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CONTENT

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Spectral properties of local and nonlocal problems for the diffusion-wave equation of fractional order

The paper investigates the issues of solvability and spectral properties of local and nonlocal problems for the fractional order diffusion-wave equation. The regular and strong solvability to problems stated in the domains, both with characteristic and non-characteristic boundaries are proved. Unambiguous solvability is established and theorems on the existence of eigenvalues or the Volterra property of the problems under consideration are proved.

Keywords: diffusion-wave equations, fractional order equations, boundary value problems, strong solution, Volterra property, eigenvalue.

1 Introduction

The theory of derivatives and integrals of non-integer (fractional) order, called fractional calculus, is becoming increasingly important both for the development of modern mathematics and for applications in various fields of natural science. Both ordinary and partial differential equations of fractional order have been used over the past few decades to model many physical and chemical processes and in engineering [1–7].

Fractional partial differential equations have become especially important for modeling the so-called anomalous diffusion processes in nature and the theory of complex systems [1]. Such equations are also associated with fractional Brownian motions, the continuous random walk in time (CTRW) method, stable Levy distributions, etc. [2, 7]. Fractional differential equations also make it possible to study the long-term and nonlocal dependence of many anomalous processes.

Since the fractional order equation generalizes the integer order equation, as well as a relatively small number of systematized analytical and numerical methods for such equations, make this direction a priority in the general theory of differential equations.

The mathematical theory of fractional differential equations is more or less fully investigated for ordinary equations [1], whereas for partial differential equations it differs from the situation for the equation of one variable. In the scientific literature, analogs of the initial data problem and initial boundary value problems for the simplest partial differential equations of fractional order were considered mainly. Methods for solving such problems are considered in [1, 8–10].

The issues of solvability of local and non-local problems for various fractional order equations are considered in [11–16].

Spectral properties, including Volterra property and the existence of eigenvalues, for a mixed fractional order equation, as far as we know, are almost not studied. Note that the solvability issues and spectral properties of local and nonlocal problems for a mixed parabolic-hyperbolic equation of the second and third orders are studied in [17–24].

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The work is devoted to the study of the solvability and spectral properties of local and nonlocal problems for the diffusion-wave equation of fractional order. The regular and strong solvability of the tasks set in the domains with both characteristic and non-characteristic boundaries of the domain is proved. The unambiguous solvability of the problem is established, theorems on the existence of eigenvalues are proved, or the Volterra nature of the problems under consideration.

Consider equation

$$Lu(x,y) = f(x,y), \tag{1}$$

where

$$Lu(x,y) = \begin{cases} {}_cD_{0x}^{\alpha}u(x,y) - u_{yy}(x,y), \; y > 0, \\ u_{xx}(x,y) - u_{yy}(x,y), \; y < 0, \end{cases} \tag{2}$$

$${}_cD_{0x}^{\alpha}u\,(x,y) = \frac{1}{\Gamma\,(1-\alpha)}\int\limits_0^x \frac{u_x\,(t,y)}{(x-t)^{\alpha}}dt, \;\; 0 < \alpha < 1.$$

$\Gamma(x)$ is Euler's gamma-function, (2) is an integral-differential operator of fractional order α in the sense of Caputo [1; 92], $f(x,y)$ is a given function.

2 Solvability and Volterra property of local and nonlocal problems for the diffusion-wave equation

Let $\Omega = \Omega_0 \cup \Omega_1 \cup AB$ be a domain, where Ω_0 is a rectangle ABB_0A_0 with vertices $A\,(0,0)$, $B\,(1,0)$, $B_0\,(1,1)$, $A_0\,(0,1)$, Ω_1 is a domain bounded by segments AB and smooth curve $AD: y = -\gamma\,(x)$, $0 < x < l$, where $0,5 < l \leq 1$; $\gamma\,(0) = 0$, $l + \gamma\,(l) = 1$, and characteristic $BD: x - y = 1$ of equation (1), if $l < 1$ and $\gamma(l) = 0$, if $l = 1$ (when $D = B$), located inside the characteristic triangle $0 < x + y \leq x - y < 1$.

With respect to the curve $\gamma\,(x)$, we suppose that $\gamma\,(x)$ is twice continuously differentiable function and $x \pm \gamma\,(x)$ are monotonically increasing functions, and $0 < \gamma'(x) < 1$, $\gamma(x) > 0$, $x > 0$.

Problem M_1A. Find a solution to equation (1) satisfying conditions:

$$u(0,y) = 0, \; 0 \leq y \leq 1, \tag{3}$$

$$u(x,1) = 0, \; 0 \leq x \leq 1, \tag{4}$$

$$(u_x - u_y)\,|_{AD} = 0. \tag{5}$$

Definition 1. The regular solution to the problem M_1A in the domain Ω will be called the function $u(x,y) \in V$, where

$$V = \left\{u(x,y):\, u(x,y) \in C(\bar{\Omega}) \cap C^{1.1}(\Omega \cup AC),\, D_{0x}^{\alpha}u(x,y),\, u_{yy}(x,y) \in C(\Omega_0),\, u(x,y) \in C^{2.2}(\Omega_1)\right\},$$

satisfying the equation (1) in $\Omega_0 \cup \Omega_1$ and conditions (3)–(5).

In domain $\Omega_0$consider the following auxiliary problem:

Problem C_1. Find a solution to equation (1) for $y > 0$ satisfying conditions (3), (4) and

$$u_x(x,0) - u_y(x,0) = \delta(x), \; 0 < x < 1, \tag{6}$$

where $\delta(x)$ is a given function.

Lemma 1. Let be $\delta(x) \in C^1[0,1]$. Then for any function $f(x,y) \in C^1(\bar{\Omega}_0)$ is a solution to problem C_1 allows a priori estimates.

$$D_{0x}^{\alpha-1}\,\|u(x,y)\|^2_{L_2(0,1)} + 2\int\limits_0^x \|u_y(t,y)\|^2_{L_2(0,1)}\,dt \leq C\left[\int\limits_0^x \|f(t,y)\|^2_{L_2(0,1)}\,dt + \int\limits_0^x \delta^2(t)dt\right], \tag{7}$$

where $\|f(x,y)\|^2_{L_2(0,1)} = \int\limits_0^1 f^2(x,y)dy$. Hereinafter symbol will denote a positive constant that does not depend on $u(x,y)$, not necessarily the same.

Proof of Lemma 1. We multiply equation (1) for $y > 0$ by $u(x,y)$ and integrating from 0 to 1 over y and taking into account conditions (3),(4) after some transformations we have

$$\int\limits_0^1 u(x,y)D^\alpha_{0x}u(x,y)dy + \int\limits_0^1 u^2_y(x,y)dy + \tau(x)\nu(x) = \int\limits_0^1 f(x,y)u(x,y)dy, \tag{8}$$

where

$$\tau(x) = u(x,0), \;\; 0 \le x \le 1, \tag{9}$$

$$\nu(x) = u_y(x,0), \;\; 0 < x < 1. \tag{10}$$

It is known [10], that

$$\int\limits_0^1 u(x,y)\cdot D^\alpha_{0x}u(x,y)dy \ge \frac{1}{2}\int\limits_0^1 D^\alpha_{0x}u^2(x,y)dy.$$

By virtue of the last inequality, from (8), taking into account (6) and the notations (9), (10) we obtain

$$\int\limits_0^1 D^\alpha_{0x}u^2(x,y)dy + 2\int\limits_0^1 u^2_y(x,y)dy + 2\tau(x)\tau'(x) \le 2\int\limits_0^1 u(x,y)f(x,y)dy + 2\tau(x)\delta(x). \tag{11}$$

Integrating (11) over t from 0 to x, taking into account $\tau(0) = 0$ and using known inequalities we have

$$D^{\alpha-1}_{0x}\|u(x,y)\|^2_{L_2(0,1)} + 2\int\limits_0^x \|u_y(t,y)\|^2_{L_2}dt + \tau^2(x) \le \int\limits_0^x \left[\|u(t,y)\|^2_{L_2(0,1)} + \|f(t,y)\|^2_{L_2(0,1)} + \tau^2(t) + \delta^2(t)\right]dt. \tag{12}$$

In the left part of (12), omitting the first two terms and applying the Gronwall-Bellman inequality, we will have

$$\int\limits_0^x \tau^2(t)dt \le C\int\limits_0^x \left[\|u(t,y)\|^2_{L_2(0,1)} + \|f(t,y)\|^2_{L_2(0,1)} + \delta^2(t)\right]dt.$$

Taking into account the last from (12) we have

$$D^{\alpha-1}_{0x}\|u(x,y)\|^2_{L_2(0,1)} + 2\int\limits_0^x \|u_y(t,y)\|^2_{L_2(0,1)} \le C\int\limits_0^x \left[\|u(t,y)\|^2_{L_2(0,1)} + \|f(t,y)\|^2_{L_2(0,1)} + \delta^2(t)\right]dt. \tag{13}$$

Similarly as above, omitting the second term of the left part in (13) and applying Lemma 1 in [10] we have

$$D^{-\alpha-1}_{0x}\|f(x,y)\|^2_{L_2(0,1)} \le \frac{x^\alpha}{\Gamma(1+\alpha)}\int\limits_0^x \|f(t,y)\|^2_{L_2(0,1)}dt$$

we have

$$\int\limits_0^x \|u(t,y)\|^2_{L_2(0,1)}dt \le C\int\limits_0^x \left[\|f(t,y)\|^2_{L_2(0,1)} + \delta^2(t)\right]dt. \tag{14}$$

From (12)–(14) it is followed the validity of the a priori estimate (7). *Lemma 1 is proved.*

Now consider equation (1) in the domain Ω_1. By virtue of the unambiguous solvability of the Cauchy problem (1), (9), (10) for the wave equation, any regular solution of the в problem in the domain Ω_1 is represented as

$$u(x,y) = \frac{1}{2}\left[\tau(\xi) + \tau(\eta) - \int_{\xi}^{\eta} \nu(t)\,dt\right] - \int_{\xi}^{\eta} d\xi_1 \int_{\xi_1}^{\eta} f_1(\xi_1,\eta_1)\,d\eta_1, \tag{15}$$

where $\xi = x+y,\;\; \eta = x-y,\quad 4f_1(\xi,\eta) = f\left(\frac{\xi+\eta}{2}, \frac{\xi-\eta}{2}\right)$. Due to the conditions imposed on the function $\gamma(x)$, equation of the curve AD in characteristic variables ξ, η allows representation

$$\xi = \lambda(\eta),\;\; 0 \le \eta \le 1,\;\; and\; \lambda(\eta) < \eta.$$

In (15) satisfying condition (5) after some simple transformations we have

$$\nu(x) = \tau'(x) - 2\int_{\lambda(x)}^{x} f_1(\xi,x)d\xi,\;\; 0 < x < 1. \tag{16}$$

The ratio (16) is the main functional relationship between $\tau(x)$ and $\nu(x)$ brought to the segment AB from hyperbolic domain Ω_1.

Substituting obtained expression $\nu(x)$ into (15), after some transformations we get presentation of the solution $u(\xi,\eta)$ in domain Ω_1.

$$u(x,y) = \tau(\xi) + \int_{\xi}^{\eta} d\eta_1 \int_{\lambda(\eta_1)}^{\xi} f_1(\xi_1,\eta_1)\,d\xi_1. \tag{17}$$

Now in (7) assuming that $\delta(x) = 2\int_{\lambda(x)}^{x} f_1(\xi,x)d\xi$ it is not difficult to establish the validity of the following lemma.

Lemma 2. For any function $f(x,y) \in C^1(\bar{\Omega})$, $f(0,0) = 0$ the solution to problem M_1B allows a priori estimate

$$D_{0x}^{\alpha-1}\|u(x,y)\|^2_{L_2(0,1)} + \int_0^x \|u_y(t,y)\|^2_{L_2(0,1)}\,dt \le C\left[\int_0^x \|f(t,y)\|^2_{L_2(0,1)}\,dt + \int_0^x d\xi \int_{\xi}^{x} |f(\xi,t)|^2 dt\right]. \tag{18}$$

Lemma 2 implies the validity of the following estimate

$$\|u(x,y)\|_{L_2(\Omega_0)} + \|u_y(x,y)\|_{L_2(\Omega_0)} \le C\,\|f(x,y)\|_{L_2(\Omega)}, \tag{19}$$

where $L_2(\Omega)$ is quadratically summable functions in Ω.

Consider the following auxiliary problem C_2. In domain Ω_0 find a solution of equation (1), satisfying conditions (3), (4) and (9).

The solution of equation (1), satisfying conditions (3), (4) and (9) in domain Ω_0 can be presented in a form [8]

$$u(x,y) = \int_0^x E_{y_1}(x-x_1,y,0)\,\tau(x_1)\,dx_1 + \int_0^x dx_1 \int_0^1 E(x-x_1,y,y_1)\,f(x_1,y_1)\,dy_1, \tag{20}$$

where

$$E(x,y,y_1)=\frac{x^{\beta-1}}{2}\sum_{n=-\infty}^{+\infty}\left[e^{1,\beta}_{1,\beta}\left(-\frac{|y-y_1+2n|}{x^\beta}\right)-e^{1,\beta}_{1,\beta}\left(-\frac{|y+y_1+2n|}{x^\beta}\right)\right],\ \beta=\frac{\alpha}{2},$$

$e^{1,\beta}_{1,\beta}(t)=\sum\limits_{n=0}^{\infty}\frac{t^n}{n!\Gamma(\beta-\beta n)}$ is Wright type function [8]. Differentiating (20) over y we have

$$u_y(x,y)=\int\limits_0^x E_{y_1y}(x-x_1,y,0)\,\tau(x_1)\,dx_1+\int\limits_0^x dx_1\int\limits_0^1 E_y(x-x_1,y,y_1)\,f(x_1,y_1)\,dy_1 \tag{21}$$

and using known formulas [8], [18] after some calculations, going to limit in (21) for $y\to 0$ we have:

$$\nu(x)=-\int m(x-x_1)\,\tau'(x_1)dx_1+\int\limits_0^x dx_1\int\limits_0^1 E_y(x-x_1,0,y_1)\,f(x_1,y_1)\,dy_1, \tag{22}$$

where

$$m(x)=\sum_{n=-\infty}^{+\infty}x^{-\beta}e^{1,1-\beta}_{1,\beta}\left(-\frac{|2n|}{x^\beta}\right)=\frac{1}{\Gamma(1-\beta)}x^{-\beta}+2x^{-\beta}\sum_{n=1}^{+\infty}e^{1,1-\beta}_{1,\beta}\left(-\frac{2n}{x^\beta}\right). \tag{23}$$

Note that (22) is the main functional rate between $\tau'(x)$ and $\nu(x)$, brought to the segment from domain Ω_0.

Excluding from the functional relations (16) and (22) the function $\nu(x)$, with respect to $\tau'(x)$ we obtain the equation

$$\tau'(x)+\int\limits_0^x m(x-t)\,\tau'(t)dt=Q(x),\ \ 0\le x\le 1, \tag{24}$$

where

$$Q(x)=2\int\limits_{\lambda(x)}^x f_1(\xi,x)d\xi+\int\limits_0^x dx_1\int\limits_0^1 \mathrm{E}_y(x-x_1,0,y_1)f(x_1,y_1)dy_1. \tag{25}$$

Lemma 3. [8] Let be $0<\theta\le 1$. Then for functions $\mathrm{E}(x,y,y_1)$ and $\mathrm{E}_y(x,y,y_1)$ the following estimates take place

$$|\mathrm{E}(x,y,y_1)|\le Cx^{(2+\theta)\beta-1},\ \ 0<x\le 1,\ \ 0\le y_1<y\le 1,\ \ 0<\theta\le 1, \tag{26}$$

$$|\mathrm{E}_y(x,y,y_1)|\le Cx^{\beta(1+\theta)-1},\ \ 0<x\le 1,\ \ 0\le y_1<y\le 1,\ \ 0<\theta\le 1. \tag{27}$$

The proof of Lemma 3 is carried out using the inequality

$$\left|y^{p-1}t^{\delta-1}e^{p,\delta}_{\omega,\tau}(-y^\omega t^{-\tau})\right|<Cy^{p-\omega\theta-1}\cdot t^{\delta+\theta\tau-1},\ \ 0<\theta\le 1.$$

By virtue of Lemma 3 and $\gamma(x)\in C^2[0,l]$, $f(x,y)\in C^1(\bar\Omega)$, $f(0,0)=0$ from (25) it is not difficult to establish that

$$Q(x)\in C^1[0,1]\ \text{ and }\ Q(0)=0. \tag{28}$$

Thus, by virtue of (23), the problem M_1A is equivalently (in the sense of unambiguous solvability) reduced to a Volterra type integral equation of the second kind with a weak singularity (24). Therefore, by virtue of (28), there is a unique solution of equation (24) from the class $C^1[0,1]$ and it is representable as

$$\tau'(x) = Q(x) + \int_0^x R(x-t)\,Q(t)\,dt, \tag{29}$$

where $R(x)$ is the resolvent of the integral equation (24)

$$R(x) = \sum_{n=1}^{\infty}(-1)^n m_n(x)\,,\ m_1(x) = m(x),\ m_{n+1}(x) = \int_0^x m_1(x-t)\,m_n(t)\,dt.$$

From (29) taking into account $\tau(0) = 0$, we have

$$\tau(x) = \int_0^x R_1(x-t)\,Q(t)\,dt,\ where\ R_1(x) = 1 + \int_0^x R(t)\,dt. \tag{30}$$

Substituting (30) in (17) and (20), taking into account (25) after some transformations we have

$$u(x,y) = \iint_\Omega M_1(x,y,x_1,y_1)f(x_1,y_1)dxdy, \tag{31}$$

where

$$M_1(x,y,x_1,y_1) = \theta(x-x_1)\left[\theta(y)\,M_{01}(x,y,x_1,y_1) + \theta(-y)\,M_{11}(x,y,x_1,y_1)\right], \tag{32}$$

$$M_{01}(x,y,x_1,y_1) = \theta(y_1)\left[E(x-x_1,y,y_1) + \int_{x_1}^x dz \int_{x_1}^z E_{y_1}(x-z,y,0)\,R_1(z-t)\,E_y(t-x_1,0,y_1)\,dt\right] +$$

$$+\theta(-y_1)\int_{\eta_1}^x E_{y_1}(x-t,y,0)\,R_1(t-\eta_1)\,dt,$$

$$M_{11}(x,y,x_1,y_1) = \theta(y_1)\int_0^\xi R_1(\xi-t)\,E_y(t-x_1,0,y_1)\,dt +$$

$$+\frac{1}{2}\theta(-y_1)\left[\theta(\xi-\eta_1)\,R_1(\xi-\eta_1) + \theta(-y_1)\,\theta(\eta-\eta_1)\,\theta(\eta_1-\xi)\,\theta(\xi-\xi_1)\right],$$

where $\xi_1 = x_1 + y_1,\ \eta_1 = x_1 - y_1,\ \xi = x + y,\ \eta = x - y,\ \theta(y) = 1,\ y > 0$ and $\theta(y) = 0, y < 0$.

Taking into account explicit types of functions

$$M_{01}(x,y,x_1,y_1)\,,\ M_{11}(x,y,x_1,y_1)$$

it is not difficult to establish that in (32) all terms are bounded, with the exception of the first – $M_{01}(x,y,x_1,y_1)$, in which by virtue of Lemma 3, the summand may not be limited $E(x-x_1,y,y_1)$. Therefore, it is enough to show that

$$\theta(x-x_1)\,\theta(y_1)\,\theta(y)\,E(x-x_1,y,y_1) \in L_2(\Omega\times\Omega)\,.$$

By virtue of Lemma 3 from estimation (26) by direct calculation we have

$$\|\theta(x-x_1)\mathrm{E}(x-x_1,\ y,\ y_1)\|^2_{L_2(\Omega\times\Omega)} \le C\{(2+\theta)\beta\,[1+(2+\theta)\beta]\}^{-1}.$$

Therefore, $M_1(x,y,x_1,y_1) \in L_2(\Omega\times\Omega)$.

Lemma 4. If $f(x,y) \in L_2(\Omega)$, then $Q(x) \in L_2[0,1]$ and $\|Q(x)\|^2_{L_2(0,1)} \le C\|f(x,y)\|^2_{L_2(\Omega)}$.

Proof of Lemma 4 taking into account (25), (27) It is carried out by direct calculation using the well-known Cauchy-Bunyakovsky inequality. From (29) we have

$$\left\|\tau'(x)\right\|_{L_2(0,1)} \le C\|Q(x)\|_{L_2(0,1)} \le C\|f(x,y)\|_{L_2(\Omega)}. \tag{33}$$

From (17) by virtue (33) by direct calculation it is not difficult to establish that

$$\|u(x,y)\|_{W_2^1(\Omega_1)} \le C\|f(x,y)\|_{L_2(\Omega)}, \tag{34}$$

where $W_2^1(\Omega)$ is S.L. Sobolev's space. From (18) and (34) we have

$$\begin{gathered} D_{0x}^{\alpha-1}\|u(x,y)\|^2_{L_2(0,1)} + \int\limits_0^x \|u_y(t,y)\|^2_{L_2(0,1)}dt + \|u(x,y\|^2_{W_2^1(\Omega_2)} \le \\ \le C\left[\int\limits_0^x \|f(t,y)\|^2_{L_2(0,1)} + \int\limits_0^x d\xi \int\limits_\xi^1 |f(\xi,x)|^2 dt + \|f(x,y)\|^2_{L_2(\Omega)}\right]. \end{gathered} \tag{35}$$

Thus, summarizing the above statements, the following theorem is proved.

Theorem 1. For any function $f(x,y) \in C^1(\bar{\Omega})$, $f(A) = 0$ there is a unique regular solution to the problem M_1A (1), (3)-(5) and it is represented in the form (31) and satisfies the inequality (35). From (35) or (19) and (34) it is followed the the validity of the estimate

$$\|u(x,y)\|_{L_2(\Omega_0)} + \|u_y(x,y)\|_{L_2(\Omega_0)} + \|u(x,y)\|_{W_2^1(\Omega_1)} \le C\|f(x,y)\|_{L_2(\Omega)}. \tag{36}$$

Definition 2. The function $u(x,y) \in L_2(\Omega)$ is called a strong solution to problem M_1A, if there is a sequence of functions $\{u_n(x,y)\}$, $u_n(x,y) \in V$, satisfying conditions (3)–(5), such that

$$\|u_n(x,y) - u(x,y)\|_{L_2(\Omega)} \to 0, \ \ \|Lu_n(x,y) - f(x,y)\|_{L_2(\Omega)} \to 0 \ \text{ for } \ n \to \infty.$$

Theorem 2. For any function $f(x,y) \in L_2(\Omega)$ there is a unique strong solution $u(x,y)$ to the problem M_1A. This solution can be represented as (31) and satisfies the estimate (36).

The proof of Theorem 2 in the presence of a representation of the solution (31) and the estimate (36) is proved in the same way as in [22–24].

By B_1 we denote a closure in space $L_2(\Omega)$, of fractional differential operator given at the set of functions V, satisfying conditions (3)–(5), with expression (2).

According to the definition of a strong solution to the problem M_1A, $u(x,y)$ is a strong solution to the problem M_1A only and only then, when $u(x,y) \in D(B_1)$, where $D(B_1)$ is a definition domain of operator B_1.

From theorem 2 it follows that operator B_1 is closed and its definition domain is dense in $L_2(\Omega)$; there exists an inverse operator B_1^{-1}, it is defined in all $L_2(\Omega)$ and quite continuous.

In this regard, a natural question arises: is there an eigenvalue of the operator B_1^{-1}, and therefore to the problem? The main result is the theorem on the absence of eigenvalues of the operator B_1^{-1}.

Theorem 3. Integral operator

$$B_1^{-1}f(x,y) = \iint\limits_\Omega M_1(x,y,x_1,y_1)f(x_1,y_1)dx_1dy_1, \tag{37}$$

where $M_1(x,y,x_1,y_1) \in L_2(\Omega \times \Omega)$ is Volterra in $L_2(\Omega)$.

