, these functions have to meet not only geometrical boundary conditions, but also static. Otherwise, they must be linearly independent and form a complete system. Later V.G. Galerkin refused demands orthogonal coordinate functions [2].

Let us now consider the application of Bubnov-Galerkin method to the problem of bending of rectangular plates. At the solution of the equation of a bend of a plate (1) we will set required function as follows $W(x,y)$:

$$W(x,y) = \sum_{i=1}^{n} a_i\varphi_i(x,y), \tag{11}$$

where $\varphi_i(x,y)$ — the approximating functions meeting all kinematic and static conditions on a plate contour. Substituting (11) in (1), we will multiply both sides by $\varphi_k(x,y)$ and integrate the resulting expression of the entire surface of the plate. As a result we will get:

$$\iint\limits_s \Big[D\sum a_i\nabla\nabla\varphi_i(x,y)\Big]\cdot\varphi_k(x,y)dxdy = \iint\limits_s q(x,y)\varphi_k(x,y)dxdy; \tag{12}$$

$$\delta_{ik} = \delta_{ki} = \iint\limits_s \nabla\nabla\varphi_i(x,y)]\cdot\varphi_k(x,y)dxdy;$$

$$\triangle_{kq} = \iint\limits_s \frac{q(x,y)}{D}\varphi_k(x,y)dxdy;$$

$$\begin{cases} \delta_{11}a_1 + \delta_{12}a_2 + \delta_{13}a_3 + ... + \triangle_{1q} = 0; \\ \delta_{21}a_1 + \delta_{22}a_2 + \delta_{23}a_3 + ... + \triangle_{2q} = 0; \\ \delta_{31}a_1 + \delta_{32}a_2 + \delta_{33}a_3 + ... + \triangle_{3q} = 0; \\ \text{...} \\ \delta_{n1}a_1 + \delta_{n2}a_2 + \delta_{n3}a_3 + ... + \triangle_{nq} = 0. \end{cases} \quad (13)$$

The reciprocity of δ_{ik} and δ_{ki} coefficients is also easily proved by integration in parts provided that φ_i functions meet all boundary statements of the problem. The decision of system (13) gives all a_i unknown numbers and defines, thus, the approximate solution of the allowing equation (1). It should be noted that equation (12) can be given a clear mechanical meaning when functions $\varphi_k(x, y)$ are viewed as possible movements. In this case each of the equations of system (12) can be interpreted as a result of the principle of possible movements [3].

As an example we will consider the rectangular plate of ABCB jammed on all contour and loaded with uniform loading of q.

The boundary conditions in this case are:

$$x = \pm a; \quad W(x, y) = \frac{\partial W}{\partial x} = 0; \quad (14)$$

$$y = \pm b; \quad W(x, y) = \frac{\partial W}{\partial y} = 0.$$

We will take the following decomposition for $W(x, y)$ function, taking into account symmetry of a task:

$$W(x, y) = a_1(x^2 - a^2)^2(y^2 - b^2)^2 + a_2(x^2 - a^2)^2(y^2 - b^2)^3+$$
$$+a_3(x^2 - a^2)^3(y^2 - b^2)^2 + a_4(x^2 - a^2)^3(y^2 - b^2)^3 + ...,$$

where $\varphi_1(x, y) = (x^2 - a^2)^2(y^2 - b^2)^2; \quad \varphi_2(x, y) = (x^2 - a^2)^2(y^2 - b^2)^3; ...$ etc.