Proof. To prove Theorem 3, we need to show that the operator B_1^{-1} defined by formula (37) is completely continuous and quasinilpotent. Since the complete continuity of this operator follows from the fact that $M_1(x,y,x_1,y_1) \in L_2(\Omega \times \Omega)$, show that B_1^{-1} is quasinilpotent, i.e.

$$\lim_{n\to\infty} \left\|B_1^{-1}\right\|^{\frac{1}{n}}_{L_2(\Omega)\to L_2(\Omega)} = 0, \tag{38}$$

where $B_1^{-n} = B_1^{-1}\left[B_1^{-(n-1)}\right],\ n = 1.2...$

From (37) by direct calculation, taking into account (32) is not difficult to obtain that

$$B_1^{-n} f(x,y) = \iint\limits_{\Omega} M_n(x,y,x_1,y_1) f(x_1,y_1)dx_1dy_1, \tag{39}$$

where

$$M_n(x,y,x_1,y_1) = \iint\limits_{\Omega} M_1(x,y,x_2,y_2)M_{(n-1)}(x_2,y_2,x_1,y_1)dx_2dx_1,\ \ n = 2.3...$$

Lemma 5. For iterated kernels $M_n(x,y,x_1,y_1)$ there is an assessment

$$|M_n(x,y,x_1,y_1)| \leq \left(\frac{3}{2}\right)^{n-1} N^n \frac{\Gamma^n(\gamma)}{\Gamma(n\gamma)}(x-x_1)^{n\gamma-1}, \tag{40}$$

where $\gamma = (2+\theta)\beta,\ \ N = Cd$, C is coefficient from the assessment (26),

$$d = \max_{\substack{(x,y)\in\Omega \\ (x_1,y_1)\in\Omega}} \left|(x-x_1)^{1-\gamma}M_1(x,y,x_1,y_1)\right|,\ if\ \gamma < 1.$$

$$d = \max_{\substack{(x,y)\in\Omega \\ (x_1,y_1)\in\Omega}} |M_1(x,y,x_1,y_1)|,\ if\ \gamma \geq 1.$$

The proof of Lemma 5 we carry out by induction method over n.

For $n = 1$ the inequality

$$|M_1(x,y,x_1,y_1)| \leq N(x-x_1)^{\gamma-1}$$

follows from representation (32) taking into account estimate (26).

Let be (40) valid for $n = k-1$. Let 's prove the validity of this formula for $n = k$.

Using inequality (40) for $n = 1$ and $n = k-1$ we have

$$|M_k(x,y,x_1,y_1)| = \left|\iint\limits_{\Omega} M_1(x,y,x_2,y_2)\cdot M_{(k-1)}(x_2,y_2,x_1y_1)dx_2dy_2\right| \leqslant$$

$$\leqslant \iint\limits_{\Omega} |M_1(x,y,x_2,y_2)| \cdot \left|M_{(k-1)}(x_2,y_2,x_1,y_1)\right| dx_2dy_2 \leqslant$$

$$\leqslant \iint\limits_{\Omega} \theta(x-x_2)N(x-x_2)^{\gamma-1}\theta(x_2-x_1)\left(\frac{3}{2}\right)^{k-2}N^{k-1}\frac{\Gamma^{k-1}(\gamma)}{\Gamma\left[(k-1)\gamma\right]}(x_2-x_1)^{(k-1)\gamma-1}dx_2dy_2 \leqslant$$

$$\leq \left(\frac{3}{2}\right)^{k-1} N^k \frac{\Gamma^{k-1}(\gamma)}{\Gamma\left[(k-1)\gamma\right]} \int\limits_{x_1}^{x} (x-x_2)^{\gamma-1}(x_2-x_1)^{(k-1)\gamma-1}dx_2 =$$

$$= \left(\frac{3}{2}\right)^{k-1} N^k \frac{\Gamma^{k-1}(\gamma)}{\Gamma\left[(k-1)\gamma\right]} (x-x_1)^{k\gamma-1} \int\limits_0^1 \sigma^{\gamma-1}(1-\sigma)^{(k-1)\gamma-1} d\sigma = \left(\frac{3}{2}\right)^{k-1} N^k \frac{\Gamma^k(\gamma)}{\Gamma(k\gamma)} (x-x_1)^{k\gamma-1},$$

which proves Lemma 5.

Using the consistently known Schwarz inequality and Lemma 5 from the representation (39) we have

$$\left\|B_1^{-n} f(x,y)\right\|^2_{L_2(\Omega)} = \iint\limits_{\Omega} \left|B_1^{-n} f(x,y)\right|^2 dxdy = \iint\limits_{\Omega} \left[\iint\limits_{\Omega} M_n(x,y,x_1,y_1) f(x_1,y_1) dx_1 dy_1\right]^2 dxdy \leqslant$$

$$\leqslant \iint\limits_{\Omega} \left[\left(\iint\limits_{\Omega} |M_n(x,y,x_1,y_1)|^2 dx_1 dy_1\right)\left(\iint\limits_{\Omega} |f(x_1,y_1)|^2 dx_1 dy_1\right)\right] dxdy \leqslant$$

$$\leq \left(\frac{3}{2}N\right)^{2n} \frac{\Gamma^{2n}(\gamma)}{\left[(2n\gamma-1)\right](2n\gamma)\Gamma^2(n\gamma)} \left\|f(x,y)\right\|^2_{L_2(\Omega)}.$$

From here we get

$$\left\|B_1^{-n}\right\|_{L_2(\Omega)\to L_2(\Omega)} \leq \left(\frac{3N}{2}\right)^n \left(4-\frac{2}{n\gamma}\right)^{-\frac{1}{2}} \frac{\Gamma^n(\gamma)}{\Gamma(1+n\gamma)}.$$

From the latter it is not difficult to establish equality (38). *Theorem 3 is proved.*

Corollary 1. Problem M_1A is Volterra nature problem.

Corollary 2. For any complex number λ the equation $B_1 u(x,y) - \lambda u(x,y) = f(x,y)$ is unambiguously solvable at all $f(x,y) \in L_2(\Omega)$.

Let now Ω_1 is a domain bounded by segments AB and characteristics $AC: x+y=0$, $BC: x-y=1$ of equation (1) and smooth curve $AD: y=-\gamma(x)$, $0<x<l$, where $0,5<l\leq 1$; $\gamma(0)=0$, $l+\gamma(l)=1$, if $l<1$ and $\gamma(l)=0$, if $l=1$ is located inside the characteristic triangle $0<x+y\leq x-y<1$.

A generalization of the problem in the domain Ω is the following non-local problem for equation (1), where in the hyperbolic part of the mixed domain, the non-local condition pointwise connects the values of the tangent derivative of the desired solution on the characteristic AC with the derivatives in the direction of the characteristic of the desired function on an arbitrary curve AD lying inside the characteristic triangle, with the ends at the origin and on the characteristic BC (at a point B).

Problem M_1B. Find a solution of equation (1) satisfying the conditions (3), (4) and

$$\left[u_x-u_y\right]\left[\theta_0(t)\right] + \mu(t)\left[u_x-u_y\right]\left[\theta^*(t)\right] = 0,\ 0<t<1, \tag{41}$$

where $\theta_0(t), (\theta^*(t))$ is an affix of the intersection point of the characteristic AC (curve AD) with the characteristic coming out of the point $(t,0)$, 0<t<1, $\mu(t)$ is a given function.

In the case when $\alpha=1$, the problem M_1B coincides with nonlocal problem for mixed parabolic and hyperbolic equation with non-characteristic line of type change. In this case, regular and strong solvability issues and Volterra property of problem M_1B are investigated in [21–24]. Note that the problem M_1B, when $\mu(x)=0$ coincides with the problem of Tricomi for diffusion and wave equation, and in the case when $\mu(t)=\infty$ coincides with the problem M_1A.

Similarly, as in the case of the problem M_1A, the concept of a regular and strong solution to the problem is introduced. Applying the methodology of proofs of theorems 1–3, the following theorem is proved.

Theorem 4. Let be $\mu(t) \in C^1[0,1]$ and $\mu(x) \neq -1$, $0 \leq x \leq 1$. Then :

a) for any function $f(x,y) \in C^1(\bar{\Omega})$, $f(A)=0$ there is a unique regular solution to the problem M_1B (1), (3), (4), (41) and it is represented in the form (31) and satisfies the inequality (35);

b) for any function $f(x,y) \in L_2(\Omega)$ there exists a unique strong solution $u(x,y)$ to problem M_1B. This solution can be presented in the form (31) and satisfies estimate (36);

c) the problem M_1B is Volterra nature problem.

3 Solvability and existence of eigenvalues of local and nonlocal problems for the diffusion-wave equation

In domain Ω of considered section 2 we investigate the following problem: **Problem** M_2A. Find a solution of equation (1) satisfying the conditions

$$u\left.\right|_{AA_0 \cup A_0B_0} = 0\,, \tag{42}$$

$$u_x + u_y \left.\right|_{AD \cup BD} = 0. \tag{43}$$

Definition 3. The regular solution to the problem M_2A in the domain Ω will be called the function $u(x,y) \in W$, where $W = \{(x,y):\ u(x,y) \in C(\bar{\Omega}) \cap C^{1.1}(\Omega \cup AD \cup BD),\ D_{0x}^{\alpha} u(x,y),\ u_{yy}(x,y) \in C(\Omega_0),\ u(x,y) \in C^{2.2}(\Omega_1)\}$, satisfying equation (1) in $\Omega_0 \cup \Omega_1$ and conditions (42)–(43).

Definition 4. The function $u(x,y) \in L_2(\Omega)$ is called a strong solution to the problem M_2A, if there exists $\{u_n(x,y)\}$, $u_n(x,y) \in W$, satisfying conditions (42)–(43), such that $\|u_n(x,y) - u(x,y)\|_{L_2(\Omega)} \to 0$, $\|Lu_n(x,y) - f(x,y)\|_0 \to 0$, for $n \to \infty$.

Similarly, as in section 2, the regular solvability of the problem M_2A.

Theorem 5. For any function $f \in L_2(\Omega)$ there is a unique strong solution $u(x,y)$ to problem M_2A. This solution can be presented in the form

$$u(x,y) = \iint\limits_{\Omega} K(x,y;x_1,y_1) f(x_1,y_1)\, dx_1 dy_1, \tag{44}$$

where $K(x,y;x_1,y_1) \in L_2(\Omega \times \Omega)$, and satisfies estimate (36).

Similarly as in the problem M_1A, the solution to problem M_2A in domain Ω_1 we seek in the form (15). Based on (43) from (15) we find

$$v(\xi) = -\tau'(\xi) - 2\int\limits_{\xi}^{\varphi(\xi)} f_1(\xi,\eta_1)\, d\eta_1, \quad 0 \le \xi \le 1, \tag{45}$$

where $\eta = \varphi(\xi)$, $0 \le \xi \le \xi_0$, $\varphi(\xi_0) = 1$ is an equation of the curve AD in characteristic variables ξ, η and $\varphi(\xi) \equiv 1$, $\xi_0 \le \xi \le 1$ in the case when $D \ne B$ and $\eta = \varphi(\xi)$, $0 \le \xi \le 1$ when $D = B$.

Substituting the resulting expression $v(\xi)$ into (15), we obtain

$$u(x,y) = \tau(\eta) + \int\limits_{\xi}^{\eta} d\xi_1 \int\limits_{\eta}^{\varphi(\xi_1)} f_1(\xi_1,\eta_1)\, d\eta_1. \tag{46}$$

The formula (45) gives an integro-differential relation between $\tau(x)$ and $\nu(x)$, brought to the segment AB from hyperbolic part Ω_1.

Taking into account (22) and (45), it is not difficult to establish that the problem M_2A is equivalent to the following Volterra integral equation of the second kind

$$\tau'(x) - \int_0^x m(x-t)\,\tau'(t)\,dt = \Phi(x), \quad 0 \le x \le 1, \tag{47}$$

where $\Phi(x) = -2\int\limits_x^{\varphi(x)} f_1(x,\eta_1)\,d\eta_1 - \int\limits_0^x dx_1 \int\limits_0^1 \mathrm{E}_y(x-x_1,0,y_1) f(x_1,y_1) dy_1$.

Since $m(x-t)$ is a kernel with a weak feature, then there is a unique strong solution to equation (47), and it is representable as

$$\tau'(x) = \Phi(x) + \int_0^x \Gamma(x-t)\,\Phi(t)\,dt, \tag{48}$$

where $\Gamma(x)$ is a resolvent of equation (48):

$$\Gamma(x) = \sum_{j=1}^{\infty} m_j(x),\; m_1(x) = m(x)\,,\; m_{j+1}(x) = \int_0^x m_1(x-t)\,m_j(t)\,dt.$$

From (48), taking into account $\tau(0) = 0$, we have

$$\tau(x) = -2\int_0^x d\xi_1 \int_{\xi_1}^{\varphi(\xi_1)} \Gamma_1(x-\xi_1)\,f(\xi_1,\eta_1)\,d\eta_1 - \int_0^x dx_1 \int_0^1 E_1(x-x_1,y_1)\,f(x_1,y_1)\,dy_1, \tag{49}$$

where

$$\Gamma_1(x) = 1 + \int_0^x \Gamma(t)\,dt,\; E_1(x,y_1) = \int_0^x E_y(t,0,y_1)\Gamma_1(x-t)\,dt. \tag{50}$$

Substituting (49) into (20) and (46), we obtain

$$u(x,y) = \int_0^x dx_1 \int_0^1 E_2(x-x_1,y,y_1)\,f(x_1,y_1) dy_1 - 2\int_0^x d\xi_1 \int_{\xi_1}^{\varphi(\xi_1)} E_1(x-\xi_1,y)\,f_1(\xi_1,\eta_1)\,d\eta_1,\; y > 0, \tag{51}$$

$$\begin{aligned} u(x,y) = &\int_\xi^\eta d\xi_1 \int_\eta^{\varphi(\xi_1)} f_1(\xi_1,\eta_1)\,d\eta_1 - 2\int_0^\eta d\xi_1 \int_{\xi_1}^{\varphi(\xi_1)} \Gamma_1(\eta-\xi_1)\,f_1(\xi_1,\eta_1)\,d\eta_1 - \\ &- \int_0^\eta dx_1 \int_0^1 E_1(\eta-x_1,y_1)\,f(x_1,y_1)\,dy_1,\; y < 0, \end{aligned} \tag{52}$$

where

$$E_2(x,y,y_1) = E(x,y,y_1) - \int_0^x E_y(x,0,y_1) E_1(x-t,y_1)\,dt.$$

From (51) and (52) we get (44), where the kernel has the form

$$
\begin{gathered}
K\left(x, y ; x_{1}, y_{1}\right)=\theta\left(y\right)\left\{\theta\left(y_{1}\right) \theta\left(x-x_{1}\right)\ E_{2}\left(x-x_{1}, y, y_{1}\right)-\right. \\
-\theta\left(-y_{1}\right) \theta\left(x-\xi_{1}\right) E_{1}\left(x-\xi_{1}, y\right)\}+\theta\left(-y\right)\left\{-\theta\left(y_{1}\right) \theta\left(\eta-x_{1}\right) E_{1}\left(\eta-x_{1}, y_{1}\right)+\right. \\
\left.+\theta\left(-y_{1}\right)\left[\frac{1}{2} \theta\left(\xi_{1}-\xi\right) \theta\left(\eta-\xi_{1}\right) \theta\left(\eta_{1}-\eta\right)-\theta\left(\eta-\xi_{1}\right) \Gamma_{1}(\eta-\xi_{1})\right]\right\} .
\end{gathered} \tag{53}
$$

From (44), (51), (52) and properties of the solution to the first initial boundary value problem for the diffusion equation [8], as in Theorem 2 it follows all statements of Theorem 5.

By B_2 we denote a closure in $L_2(\Omega)$ of the operator given on a set of functions from W, satisfying conditions (42),(43), with expression (2).

Theorem 6. Let be $\gamma(x) \not\equiv 0$. Then there exists $\lambda \in C$ such that equation $B_2 u(x,y) = \lambda u(x,y)$ has non-trivial solution $u(x,y) \in W$.

Proof. From theorem 5 it is followed, that B_2 is inversible and B_2^{-1} is an operator of Hilbert–Schmidt, defined by the formula (44). Then $B_2^{-2} \equiv \left(B_2^{-1}\right)^2$ kernel operator in $L_2(\Omega)$.

Therefore, for the operator B_2^{-2} we apply the result of V.B. Lidskii [25] on the coincidence of matrix and spectral traces. It is also known that for the kernel operator, represented as the product of two Hilbert-Schmidt operators, the Gaal formula [26] trace calculation takes place. Using formula of Gaal, we calculate the matrix trace B_2^{-2}.

$$
SpB_2^{-2} = \iint\limits_{\Omega} dxdy \iint\limits_{\Omega} K\left(x, y ; x_{1}, y_{1}\right) K\left(x_{1}, y_{1} ; x, y\right) dx_{1} dy_{1}. \tag{54}
$$

Taking into account the representation (53) from (54), after simple transformations, we obtain

$$
\begin{gathered}
SpB_2^{-2} = \int_0^1 dx \int_0^1 dy \int_0^x E_1\left(\xi_2, y\right) d\xi_2 \theta\left[\varphi\left(x-\xi_2\right)-x\right] \int_0^{\varphi(x-\xi_2)-x} E_1\left(\eta_2, y\right) d\eta_2 + \\
+\frac{1}{4} \int_0^1 d\xi \int_{\xi}^{\varphi(\xi)} d\eta \int_{\xi}^{\eta} d\xi_1 \int_{\eta}^{\varphi(\xi_1)} \theta\left(\eta_1-\xi\right) \Gamma_1\left(\eta_1-\xi\right)\left[\Gamma_1\left(\eta-\xi_1\right)-\theta\left(\eta_1-\eta\right)\right] d\eta_1 = A + B.
\end{gathered}
$$

We will show that $A + B > 0$. Indeed, taking into account (50) and $\varphi(t) \not\equiv t$ will take place $A \geq 0$, if

$$
E_y\left(t, 0, y_1\right) > 0. \tag{55}
$$

We represent the function $E_y(t, 0, y_1)$ in the form

$$
E_y\left(t, 0, y_1\right) = \frac{1}{2t} \sum_{n=0}^{+\infty}\left[e_{1,\beta}^{1,0}\left(-\frac{\left|2n+y_1\right|}{t}\right)+e_{1,\beta}^{1,0}\left(-\frac{\left|2(n+1)-y_1\right|}{t}\right)\right].
$$

Due to the properties of the Wright function [8; 46] $e_{1,\beta}^{1,0}(-z) > 0,\ z > 0$, therefore, from the latter we get the justice of inequality (55). Also, from (50) it easily follows that

$$
\Gamma_1\left(\eta-\xi_1\right)-\theta\left(\eta_1-\eta\right) \geq 0,
$$

therefore $B \geq 0$.

Thus, $A + B > 0$, as an integral in the positive direction of a non-negative and identically non - zero function. From here we get that $SpB_2^{-2} > 0$. Further, applying the results of [25], we have

$$\sum_{k=1}^{\infty} \lambda_k \left(B_2^{-2}\right) = \sum_{k=1}^{\infty} \lambda_k^2 \left(B_2^{-1}\right) > 0,$$

where $\lambda_k \left(B_2^{-2}\right)$ are eigenvalues of operator B_2^{-2}. It means that $\sum\limits_{k=1}^{\infty} \frac{1}{\lambda_k^2} > 0$, where λ_k are eigenvalues of the problem (1), (42) and (43). This implies the existence of the eigenvalues of the problem M_2A for the diffusion-wave equation of fractional order. *Theorem 6 is proved.*

In conclusion, we note that the most interesting is the fact that in problems M_1A and M_2A, in the case when point D coincides with point B, the Volterra property or existence of the problems eigenvalues depend on the derivative directions of the desired function given in the non-characteristic curve of the hyperbolic part of the boundary.

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Бөлшек ретті диффузиялық-толқындық теңдеу үшін локальді және локальді емес есептердің спектрлік қасиеттері

Мақалада бөлшек ретті диффузиялық-толқындық теңдеу үшін локальді және локальді емес есептердің шешімділік мәселелері мен спектрлік қасиеттері зерттелген. Сипаттауыш және сипаттауыш емес шекаралары бар облыстарда қойылған есептердің регуляр және күшті шешімділігі дәлелденді. Есептердің бірегей шешімділігі дәлелденіп, меншікті мәндердің бар екендігі немесе Вольтерра типіндегі есеп екендігі туралы теоремалар дәлелденген.

Кілт сөздер: диффузиялық-толқындық теңдеу, бөлшек ретті теңдеулер, шекаралық есептер, күшті шешім, Вольтерра қасиеті, меншікті мән.

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Спектральные свойства локальных и нелокальных задач для диффузионно-волнового уравнения дробного порядка

В статье исследованы вопросы разрешимости и спектральные свойства локальных и нелокальных задач для диффузионно-волнового уравнения дробного порядка. Доказаны регулярная и сильная разрешимости поставленных задач в областях, как с характеристической, так и с нехарактеристической границей области. Установлена однозначная разрешимость задач, и доказаны теоремы о существовании собственных значений либо вольтерровости рассматриваемых задач.

Ключевые слова: диффузионно-волновое уравнение, уравнения дробного порядка, краевые задачи, сильное решение, вольтерровость, собственное значение.

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Controllability and optimal speed-in-action of linear systems with boundary conditions

The paper proposes a method for solving the problem of optimal performance for linear systems of ordinary differential equations in the presence of phase and integral restrictions, when the initial and final states of the system are elements of given convex closed sets, taking into account the control value restriction. The presented work refers to the mathematical theory of optimal processes from L.S. Pontryagin and his students and the theory of controllability of dynamic systems from R.E. Kalman. We study the problem of optimal speed for linear systems with boundary conditions from given sets close to the presence of phase and integral constraints, as well as constraints on the control value. A theory of the boundary value problem has been created and a method for solving it based on the study of solvability and the construction of a general solution to the Fredholm integral equation of the first kind has been developed. The main results are the distribution of all controls' sets, each subject of which transfers the trajectory of the system from any initial state to any final state; reducing the initial boundary point to a special initial optimal control problem; constructing a system of algorithms for the gamma-algorithm study on the derivation of problems and rational execution with restrictions on the solution of the optimal speed' problem with restrictions.

Keywords: optimal performance, integrity constraints, functional gradient, integral equation.

Introduction

Methods are proposed for constructing program and positional controls for processes described by linear ordinary differential equations in the presence of boundary conditions, as well as phase and integral constraints, taking into account constraints on controls. Two problems were solved: the problem of a control existence and the problem of constructing a set of all controls that transfers the trajectory of the system from any initial state to a given final state [1–2]. The proposed methods for constructing programs and positional controls are based on the Fredholm integral equation of the first kind. A necessary and sufficient condition for the existence of a solution to a linear integral equation is obtained. A general solution is found for a class of Fredholm integral equations of the first kind [3–5]. It is shown that the boundary value problems of linear ordinary differential equations are reduced to the original optimal control problems with a quadratic functional. Algorithms for constructing minimizing sequences and estimating their convergence are given [6]. Algorithms for solving the optimal performance problem based on solving the controllability problem are presented [7–8]. One of the complex and unsolved problems of control theory is the existence of a solution to the boundary value problem of optimal control in the presence of phase and integral constraints. To solve the problem of the existence of a solution, it is necessary to create a general theory of controllability of dynamical systems. This work is devoted to solving problems of controllability of complex dynamic systems with boundary conditions and constraints [9]. It should be noted that in these works special cases of the general problem of controllability and speed of dynamic systems without phase and integral constraints were studied [10–12]. Actual and unsolved problems of controllability and optimal performance are: obtaining necessary and sufficient conditions for the solvability of general problems of controllability and performance; development of constructive methods for constructing solutions to general problems of controllability and optimality of ordinary differential equations.

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1 *Statement of the problem*

Consider a controlled process described by a linear ordinary differential equation with an integral and a control of the following form:

$$\dot{x} = A(t)x + B(t)u(t) + C(t)\int_a^b K(t,\tau)v(\tau)d\tau + \mu(t), \quad t \in I = [t_0, t_1], \tag{1}$$

$$\tau \in I_2 = [a, b]$$

with boundary conditions

$$(x(t_0) = x_0) \in S_0, \ (x(t_1) = x_1) \in S_1, \ S_0 \subset R^n, \ S_1 \subset R^n \tag{2}$$

as well as restrictions on control values

$$u(t) \in U(t) = \{u(\cdot) \in L_2(I_1, R^m) \mid u(t) \in U_1(t) \subset L_2(I_1, R^m), \ a.e., \ t \in I_1\}, \tag{3}$$

$$v(\tau) \in V(\tau) = \{v(\cdot) \in L_2(I_2, R^{n_1}) \mid v(\tau) \in V_1(\tau) \subset L_2(I_2, R^{n_1}), \ a.e., \tau \in I_2\}. \tag{4}$$

Here $A(t)$, $B(t)$, $C(t)$, $t \in I_1$ are matrices of orders $n\times n$, $n\times m$, $n\times m_1$ respectively, with piecewise continuous elements; $K(t,\tau)$ is a known matrix of order $m_1 \times n_1$ with elements from L_2, $\mu(t) \in L_2(I_2, R^n)$ of a given function $S_0 \subset R^n$, $S_1 \subset R^n$ of given convex closed sets, which defines restrictions on the initial and final state of the phase variables $U_1(t) \subset L_2(I_1, R^m)$, $V_1(\tau) \subset L_2(I_2, R^{n_1})$ of given convex closed sets. In particular, the sets

$$S_0 = \{x_0 \in R^n \mid |x_0 - \overline{x_1}| \leq r\}, \quad S_0 = \{x_0 \in R^n \mid c_i \leq x_{0i} \leq d_i, \, i = \overline{1,n}\}$$

$$S_1 = \{x_1 \in R^n \mid |x_1 - \overline{x_1}| \leq R\}, \quad S_1 = \{x_1 \in R^n \mid \overline{c_i} \leq x_{1i} \leq \overline{d_i}, \, i = \overline{1,n}\},$$

where $\overline{x_0} \in R^n$, $\overline{x_1} \in R^n$ are fixed vectors, r, R are given numbers, $x_0 = (x_{01}, \dots x_{0n}) \in R^n$, $x_1 = (x_{11}, \dots x_{1n}) \in R^n$, c_i, d_i, $\overline{c_i}$, $\overline{d_i}$, $i = \overline{1,n}$ are fixed numbers.

There are sets

$U_1 = \{u(\cdot) \in L_2(I_1, R^m) \mid \|u - \overline{u}\| \leq r, \text{ a.e., } t \in I_1\}$,

$U_1 = \{u(\cdot) \in L_2(I_1, R^m) \mid \alpha_i(t) \leq u_i(t) \leq \beta_i(t), \text{ a.e., } i = \overline{1,n}, \, t \in I_1\}$,

$V_1(\tau) = \{v(\cdot) \in L_2(I_1, R^{n_1}) \mid \|v - \overline{v}\| \leq R, \text{ a.e., } \tau \in I_1\}$,

$V_1(\tau) = \{v(\cdot) \in L_2(I_1, R^{n_1}) \mid \overline{\alpha_i}(\tau) \leq v_i(\tau) \leq \overline{\beta_i}(\tau), \text{ a.e. } i = \overline{1,n}, \, \tau \in I_2\}$,

where $r > 0$, $R > 0$ are given numbers, $u(t) = (u_1(t), \dots, u_m(t))$, $v(\tau) = (v_1(\tau), \dots, v_{n_1}(\tau))$, $\alpha_i(t)$, $\beta_i(t)$, $t \in I_1$, $\alpha_i(\tau)$, $\beta_i(\tau)$, $\tau \in I_2$ are given continuous functions.

There are the possible cases: 1) when the moments are fixed; 2) t_0 is fixed, to find the smallest value t_1, $t_1 > 0$ when boundary value problem (1)–(4) has a solution. Boundary value problem (1)–(4) in the second case is called the optimal performance problem.

Definition 1. Let the moments be fixed. The solution of the differential equation with subintegral control (1) is called controllable at the time of control $u_*(t) \in U(t)$, $v_*(\tau) = V(\tau)$ which transfers the trajectory of the equation (1) from point $x_{0_*}(t) = x_*(t_0) \in S_0$ at time t_0 points to $x_{1_*}(t) = x_*(t_1) \in S_1$ time t_1.

Definition 2. A quadruple $(u_*(t), \, v_*(\tau), \, x_{0_*}, \, x_{1_*}) \in U(t) \times V(\tau) \times S_0 \times S_1$ is called correct if the function $x_*(t) = x_*(t; \, t_0, \, x_{0_*}, u_*, \, v_*)$, $t \in I_1$ that is a solution of differential equation (1) satisfies condition (2). The set of all admissible quadruples is denoted by Σ.

2 Necessary and sufficient conditions for controllability

To solve problems (1)–(4), we consider the controllability problem of a linear system

$$\dot{y} = A(t)y + B(t)w_1(t) + C(t)w_2(t) + \mu(t),\ t \in I_1, \tag{5}$$

$$y(t_0) = x_0 = x(t_0) \in S_0,\ y(t_1) = x_1 = x(t_1) \in S_1, \tag{6}$$

$$w_1(\cdot) \in L_2(I_2, R^m),\ w_2(\cdot) \in L_2(I_2, R^{m_1}). \tag{7}$$

Theorem 1. The integral equation

$$Kw = \int_{t_0}^{t_1} \mathrm{K}(t_0,t)w(t)dt = \beta, \quad t \in I = [t_0, t_1], \tag{8}$$

have a solution for any fixed $\beta \in R^{n_1}$ if and only if the matrix

$$C(t_0,t_1) = \int_{t_0}^{t_1} K(t_0,t)K^*(t_0,t)dt \tag{9}$$

of order $n_1 \times n_1$ is positive definite, where $(*)$ is the transposition sign.

Proof. Sufficiency. Let the matrix $C(t_0,t_1) > 0$. Let us show that integral equation (8) have a solution for any $\beta \in R^n$. Let's choose

$$w(t) = K^*(t_0,\ t)C^{-1}(t_0,t_1)\beta,\ \ t \in I = [t_0,\ t_1].$$

Then

$$Kw = \int_{t_0}^{t_1} K(t_0,t)K^*(t_0,\ t)dt\ \ C^{-1}(t_0,t_1)\beta = \beta.$$

Thus, for $C(t_0,t_1) > 0$, integral equation (8) have at least one solution

$$w(t) = K^*(t_0,\ t)C^{-1}(t_0,t_1)\beta,\ t \in I,\ \beta \in R^n.$$

The sufficiency is proved.

Necessity. Let integral equation (8) have a solution for any fixed $\beta \in R^n$. Let's prove that the matrix $C(t_0,t_1) > 0$. Since $C(t_0,t_1) \geq 0$, then to prove $C(t_0,t_1) > 0$ it is necessary to show that the matrix $C(t_0,t_1)$ is nonsingular.

Suppose, by contradiction, that the matrix $C(t_0,t_1)$ is singular. Then there is a vector $c \in R^n$, $c \neq 0$ such that $c^*C(t_0,t_1)c = 0$. Let's define the function $\bar{v}(t) = K^*(t_0,t)c$, $t \in I$, $\bar{v}(\cdot) \in L_2(I, R^m)$. Note that

$$\int_{t_0}^{t_1} \bar{v}^*(t)\bar{v}(t)dt = c^* \int_{t_0}^{t_1} K^*(t_0,\ t)K(t_0,\ t)dt \cdot c = c^*\, C(t_0,t_1)c = 0.$$

Therefore, the function $\bar{v}(t) = 0,\ \ t \in I$. Since integral equation (8) have a solution for any $\beta \in R^n$, then, in particular, there exists function (7) such that $\bar{w}(\cdot) \in L_2(I, R^m)$ and $(\beta = c)$

$$\int_{t_0}^{t_1} K(t_0,t)\bar{w}(t)dt = c.$$

Thus the identity

$$0 = \int_{t_0}^{t_1} v^*(t)\bar{w}(t)dt = c^* \int_{t_0}^{t_1} K(t_0,\ t)w(t)dt = c^*c$$

is true. This contradicts the condition that $c \neq 0$. The necessity is proved. The theorem is proved.

Theorem 2. The existence of a control $w_*(\cdot) = w_{1*}(\cdot), w_{2*}(\cdot) \in L_2(I_2, R^m) \times L_2(I_2, R^{m_1})$ transferring the trajectory of equation (5) from the starting point $y(t_0) = x_0 \in S_0$ to the point $y(t_1) = x_1 \in S_1$ it is necessary and sufficient condition for the matrix

$$W(t_0, t_1) = \int_{t_0}^{t_1} \Phi(t_0, t)B_1(t)B_1^*(t)\Phi^*(t_0, t)dt \tag{10}$$

the order $n \times n$ be positive defined, where $B_1(t) = (B(t), C(t))$. Linear control system (5)–(7) differs from (1)–(3) in that the point, $x_0 \in S_0$, $x_1 \in S_1$. Let the matrix $W(t_0, t_1)$ determined by formula (8) be positive defined. Then a control $w_*(\cdot) = (w_{1*}(\cdot),\ w_{2*}(\cdot)) \in L_2(I_2, R^{m+m_1})$ transfers the trajectory of equation (5) from point $y_*(t_0) = x_{0*} \in S_0$ to point $y_*(t_1) = x_{1*} \in S_1$, if and only if

$$w_*(t) \in W_1 = \{w_*(\cdot) \in L_2(I_2, R^{m+m_1}) | w_*(t) = p_*(t) + \lambda_1(t,\ x_{0*},\ x_{1*}) + N_1(t)z(t_1, p_*),$$

$$x_{0*} \in S_0,\ x_{1*} \in S_1,\ \forall p_*(\cdot) = (p_{1*}(\cdot),\ p_{2*}(\cdot)) \in L_2(I_2, R^{m+m_1})\}, \tag{11}$$

where

$$\lambda_1(t,\ x_{0*},\ x_{1*}) = B_1^*(t)\Phi^*(t_0, t)W^{-1}(t_0, t_1)a, \tag{12}$$

$$a = \Phi(t_0, t_1)x_{1*} - x_{0*} - \int_{t_0}^{t_1} \Phi(t_0, t)\mu(t)dt,\ \ N_1(t) = -B_1^*(t)\Phi^*(t_0, t)W^{-1}(t_0, t_1)\Phi(t_0, t_1),$$

$$p_*(\cdot) \in L_2(I_2, R^{m+m_1}) \tag{13}$$

and the function $z(t) = z(t, p_*)$, $t \in I_1$ is a solution of the differential equation

$$\dot{z}(t) = A(t)z + B_1(t)p_*(t),\ z(t_0) = 0,\ t \in I_1.$$

The solution of differential equation (5) corresponding to the controller, is determined by the formula

$$y_*(t) = z(t,\ p_*) + \lambda_2(t,\ x_{0*},\ x_{1*}) + N_2(t)z(t_1,\ p_*),\ t \in I_1,$$

where

$$\lambda_2(t,\ x_{0*},\ x_{1*}) = \Phi(t, t_0)W(t, t_1)W^{-1}(t_0, t_1)x_{0*} + \Phi(t, t_0,)W(t, t_1)\,W^{-1}(t_0, t_1)\Phi(t, t_0)x_{1*} +$$

$$+ \int_{t_0}^{t} \Phi(t_0, \tau)\mu(\tau)d\tau - \Phi(t, t_0)W(t_0, t)\,W^{-1}(t_0, t_1) \int_{t_0}^{t_1} \Phi(t_0, t)\mu(t)dt,\ \ t \in I_1,$$

$$x_{0*} \in S_0, x_{1*} \in S_1 N_2(t) = -\Phi(t, t_0)W(t_0, t)W^{-1}(t_0, t_1)\Phi(t_0, t_1),\ \ t \in I_1.$$

Proof. Indeed, from (8) for $K(t_0, t) = \Phi(t_0, t)B_1(t)$ we have (10). Then

$$C(t_0, t_1) = \int_{t_0}^{t_1} K(t_0, t)K^*(t_0, t)dt = \int_{t_0}^{t_1} \Phi(t_0, t)B_1(t)B_1^*(t)\Phi^*(t_0, t)dt = W(t_0, t_1)$$

for the existence of a solution to integral equation (10) it is necessary and sufficient that the matrix $W(t_0, t_1) > 0$, the control $w_1(t),\ t \in I$ is determined by the formula $w(t) = v(t) + K^*(t_0,\ t)C^{-1}(t_0, t_1)\beta - K^*(t_0, t)C^{-1}(t_0, t_1)\int\limits_{t_0}^{t_1} K(t_0, t)v(t)dt,\ \ t \in I$. Then (see (9))

$$w_1(t) = v(t) + K^*(t_0, t)\, W^{-1}(t_0, t_1)a - K^*(t_0, t)\, W^{-1}(t_0, t_1) \int\limits_{t_0}^{t_1} K(t_0, t)v(t)dt =$$

$$= v(t) + B_1^*(t)\Phi^*(t_0, t)W^{-1}(t_0, t_1)[\Phi(t_0, t_1)\varepsilon_1 - \varepsilon_0 - \int\limits_{t_0}^{t_1} \Phi(t_0, t)\bar{\mu}(t)dt] -$$

$$-B_1^*(t)\Phi^*(t_0, t)W^{-1}(t_0, t_1) \int\limits_{t_0}^{t_1} \Phi(t_0, t)B_1(t)v(t)dt =$$

$$= v(t) + T_1(t)\varepsilon_0 + T_2(t)\varepsilon_1 + \bar{\mu}(t) + M_1(t)z(t_1, v),\ \ t \in I,\ \ \forall v,\ \ v(\cdot) \in L_2(I, R^m),$$

where matrices $T_1(t)$, $T_2(t)$, $M_1(t)$, $t \in I$, are defined by relations (10),

$$\int\limits_{t_0}^{t_1} \Phi(t_0, t)B_1(t)v(t)dt = \Phi(t_0, t_1)z(t_1, v), v(\cdot) \in L_2(I, R^m),$$

$z(t, v),\ \ t \in I$, is a solution of differential equations (11)–(13). The set U_1 is generated when an arbitrary function $v(\cdot) \in L_2(I, R^m)$ runs through all elements of the space $L_2(I, R^m)$. The theorem is proved.

3 Creating and solving controllability problems

Consider optimization problem (5)–(7), in the form of

$$J(\theta) = \int_{t_0}^{t_1} F_0(q(t), t)dt \to \inf, \theta \in X \subset H,$$

where $q(t) = (\theta(t), z(t_1, p))$, $p_1(t) \in L_2^{\rho}(I_1, R^{m_1}) = \{p_1(\cdot) \in L_2(I_1, R^m) | \|\rho_1\| \leq \rho\}$,

$$p_2(t) \in L_2^{\rho}(I_1, R^{m_1}) = \{p_2(\cdot) \in L_2(I_1, R^{m_1}) | \|\rho_2\| \leq \rho\},$$

$$F_0(q(t), t) = |F_1(q(t), t)|^2 + |F_2(q(t), t)|^2, F_1(q(t), t) = w_1 - u,$$

$$F_2(q(t), t) = w_1 - \int_a^b K(t, \tau)v(\tau)d\tau.$$

Note that:

1) $U(t)$, $V(t)$, S_0, S_1 are bounded convex closed sets, then X is a bounded convex closed set in a reflexive Banach space H, where $L_2(I_1, R^m)$, $L_2^{\rho}(I_1, R^{m_1})$ are bounded convex closed sets in the Hilbert space L_2.

2) the functional $J(\theta)$, $\theta \in X$ is bounded from below $J(\theta) \geq 0$, $\forall \theta \in X$. It is easy to see that the quadratic functional $J(\theta)$, $\theta \in X$ is convex since $z(t, \alpha\overline{p} + (1-\alpha)\overline{\overline{p}}) = \alpha z(t, \overline{p}) + (1-\alpha)z(t, \overline{\overline{p}}), \forall \overline{p}, \overline{\overline{p}} \in L_2^{\rho}(I_1, R^{m+m_1})$, $\alpha \in [0, 1]$.

3) It is known that a bounded convex closed set X in the reflexive Banach space H is weakly bicompact, and a continuous convex functional $J(\theta)$, $\theta \in X$ is weakly semicontinuous from below.

4) A weakly lower semicontinuous functional $J(\theta)$, $\theta \in X$ on a weekly bicompact set reaches the infimum on the set X, and hence the set.

$X_* = \theta_* \in X | J(\theta_*) = J_* = \inf_{\theta\in X} J(\theta) = \min_{\theta\in X} J(\theta)\} \neq \emptyset$ where $\emptyset$ is the empty set.

The partial derivative of the function $F_0(q,t)$ are:

$$F_{0x_0}(q,t) = 2\,T_0^*(t)F_1(q,t) + 2\,T_2^*(t)F_2(q,t),$$

$$F_{0x_1}(q,t) = 2\,T_1^*(t)F_1(q,t) + 2\,T_2^*(t)F_2(q,t),$$

$$F_{0z(t_1)} = 2\,N_{11}^*(t)F_1(q,t) + 2\,N_{12}^*(t)F_2(q,t),$$

$$F_{0p_1}(q,t) = 2\,F_1(q,t),\;\; F_{0p_2}(q,t) = 2\,F_2(q,t),\; F_{0u}(q,t) = -2\,F_1(q(t),t).$$

Theorem 3. Let the matrix $W(t_0,t_1)$ be positively defined. Then the functional under the conditions is continuously differentiable with respect to Frechet, the gradient

$$J'(\theta) = (J'_u(\theta),\, J'_v(\theta),\, J'_{p_1}(\theta),\, J'_{p_2}(\theta),\, J'_{x_0}(\theta),\, J'_{x_1}(\theta)) \in H$$

at any point $\theta \in X$ is determined by the formula

$$J'(\theta) = -F_{0u}(q(t),t),$$

$$J'_v(\theta) = 2\int_{t_0}^{t_1} K^*(t,\tau)w_2(t)dt + 2\int_{t_0}^{t_1}\int_a^b K^*(t,\tau)K(t,\tau)v(\tau)d\tau dt,$$

$$J'_{p_1}(\theta) = 2F_1(q,t) - B^*(t)\psi(t),\; J'_{p2}(\theta) = 2F_2(q,t) - C^*(t)\psi(t), \tag{14}$$

$$J'_{x_0}(\theta) = \int_{t_0}^{t_1} F_{0x_0}(q(t),t)dt,\; J'_{x_1}(\theta) = \int_{t_0}^{t_1} F_{0x1}(q(t),t)dt,$$

where $\psi(t), t \in I_1$ is the solution of the coupled system

$$\dot{\psi} = -A^*(t)\psi(t),\; \psi(t_1) = -\int_{t_0}^{t_1} F_{0z_{(t_1)}}(q(t),t)dt, \tag{15}$$

$$F_{0z(t_2)}(q(t),t) = 2\,N_{11}^*(t)F_1(q(t),t) + 2\,N_{12}^*(t)F_2(q(t),t),\; t \in I_1,$$

function $z(t) = z(t,p)$, $t \in I_1$ solution of the differential equation (13).

In addition, the gradient satisfies $J'(\theta)$, $\theta \in X$ the Lipschitz condition

$$\|J'(\theta_1) - J'(\theta_2)\| \le K\|\theta_1 - \theta_2\|,\; \forall\theta_1,\theta_2 \in X. \tag{16}$$

Proof. Let $\theta, \theta + \Delta\theta \in X$, $\Delta\theta = (\Delta u,\, \Delta v,\, \Delta p_1,\, \Delta p_2,\, \Delta x_{0*},\, \Delta x_{1*})$. As in the proof of Theorem 3, the functional increment can be represented as

$$\Delta J = J(\theta + \Delta\theta) - J(\theta) = \int_{t_0}^{t_1} \{\Delta u^*(t)F_{0u}(q(t) + \Delta p_1^*(t)[F_{0p1}(q,t) - B^*(t)\psi(t)] +$$

$$+\Delta p_2^*(t)[F_{0p2}(q,t) - C^*(t)\psi(t)] + \Delta x_0^*(t)F_{0x0}(q,t) + \Delta x_1^*(t)F_{0x_1}(q,t)\}dt +$$

$$+\int_a^b \Delta v^*(\tau)J'_v d\tau + R_1 + R_2 + R_3 + R_4, \tag{17}$$

where

$$R_1 = \int_{t_0}^{t_1} |\Delta w_1 - \Delta v|^2 dt,\; R_2 = \int_{t_0}^{t_1} |\int_a^b K(t,\tau)\Delta v(\tau)d\tau - \Delta w_2(t)|^2 dt,$$

$$R_3=\int_{t_0}^{t_1}\Delta x_0^*[F_{0x0}(q+\Delta q,t)+F_{0x_0}(q,t)]dt,\ R_4=\int_{t_0}^{t_1}\Delta x_1^*[F_{0x_1}(q+\Delta q,t)-F_{0x_1}(q,t)]dt,$$

$$|R|\le c_1\|\Delta\theta\|^2,\ \Delta q(t)=(\Delta\theta(t),z(t_1,p)).$$

From (15)–(17) it follows that the Freschi derivative of functional (16) under conditions (15)–(17) is determined by formula (14), where $\varphi(t)$, $t\in I_1$ is a solution of differential equation (15)–(17).

Let $\theta_1=\theta+\Delta\theta$, $\theta_2=\theta$. Then

$$J'(\theta_1)-J'(\theta_2)=F_{0u}(q+\Delta q,t)-F_{0u}(q,t),\ J'_v(\theta+\Delta\theta)-J'_v(\theta),$$

$$2F_1(q+\Delta q,t)-2F_1(q,t)-B^*(t)\Delta\psi(t),$$

$$2F_2(q+\Delta q,t)-2F_2(q,t)-C^*(t)\Delta\psi(t),$$

$$\int_{t_0}^{t_1}[F_{0x0}(q+\Delta q,t)-F_{0x_0}(q,t)]dt,\int_{t_0}^{t_1}[F_{0x_1}(q+\Delta q,t)-F_{0x_1}(q,t)]dt.$$

$$|J'(\theta_1)-J'(\theta_2)|\le L_1|\Delta q(t)|+L_2|\Delta\psi(t)|+L_3\|\Delta q\|,$$

$$\|J'(\theta_1)-J'(\theta_2)\|^2=\int_{t_0}^{t_1}|J'(\theta_1)-J'(\theta_2)|^2dt\le L_4\|\Delta q\|^2+L_5\int_{t_0}^{t_1}|\Delta\psi(t)|^2dt. \tag{18}$$

Since

$$\Delta\dot{\psi}=-A^*(t)\Delta\psi(t),\ t\in I_1,\ \Delta\psi(t_1)=-\int_{t_0}^{t_1}[F_{0z_{(t_1)}}(q+\Delta q,t)-F_{0z_{(t_1)}}(q,t)]dt,$$

that

$$\Delta\psi(t)=\Delta\psi(t_1)+\int_t^{t_1}A^*(t)\Delta\psi(t),\ t\in I_1,$$

$$|\Delta\psi(t)|\le|\Delta\psi(t_1)|+A^*_{\max}\int_t^{t_1}|\Delta\psi(\tau)|d\tau\le L_6\int_{t_0}^{t_1}|\Delta q(t)|dt+A^*_{\max}\int_t^{t_1}|\Delta\psi(\tau)|d\tau,$$

$$\|\Delta q(t)\|\le c_2\|\Delta\theta\|,\ |\Delta z(t,p)|\le c_3\|\Delta p_1\|+c_4\|\Delta p_2\|.$$

Then, applying the Gronwall lemma, we obtain

$$|\Delta\psi(t)|\le L_7e^{A^*_{\max}(t_1-t_2)}\|\Delta\theta\|. \tag{19}$$

From (18), (19) follows estimate (16).

Based on the results of Theorem 3, we construct the sequences $\{\theta_n\}=\{u_n,v_k,p_{1n},p_{2n},x_{0n},x_{1n}\}\subset X$ by algorithm

$$\begin{array}{l} u_{n+1}=P_U[u_n-\alpha_nJ'_u(\theta_n)],\quad v_{n+1}=P_V[v_n-\alpha_nJ'_v(v_n)],\\ p_{1n+1}=P_{L_2^p}[p_{1n}-\alpha_nJ'_{p_1}(\theta_n)],\quad p_{2n+1}=P_{L_2^p}[p_{2n}-\alpha_nJ'_{p_2}(\theta_n)],\\ x_{0n+1}=P_{S_0}[x_{0n}-\alpha_nJ'_{x0}(\theta_n)],\quad x_{1n+1}=P_{S_1}\left[x_{1n}-\alpha_nJ'_{x1}(\theta_n)\right],\\ n=0,1,2,\ldots,\quad 0<\xi_0\le\alpha_n\le\frac{2}{K+2\varepsilon_1},\ \varepsilon_1>0, \end{array} \tag{20}$$

where $K>0$ is the Lipschitz constant of equation (14), in particular, $\varepsilon_1=\frac{K}{2}$ in the case of $\varepsilon_0=\alpha_n=\frac{1}{K}$. We get that U,V,S_0,S_1 are bounded convex closed sets, $P_\Omega[\cdot]$ is the projection of a point onto the set Ω. Any point has a unique projection onto a convex closed set.