We can make sure that at such choice of $\varphi_i(x, y)$ functions boundary conditions (14) are satisfied precisely. For creation of the approximate solution we will be limited to the first approach, i.e.

$$W(x, y) = a_1(x^2 - a^2)^2(y^2 - b^2)^2.$$

Then the only one equation will be left from the system (13)

$$\delta_{11}a_1 = \Delta_{1q}, \quad \delta_{11} = \int_{-a}^{a}\int_{-b}^{b} \nabla^4\varphi_1\varphi_1 dxdy, \quad \triangle_{1q} = \int_{-a}^{a}\int_{-b}^{b} \frac{q}{D}\varphi_1 dxdy. \quad (15)$$

Further we have

$$\frac{\partial^4\varphi_1}{\partial x^4} = 24(y^2 - b^2)^2; \quad \frac{\partial^4\varphi_1}{\partial y^4} = 24(y^2 - a^2)^2; \quad \frac{\partial^4\varphi_1}{\partial x^2 \partial y^2} = 16(3x^2 - a^2(3y^2 - b^2).$$

Hence

$$\delta_{11} = \int_{-a}^{a}\int_{-b}^{b} \frac{\partial^4\varphi_1}{\partial x^4} + 2\frac{\partial^4\varphi_1}{\partial x^2\partial y^2} + \frac{\partial^4\varphi_1}{\partial y^4}\varphi_1 dxdy = 4\int_{0}^{a}\int_{0}^{b}[24(x^2-a^2)^2(y^2-b^2)^4+$$

$$+32(3x^2-a^2)(3y^2-b^2)(x^2-a^2)^2(y^2-b^2)^2+24(x^2-a^2)^4(y^2-b^2)^2]dxdy =$$

$$= \frac{4\cdot 128\cdot 64}{9\cdot 7\cdot 5\cdot 5}(b^4+\frac{4}{7}a^2b^2+a^4)a^5b^5;$$

$$\triangle_{1q} = \int_{-a}^{a}\int_{-b}^{b}\frac{q}{D}(x^2-a^2)^2(y^2-b^2)^2dxdy = 4\frac{q}{D}\frac{64}{225}a^5b^5.$$

From the first equation (15) we find that

$$a_1 = \frac{\triangle_{1q}}{\delta_{11}} = \frac{7q}{128(a^4+4/7a^2b^2+b^4)D}$$

and, therefore,

$$W(x,y) = a_1\varphi_1 = \frac{7q}{128(a^4+4/7a^2b^2+b^4)D}(x^2-a^2)^2(y^2-b^2)^2.$$

The greatest deflection in the middle when x = 0, y = 0 and a = b, i.e. for a square plate is

$$W_{max} = 0.0213\frac{qa^4}{D}.$$

The exact solution obtained using the series gives

$$W_{max}^{ex} = 0.0202\frac{qa^4}{D}.$$

We receive the greatest bending moment in the middle of side of a square from the following formula

$$x = \pm a; \quad y = 0, \quad M_{max} = -D\left(\frac{\partial^2 W}{\partial x^2} + v\frac{\partial^2 W}{\partial y^2}\right)$$

$$M_{max} = -0.171qa^2.$$

Exact solution is

$$M_{max}^{ex} = -0.205qa^2.$$

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Майысу функцияларын қатарларға жіктеу арқылы пластиналарды есептеу

Мақалада пластинаның майысу функциясын қатарға жіктеу арқылы тіктөртбұрышты пластиналарды есептеу көрсетілген. Қарастырылған тіктөртбұрышты пластиналар үшін Ритц вариациялық және Галеркин-Бубнов әдістері қолданылған. Берілген зерттеулердің нәтижелерін көрсету үшін бірқалыпты таралған жүктеме әсер ететін барлық жағы қатты бекітілген және топсалы тірелген тіктөртбұрышты және квадрат пластиналар есебі шығарылған.

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О расчете пластин разложением в ряд функции прогибов

В статье представлены расчеты прямоугольных пластин разложением в ряд функции прогиба пластины. Для рассматриваемой прямоугольной пластины проведено исследование путем отыскания минимума потенциальной энергии пластины по вариационному методу Ритца и непосредственной подстановкой разложения функции прогибов в уравнение равновесия пластины по методу Бубнова-Галеркина. Для иллюстрации изложенного исследования приведены конкретные примеры расчета для квадратной шарнирно-опертой пластины и прямоугольной пластины, защемленной по всему контуру и нагруженной равномерной нагрузкой заданной интенсивности.

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