Theorem 4. Let the matrix $W(t_0, t_1) > 0$, the sequence $\{\theta_n\}$ be defined by the formula (20). Then:
1. the numeric sequence $\{J(\theta_n)\}$ is strictly decreasing;
2. $\|\theta_n - \theta_{n+1}\| \to 0$ when $n \to \infty$;
3. the sequence $\{\theta_n\} \subset X$ is minimized: $\lim\limits_{n\to\infty} J(\theta_n) = J_* = \inf\limits_{\theta\to X} J(\theta)$;
4. the set $X_* = \{\theta_* \in X | J(\theta_*) = J_* = \inf\limits_{\theta\to X} J(\theta) = \min\limits_{\theta\to X} J(\theta)\}$ is not empty, the lower bound functional $J(\theta)$, $\theta \in X$ is reached on the set X;
5. the sequence $\{\theta_n\} \subset X$ converges weakly to the set X_*, $u_n \overset{weak}{\to} u_*$, $v_n \overset{weak;}{\to} v_*$, $p_{2n} \overset{weak}{\to} p_{2*}$, $x_{0n} \overset{weak}{\to} x_{0*}$, $x_{1n} \overset{weak}{\to} x_{1*}$ at $n \to \infty$, where $(u_*,\ v_*, p_{1*}, p_{2*}, x_{0*}, x_{1*}) \in X_*$;
6. the following convergence rate estimate is valid

$$0 \le J(\theta_n) - J_* \le \frac{m_0}{n}\ n = 1, 2,\ \ldots,\ m_0 = const > 0,$$

where $J(\theta_*) = J_*$;
7. controllability problem (1)–(4) has a solution if and only if the value $J(\theta_*) = 0$. In this case, the solution of controllability problem (1)–(4) is the function

$$x_*(t) = z(t, p_*) + \lambda_*(t, x_{0*}, x_{1*}) + N_2(t)z(t_1, p_*),\ t \in I_1.$$

If $J(\theta_*) > 0$, then controllability problem (1)–(4) has no solution, $x_*(t)$, $t \in I_1$ is the best necessary solution to controllability problem (1)–(4).

Proof. From the property for the projection of a point onto a set, we have

$$< J'(\theta_n),\ \theta - \theta_{n-1} > \ge \frac{1}{\alpha_n} < \theta_n - \theta_{n-1},\ \theta - \theta_{n-1},\ \ \forall\theta,\ \theta \in X. \tag{21}$$

Since $J'(\theta) \in C^{1,1}(X), X$ is a convex set, the estimate is true

$$J(\theta_1) - J(\theta_2) \ge < J'(\theta_1),\ \theta_1 - \theta_2 > -\frac{K}{2}\|\theta_1 - \theta_2\|^2,\ \forall\theta_1,\ \theta_2 \in X. \tag{22}$$

From (16) and (17) $\theta = \theta_n$, $\theta_1 = \theta_n$, $\theta_2 = \theta_{n+1}$, we get

$$J(\theta_n) - J_1(\theta_{n+1}) \ge (\frac{1}{\alpha_n} - \frac{K}{2})\,\|\theta_n - \theta_{n+1}\|^2 \ge \xi_1\|\theta_n - \theta_{n+1}\|^2,\ \frac{1}{\alpha_n} - \frac{K + 2\varepsilon_1}{2}. \tag{23}$$

It follows from equality (23) that the numerical sequence $\{J(\theta_n)\}$ is strictly decreasing, and also because of the limited value of the functional at $\|\theta_n - \theta_{n+1}\| \to 0$ by $n \to \infty$. Thus, assertions 1) and 2) of the theorems are proved.

The functional $J(\theta), \theta \in M_0$ is weakly lower semicontinuous on a weakly bicompact set X, then the set is empty. The sequence $\{\theta_n) \subset M_0$. Then, due to the weakly bicompactness of the set M_0 it follows that $\theta_n \overset{weak}{\to} \theta_*$, $n \to \infty$, $\theta_* \in X_*$. Thus, statements 4), 5) of the theorem are proved.

For convex functional $J(\theta) \in C^{1,1}(M_0)$, the following inequality holds

$$\begin{aligned}
&J(\theta_n) - J(\theta_*) \le < J'(\theta_n),\ \theta_n - \theta_* > = < J'(\theta_n),\ \theta_n - \theta_{n+1} + \theta_{n+1} - \theta_* > = \\
&= < J'(\theta_n),\ \theta_n - \theta_{n+1} > - < J'(\theta_n), \theta_* - \theta_{n-1} > .
\end{aligned}$$

Hence, taking into account the inequality for $\theta \in \theta_*$, we have

$$\begin{aligned}
&0 \le J'(\theta_n) - J_* \le < J'(\theta_n), \theta_n - \theta_{n+1} > -\tfrac{1}{\alpha_n} < \theta_n - \theta_{n+1}, \theta_* - \theta_{n+1} > = \\
&= < J'(\theta_n) - \tfrac{1}{\alpha_n}(\theta_* - \theta_{n+1}), \theta_n - \theta_{n+1} > \le \\
&\le \|J'(\theta_n) - \tfrac{1}{\alpha_n}(\theta_* - \theta_{n+1})\|\|\theta_n - \theta_{n+1}\| \le \\
&\le (\|J'(\theta_n)\| + \tfrac{1}{\alpha_n})\|\theta_* - \theta_{n+1}\| = \|\theta_n - \theta_{n+1}\|c_0\|\theta_n - \theta_{n+1}\|,
\end{aligned} \tag{24}$$

where $\|\theta_* - \theta_{n+1}\| \le D$, $\frac{1}{\alpha_n} \le \frac{r}{\xi_0}$, $c_0 = \sup \|J'(\theta_n)\| + \frac{D}{\xi_0}$, D is the diameter of the set M_0. Since with $\|\theta_n - \theta_{n+1}\| \to 0$, then $n \to \infty$ that $\lim\limits_{n\to\infty} J(\theta_n) = J_* = J(\theta_*)$. This means that the $\{\theta_n\} \subset M_0$ sequence reaches a minimum.

It follows from inequalities (23), (24) that $J(\theta_n) - J(\theta_{n+1}) = a_n - a_{n+1} \ge \varepsilon_1 \|\theta_n - \theta_{n+1}\|^2$, $a_n - a_{n+1} \ge c_0\|\theta_n - \theta_{n+1}\|$, $a_n = J(\theta_n) - J_* = J(\theta_n) - J(\theta_*)$. Then $a_n > 0$,$a_n - a_{n+1} \ge \frac{\varepsilon_1}{c_0^2} a_n^2$,$n = 1, 2, ...$,$m_0 \ge \frac{c_0^2\varepsilon_1}{\varepsilon_1}$. The theorem is proved.

4 Solving a model problem

Consider a controlled process described by a differential equation with an integral equation of the form

$$\dot{x_1} = x_2,\ \dot{x_2} = u + \int_1^2 e^{(t+1)\tau} v(\tau)d\tau,\ t \in I_1 = [0, 2],\ \tau \in I_2 = [1, 2], \tag{25}$$

$$(x_{10}(0),\ x_{20}(0)) \in S_0 = \{-1 \le x_{10} \le 1,\ 1 \le x_{20} \le 2\},$$

$$(x_{11}(2),\ x_{21}(2)) \in S_1 = \{-1 \le x_{11}(2) \le 1,\ -2 \le x_{21}(0) \le -1\},$$

$$u(t) \in U = \{u(\cdot) \in L_2(I_2,\ R^1)|\ -1 \le u(t) \le 1,\ a.e.\ t \in I_1\},$$

$$v(t) \in V = \{v(\cdot) \in L_2(I_2,\ R^1)|\ \tau \le v(\tau) \le 2\tau,\ a.e.\ \tau \in I_2\}. \tag{26}$$

1. The necessary sufficient conditions satisfy controllability defined by the ratios:

a) Matrix $W(0, 2) = \begin{pmatrix} 16/3 & -4 \\ -4 & 4 \end{pmatrix} > 0$;

b) $u_*(t) = w_{1*}(t) = p_{1*}(t)T_0(t)x_{0*} + T_1(t)x_{1*} + \mu_{11}(t) + N_{11}(t)z(t_1,\ p_*),\ t \in I_1$,

$$T_0(t) = (\frac{3}{4}(t-1),\ \frac{3t-4}{4}),\ T_1(t) = (\frac{3t-4}{4},\ \frac{(3t-4)}{4}),\ \mu_{11}(t) \equiv 0,$$

$$N_{11}(t) = (\frac{3(t-1)}{4},\ \frac{(2-3t)}{4}),\ p_{1*}(\cdot) \in L_2(I_2,\ R^1),\ x_{0*} \in (x_{10*},\ x_{20*}) \in S_0,$$

$$(x_{11*},\ x_{21*}) \in S_1,\ u_*(t) \in U,\ v_*(t) \in V;$$

c) $\int_1^2 e^{(t+1)\tau} v_*(\tau)d\tau = w_{2*}(t) = p_{2*}(t) + T_2(t)x_{0*} + T_3(t)x_{1*} + \mu_{12}(t) + N_{12}(t)z(t_1,\ p_*)$,

$$T_2(t) = (\frac{3}{4}(t-1),\ \frac{3t-4}{4}),\ T_3(t) = (\frac{3(1-t)}{4},\ \frac{3t-2}{4}),\ N_{12}(t) = (\frac{3(t-1)}{4},\ \frac{(2-3t)}{4}),$$

$$x_{0*} \in (x_{10*},\ x_{20*}) \in S_0,\ x_{1*} = (x_{11*},\ x_{21*}) \in S_1,\ p_{2*}(\cdot) \in L_2(I_2,\ R^1), \mu_{12}(t) \equiv 0,$$

$$S_0 = S_{10},\ S_{20},\ S_{10} = \{x_{10} \in R^1|-1 \le x_{10} \le 1,\ S_{20} = \{x_{20} \in R^1|1 \le x_{20} \le 2\};$$

$$S_1 = S_{11},\ S_{21},\ S_{11} = \{x_{11}(2) \in R^1|-1 \le x_{11}(2) \le 1\},\ S_{21} = \{x_{21}(2) \in R^1|-2 \le x_{21}(2) \le -1\}.$$

2. Construction of a solution to the controllability problem. The desired controls $u_*(t) \in U$, $v_*(\tau) \in V$, $p_{1*}(t) \in L_2^{\rho}(I_1,\ R^1)$, $p_{2*}(t) \in L_2^{\rho}(I_1,\ R^1)$, $x_{0*} \in S_0$, $x_{1*} \in S_1$, can be found when solving the optimal control problem: minimize the functional

$$J(u,\ v, p_1, p_2, x_{10}, x_{20}, x_{11}, x_{21}) = \int_{t_0}^{t_1} \{|w_1(t) - u(t)|^2 + |w_2(t) - \int_a^b K(t,\tau)v(\tau)d\tau|^2\}dt \to \inf \tag{27}$$

under conditions

$$u(t) \in U,\ v(\tau) \in V,\ p_1(t) \in L_2^{\rho}(I_2, R^1),\ p_2(t) \in L_2(I_1, R^1),\ x_{10} \in S_{10},\ x_{20} \in S_{20},\ x_{11} \in S_{11},\ x_{21} \in S_{21}, \tag{28}$$

$$w_1(t) = p_1(t) + T_{10}(t)x_{10} + T_{20}(t)x_{20} + T_{11}(t)x_{11} + T_{21}(t)x_{21} + N_{11}(t)z(t_1, p),\ t \in I_1,$$
$$w_2(t) = p_2(t) + T_{20}(t)x_{10} + T_{30}(t)x_{20} + T_{31}(t)x_{11} + T_{41}(t)x_{21} + N_{12}(t)z(t_1, p),\ t \in I_1,$$
$$T_{10} = T_{10}(t) = \frac{3}{4}(t-1),\ T_{20} = T_{20}(t) = \frac{3t-4}{4}, T_{11} = T_{11}(t) = \frac{3t-4}{4},\ T_{21} = T_{21}(t) = \frac{3t-2}{4},$$
$$T_{20} = T_{20}(t) = \frac{3}{4}(t-1),\ T_{30} = T_{30}(t) = \frac{3t-4}{4}, T_{31} = T_{31}(t) = \frac{3(t-1)}{4},\ T_{41} = T_{41}(t) = \frac{3t-2}{4},$$

where a function $z(t,p)$, $t \in I_1$ is a solution of the differential equation

$$\dot{z_1} = z_2,\ \dot{z_2} = p_1(t) + p_2(t),\ z_1(0) = 0,\ z_2(0) = 0,\ t \in I_1.$$

Let us calculate at $\theta = (u,\ v, p_1, p_2, x_{10}, x_{20}, x_{11}, x_{21})$, $q = (\theta, z(2))$ a gradient of functional (24) under conditions (25)–(28):

a) $J'_u(\theta) = F_{0u}(q,t) = -2(w_1 - u)$, $F_0(q,t) = |F_1(q,t)|^2 + |F_2(q,t)|^2$,

$$F_1(q,t) = (w_1 - u),\ F_2(q,t) = w_2 - \int_1^2 e^{(t+1)\tau} w(\tau)d\tau,$$
$$J'_v(\theta) = -2\int_0^2 e^{(t+1)\tau} w_2(t)dt + 2\int_0^2 \int_1^2 e^{(t+1)\tau} e^{(t+1)\tau} v(\tau)d\tau dt,$$
$$J'_{\rho_1}(\theta) = 2F_1(q,t) - B^*(t)\psi(t),\ J'_{\rho_2}(\theta) = 2F_2(q,t) - C^*(t)\psi(t),\ t \in I_1,$$
$$J'_{x_{10}}(\theta) = \int_0^2 [2T_{10}(t)F_1(q,t) + 2T_{20}(t)F_2(q,t)]dt,\ J'_{x_{20}}(\theta) = \int_0^2 [2T_{20}(t)F_1(q,t) + 2T_{30}(t)F_2(q,t)]dt,$$
$$J'_{x_{11}}(\theta) = \int_0^2 [2T_{11}(t)F_1(q,t) + 2T_{31}(t)F_2(q,t)]dt,\ J'_{x_{21}}(\theta) = \int_0^2 [2T_{21}(t)F_1(q,t) + 2T_{40}(t)F_2(q,t)]dt.$$

b) partial derivative

$$F_{0z_{(t_1)}}(q,t) = \begin{pmatrix} \frac{3(t-1)}{4} \\ \\ \frac{2-3t)}{4} \end{pmatrix},\ F_1(q,t) = \begin{pmatrix} \frac{3(t-1)}{4} \\ \\ \frac{2-3t)}{4} \end{pmatrix}.$$

c) coupled system

$$\dot{\psi}_1 = 0,\ \dot{\psi}_2 = -\psi_1,\ \ \psi(2) = \begin{pmatrix} \psi_1(2) \\ \psi_2(2) \end{pmatrix} = -\int_{t_0}^{t_1} F_{0z(t_1)}\left(q\left(t\right), t\right) dt.$$

d) minimizing sequences are:

$$\begin{array}{l} u_{n+1} = P_U[u_n - \alpha_n J'_u(\theta_n)],\ \ p_{n+1} = P_V[v_n - \alpha_n J'_v(\theta_n)], \\ p_{1n+1} = P_{L_2^p}[p_{1n} - \alpha_n J'_{p_1}(\theta_n)],\ \ p_{2n+1} = P_{L_2^p}[p_{2n} - \alpha_n J'_{p_2}(\theta_n)], \\ x_{10}^{n+1} = P_{S_{10}}[x_{10}^{(n)} - \alpha_n J'_{x_{10}}(\theta_n)],\ \ x_{20}^{n+1} = P_{S_{20}}[x_{20}^{(n)} - \alpha_n J'_{x_{20}}(\theta_n)], \\ x_{11}^{n+1} = P_{S_{11}}[x_{11}^{(n)} - \alpha_n J'_{x_{11}}(\theta_n)],\ \ x_{20}^{n+1} = P_{S_{21}}[x_{21}^{(n)} - \alpha_n J'_{x_{21}}(\theta_n)], \\ n = 0, 1, 2, \ldots,\ \ \alpha_n \le \frac{1}{K}. \end{array}$$

e) projections of a point onto sets

$$P_V[v_n - \alpha_n J'{}_v(\theta_n)] = \begin{cases} \tau,\ \text{if } v_n - \alpha_n J'{}_v(\theta_n) < \tau; \\ v_n - \alpha_n J'{}_v(\theta_n,\ \text{if } \tau \le v_n - \alpha_n J'{}_v(\theta_n) \le 2\tau; \\ 2\tau,\ \text{if } v_n - \alpha_n J'{}_v(\theta_n) > 2\tau; \end{cases}$$

$$P_{L_2^p}[p_{1n} - \alpha_n {J'}_{p_1}(\theta_n)] = p_{1n} - \alpha_n {J'}_{p_1}(\theta_n),\ if\ \ \|p_{1n} - \alpha_n {J'}_{p_1}(\theta_n)\| \le \rho,$$

$$P_{L_2^p}[p_{2n} - \alpha_n {J'}_{p_2}(\theta_n)] = p_{2n} - \alpha_n {J'}_{p_2}(\theta_n),\ if\ \ \|p_{2n} - \alpha_n {J'}_{p_2}(\theta_n)\| \le \rho,$$

$\rho > 0$ is quite large;

$$P_{S_{10}}[x_{10}^{(n)} - \alpha_n {J'}_{x_{10}}(\theta_n)] = \begin{cases} -1, \text{ if } x_{10}^{(n)} - \alpha_n {J'}_{x_{10}}(\theta_n) < -1; \\ x_{10}^{(n)} - \alpha_n {J'}_{x_{10}}(\theta_n), \text{ if } -1 \le x_{10}^{(n)} - \alpha_n {J'}_{x_{10}}(\theta_n) \le 1; \\ 1, \text{ if } x_{10}^{(n)} - \alpha_n {J'}_{x_{10}}(\theta_n) > 1; \end{cases}$$

$$P_{S_{20}}[x_{20}^{(n)} - \alpha_n {J'}_{x_{20}}(\theta_n) = \begin{cases} 1, \text{ if } x_{20}^{(n)} - \alpha_n {J'}_{x_{20}}(\theta_n) < 1; \\ x_{20}^{(n)} - \alpha_n {J'}_{x_{20}}(\theta_n), \text{ if } 1 \le P_{S_{20}}[x_{20}^{(n)} - \alpha_n {J'}_{x_{20}}(\theta_n) \le 2; \\ 2, \text{ if } P_{S_{20}}[x_{20}^{(n)} - \alpha_n {J'}_{x_{20}}(\theta_n) > 2; \end{cases}$$

$$P_{S_{11}}[x_{11}^{(n)} - \alpha_n {J'}_{x_{11}}(\theta_n)] = \begin{cases} -1, \text{ if } x_{11}^{(n)} - \alpha_n {J'}_{x_{11}}(\theta_n) < -1; \\ x_{11}^{(n)} - \alpha_n {J'}_{x_{11}}(\theta_n), \text{ if } -1 \le x_{11}^{(n)} - \alpha_n {J'}_{x_{11}}(\theta_n) \le 1; \\ -2, \text{ if } x_{11}^{(n)} - \alpha_n {J'}_{x_{11}}(\theta_n) > -2; \end{cases}$$

$$P_{S_{21}}[x_{21}^{(n)} - \alpha_n {J'}_{x_{21}}(\theta_n)] = \begin{cases} -2, \text{ if } x_{21}^{(n)} - \alpha_n {J'}_{x_{21}}(\theta_n) < -2; \\ x_{21}^{(n)} - \alpha_n {J'}_{x_{21}}(\theta_n), \text{ if } -2 \le x_{21}^{(n)} - \alpha_n {J'}_{x_{21}}(\theta_n) \le -1; \\ -1, \text{ if } x_{21}^{(n)} - \alpha_n {J'}_{x_{21}}(\theta_n) > -1. \end{cases}$$

f) limit points of minimizing sequences:

$$u_n(t) \overset{weak}{\to} u_*(t),\ v_n(\tau) \overset{weak}{\to} v_*(\tau),\ p_1(t) \overset{weak}{\to} p_{1*}(t),\ p_2(t) \overset{weak}{\to} p_{2*}(t),\ t \in I_1,$$

$$x_{10}^{(n)} \to x_{10}^*,\ x_{20}^{(n)} \to x_{20}^*,\ x_{11}^{(n)} \to x_{11}^*,\ x_{21}^{(n)} \to x_{21}^*.$$

g) solvability of the controllability problem (22), (23):

1) if $J(u_*,\ v_*, p_{1*}, p_{2*}, x_{10}^*, x_{20}^*, x_{11}^*, x_{21}^*) = 0$, the solution of problem (21)–(23) is a function

$$x_*(t) = z(t, p_*) + \lambda(t, x_{10}^*, x_{20}^*, x_{11}^*, x_{21}^*) + N_2(t) z(t_1, p_*),\ t \in I_1;$$

2) if $J(u_*,\ v_*, p_{1*}, p_{2*}, x_{10}^*, x_{20}^*, x_{11}^*, x_{21}^*) > 0$, then the controllability problem (22), (23) has no solution. In this case, the function $x_*(t)$, $t \in I_1$, is a given approximation of the controllability problem.

5 *Conclusion*

The main results obtained in this work are: the choice of a set of program and positional controls for the process described by a linear ordinary differential equation, in the absence of restrictions on the values of the controls, by constructing a general solution of the Fredholm integral equation of the first kind; determination of program and positional control, as well as solving problems of optimal performance in the presence of restrictions on the control values and phase and integral restrictions; reduction of the initial-boundary value problem with restrictions to a special initial-boundary value problem of the optimal control and the construction of minimizing sequences and successive narrowing of the area of admissible controls solution of the optimal performance problem.

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Шектік шарттары бар сызықтық жүйелердің басқарылуы және оңтайлы әсері

Мақалада фазалық және интегралдық шектеулер болған кезде жай дифференциалдық теңдеулердің сызықтық жүйелері үшін оңтайлы жылдамдық әсерін шешу әдісі ұсынылған, мұнда жүйенің бастапқы және соңғы күйі басқару мәнінің шектеулілігін ескере отырып, берілген дөңес тұйық жиындардың элементтері болып табылады. Ұсынылған жұмыс Л.С. Понтрягин мен оның шәкірттерінің оңтайлы процестерінің математикалық теориясына, сонымен бірге Р.Е. Кальманның динамикалық жүйелерін басқару теориясына жатады. Фазалық және интегралдық шектеулер, сондай-ақ басқару шектеулері болған кезде берілген жиындардың шектік шарттары бар сызықтық жүйелер үшін оңтайлы жылдамдық әсері зерттелді. Шектік есептің теориясы құрылуы және оны шешу әдісі шешімділікті зерттеу, бірінші типтегі Фредгольм интегралдық теңдеуінің жалпы шешімін құру негізінде жасалды. Негізгі нәтижелер: жүйенің траекториясын кез келген бастапқы күйден кез келген қажетті соңғы күйге ауыса алатын әрбір элементті барлық басқару жиындарынан бөліп алу; алынған басқарудың қажетті және жеткілікті шарттарының бар болуы; шектеулері бар оңтайлы жылдамдық әсерінің мәселесін шешудің алгоритмі.

Кілт сөздер: оңтайлы тиімділік, толығымен шектеу, функционалды градиент, интегралдық теңдеу.

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Управляемость и оптимальное быстродействие линейных систем с граничными условиями

В статье предложен метод решения задачи оптимальной скорости для линейных систем обыкновенных дифференциальных уравнений при наличии фазовых и интегральных ограничений, когда начальное и конечное состояния системы являются элементами заданных выпуклых замкнутых множеств с учетом ограничения контрольного значения. Представленная работа относится к математической теории оптимальных процессов Л.С. Понтрягина и его учеников и теории управляемости динамических систем Р.Е. Кальмана. Исследована задача оптимальной скорости для линейных систем с граничными условиями из заданных множеств, близких к наличию фазовых и интегральных ограничений, а также ограничения по управляющему значению. Создана теория граничной задачи, и разработан метод ее решения на основе изучения разрешимости и построения общего решения интегрального уравнения Фредгольма первого рода. Основными результатами являются распределение всех наборов элементов управления, каждый субъект которых переводит траекторию системы из любого начального состояния в любое конечное состояние; сведение начальной граничной точки к специальной исходной задаче оптимального управления; построение системы алгоритмов гамма-алгоритма учения о выводе задач и рациональном выполнении с ограничениями решения задачи оптимальной скорости с ограничениями.

Ключевые слова: оптимальная производительность, ограничения целостности, функциональный градиент, интегральное уравнение.

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On Some Non-local Boundary Value and Internal Boundary Value Problems for the String Oscillation Equation

The work is devoted to the problem of setting new boundary and internal boundary value problems for hyperbolic equations. The consideration of these settings is given on the example of a wave equation. The research involves the d'Alembert method, the mean value theorem and the method of successive approximations. The paper formulates and studies a number of non-local problems summarizing the classical Goursat and Dardu tasks. Some of them are marginal, and the other part is internal-marginal, and in both cases both characteristic and uncharacteristic displacements are considered. It should also be noted that a number of problems discussed below arose as a special case in the construction of the theory of correct problems for the model loaded equation of string oscillation.

Keywords: Wave equation, general solution, Cauchy problem, Goursat problem, Darboux problem, problem with characteristic shift, problem with uncharacteristic displacement.

Introduction

Boundary value problems for partial differential equations with nonlocal conditions present a class of problems solvability of which is important both for differential equations development and for their applications. They can be used in mathematical modeling as well as in the theory of loaded equations and coefficient inverse problems.

In 1969, in the work [1] A.M. Nakhushev proposed a number of problems of a new type, which entered the mathematical literature under the name of problems with displacement. These tasks were announced as part of the implementation of the problem of finding correctly posed problems for second-order mixed-type equations with two independent variables, put forward in the 60s of the last century by A.V. Bitsadze.

In accordance with the classification proposed by him in [2, 3], these problems, bounded by two intersecting characteristics of a given hyperbolic equation and one characteristic line, are non-local and with an edge offset. The bibliography of works devoted to regular local and nonlocal boundary value problems for strictly hyperbolic equations is very extensive and is most fully given in the monograph [4]. Let us note some of them that are closest to the problems discussed in this paper [1–11].

In this paper, a number of non-local problems generalizing the classical problems of Goursat and Darboux are formulated and investigated. Some of them are marginal, and the other part is internally marginal, and in both cases both characteristic and non-characteristic displacements are considered. It should also be noted that a number of the problems discussed below arose as a special case when constructing the theory of correct problems by analogy with [7] for the model loaded string oscillation equation.

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1 The main part

In this paper, the wave equation is considered as a model equation with second-order partial derivatives of two independent variables x and y a hyperbolic equation

$$u_{xx} - u_{yy} = 0. \tag{1}$$

Let Ω_1 be a simply connected domain of the plane of a complex variable $z = x + iy$, bounded by the characteristics AB, BC, CD and DA equation (1), coming out of the points, respectively $A\,(0,\,0)$, $B\left(\frac{1}{2},\,\frac{1}{2}\right)$, $C\,(1,\,0)$ and $D\left(\frac{1}{2},\,-\frac{1}{2}\right)$; $AC = \bar{J}$. By Ω_2 we denote the area bounded by the characteristics AB, BC and the segment AC of the straight line $y = 0$.

By the regular solution of the equation (1) we will understand any function $u\,(x, y) \in C\left(\bar{\Omega}_i\right) \cap C^2\left(\Omega_i\right),\ i = 1, 2$ satisfying equation (1).

2 Non-local problems with edge displacement

Problem 2.1. Find regular in the area Ω_1 decision $u\,(x, y)$ equation (1), satisfying conditions

$$u\left(\frac{x}{2}, \frac{x}{2}\right) + \alpha_1\,(x)\; u\left(\frac{1+x}{2},\; \frac{x-1}{2}\right) = \gamma_1\,(x)\,,\;\; x \in \bar{J},$$

$$u\left(\frac{x}{2}, -\frac{x}{2}\right) + \alpha_2\,(x)\; u\left(\frac{1+x}{2},\; \frac{1-x}{2}\right) = \gamma_2\,(x)\,,\;\; x \in \bar{J},$$

where

$$\alpha_1\,(0) \cdot \alpha_1\,(1) \neq \alpha_2\,(0) \cdot \alpha_2\,(1)\,, \tag{2}$$

$$\alpha_1\,(x) \cdot \alpha_2\,(x) \neq 1, \forall x \in \bar{J}, \tag{3}$$

$$\alpha_i\,(x)\,,\;\gamma_i\,(x) \in C\left(\bar{J}\right) \cap C^2\,(J)\,,\;\; i = 1, 2.$$

Based on the Asgeirsson principle for characteristic quadrilaterals with vertices respectively at point

$$(0,0)\,, \left(\frac{x+1}{2},\; \frac{x-1}{2}\right), \left(\frac{x}{2},\; \frac{x}{2}\right), \left(\frac{1}{2},\; -\frac{1}{2}\right)$$

and

$$(0,0)\,, \left(\frac{1+x}{2},\; \frac{1-x}{2}\right), \left(\frac{x}{2},\; -\frac{x}{2}\right), \left(\frac{1}{2},\; \frac{1}{2}\right)$$

it is easy to verify the equivalence of problem 2.1 to the following algebraic system

$$u\left(\frac{x}{2},\; \frac{x}{2}\right) + \alpha_1\,(x)\; u\;\left(\frac{x}{2}, -\;\frac{x}{2}\right) = \gamma_1\,(x) - \alpha_1\,(x)\;\left[u\left(\frac{1}{2},\; \frac{1}{2}\right) - u\,(0,0)\right],$$

$$\alpha_2\,(x)\;\; u\;\left(\frac{x}{2},\; \frac{x}{2}\right) + u\;\left(\frac{x}{2}, -\;\frac{x}{2}\right) = \gamma_2\,(x) - \alpha_2\,(x)\;\left[u\left(\frac{1}{2}, -\frac{1}{2}\right) - u\,(0,0)\right],$$

which by virtue of (2), (3) is unambiguously and unconditionally solvable in the class $C\left(\bar{J}\right) \cap C_2^2\,(J)$.

Therefore, problem 2.1 is reduced in an equivalent way to the Goursat problem with data on AB and AD, which is known to be correct.

Problem 2.2. Find a regular Ω_1 solution $u\;(x,\;y)$ of equation (1), the domain satisfying the conditions

$$u\;\left(\frac{x}{2},\; \frac{x}{2}\right) + \alpha_1\,(x)\; u\;\left(\frac{1+x}{2},\; \frac{1-x}{2}\right) = \gamma_1\,(x)\,,\;\;\; x \in \bar{J},$$

$$u\left(\frac{x}{2}, -\frac{x}{2}\right) + \alpha_2(x)\, u\left(\frac{x+1}{2}, \frac{x-1}{2}\right) = \gamma_2(x), \quad x \in \bar{J},$$

where

$$\alpha_1(0) \cdot \alpha_1(1) \neq \alpha_2(0) \cdot \alpha_2(1), \tag{4}$$

$$1 + \alpha_1(x) \neq 0, 1 + \alpha_2(x) \neq 0 \ \forall x \in \bar{J} \tag{5}$$

$$\alpha_i(x),\ \gamma_i(x) \in C\left(\bar{J}\right) \cap C^2(J),\ i = 1, 2.$$

Based on the Asgeirsson principle for characteristic quadrilaterals with vertices respectively at points

$$\left(\frac{x}{2}, \frac{x}{2}\right), (1, 0), \left(\frac{x+1}{2}, \frac{x-1}{2}\right), \left(\frac{1}{2}, \frac{1}{2}\right)$$

and

$$\left(\frac{x}{2}, -\frac{x}{2}\right), (1, 0), \left(\frac{1+x}{2}, \frac{1-x}{2}\right), \left(\frac{1}{2}, -\frac{1}{2}\right)$$

we are convinced of the equivalence of problem 2.2 to the following two algebraic systems for finding $u\left(\frac{x}{2}, \frac{x}{2}\right)$, $u\left(\frac{x}{2}, -\frac{x}{2}\right)$, $u\left(\frac{x+1}{2}, \frac{x-1}{2}\right)$ and $u\left(\frac{1+x}{2}, \frac{1-x}{2}\right)$

$$\begin{cases} u\left(\frac{x}{2}, \frac{x}{2}\right) + \alpha_1(x)\, u\left(\frac{x+1}{2}, \frac{x-1}{2}\right) = \gamma_1(x), \\ u\left(\frac{x}{2}, \frac{x}{2}\right) - u\left(\frac{x+1}{2}, \frac{x-1}{2}\right) = u\left(\frac{1}{2}, \frac{1}{2}\right) - u(1, 0) \end{cases}$$

and

$$\begin{cases} u\left(\frac{x}{2}, -\frac{x}{2}\right) + \alpha_2(x)\, u\left(\frac{1+x}{2}, \frac{1-x}{2}\right) = \gamma_2(x), \\ u\left(\frac{x}{2}, -\frac{x}{2}\right) - u\left(\frac{1+x}{2}, \frac{1-x}{2}\right) = u\left(\frac{1}{2}, -\frac{1}{2}\right) - u(1, 0), \end{cases}$$

which, by virtue of conditions (4), (5) are uniquely solvable.

Therefore, problem 2.2 is reduced in an equivalent way, as in the case of problem 2.1, to the Goursat problem.

Problem 2.3. Find a regular Ω_1 solution $u(x, y)$ of equation (1), in the domain of equation (1), satisfying the conditions

$$u\left(\frac{x}{2}, \frac{x}{2}\right) + \alpha_1(x)\, u\left(\frac{1-x}{2}, -\frac{1-x}{2}\right) = \gamma_1(x),\ x \in \bar{J}, \tag{6}$$

$$u\left(\frac{x}{2}, -\frac{x}{2}\right) + \alpha_2(x)\, u\left(\frac{1-x}{2}, \frac{1-x}{2}\right) = \gamma_2(x),\ x \in \bar{J}, \tag{7}$$

where

$$1 - \alpha_1(1-x)\ \alpha_2(x) \neq 0, x \in \bar{J}, \tag{8}$$

$$\alpha_i(x),\ \gamma_i(x) \in C\left(\bar{J}\right) \cap C^2(J),\ i = 1, 2.$$

Replacing in (6) everywhere x by $1-x$, to find $u\left(\frac{x}{2}, -\frac{x}{2}\right)$ we obtain the following algebraic system

$$\alpha_1(1-x)\ u\left(\frac{x}{2}, -\frac{x}{2}\right) + u\left(\frac{1-x}{2}, \frac{1-x}{2}\right) = \gamma_1(1-x),$$

$$u\left(\frac{x}{2}, -\frac{x}{2}\right) + \alpha_2(x)\, u\ \left(\frac{1-x}{2}, \frac{1-x}{2}\right) = \gamma_2(x).$$

Similarly, replacing in (7) x by $1-x$, to find $u\left(\frac{x}{2},\,\frac{x}{2}\right)$ we get

$$u\left(\frac{x}{2},\,\frac{x}{2}\right)+\alpha_1\left(x\right)u\left(\frac{1-x}{2},-\frac{1-x}{2}\right)=\gamma_1\left(x\right),$$

$$\alpha_2\left(1-x\right)\,u\left(\frac{x}{2},\,\frac{x}{2}\right)+u\left(\frac{1-x}{2},\,-\frac{1-x}{2}\right)=\gamma_2\left(1-x\right).$$

The last two systems are unconditionally and unambiguously solvable under the conditions (8). Therefore, problem 2.3 is reduced equivalently, as in the case of problem 2.1, to the Goursat problem.

Problem 2.4. Find regular in the area Ω_1 solution $u\left(x,y\right)$ of equation (1), satisfying conditions

$$u\left(\frac{x}{2},\,\frac{x}{2}\right)+\alpha_1\left(x\right)\,u\left(\frac{2-x}{2},\,\frac{x}{2}\right)=\gamma_1\left(x\right),\;x\in\bar{J},$$

$$u\left(\frac{x}{2},-\frac{x}{2}\right)+\alpha_2\left(x\right)u\left(\frac{2-x}{2},-\frac{x}{2}\right)=\gamma_2\left(x\right),\;x\in\bar{J},$$

where $\alpha_i\left(x\right),\,\gamma_i\left(x\right)\in C\left(\bar{J}\right)\cap C^2\left(J\right),\;i=1,2.$

Based on Asgeirsson's principle for characteristic quadrilaterals with vertices respectively at points

$$\left(0,0\right),\,\left(\frac{1}{2},\frac{1}{2}\right),\left(\frac{2-x}{2},\frac{x}{2}\right),\left(\frac{1-x}{2},\,-\frac{1-x}{2}\right)$$

and

$$\left(0,0\right),\,\left(\frac{1}{2},\,-\frac{1}{2}\right),\left(\frac{2-x}{2},\,-\frac{x}{2}\right),\left(\frac{1-x}{2},\,\frac{1-x}{2}\right),$$

we are convinced of the equivalence of problem 2.4 to the following algebraic system

$$u\left(\frac{x}{2},\,\frac{x}{2}\right)+\alpha_1\left(x\right)\,u\left(\frac{1-x}{2},-\frac{1-x}{2}\right)=\gamma_1\left(x\right)-\alpha_1\left(x\right)\,\left[u\left(\frac{1}{2},\,\frac{1}{2}\right)-u\left(0,0\right)\right],$$

$$u\left(\frac{x}{2},-\frac{x}{2}\right)+\alpha_2\left(x\right)\,u\left(\frac{1-x}{2},\,\frac{1-x}{2}\right)=\gamma_2\left(x\right)-\alpha_2\left(x\right)\,\left[u\left(\frac{1}{2},-\frac{1}{2}\right)-u\left(0,0\right)\right].$$

Therefore, problem 2.4 is equivalently reduced to problem 2.3.

Problem 2.5. Find regular in the area Ω_1 solution $u\left(x,y\right)$ of equation (1), satisfying conditions

$$u\left(\frac{x}{2},\,\frac{x}{2}\right)+\alpha_1\left(x\right)\,u\,\left(\frac{2-x}{2},\,-\frac{x}{2}\right)=\gamma_1\left(x\right),\quad x\in\bar{J},$$

$$u\left(\frac{x}{2},\,-\frac{x}{2}\right)+\alpha_2\left(x\right)\,u\,\left(\frac{2-x}{2},\,\frac{x}{2}\right)=\gamma_2\left(x\right),\quad x\in\bar{J},$$

where

$$\alpha_1\left(0\right)\neq\alpha_2\left(0\right),\,\alpha_1\left(1\right)\,\cdot\alpha_2\left(1\right)\neq 1, \tag{9}$$

$$\alpha_1\left(x\right)\cdot\alpha_1\left(1-x\right)\neq 1,\alpha_2\left(x\right)\cdot\alpha_2\left(1-x\right)\neq 1\;\forall\,x\in\bar{J}, \tag{10}$$

$$\alpha_i\left(x\right),\,\gamma_i\left(x\right)\in C\left(\bar{J}\right)\cap C^2\left(J\right),\;i=1,2.$$

Using the same reasoning as in problem 2.4, we are convinced that problem 2.5 is equivalent to the following algebraic system

$$u\left(\frac{x}{2},\,\frac{x}{2}\right)+\alpha_1\left(x\right)\,u\left(\frac{1-x}{2},\,\frac{1-x}{2}\right)=\gamma_1\left(x\right)-\alpha_1\left(x\right)\,\left[u\left(\frac{1}{2},-\frac{1}{2}\right)-u\left(0,0\right)\right],$$

$$u\left(\frac{x}{2}, -\frac{x}{2}\right) + \alpha_2(x)\, u\left(\frac{1-x}{2}, \frac{x-1}{2}\right) = \gamma_2(x) - \alpha_2(x)\, \left[u\left(\frac{1}{2}, \frac{1}{2}\right) - u(0,0)\right],$$

which is apparently equivalently reduced to the next two algebraic systems

$$\begin{cases} u\left(\frac{x}{2}, \frac{x}{2}\right) + \alpha_1(x)\, u\left(\frac{1-x}{2}, \frac{1-x}{2}\right) = \gamma_1(x) - \alpha_1(x)\, \left[u\left(\frac{1}{2}, -\frac{1}{2}\right) - u(0,0)\right], \\ \alpha_1(1-x)\, u\left(\frac{x}{2}, \frac{x}{2}\right) + u\left(\frac{1-x}{2}, \frac{1-x}{2}\right) = \gamma_1(1-x) - \alpha_1(1-x)\, \left[u\left(\frac{1}{2}, -\frac{1}{2}\right) - u(0,0)\right] \end{cases}$$

and

$$\begin{cases} u\left(\frac{x}{2}, -\frac{x}{2}\right) + \alpha_2(x)\, u\left(\frac{1-x}{2}, \frac{x-1}{2}\right) = \gamma_2(x) - \alpha_2(x)\, \left[u\left(\frac{1}{2}, \frac{1}{2}\right) - u(0,0)\right], \\ \alpha_2(1-x)\, u\left(\frac{x}{2}, -\frac{x}{2}\right) + u\left(\frac{1-x}{2}, \frac{x-1}{2}\right) = \gamma_2(1-x) - \alpha_2(1-x)\, \left[u\left(\frac{1}{2}, \frac{1}{2}\right) - u(0,0)\right]. \end{cases}$$

If the conditions (9) and (10) are met, the last two systems are unambiguously and unconditionally solvable. Therefore, problem 2.5, as in the case of problem 2.1, is equivalently reduced to Goursat problem.

3 Non-local problems with intra-boundary displacement

Problem 3.1. Find regular in the area Ω_1 solution $u(x,y)$ of equation (1), satisfying conditions

$$u\left(\frac{x}{2}, \frac{x}{2}\right) + \alpha\, u\left(\frac{1}{2}, \frac{1}{2} - x\right) = \gamma(x), \;\; x \in \bar{J},$$

$$u(x,0) = \tau(x), \;\; x \in \bar{J}, \tag{11}$$

where

$$\alpha \neq \frac{1}{2},\; \alpha\left[\gamma(1) - \alpha\tau(1)\right] = \left(\alpha^2 + \alpha - 1\right)\tau(0) - (1-\alpha)\,\gamma(0), \tag{12}$$

$$\tau(x),\, \gamma(x) \in C\left(\bar{J}\right) \cap C^2(J). \tag{13}$$

Based on Asgeirsson's principle for characteristic quadrilaterals with vertices respectively at points

$$\left(\frac{x}{2}, \frac{x}{2}\right), \left(\frac{1}{2}, \frac{1}{2} - x\right), \left(\frac{1-x}{2}, \frac{1-x}{2}\right) \text{ and } (x,0),$$

seeing the equivalence of problem 3.1 to the following algebraic system

$$u\left(\frac{x}{2}, \frac{x}{2}\right) + \alpha u\left(\frac{1}{2}, \frac{1}{2} - x\right) = \gamma(x),$$

$$u\left(\frac{x}{2}, \frac{x}{2}\right) + u\left(\frac{1}{2}, \frac{1}{2} - x\right) = u\left(\frac{1-x}{2}, \frac{1-x}{2}\right) + u(x,0).$$

From where

$$(1-\alpha)\, u\left(\frac{x}{2}, \frac{x}{2}\right) + \alpha\, u\left(\frac{1-x}{2}, \frac{1-x}{2}\right) = \gamma(x) - \alpha\,\tau(x). \tag{14}$$

Changing everywhere x on $1-x$, from the last equation we get

$$\alpha u\left(\frac{x}{2}, \frac{x}{2}\right) + (1-\alpha)\, u\left(\frac{1-x}{2}, \frac{1-x}{2}\right) = \gamma(1-x) - \alpha\,\tau(1-x). \tag{15}$$

From the system (14), (15) we find that

$$u\left(\frac{x}{2}, \frac{x}{2}\right) = \frac{1-\alpha}{1-2\alpha}\left[\gamma(x) - \alpha\tau(x)\right] - \frac{\alpha}{1-2\alpha}\left[\gamma(1-x) - \alpha\tau(1-x)\right]. \tag{16}$$

Consequently, the solution of problem 3.1 is equivalently reduced to the problem of Darboux (11), (16), the regular solution of which, when the conditions (12), (13) are met, exists only. Next, obtained in the area Ω_2 the solution can be naturally continued throughout Ω_1.

Problem 3.2. Find regular in the area Ω_2 solution $u(x, y)$ of equation (1), satisfying conditions

$$u\left(\frac{x}{2}, \frac{x}{2}\right) + \alpha u\left(\frac{1}{2}, \left|\frac{1}{2} - x\right|\right) = \gamma(x), \ x \in \bar{J},$$

$$u(x, 0) = \tau(x), \ x \in \bar{J}.$$

By virtue of the Asgeirsson principle $0 < x^* < \frac{1}{2}$ fair ratio

$$u\left(\frac{x^*}{2}, \frac{x^*}{2}\right) + u\left(\frac{1}{2}, \left|\frac{1}{2} - x^*\right|\right) = u(x^*, 0) + u\left(\frac{1 - x^*}{2}, \frac{1 - x^*}{2}\right).$$

For a symmetric point with respect to zero, the following ratio is valid

$$u\left(\frac{x^*}{2}, \frac{x^*}{2}\right) + u\left(\frac{1}{2}, \left|\frac{1}{2} - x^*\right|\right) = u(1 - x^*, 0) + u\left(\frac{1 - x^*}{2}, \frac{1 - x^*}{2}\right).$$

This suggests that the mean theorem, which is valid for all $x \in \bar{J}$ we can only use it when

$$u(x, 0) = u(1 - x, 0).$$

Theorem 3.1. Let $\tau(x), \gamma(x) \in C\left(\bar{J}\right) \cap C^2(J)$ and the conditions are met

$$\alpha \neq 1, \tau(x) \equiv \tau(1 - x) \ \forall x \in \bar{J}. \tag{17}$$

Then there is only one regular in the field Ω_2 problem solving 3.2.

When the conditions (17) are met, it is sufficient to repeat the same reasoning and calculations as in the study of problem 3.1 to prove the theorem 3.1.

Problem 3.3. Find a regular solution in the region of Ω_2 of the $u(x, y)$ equation (1) satisfying the conditions

$$u\left(\frac{x}{2}, \frac{x}{2}\right) + \beta u\left(\frac{1}{2}, \frac{x}{2}\right) = \gamma(x), \ x \in \bar{J}, \tag{18}$$

$$u(x, 0) = \tau(x), \ x \in \bar{J}. \tag{19}$$

Theorem 3.2. Let $|\beta| < 1$, $\tau(x), \gamma(x) \in C\left(\bar{J}\right) \cap C^2(J)$ and the condition is fulfilled

$$\tau(0) + \beta\tau\left(\frac{1}{2}\right) = \gamma(0).$$

Then problem 3.3 is uniquely solvable, its solution is represented as

$$\begin{aligned} u(x, y) = {} & \tau(x - y) + \beta\left[\tau\left(\frac{1 - x + y}{2}\right) - \tau\left(\frac{1 - x - y}{2}\right)\right] - \gamma(x - y) + \gamma(x + y) - \\ & - \sum_{j=1}^{\infty}\sum_{i=j}^{\infty}(-1)^i\beta^i\left[\gamma\left(\frac{(2j-1) + x - y}{2^i}\right) - \gamma\left(\frac{(2j-1) + x + y}{2^i}\right) - \right. \\ & -\gamma\left(\frac{(2j-1) - x + y}{2^i}\right) + \gamma\left(\frac{(2j-1) - x - y}{2^i}\right) + \\ & +\beta\tau\left(\frac{2(2j-1) - 1 + x - y}{2^{i+1}}\right) - \beta\tau\left(\frac{2(2j-1) + 1 + x + y}{2^{i+1}}\right) - \\ & \left. -\beta\tau\left(\frac{2(2j-1) - 1 - x + y}{2^{i+1}}\right) + \beta\tau\left(\frac{2(2j-1) + 1 - x - y}{2^{i+1}}\right)\right]. \end{aligned} \tag{20}$$

Proof. It is known that any regular solution of the equation (1) can be represented as

$$u(x,y)=f(x-y)+g(x+y), \tag{21}$$

where $f(x)$, $g(x)\in C^2(\Omega)\cap C(\bar{\Omega})$.

Satisfying (21) conditions (18), (19) for finding $f(x)$ and $g(x)$ we obtain the following system of functional equations

$$f(x)=\tau(x)-g(x), \tag{22}$$

$$g(x)-\beta g\left(\frac{1-x}{2}\right)+\beta g\left(\frac{1+x}{2}\right)=\gamma(x)-\beta\tau\left(\frac{1-x}{2}\right). \tag{23}$$

The solution of equation (23) under the condition $|\beta|<1$ can be constructed by the iteration method. Indeed

$$g_0(x)=\gamma(x)-\beta\tau\left(\frac{1-x}{2}\right),$$

$$g_1(x)=\gamma(x)-\beta\tau\left(\frac{1-x}{2}\right)+\beta\gamma\left(\frac{1-x}{2}\right)-\beta\gamma\left(\frac{1+x}{2}\right)-$$
$$-\beta^2\tau\left(\frac{1+x}{4}\right)+\beta^2\tau\left(\frac{3-x}{4}\right)$$

etc.

Continuing this process indefinitely, we get

$$\begin{gathered} g(x)=\gamma(x)-\beta\tau\left(\frac{1-x}{2}\right)+ \\ +\sum_{j=1}^{\infty}\sum_{i=j}^{\infty}(-1)^i\beta^i\left[\gamma\left(\frac{(2j-1)+x}{2^i}\right)-\gamma\left(\frac{(2j-1)-x}{2^i}\right)+\right. \\ \left.+\beta\tau\left(\frac{2(2j-1)-1+x}{2^{i+1}}\right)-\beta\tau\left(\frac{2(2j-1)+1-x}{2^{i+1}}\right)\right]. \end{gathered} \tag{24}$$

It is easily verified that the series itself (24) the series obtained after differentiation converge uniformly. Substituting (24) into (22) we find $f(x)$, and hence $u(x,y)$ by formula (21).

It should be noted that formula (20) in the case of $\beta=0$ coincides with the formula for solving the Darboux problem.

Conclusion

In this paper, all problems are considered in the characteristic quadrilateral; therefore, the choice of an inner manifold, the points of which are associated with the boundary manifold, is small. In the case of sufficiently derived domains (for example, a rectangular noncharacteristic domain), a problem arises both in the nonlocal condition itself and in the corresponding inner manifold. This method is of particular interest for weakly hyperbolic equations.

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Ішек тербелісінің теңдеуі үшін кейбір бейлокал шеттік және ішкі шеттік есептер

Жұмыста классикалық Гурса және Дарбу есептерін жалпылайтын бірқатар бейлокал есептер тұжырымдалған және зерттелген. Олардың кейбіреулері шеттік, ал екінші бөлігі ішкі шеттік болып табылады және екі жағдайда да сипаттамалық және сипаттамалық емес ығысулар қарастырылған. Сондай-ақ, мақалада қарастырылған бірқатар есептер ішек тербелісінің модельдік жүктелген теңдеуі үшін корректілі қойылған есептер теориясын құруда ерекше жағдай ретінде туындағанын атап өткен жөн.

Кілт сөздер: толқындық теңдеу, жалпы шешім, Коши есебі, Гурса есебі, Дарбу есебі, сипаттамалық ығысу есептері, сипаттамалық емес ығысу есептері.

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О некоторых нелокальных краевых и внутренних краевых задачах для уравнения колебания струны

В работе сформулированы и исследованы некоторые нелокальные задачи, обобщающие классические задачи Гурса и Дарбу. Часть из них являются краевыми, а другая — внутренними краевыми, причем в обоих случаях рассмотрены как характеристические, так и нехарактеристические смещения. Следует также отметить, что ряд задач, рассмотренных в статье, возникли как частный случай при построении теории корректных задач для модельного нагруженного уравнения колебания струны.

Ключевые слова: волновое уравнение, общее решение, задача Коши, задача Гурса, задача Дарбу, задача с характеристическим смещением, задача с нехарактеристическим смещением.

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On quasi-identities of finite modular lattices. II

The existence of a finite identity basis for any finite lattice was established by R. McKenzie in 1970, but the analogous statement for quasi-identities is incorrect. So, there is a finite lattice that does not have a finite quasi-identity basis and, V.A. Gorbunov and D.M. Smirnov asked which finite lattices have finite quasi-identity bases. In 1984 V.I. Tumanov conjectured that a proper quasivariety generated by a finite modular lattice is not finitely based. He also found two conditions for quasivarieties which provide this conjecture. We construct a finite modular lattice that does not satisfy Tumanov's conditions but quasivariety generated by this lattice is not finitely based.

Keywords: lattice, finite lattice, modular lattice, quasivariety, variety, quasi-identity, identity, finite basis of quasi-identities, Tumanov's conditions.

Introduction

In 1970 R. McKenzie [1] proved that any finite lattice has a finite basis of identities. However the similar result for quasi-identities is not true. That is, there is a finite lattice that has no finite basis of quasi-identities (V.P. Belkin [2]). The problem "Which finite lattices have finite basis of quasi-identities?" was suggested by V.A. Gorbunov and D.M. Smirnov in [3]. V.I. Tumanov [4] found a sufficient condition consisting of two parts under which a locally finite quasivariety of lattices has no finite (independent) basis of quasi-identities. He also conjectured that a finite (modular) lattice has a finite (independent) basis of quasi-identities if and only if a quasivariety generated by this lattice is a variety. In general, the conjecture is not true. W. Dziobiak [5] found a finite lattice that generates finitely axiomatizable proper quasivariety. We also would like to point out that Tumanov's problem is still unsolved for modular lattices.

The main goal of the paper is to present a finite modular lattice that has no finite basis of quasi-identities and does not satisfy conditions of Tumanov's theorem.

1 Basic concepts and preliminaries

We recall some basic definitions and results for quasivarieties that we will refer to. For more information on the basic notions of universal algebra and lattice theory introduced below and used throughout this paper, we refer to [6] and [7].

A *quasivariety* is a class of algebras that is closed with respect to subalgebras, direct products, and ultraproducts. Equivalently, a quasivariety is the same thing as a class of algebras axiomatized by a set of quasi-identities. A *quasi-identity* means a universal Horn sentence with the non-empty positive part, that is of the form

$$(\forall \bar{x})[p_1(\bar{x}) \approx q_1(\bar{x}) \wedge \cdots \wedge p_n(\bar{x}) \approx q_n(\bar{x}) \to p(\bar{x}) \approx q(\bar{x})],$$

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where $p, q, p_1, q_1, \dots, p_n, q_n$ are terms. A *variety* is a quasivariety which is closed under homomorphisms. According to Birkhoff's theorem [8], a variety is a class of similar algebras axiomatized by a set of identities, where by an identity we mean a sentence of the form $(\forall \bar{x})[s(\bar{x}) \approx t(\bar{x})]$ for some terms $s(\bar{x})$ and $t(\bar{x})$.

By $\mathbf{Q}(\mathbf{K})$ ($\mathbf{V}(\mathbf{K})$) we denote the smallest quasivariety (variety) containing a class $\mathbf{K}$. If $\mathbf{K}$ is a finite family of finite algebras then $\mathbf{Q}(\mathbf{K})$ is called finitely generated. In case when $\mathbf{K} = \{A\}$ we write $\mathbf{Q}(A)$ instead of $\mathbf{Q}(\{A\})$. By Maltsev-Vaught theorem [9], $\mathbf{Q}(\mathbf{K}) = \mathbf{SPP}_u(\mathbf{K})$, where $\mathbf{S}$, $\mathbf{P}$ and $\mathbf{P}_u$ are operators of taking subalgebras, direct products and ultraproducts, respectively.

Let $\mathbf{K}$ be a quasivariety. A congruence α on an algebra A is called a $\mathbf{K}$-*congruence* or *relative congruence* provided $A/\alpha \in \mathbf{K}$. The set $\mathrm{Con}_{\mathbf{K}}A$ of all $\mathbf{K}$-congruences of A forms an algebraic lattice with respect to inclusion $\subseteq$ which is called a *relative congruence lattice*.

For a quasivariety $\mathbf{K}$, an algebra $A \in \mathbf{K}$ is said to be *subdirectly* $\mathbf{K}$-*irreducible* if the least congruence 0_A is completely meet irreducible in $\mathrm{Con}_{\mathbf{K}}A$. By Birkhoff's theorem for a quasivariety, every algebra of a quasivariety $\mathbf{K}$ is a subdirect product of subdirectly $\mathbf{K}$-irreducible algebras ([7,8]). By $\mathbf{K}_{SI}$ we denote the class of all subdirectly $\mathbf{K}$-irreducible algebras in $\mathbf{K}$. Since $\mathbf{Q}(\mathbf{K}) = \mathbf{SPP}_u(\mathbf{K}) = \mathbf{P}_s\mathbf{SP}_u(\mathbf{K})$, where $\mathbf{P}_s$ is operator of taking subdirect products, we have $\mathbf{K}_{SI} \subseteq \mathbf{SP}_u(\mathbf{K})$. Thus, for finitely generated quasivariety $\mathbf{Q}(A)$, every subdirectly $\mathbf{Q}(A)$-irreducible algebra is isomorphic to some subalgebra of A.

The least $\mathbf{K}$-congruence $\theta_{\mathbf{K}}(a, b)$ on an algebra $A \in \mathbf{K}$ containing pair $(a, b) \in A \times A$ is called a *principal* $\mathbf{K}$-*congruence* or a *relative principal congruence*. In case when $\mathbf{K}$ is a variety, relative congruence $\theta_{\mathbf{K}}(a, b)$ is a usual principal congruence that we denote by $\theta(a, b)$.

Let $(a] = \{x \in L \mid x \leq a\}$ ($[a) = \{x \in L \mid x \geq a\}$) be a principal ideal (coideal) of a lattice L. A pair $(a, b) \in L \times L$ is called *dividing* (*semi-dividing*) if $L = (a] \cup [b)$ and $(a] \cap [b) = \varnothing$ ($L = (a] \cup [b)$ and $(a] \cap [b) \neq \varnothing$).

For any semi-dividing pair (a, b) of a lattice M we define a lattice

$$M_{a-b} = \langle \{(x, 0), (y, 1) \in M \times 2 \mid x \in (a], y \in [b)\}; \vee, \wedge \rangle \leq_s M \times \mathbf{2},$$

where $\mathbf{2} = \langle \{0, 1\}; \vee, \wedge \rangle$ is a two element lattice.

Theorem 1. (Tumanov's theorem [4])

Let $\mathbf{M}$, $\mathbf{N}$ ($\mathbf{N} \subset \mathbf{M}$) be locally finite quasivarieties of lattices satisfying the following conditions:

a) in any finitely subdirectly $\mathbf{M}$-irreducible lattice $M \in \mathbf{M} \backslash \mathbf{N}$ there is a semi-dividing pair (a, b) such that $M_{a-b} \in \mathbf{N}$;

b) there exists a finite simple lattice $P \in \mathbf{N}$ which is not a proper homomorphic image of any subdirectly $\mathbf{N}$-irreducible lattice.

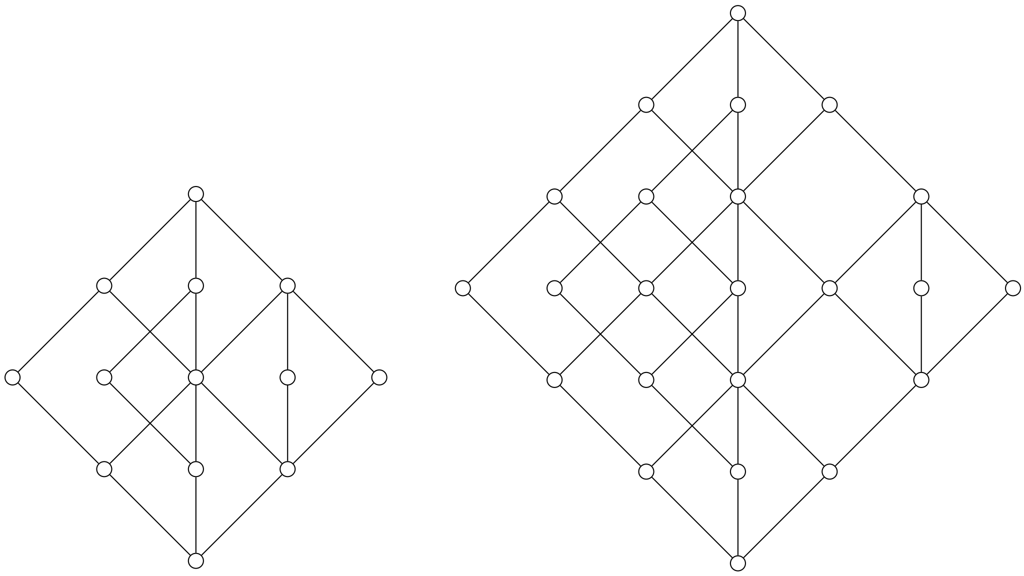
Then the quasivariety $\mathbf{N}$ has no coverings in the lattice of subquasivarieties of $\mathbf{M}$. In particular, $\mathbf{N}$ has no finite (independent) basis of quasi-identities provided $\mathbf{M}$ is finitely axiomatizable.

A subalgebra B of an algebra A is called *proper* if $B \not\cong A$. We will use the following folklore criterion of non-finite axiomatizability of quasivarieties (see [7]).

Lemma 1. A locally finite quasivariety $\mathbf{K}$ is not finitely axiomatizable if for any positive integer n there is a finite algebra L_n such that $L_n \notin \mathbf{K}$ and every proper subalgebra of L_n belongs to $\mathbf{K}$.

2 *Main result*

In this chapter we show that there are two locally finite quasivarieties of modular lattices $\mathbf{N}$ and $\mathbf{M}$, $\mathbf{N} \subset \mathbf{M}$, that do not satisfy conditions *a*) and *b*) of Tumanov's theorem, however, $\mathbf{N}$ is not finitely axiomatizable. Note that in our example we do not need to require that $\mathbf{M}$ be finitely axiomatizable. We also note that the first example of a finite lattice that does not satisfy the condition *b*) and has no finite basis of quasi-identities was provided in [10].



A' A

Figure 1. Lattices A' and A

Let A' and A are the modular lattices displayed in Figure 1. And let $\mathbf{Q}(A)$ and $\mathbf{V}(A)$ are quasivariety and variety generated by A, respectively. Since every subdirectly $\mathbf{Q}(A)$-irreducible lattice is a sublattice of A, and A' is simple, a homomorphic image of A and is not a sublattice of A we have $pA' \in \mathbf{V}(A)\backslash\mathbf{Q}(A)$, that is $\mathbf{Q}(A)$ is a proper quasivariety. One can check that A' has no semi-dividing pair. Thus, the condition $a)$ of Tumanov's theorem does not hold on the quasivariety $\mathbf{Q}(A)$. It is easy to see that M_3 is a unique simple lattice in $\mathbf{Q}(A)_{SI}$ and it is a homomorphic image of A. Hence, the condition $b)$ of Tumanov's theorem is not valid for quasivarieties $\mathbf{Q}(A)$ and $\mathbf{V}(A)$. Thus, to establish our main result we have to prove.

Theorem 2. Quasivariety $\mathbf{Q}(A)$ generated by the lattice A is not finitely based.

Proof. To prove the theorem we modify the proof of the second part of Theorem 3.4 from [11] (also see [10]).

According to Lemma 1 we will construct an infinite set $\{L_n \mid n \geq 0\}$ of finite modular lattices such that $L_n \in \mathbf{V}(A)\backslash\mathbf{Q}(A)$ and every n-generated subalgebra of L_n belongs to $\mathbf{Q}(A)$.

Let S be a non-empty subset of a lattice L. Denote by $\langle S\rangle$ the sublattice of L generated by S.

We define a modular lattice L_n by induction:

$n = 0$. $L_0 \cong M_{3-3}$ and $L_0 = \langle\{a_0, b_0, c_0, a^0, b^0, c^0\}\rangle$ (see Fig. 2).

$n = 1$. L_1 is a modular lattice generated by $L_0 \cup \{a_1, b_1, c_1, a^1, b^1, c^1\}$ such that $\langle\{a_1, b_1, c_1, a^1, b^1, c^1\}\rangle \cong M_{3-3}$, and $c_0 = a^1$, $a^0 \wedge b^0 = c_0 \vee b^1 = c_0 \vee c_1$ (see Fig. 3).

$n > 1$. L_n is a modular lattice generated by the set $L_{n-1} \cup \{a_n, b_n, c_n, a^n, b^n, c^n\}$ such that $\langle\{a_n, b_n, c_n, a^n, b^n, c^n\}\rangle \cong M_{3-3}$, and $c_{n-1} = a^n$, $a^0 \wedge b^0 = c_0 \vee b^n = c_0 \vee c_n$ (see Fig. 4).

Claim 1. For any $n > 0$, the lattice L_n does not belong to $\mathbf{Q}(A)$.

Proof. We prove by induction on $n > 0$.

$n = 1$. Assume that $L_1 \in \mathbf{Q}(A)$. At first we note that $M_{3,3}$ is a sublattice of $L_1/\theta(a_1, b_1)$ and $L_1/\theta(a_1, a^1 \wedge b^1)$. Hence, $(a_0, b_0) \in \theta_{\mathbf{Q}(A)}(a_1, b_1) \cap \theta_{\mathbf{Q}(A)}(a_1, a^1 \wedge b^1)$. One can also see that any non-trivial congruence contains (a_1, b_1) or $(a_1, a^1 \wedge b^1)$ or (a_0, b_0). Therefore, intersection of any two different non-trivial $\mathbf{Q}(A)$-congruences contains (a_0, b_0). It means that L_1 is subdirectly $\mathbf{Q}(A)$-irreducible. In this event, L_1 is a sublattice of A because $\mathbf{Q}(A)_{SI} \subseteq \mathbf{S}(A)$. One can check that L_1 is not a sublattice of A. Thus, L_1 does not belong to $\mathbf{Q}(A)$. Also we have $(a_0, b_0) \in \theta$ for any non-trivial $\theta \in \mathrm{Con}_{\mathbf{Q}(A)} L_1$.

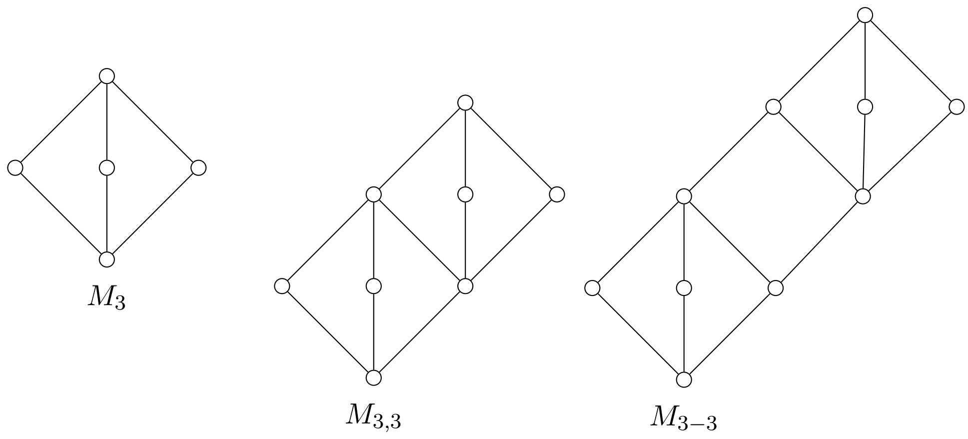


Figure 2. Lattices M_3, $M_{3,3}$ and M_{3-3}

$n > 1$. By induction, we have $L_{n-1} \notin \mathbf{Q}(A)$ and $(a_0, b_0) \in \theta$ for any non-trivial $\theta \in \mathrm{Con}_{\mathbf{Q}(A)} L_{n-1}$.

Assume that $L_n \in \mathbf{Q}(A)$. We note that $M_{3,3}$ is a sublattice of $L_n/\theta(a_i, a^i \wedge b^i)$, L_{n-1} is a sublattice of $L_n/\theta(a_i, b_i) \leq_s L_{n-1} \times \mathbf{2}$ and $L_n/\theta(a_i, a^i) \cong L_{n-1}$, for all $0 < i \leq n$. Hence, $(a_i, a^i) \in \theta_{\mathbf{Q}(A)}(a_i, a^i \wedge b^i)$. It means that any homomorphic image of L_n that belongs to $\mathbf{Q}(A)$ is a homomorphic image of L_{n-1} or some $S \leq_s L_{n-1} \times \mathbf{2}$ or $L_n^- \cong L_n/\theta(a_0, b_0)$.

Let $\theta \in \mathrm{Con}_{\mathbf{Q}(A)} L_n$. If $(a_i, a^i) \in \theta$ then $\theta/\theta(a_i, a^i) \in \mathrm{Con}_{\mathbf{Q}(A)} L_n/\theta(a_i, a^i) \cong \mathrm{Con}_{\mathbf{Q}(A)} L_{n-1}$. By induction, $(a_0/\theta(a_i, a^i), b_0/\theta(a_i, a^i)) \in \theta/\theta(a_i, a^i)$. Since the congruence classes $a_0\theta(a_i, a^i)$ and $b_0\theta(a_i, a^i))$ consist of one elements $\{a_0\}$ and $\{b_0\}$, respectively, we get $(a_0, b_0) \in \theta$.

If $(a_i, b_i) \in \theta$ then $L_n/\theta(a_i, b_i) \leq_s L_{n-1} \times \mathbf{2}$. Since $L_{n-1} \leq L_n/\theta(a_i, b_i)$ then $L_{n-1}/(\theta \cap L_{n-1}^2) \in \mathbf{Q}(A)$. By induction, $(a_0/(\theta \cap L_{n-1}^2), b_0/(\theta \cap L_{n-1}^2)) \in (\theta \cap L_{n-1}^2)$. By argument above, $(a_0, b_0) \in \theta$.

Thus, we have that $(a_0, b_0) \in \theta$ for each non-trivial $\mathbf{Q}(A)$-congruence θ. It means that L_n is relative subdirectly irreducible. Hence, $L_n \leq A$. Contradiction. Therefore, $L_n \notin \mathbf{Q}(A)$.

Let L_n^- be a sublattice of L_n generated by the set $\{a_i, b_i, c_i, a^i, b^i, c^i \mid 0 < i \leq n\}$. One can see that $L_n^- \cong L_n/\theta(a_0, b_0)$ and $L_n^- \leq_s M_{3-3}^n$. Hence, $L_n^- \in \mathbf{Q}(A)$.

Claim 2. Every proper sublattice of L_n belongs to $\mathbf{Q}(A)$.

Proof. It is enough to prove the claim for arbitrary maximal proper sublattices of L_n. Since L_n is generated by the set of double irreducible elements $S = \{a_0, b_0, b^0, c^0, c_n\} \cup \{b_i, b^i \mid 0 < i \leq n\}$ then every maximal proper sublattices L of L_n generated by $S - \{x\}$ for some $x \in S$, that is $L = \langle S - \{x\}\rangle$.

Suppose that $x \in \{a_0, b_0, b^0, c^0\}$. Then the lattice $\langle\{a_0, b_0, c_0, a^0, b^0, c^0\} - \{x\}\rangle/\theta(c_0, a^0 \wedge b^0)$ is a homomorphic image of L with the kernel $\alpha = \theta(a_1, c_n)$ and belongs to $\mathbf{Q}(A)$.

One can see that for $\beta = \theta(a_0, b_0)$ if $x \in \{b^0, c^0\}$ and $\beta = \theta(b^0, c^0)$ if $x \in \{a_0, b_0\}$, L/β is isomorphic to L_n^- or $L_n^- \times \mathbf{2}$ and both these lattices belong to $\mathbf{Q}(A)$. Thus, α and β are $\mathbf{Q}(A)$-congruences. One can check that $\alpha \cap \beta = 0$. Hence $L \leq_s L/\alpha \times L/\beta$. Therefore, $L \in \mathbf{Q}(A)$.

Suppose that $x \in \{b_i, b^i \mid 0 < i \leq n\} \cup \{c_n\}$. For sake of brevity, we assume that $x = b_n$. Let $\alpha = \theta(c_0, c_{n-1})$. Then L/α is isomorphic to the sublattice S of L_1 generated by the set $\{a_0, b_0, c_0, a^0, b^0, c^0, a_1, b_1, b^1\}$. Since the lattice $P = \langle\{a_0, b_0, c_0, a^0, b^0, c^0, b^1, c^1\}\rangle$ is a sublattice of A and $S \leq_s P \times \mathbf{2}^2$ we get $S \in \mathbf{Q}(A)$. On the other hand, $L/\theta(a_0, b_0)$ is a sublattice of L_n^-. Since $L_n^- \in \mathbf{Q}(A)$ then $L/\theta(a_0, b_0) \in \mathbf{Q}(A)$. One can see that $\alpha \cap \theta(a_0, b_0) = 0$. Hence, L is a subdirect product of two lattices from $\mathbf{Q}(A)$. Therefore, $L \in \mathbf{Q}(A)$.

Thus, we obtain that $L_n \notin \mathbf{Q}(A)$ and every its proper sublattice belongs to $\mathbf{Q}(A)$. By Lemma 1, the quasivariety generated by the lattice A is not finitely based.

From the proof of Theorem 2 we have more general result:

Theorem 3. Suppose that $\mathbf{K}$ is a locally finite quasivariety and

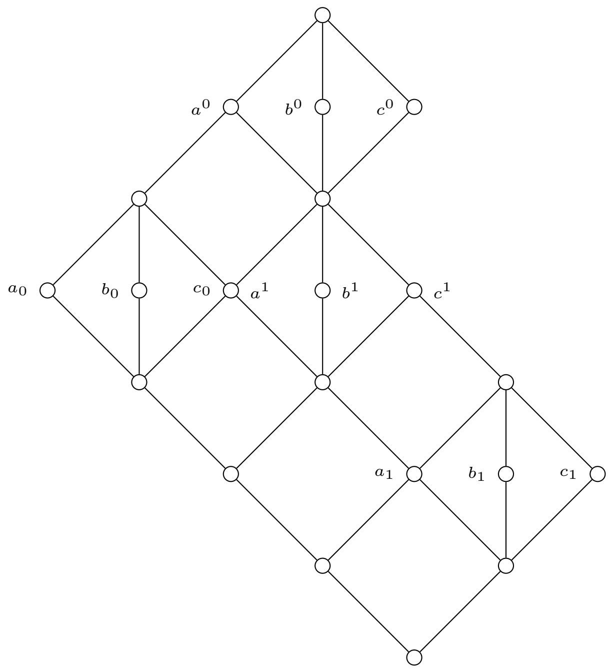


Figure 3. Lattice L_1

a) $M_{3,3} \notin \mathbf{K}$,
b) every proper sublattice of L_n belongs to $\mathbf{K}$,
c) $L_n \notin \mathbf{K}$ for all $n > 0$.
Then the quasivariety $\mathbf{K}$ is not finitely axiomatizable.

Corollary 1. There is an infinite number of finite lattices which do not satisfy conditions of Tumanov's theorem and have no finite basis of quasi-identities.

Indeed, the lattice A completed by n atoms $e_1, \ldots, e_n$ such that $e_i \vee e_j = a_0 \vee b_0$, $i \neq j \leq n$, satisfies the conditions of Theorem 2.

We note that the variety lattice of a variety $\mathbf{V}(A)$ is finite because it contains a finite number of subdirectly irreducible lattices by Johnson's Lemma [12]. G. Grätzer and H. Lasker [13] shown that the quasivariety lattice of a variety $\mathbf{V}(M_{3,3})$ is continuum. Since $M_{3,3} \in \mathbf{V}(A)$ we have that the quasivariety lattice of $\mathbf{V}(A)$ is continuum.

We would like to point out that V.I. Tumanov also provided that in his theorem the quasivariety $\mathbf{N}$ has no independent basis. Our proof does not allow to prove that $\mathbf{Q}(A)$ has no independent basis of quasi-identities. On the other hand, our proof holds on $\mathbf{K}$ that is not necessarily included in the finitely axiomatizable locally finite quasivariety.

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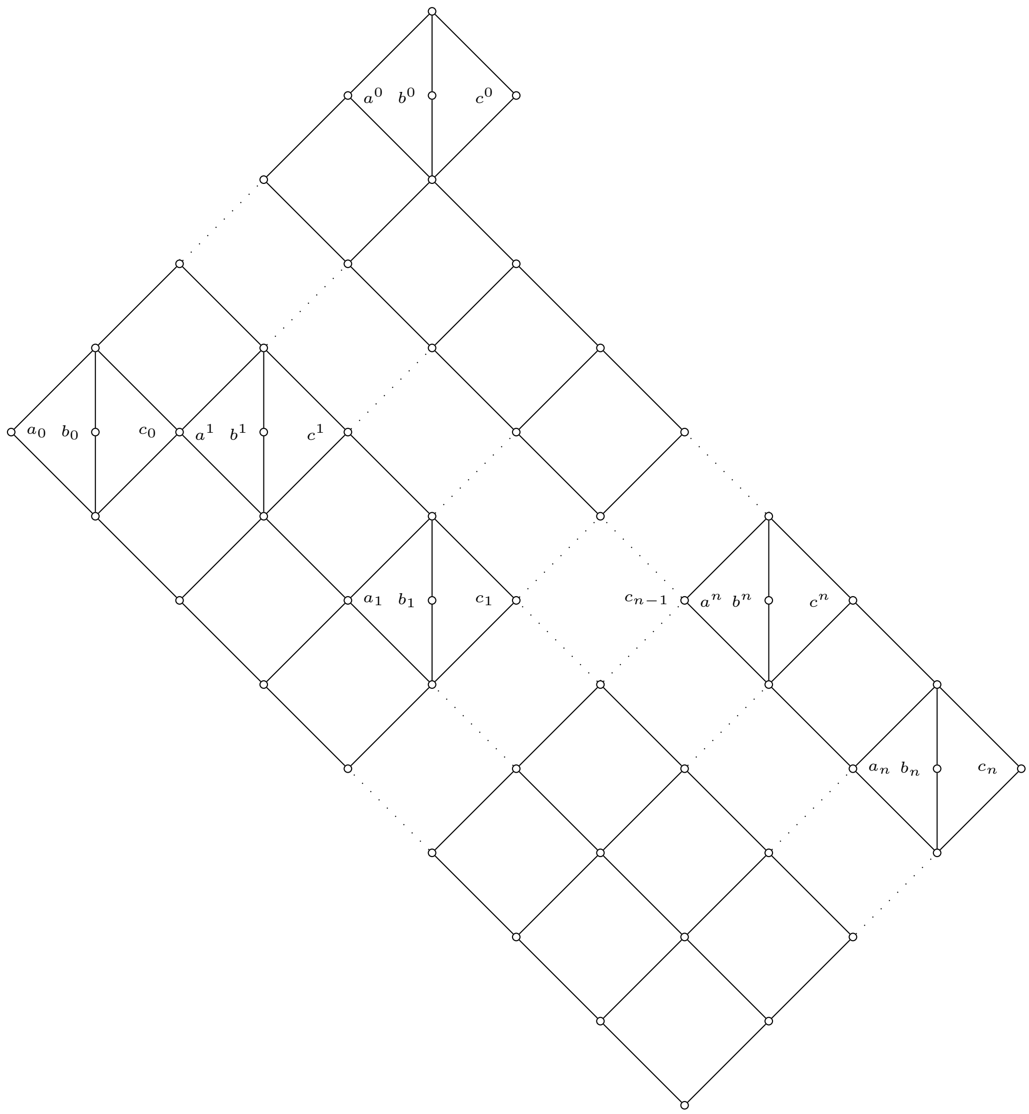

Figure 4. Lattice L_n, $n \geq 2$

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Соңғы модулярлық торлардың квазитепе-теңдіктері туралы. II

1970 жылы Р. Маккензи кез келген шекті тордың ақырлы базисті тепе-теңдіктері болатынын дәлелдеді. Дегенмен, квазитепе-теңдіктерге қатысты бұл мәлімдеме дұрыс емес. Сонымен, ақырлы базисі жоқ квазитепе-теңдіктерден шекті торлар бар. В.А. Горбунов пен Д.М. Смирнов келесі мәселені қозғады: «Ақырлы базисі бар квазитепе-теңдіктерден тұратын қандай шекті торлар бар?». 1984 жылы В.И. Туманов шекті модулярлы тордан туындаған өздік квазикөпбейненің ақырлы базисі жоқ деген болжам айтты. Ол сондай-ақ осы болжамды қамтамасыз ететін квазикөпбейнелердің екі шартын тапты. Ал біз Тумановтың шарттарын қанағаттандырмайтын шекті модулярлы торды құрастырдық, бірақ бұл тордан туындаған квазикөпбейненің ақырлы базисі жоқ.

Кілт сөздер: тор, соңғы тор, модулярлық тор, квазикөпбейне, көпбейне, квазитепе-теңдік, тепе-теңдік, квазитепе-теңдіктің соңғы базисі, Тумановтың шарттары.

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О квазитождествах конечных модулярных решеток. II

В 1970 г. Р. Маккензи доказал, что любая конечная решетка имеет конечный базис тождеств. Однако аналогичное утверждение для квазитождеств неверно. Итак, существуют конечные решетки, которые не имеют конечного базиса квазитождеств. В.А. Горбунов и Д.М. Смирнов озвучили следующую проблему: «Какие конечные решетки имеют конечный базис квазитождеств?» В 1984 г.

В.И. Туманов предположил, что собственное квазимногообразие, порожденное конечной модулярной решеткой, не является конечно базируемым. Он также нашел два условия для квазимногообразий, которые подтверждают эту гипотезу. Мы же построили конечную модулярную решетку, которая не удовлетворяет условиям Туманова, но квазимногообразие, порожденное этой решеткой, не является конечно базируемым.

Ключевые слова: решетка, конечная решетка, модулярная решетка, квазимногообразие, многообразие, квазитождество, тождество, конечный базис квазитождеств, условия Туманова.

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Cones generated by a generalized fractional maximal function

The paper considers the space of generalized fractional-maximal function, constructed on the basis of a rearrangement-invariant space. Two types of cones generated by a nonincreasing rearrangement of a generalized fractional-maximal function and equipped with positive homogeneous functionals are constructed. The question of embedding the space of generalized fractional-maximal function in a rearrangement-invariant space is investigated. This question reduces to the embedding of the considered cone in the corresponding rearrangement-invariant spaces. In addition, conditions for covering a cone generated by generalized fractional-maximal function by the cone generated by generalized Riesz potentials are given. Cones from non-increasing rearrangements of generalized potentials were previously considered in the works of M. Goldman, E. Bakhtigareeva, G. Karshygina and others.

Keywords: rearrangement-invariant spaces, non-increasing rearrangements of functions, cones generated by generalized fractional-maximal function, covering of cones.

Introduction

In this work two types of cones of non-negative monotonically non-increasing functions on the positive semiaxis generated by generalized fractional maximal functions and equipped with corresponding positively homogeneous functionals are introduced. We give the conditions on the function Φ, under which there are pointwise mutual covering of these cones.

In the work of Hakim D.I., Nakai E., Sawano Y. [1], Kucukaslan A. [2], Mustafayev R., Bilgicli N. [3], Gogatishvili A. [4] a generalized fractional-maximal functions of another type were defined, a particular case of which is the classical fractional-maximal function.

It is known that the maximal function is a very important operator in the theory of functions. With their help, many important issues of the theory of function and harmonic analysis are solved. The generalized fractional-maximal functions are also closely related to the generalized Riesz potentials, considered in the works of Goldman M.L. [5–7] (see also [8–10]).

The study of various properties of operators using a generalized fractional-maximal function is sometimes easier than the study of such operators using a generalized potential.

In this paper, we aim to determine the cones of non-negative measurable functions generated by a generalized fractional-maximal function and to investigate the properties of such cones.

1 Definitions, notation, and auxiliary statements

Let (S, Σ, μ) be space with a measure. Here is Σ is σ-algebra of subsets of the set S, on which is determined a non-negative σ-finite, σ-additive measure μ. By $L_0 = L_0(S, \Sigma, \mu)$ denotes the set of μ-measurable real-valued functions $f : S \to R$, and by L_0^+ a subset of the set L_0 consisting of non-negative functions:

$$L_0^+ = \{f \in L_0 : f \geq 0\}.$$

By $L_0^+(0, \infty; \downarrow)$ we denote the set of all non-increasing functions belonging to L_0^+.

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Definition 1. [11] A mapping $\rho : L_0^+ \to [0,\infty]$ is called a *functional norm* (short: FN), if the next conditions are met for all $f, g, f_n \in L_0^+,\ \ n \in N$:

$(P1)$ $\rho(f) = 0 \Rightarrow f = 0$, $\mu-$ almost everywhere (briefly: $\mu-$ a.e.);
$\rho(\alpha f) = \alpha\rho(f),\ \alpha \geq 0;\ \rho(f+g) \leq \rho(f) + \rho(g)$ (properties of the norm);

$(P2)$ $f \leq g$, ($\mu-$ a.e.) $\Rightarrow \rho(f) \leq \rho(g)$ (monotony of the norm);

$(P3)$ $f_n \uparrow f \Rightarrow \rho(f_n) \to \rho(f)(n \to \infty)$ (the Fatou property);

$(P4)$ $0 < \mu(\sigma) < \infty \Rightarrow \int\limits_\sigma f d\mu \leq c_\sigma \rho(f),\ f \in L_0^+$ (Local integrability);

$(P5)$ $0 < \mu(\sigma) < \infty \Rightarrow \rho(\chi_\sigma) < \infty$ (finiteness of the FN for characteristic functions (χ_σ) of sets of finite measure).

Here $f_n \uparrow f$ means that $f_n \leq f_{n+1}, \lim\limits_{n\to\infty} f_n = f$ (μ-a.e.).

Definition 2. Let ρ be a functional norm. The set of functions $X = X(\rho)$ from L_0, for which $\rho(|f|) < \infty$ is called a *Banach function space* (briefly: BFS), generated by the FN ρ. For $f \in X$ we assume

$$\|f\|_X = \rho(|f|).$$

Let $L_0 = L_0(R^n)$ be the set of all Lebesgue measurable functions $f : R^n \to C$; $\dot{L}_0 = \dot{L}_0(R^n)$ be the set of functions $f \in L_0$, for which the non-increasing rearrangement of the f^* is not identical to infinity. Non-increasing rearrangement f^* is defined by the equality:

$$f^*(t) = inf\{y \in [0;\infty) :\ \ \lambda_f(y) \leq t\},\ \ t \in R_+ = (0;\infty),$$

where

$$\lambda_f(y) = \mu_n\left\{x \in R^n :\ \ |f(x)| > y\right\},\quad y \in [0,\infty)$$

is the Lebesgue distribution function. It is known that f^* is a non-negative, non-increasing and right-continuous function on $\mathbb{R}_+$; f^* is equimeasurable with $|f|$, i.e.

$$\mu_1\left\{t \in \mathbb{R}_+ :\ \ f^*(t) > y\right\} = \mu_n\left\{x \in \mathbb{R}^n :\ \ |f(x)| > y\right\},$$

here μ is the Lebesgue measure (on $\mathbb{R}^n$ or on $\mathbb{R}_+$, respectively, see [1]).

Let $f^\# : \mathbb{R}^n \to \mathbb{R}^n$ denote a symmetric rearrangement of f, i.e. a radially symmetric non-negative non-increasing right continuous function (as a function of $r = |x|$, $x \in \mathbb{R}^n$) that is equimeasurable with f. That is

$$f^\#(r) = f^*(v_n r^n);\ f^*(t) = f^\#\left((\frac{t}{v_n})^{\frac{1}{n}}\right),\ \ r, t \in \mathbb{R}_+,$$

here v_n is the volume of the n-dimensional unit ball.

The function $f^{**} : (0,\infty) \to [0,\infty]$ is defined as

$$f^{**}(t) = \frac{1}{t}\int\limits_0^t f^*(\tau)d\tau;\ \ t \in R_+.$$

It is clear that f^{**} is a non-increasing function on R_+.

Really, let $t_1 \leq t_2$, then

$$\begin{aligned} f^{**}(t_2) &= \frac{1}{t_2}\int\limits_0^{t_2} f^*(\tau)d\tau = \frac{1}{t_2}\int\limits_0^{t_1} f^*(\tau)d\tau + \frac{1}{t_2}\int\limits_{t_1}^{t_2} f^*(\tau)d\tau \leq \\ &\leq \frac{1}{t_2}\int\limits_0^{t_1} f^*(\tau)d\tau + f^*(t_1)\cdot\frac{t_2 - t_1}{t_2}. \end{aligned}$$

Hence, we have

$$f^{**}(t_2) \le \frac{1}{t_2}\int_0^{t_1} f^*(\tau)d\tau + \frac{t_2-t_1}{t_2 t_1}\int_0^{t_1} f^*(\tau)d\tau \le \left(\frac{1}{t_2}+\frac{t_2-t_1}{t_2t_1}\right)\int_0^{t_1} f^*(\tau)d\tau =$$
$$= \frac{1}{t_1}\int_0^{t_1} f^*(\tau)d\tau = f^{**}(t_1).$$

Definition 3. A functional norm ρ is said to be *rearrangement-invariant* if

$$f^* \le g^* \Rightarrow \rho(f) \le \rho(g).$$

Banach function space $X = X(\rho)$, generated by a rearrangement invariant functional norm ρ will be called a *rearrangement invariant space* (in short: RIS).

Example 1. Let $S = R^n$, $\mu \equiv \mu_n$ be the Lebesgue measure in R^n, $1 \le p \le \infty$; $u \in L_0(R^n)$, $0 < u < \infty$, (μ-a.e.); $u \in L_p^{loc}(R^n)$, $\frac{1}{u} \in L_{p'}^{loc}(R^n)$, $\frac{1}{p}+\frac{1}{p'}=1$.

The space $X = L_{p,u}(R^n)$ with a norm $f_X = f_{L_{p,u}}$ i.e.:

$$\|f\|_X = \left(\int_{R^n} |fu|^p d\mu\right)^{\frac{1}{p}},\ 1 \le p < \infty;\ \|f\|_X = \|fu\|_{L_\infty},\ p = \infty$$

is a BFS. Associated space:

$$X' = L_{p',\frac{1}{u}}(R^n).$$

Everywhere in this work, we denote rearrangement invariant space (in short: RIS) by $E = E(R^n)$, and by $E' = E'(R^n)$ the associated rearrangement-invariant space and $\tilde{E} = \tilde{E}(R_+)$, $\tilde{E}' = \tilde{E}'(R_+)$ their Luxembourg representation, i.e. such RIS that

$$\|f\|_E = \|f^*\|_{\tilde{E}},\ \ \|g\|_{E'} = \|g^*\|_{\tilde{E}'}. \tag{1}$$

Let Ω_0 be a set of all nonnegative, finite on R_+, decreasing and right continuous functions:

$$\Omega_0 = \{g : R_+ \to [0;\infty);\ \ g\downarrow,\ \ g(t+0) = g(t),\ \ t \in R_+\}.$$

Definition 4. A function $f : R_+ \to R_+$ is called *quasi-decreasing* and is denoted by $f \downarrow$ (*quasi-increasing and is denoted by $f\uparrow$)* if there exists $C > 1$, such that

$$f(t_2) < Cf(t_1) \quad \text{if } t_1 < t_2$$
$$(f(t_1) < Cf(t_2) \ \text{ if } t_1 < t_2).$$

Throughout this work we will denote by C, C_1, C_2 positive constants, generally speaking, different in different places.

By the notation $f(x) \cong g(x)$ we mean that there are constants $C_1 > 0$, $C_2 > 0$ such that

$$C_1 f(t) \le g(t) \le C_2 f(t),\ \ t \in \mathbb{R}_+.$$

Definition 5. Let $n \in \mathbb{N}$ and $R \in (0;\infty]$. We say that a function $\Phi : (0;R) \to R_+$ belongs to the class $A_n(R)$ if:

(1) Φ is non-increasing and continuous on $(0;R)$;

(2) the function $\Phi(r)r^n$ is quasi-increasing on $(0,R)$.

For example, $\Phi(t) = t^{-\alpha} \in A_n(\infty)$, $\ 0 < \alpha < n$.

Definition 6. [12] Let $n \in \mathbb{N}$ and $R \in (0;\infty]$. A function $\Phi : (0;R) \to R_+$ belongs to the class $B_n(R)$ if the following conditions hold:

(1) Φ is non-increasing and continuous on $(0;R)$;

(2) there exists $C > 0$ such that

$$\int_0^r \Phi(\rho)\rho^{n-1}d\rho \le C\Phi(r)r^n, \quad r \in (0,R). \tag{2}$$

For example,

$$\Phi(\rho) = \rho^{\alpha-n} \in B_n(\infty)\ (0<\alpha<n); \quad \Phi(\rho) = ln\frac{eR}{\rho} \in B_n(R),\ R \in R_+.$$

For $\Phi \in B_n(R)$ the following estimate also holds

$$\int_0^r \Phi(\rho)\rho^{n-1}d\rho \ge n^{-1}\Phi(r)r^n,\ r \in (0,R).$$

Therefore

$$\int_0^r \Phi(\rho)\rho^{n-1}d\rho \cong \Phi(r)r^n,\ r \in (0,R), \tag{3}$$

$$\Phi \in B_n(R) \Rightarrow \{0 \le \Phi \downarrow;\ \Phi(r)r^n.\uparrow,\ r \in (0,R)\}. \tag{4}$$

Definition 7. Let $\Phi \in A_n(\infty)$. The *generalized fractional-maximal function* $M_\Phi f$ is defined for the function $f \in L^1_{loc}(\mathbb{R}^n)$ by

$$(M_\Phi f)(x) = \sup_{r>0} \Phi(r) \int_{B(x,r)} |f(y)|dy,$$

where $B(x,r)$ is a ball with the center at the point x and radius r. That is, consider the operator M_Φ: $L^1_{loc}(R^n) \to \dot{L}_0(R^n)$.

In the case $\Phi(r) = r^{\alpha-n}$, $\alpha \in (0;n)$ we obtain the classical fractional maximal function $M_\alpha f$:

$$(M_\alpha f)(x) = \sup_{r>0} \frac{1}{r^{n-\alpha}} \int_{B(x,r)} |f(y)|dy.$$

We denote by $M_E^\Phi = M_E^\Phi(R^n)$ the set of the functions u, for which there is a function $f \in E(R^n)$ such that

$$u(x) = (M_\Phi f)(x),$$

$$\|u\|_{M_E^\Phi} = inf\{\|f\|_E : f \in E(R^n),\ \ M_\Phi f = u\} \tag{5}$$

such a space M_E^Φ will be called space of generalized fractional-maximal function.

Note that in the works of Goldman M.L., Bakhtigareeva E.G [4–5], the generalized Riesz potential was considered using the convolution operator:

$$A : E_1(R^n) \to \dot{L}_0(R^n),$$

$$Af(x) = (G * f)(x) = 2\pi^{-n/2} \int_{R^n} G(x-y)f(y)dy,$$

where the kernel $G(x)$ satisfies the conditions:

$$G(x) \cong \Phi(|x|), \ \ x \in R^n, \tag{6}$$

$$\Phi \in B_n(\infty); \ \ \exists c \in R_+.$$

The kernel of the classical Riesz potential has the form

$$G(x) = |x|^{\alpha - n}, \ \ \alpha \in (0; n).$$

Note that, unlike the operator A the operator M_Φ is not linear.

Definition 8. Define $\Im_T = \{K(T)\}$ for $T \in (0, \infty]$ as a set of cones considering from measurable non-negative functions on $(0, T)$, equipped with positive homogeneous functionals $\rho_{KM(T)} : K(T) \to [0, \infty)$ with properties:

(1) $h \in K(T), \ \ \alpha \geq 0 \Rightarrow \alpha h \in K(T), \quad \rho_{K(T)}(\alpha h) = \alpha \rho_{K(T)}(h)$;
(2) $\rho_{K(T)}(h) = 0 \Rightarrow h = 0$ almost everywhere on $(0, T)$.

Definition 9. [5] Let $K(T), M(T) \in \Im_T$. The cone $K(T)$ *covers the cone* $M(T)$ (notation: $M(T) \prec K(T)$) if there exist $C_0 = C_0(T) \in R_+$, and $C_1 = C_1(T) \in [0, \infty)$ with $C_1(\infty) = 0$ such that for each $h_1 \in M(T)$ there is $h_2 \in K(T)$ satisfying

$$\rho_{K(T)}(h_2) \leq C_0 \rho_{M(T)}(h_1), \ \ h_1(t) \leq h_2(t) + C_1 \rho_{M(T)}(h_1), \ \ t \in (0, T).$$

The equivalence of the cones means mutual covering:

$$M(T) \approx K(T) \Leftrightarrow M(T) \prec K(T) \prec M(T).$$

Let E is rearrangement-invariant space (briefly: RIS). We consider the following two cones of decreasing rearrangements of generalized fractional maximal function equipped with homogeneous functionals, respectively:

$$K_1 \equiv KM_E^\Phi := \{h \in L^+(\mathbb{R}_+) : h(t) = u^*(t), \ t \in \mathbb{R}_+, \ u \in M_E^\Phi\},$$

$$\rho_{K_1}(h) = \inf\{\|u\|_{M_E^\Phi} : u \in M_E^\Phi; \ u^*(t) = h(t), \ t \in \mathbb{R}_+\}; \tag{7}$$

$$K_2 \equiv K\widetilde{M}_E^\Phi := \{h : h(t) = u^{**}(t), \ t \in \mathbb{R}_+, \ u \in M_E^\Phi\},$$

$$\rho_{K_2}(h) = \inf\{\|u\|_{M_E^\Phi} : u \in M_E^\Phi; \ u^{**}(t) = h(t), \ t \in \mathbb{R}_+\}. \tag{8}$$

This means that the cones K_1 and K_2 consist of non-increasing rearrangements of generalized fractional maximal functions.

Note that in the works of Goldman M.L. [5], Bokayev N.A., Goldman M.L., Karshygina G.Zh. [9–10] cones generated by generalized potentials are considered. They study the space of potentials $H_E^G \equiv H_E^G(\mathbb{R}^n)$ in n-dimensional Euclidean space:

$$H_E^G(\mathbb{R}^n) = \{u = G * f : f \in E(\mathbb{R}^n)\},$$

where $E(\mathbb{R}^n)$ is an rearrangement invariant space (RIS).

$$\|u\|_{H_E^G} = \inf\{\|f\|_E : f \in E(\mathbb{R}^n); G * f = u\},$$

$$M(T) \equiv KM_E^G(T) = \{h(t) = u^*(t), t \in (0; T), u \in H_E^G\},$$

$$\rho_{M(T)}(h) = \inf\{\|u\|_{H_E^G} : u \in H_E^G; u^*(t) = h(t), t \in (0; T)\};$$

$$\widetilde{M}(T) \equiv K\widetilde{M}_E^G(T) = \{h(t) = u^{**}(t), t \in (0;T) : u \in H_E^G\},$$

$$\rho_{\widetilde{M}}(h) = \inf\{\|u\|_{H_E^G} : u \in H_E^G; u^{**}(t) = h(t)), t \in (0;T)\}.$$

In the following Theorem 1 [13] gives the estimate for a non-increasing rearrangement of a generalized fractional maximal function $(M_\Phi f)$ by non-increasing rearrangement of the function f.

Theorem 1. Let $\Phi \in A_n(\infty)$. Then there exist a positive constant C, depending from $n \in \mathbb{N}$ such that

$$(M_\Phi f)^*(t) \le C \sup_{t<s<\infty} s\Phi(s^{1/n}) f^{**}(s), \quad t \in (0,\infty),$$

for every $f \in L^1_{loc}(\mathbb{R}^n)$.

In the following theorem we give the compares of the cone generated by a generalized fractional-maximal function and the cone generated by the generalized Riesz potential.

Theorem 2. Let $\Phi \in B_n(\infty)$ and kernel $G(x)$ satisfies the condition (6). Then cone generated by the generalized potential covers the cone generated by the generalized maximal function, i.e. $KM_E^\Phi \prec KM_E^G$.

Proof. Let $h_1 \in KM_E^\Phi$, then according to the definition of KM_E^Φ there is a function $u_1 \in M_E^\Phi$, such that $h_1(t) = u_1^*(t)$. So there is a function $f \in E(R^n)$ such that

$$u_1(x) = (M_\Phi f_1)(x),$$

$$\|f_1\|_E \le 2\|u_1\|_{M_E^\Phi}.$$

Therefore

$$\|u_1\|_{K_1} \le 4\rho_{K_1}(h_1),$$

$$\|f\|_E \le C\rho_{KM}(h_1).$$

Therefore, by Theorem 1 and taking into account the monotonicity of the function $\Phi \downarrow$ and denoting

$$\psi(t) = \int_t^\infty \frac{\Phi(\xi^{1/n})}{\xi} d\xi,$$

we have:

$$\begin{aligned} h_1(t) &= (M_\Phi f)^*(t) \le C \sup_{t\le s<\infty} \Phi(s^{1/n}) \int_0^s f^*(\tau)d\tau = \\ &= \sup_{t\le s<\infty} C \left(\frac{1}{\ln 2} \int_s^{2s} \frac{d\xi}{\xi} \right) \Phi(s^{1/n}) \cdot \int_0^s f^*(\tau)d\tau \le \\ &\le C \cdot \sup_{t\le s<\infty} \frac{1}{\ln 2} \int_s^{2s} \frac{\Phi(\xi^{1/n})}{\xi} d\xi \cdot \int_0^s f^*(\tau)d\tau \le \\ &\le C \cdot \sup_{t\le s<\infty} \int_s^{2s} \frac{\Phi(\xi^{1/n})}{\xi} \cdot \int_0^s f^*(\tau)d\tau d\xi \le \end{aligned}$$

$$
\begin{aligned}
&\le \quad C\cdot\int\limits_t^\infty \frac{\Phi(\xi^{1/n})}{\xi}\cdot\int\limits_0^\infty f^*(\tau)d\tau d\xi=\int\limits_t^\infty\frac{\Phi(\xi^{1/n})}{\xi}d\xi\cdot\int\limits_0^t f^*(\tau)d\tau+\\
&+\quad \int\limits_t^\infty f^*(\tau)d\tau\cdot\int\limits_t^\infty\frac{\Phi(\xi^{1/n})}{\xi}d\xi\le C\cdot\Bigg(\psi(t)\cdot\int\limits_0^t f^*(\tau)d\tau+\\
&+\quad \int\limits_t^\infty \psi(s)f^*(\tau)d\tau\Bigg)\le C\cdot\big(G*f^{\#}\big)^*(t)=h_2(t).
\end{aligned}
$$

We put that

$$h_2(t)=C\cdot\big(G*f^{\#}\big)^*(t),t\in\mathbb{R}_+.$$

Consequently $h_1(t)\le h_2(t)$. So

$$K_1=KM_E^\Phi\prec KM_E^G,$$

$$\rho_{M(T)}(h_2)\le C\|f^{\#}\|_E=C\|f\|_E\le 2C\rho_{KM}(h_1).$$

Theorem 2 is proved.

Lemma 1. The following covering takes place

$$K_1\prec K_2.$$

Proof. Let $h_1\in K_1$. Then there is a function $u_1\in M_E^\Phi$ such that

$$h_1(t)=u_1^*(t),\quad \|u_1\|_{M_E^\Phi}\le 2\rho_{K_1}(h_1).$$

For $u_1\in M_E^\Phi$ we find $f_1\in E(\mathbb{R}^n)$ satisfying

$$u_1(x)=(M_\Phi f_1)(x)=\sup_{r>0}\Phi(r)\int\limits_{B(x,r)}f_1(\xi)d\xi,$$

$$\|f_1\|_{E(\mathbb{R}^n)}\le 2\|u_1\|_{M_E^\Phi}.$$

Hence $h_1(t)=(M_\Phi f_1)^{**}(t)$ and (see (1), (2), (3)),

$$\|f_1\|_{E(\mathbb{R}^n)}=\|f_1\|_{\widetilde{E}(R^+)}\le 4\rho_{K_1}(h_1).$$

By inequality

$$u_1^*(t)\le u_1^{**}(t),\ t\in\mathbb{R}_+.$$

We set

$$h_2(t)=u_1^{**}(t)\in K_2.$$

Then we have $h_1(t)\le h_2(t)$. Moreover (see (8), (5), (4), (7))

$$\rho_{K_2}(h_2)=\|u_1\|_{M_E^\Phi}\le\|f_1\|_{E(\mathbb{R}^n)}\le 4\rho_{K_1}(h_1).$$

We proved $K_1\prec K_2$. Lemma 1 is proved.

The following theorem shows that the embedding of the space of generalized fractional linear spaces in the RIS $X(\mathbb{R}^n)$ is reduced to the embedding of the cone $K_1=K_1M_E^\Phi$ in the space of the RIS $\widetilde{X}(\mathbb{R}_+)$.

Theorem 3. Let $\Phi \in B_n(\infty)$. The embedding

$$M_E^\Phi(\mathbb{R}^n) \hookrightarrow X(\mathbb{R}^n) \tag{9}$$

is equivalence to the next embedding

$$K_1 M_E^\Phi(\mathbb{R}_+) \mapsto \widetilde{X}(\mathbb{R}_+). \tag{10}$$

Proof. From embedding (9) it follows that there is a constant $C_1 \in \mathbb{R}_+$ such that for any $u \in M_E^\Phi$

$$\|u^*\|_{\widetilde{X}(\mathbb{R}_+)} = \|u\|_{X(\mathbb{R}^n)} \le C_1 \|u\|_{M_E^\Phi(\mathbb{R}^n)}. \tag{11}$$

For $h \in K_1 = K_1 M_E^\Phi$ we find the function $u \in M_E^\Phi$ such that $u^* = h$ and

$$\|u\|_{M_E^\Phi(\mathbb{R}^n)} \le 2\rho_{K_1}(h). \tag{12}$$

From (11) and (12) it follows that $h = u^* \in \widetilde{X}(\mathbb{R}_+)$

$$\|h\|_{\widetilde{X}(\mathbb{R}_+)} \le 2C_1 \rho_{K_1}(h),$$

i.e. holds (10).

Conversely let the embedding (10) hold. For $u \in M_E^\Phi(\mathbb{R}^n)$ and $h = u^* \in K_1 M_E^\Phi(\mathbb{R}_+)$ we get

$$\|h\|_{\widetilde{X}(\mathbb{R}_+)} \le C_0 \rho_{K_1}(h),$$

but $\rho_{K_1}(h) \le \|u\|_{M_E^\Phi(\mathbb{R}^n)}$, so the last estimate is

$$\|u\|_{X(\mathbb{R}^n)} = \|u^*\|_{\widetilde{X}(\mathbb{R}_+)} \le C_0 \|u\|_{M_E^\Phi(\mathbb{R}^n)}, \quad \forall u \in M_E^\Phi(\mathbb{R}^n).$$

That is $M_E^\Phi(\mathbb{R}^n) \hookrightarrow X(\mathbb{R}^n)$. Theorem 3 is proved.

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Жалпыланған бөлшекті-максималды функциямен туындаған конустар

Жұмыста ауыстырмалы-инварианттық кеңістік негізінде жалпыланған бөлшекті-максималды функция кеңістігі қарастырылған. Жалпыланған бөлшекті-максималды функцияның өспейтін алмастыруымен құрылған және оң біртекті функциялармен жабдықталған конустардың екі түрі құрастырылған. Жалпыланған бөлшекті-максималды функция кеңістігін ауыстырмалы-инварианттық кеңістікке енгізу мәселесі зерттелді. Бұл сұрақ қарастырылатын конусты сәйкес ауыстырмалы-инварианттық кеңістіктерге енгізуге әкеледі. Сонымен қатар, жалпыланған бөлшекті-максималды функция арқылы туындаған конусты жалпыланған Рисс потенциалы арқылы туындаған конуспен жабу шарттары берілген. Жалпыланған потенциалдардың өспейтін алмастыруларының конустары бұрын М. Гольдман, Э. Бахтигареева, Г. Каршыгина және т.б. еңбектерінде қарастырылған.

Кілт сөздер: ауыстырмалы-инварианттық кеңістіктер, функцияның өспейтін алмастырулары, жалпыланған бөлшекті-максималды функциялар арқылы туындаған конустар, конустардың жабулары.

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Конусы, порожденные обобщенной дробно-максимальной функцией

В работе рассмотрено пространство обобщенной дробно-максимальной функции на основе перестановочно-инвариантного пространства. Построены два вида конусов, порожденных невозрастающей перестановкой обобщенной дробно-максимальной функцией и снабженных положительными однородными функционалами. Исследован вопрос о вложении пространства обобщенных дробно-максимальных функций в перестановочно-инвариантное пространство. Этот вопрос сводится к вложению рассматриваемого конуса в соответствующие перестановочно-инвариантные пространства. Кроме того, приведены условия для покрытия конуса, порожденного обобщенной дробно-максимальной функцией, конусом, порожденным обобщенным потенциалом Рисса. Конусы из невозрастающих перестановок обобщенных потенциалов были изучены ранее в работах М. Гольдмана, Э. Бахтигареевой, Г. Каршыгиной и других.

Ключевые слова: перестановочно-инвариантные пространства, невозрастающие перестановки функций, конусы, порожденные обобщенными дробно-максимальными функциями, покрытие конусов.

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Boundary control problem for the heat transfer equation associated with heating process of a rod

In this paper, we consider a boundary control problem for a parabolic equation in a segment. In the part of the domain's bound it is a given value of the solution and it is required to find controls to get the average value of the solution. The given control problem is reduced to a system of Volterra integral equations of the first kind. By the mathematical-physics methods it is proved that like this control functions exist over some domain, the necessary estimates were found and obtained.

Keywords: Heat conduction equation, system of integral equations, initial-boundary value problem, Laplace transform.

1 Introduction and statement of the Problem

Consider the following heat exchange process along the domain $\Omega = \{(x,t) : 0 < x < l, \quad t > 0\}$:

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(k(x)\frac{\partial u}{\partial x}\right), \quad (x,t) \in \Omega, \tag{1}$$

with boundary value conditions

$$u(0,t) = \mu_1(t), \quad u(l,t) = \mu_2(t), \quad t > 0, \tag{2}$$

and an initial value condition

$$u(x,0) = 0, \quad 0 \le x \le l. \tag{3}$$

Assume that the function $k(x) \in C^1\big([0,l]\big)$ satisfies a condition

$$k(x) > 0, \quad 0 \le x \le l.$$

Let $M_j > 0$ be some given constants. We say that the functions $\mu_j(t)$ are an *admissible control* if this functions are differentiable on the half-line $t \ge 0$ and satisfies the following constraints

$$\mu_j(0) = 0, \quad |\mu_j(t)| \le M_j, \quad j = 1, 2.$$

Consider the following eigenvalue problem

$$\frac{d}{dx}\left(k(x)\frac{dv_k(x)}{dx}\right) + \lambda_k\, v_k(x) = 0, \quad 0 < x < l, \tag{4}$$

with boundary value conditions

$$v_k(0) = v_k(l) = 0, \quad 0 \le x \le l. \tag{5}$$

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It is well-known that this problem is self-adjoint in $L_2(\Omega)$ and there exists a sequence of eigenvalues $\{\lambda_k\}$ so that

$$\lambda_1 \le \lambda_2 \le ... \le \lambda_k \to \infty, \quad k \to \infty.$$

The corresponding eigenfunctions v_k form a complete orthonormal system $\{v_k(x)\}_{k \epsilon N}$ in $L_2(\Omega)$ and these function belong to $C(\bar{\Omega})$, where $\bar{\Omega} = \Omega \cup \partial\Omega$ (see, [1]).

Problem A. For the given functions $\theta_j(t)$ Problem A consists in looking for the admissible controls $\mu_j(t)$ such that the solution $u(x,t)$ of initial-boundary value problem (1)-(3) exists and for all $t > 0$ satisfies the equations

$$\int_0^l v_j(x)\, u(x,t) dx = \theta_j(t), \quad j = 1, 2. \tag{6}$$

We recall that the time-optimal control problem for partial differential equations of the parabolic type was first investigated in [2] and [3]. More recent results concerned with this problem were established in [4–13]. Detailed information on the optimal control problems for a distributed parameter systems is given in [14] and in monographs [15, 16] and [17].

General numerical optimization and optimal boundary control have been studied in a great number of publications such as [18]. The practical approaches to optimal control of the heat equation are described in publications like [19].

2 *System of integral equations*

Definition 1. By the solution of problem (1)–(3) we understand the function $u(x,t)$ represented in the form

$$u(x,t) = \mu_1(t) + \frac{x}{l}[\mu_2(t) - \mu_1(t)] - v(x,t), \tag{7}$$

where the function $v(x,t) \in C^{2,1}_{x,t}(\Omega) \cap C(\bar{\Omega})$, $v_x \in C(\bar{\Omega})$ is the solution to the problem:

$$v_t = \frac{\partial}{\partial x}\left(k(x)\,\frac{\partial v}{\partial x}\right) + \mu_1'(t) + \frac{x}{l}\,[\mu_2'(t) - \mu_1'(t)] + \frac{k'(x)}{l}\,[\mu_1(t) - \mu_2(t)],$$

with the boundary value conditions

$$v(0,t) = 0, \quad v(l,t) = 0,$$

and the initial value condition

$$v(x,0) = 0, \quad 0 \le x \le l.$$

Set

$$a_k = \int_0^l v_k(x) dx, \quad b_k = \int_0^l \frac{x}{l}\, v_k(x) dx, \quad c_k = \int_0^l \frac{k'(x)}{l}\, v_k(x) dx. \tag{8}$$

Consequently,

$$v(x,t) = \sum_{k=1}^{\infty} v_k(x) \times$$

$$\times \int_0^t e^{-\lambda_k(t-s)} \big(a_k\, \mu_1'(s) + b_k\, [\mu_2'(s) - \mu_1'(s)] + c_k\, [\mu_1(s) - \mu_2(s)]\big)\, ds, \tag{9}$$

where a_k, b_k and c_k are defined by (8).

From (7) and (9), we get the solution of the problem (1)–(3) (see, [1]):

$$u(x,t) = \mu_1(t) + \frac{x}{l}\,[\mu_2(t) - \mu_1(t)] - \sum_{k=1}^{\infty} v_k(x)\times$$

$$\times \int_0^t e^{-\lambda_k(t-s)}\big(a_k\,\mu_1'(s) + b_k\,[\mu_2'(s) - \mu_1'(s)] + c_k\,[\mu_1(s) - \mu_2(s)]\big)\,ds.$$

We know that the eigenvalues λ_k of boundary value problem (4), (5) satisfies the following inequalities

$$\lambda_k \geq 0, \quad k = 1, 2, ...$$

Indeed, since

$$\frac{d}{dx}\left(k(x)\frac{dv_k(x)}{dx}\right) + \lambda_k\,v_k(x) = 0, \quad 0 < x < l,$$

then we have

$$\lambda_k = -\int_0^l \frac{d}{dx}\left(k(x)\frac{dv_k(x)}{dx}\right) v_k(x)dx = \int_0^l k(x)\,|v_k'(x)|^2 dx \geq 0. \tag{10}$$

According to Jentsch's theorem $v_1(x) > 0$ (see, [20, 21]). Then, from $k(x) > 0$ and the estimate (10), we have

$$\lambda_1 > 0.$$

From condition (6) and the solution of the problem (1)–(3), we write

$$\theta_j(t) = \int_0^l v_j(x)\,u(x,t)dx =$$

$$= \int_0^l \left(\mu_1(t) + \frac{x}{l}\,[\mu_2(t) - \mu_1(t)]\right) v_j(x)\,dx - \sum_{k=1}^{\infty}\int_0^l v_j(x)\,v_k(x)\,dx\times$$

$$\times \int_0^t e^{-\lambda_k(t-s)}\big(a_k\,\mu_1'(s) + b_k\,[\mu_2'(s) - \mu_1'(s)] + c_k\,[\mu_1(s) - \mu_2(s)]\big)\,ds =$$

$$= \int_0^l \left(\mu_1(t) + \frac{x}{l}\,[\mu_2(t) - \mu_1(t)]\right) v_j(x)dx -$$

$$-\int_0^t e^{-\lambda_j(t-s)}\big(a_j\,\mu_1'(s) + b_j\,[\mu_2'(s) - \mu_1'(s)] + c_j\,[\mu_1(s) - \mu_2(s)]\big)\,ds =$$

$$= \int_0^l \left(\mu_1(t) + \frac{x}{l}\,[\mu_2(t) - \mu_1(t)]\right) v_j(x)dx - a_j\mu_1(t) - b_j\,[\mu_2(t) - \mu_1(t)] +$$

$$+\int_0^t (a_j\,\lambda_j - b_j\,\lambda_j + c_j)\,e^{-\lambda_j(t-s)}\,\mu_1(s)ds + \int_0^t (b_j\,\lambda_j - c_j)\,e^{-\lambda_j(t-s)}\,\mu_2(s)ds. \qquad (11)$$

Note that

$$\int_0^l \left(\mu_1(t) + \frac{x}{l}\,[\mu_2(t) - \mu_1(t)]\right) v_j(x)dx = a_j\mu_1(t) + b_j\,[\mu_2(t) - \mu_1(t)], \qquad (12)$$

where a_j and b_j are defined by (8).

As a result, from (11) and (12), we obtain

$$\theta_j(t) = \int_0^t (a_j\,\lambda_j - b_j\,\lambda_j + c_j)\,e^{-\lambda_j(t-s)}\,\mu_1(s)ds+$$

$$+\int_0^t (b_j\,\lambda_j - c_j)\,e^{-\lambda_j(t-s)}\,\mu_2(s)ds.$$

Let

$$B_{1j}(t) = \alpha_j\,e^{-\lambda_j t}, \quad B_{2j}(t) = \beta_j\,e^{-\lambda_j t}, \quad j = 1, 2, \qquad (13)$$

where

$$\alpha_j = a_j\,\lambda_j - b_j\,\lambda_j + c_j, \quad \beta_j = b_j\,\lambda_j - c_j. \qquad (14)$$

Then we get a system of the main integral equations

$$\int_0^t B_{1j}(t-s)\,\mu_1(s)ds + \int_0^t B_{2j}(t-s)\,\mu_2(s)ds = \theta_j(t), \quad t > 0, \quad j = 1, 2. \qquad (15)$$

Denote by $W(M_0)$ the set of functions $\theta \in W_2^2(-\infty, +\infty)$, $\theta(t) = 0$ for $t \le 0$ which satisfy the condition

$$\|\theta\|_{W_2^2(R_+)} \le M_0.$$

Theorem 1. There exists $M_0 > 0$ such that for any functions $\theta_j \in W(M_0)$ the solution $\mu_j(t)$ of system (15) exists and satisfies conditions

$$|\mu_j(t)| \le M_j, \quad j = 1, 2.$$

3 Proof of the Theorem 1

To solve system (15), we use the Laplace transform method. We introduce the notation

$$\widetilde{\mu}_j(p) = \int_0^\infty e^{-pt}\,\mu_j(t)\,dt, \quad p = a + i\,\xi, \quad a > 0.$$

Then, we use the Laplace transform

$$\widetilde{\theta}_j(p) = \int_0^\infty e^{-pt}\,dt \int_0^t B_{1j}(t-s)\mu_1(s)ds + \int_0^\infty e^{-pt}\,dt \int_0^t B_{2j}(t-s)\mu_2(s)ds =$$

$$= \widetilde{B}_{1j}(p)\,\widetilde{\mu}_1(p) + \widetilde{B}_{2j}(p)\,\widetilde{\mu}_2(p). \tag{16}$$

According to (13), we get

$$\widetilde{B}_{1j}(p) = \int\limits_0^\infty B_{1j}(t)e^{-pt}\,dt = \frac{\alpha_j}{p+\lambda_j}, \tag{17}$$

and

$$\widetilde{B}_{2j}(p) = \int\limits_0^\infty B_{2j}(t)e^{-pt}\,dt = \frac{\beta_j}{p+\lambda_j}, \quad j=1,2, \tag{18}$$

where α_j, β_j are defined by (14).

Assume that the α_j, β_j $(j=1,2)$ satisfies the following condition

$$\alpha_1\,\beta_2 - \alpha_2\,\beta_1 \neq 0.$$

Consequently, from system (16) and (17), (18), we can obtain

$$\widetilde{\mu}_1(p) = \frac{\beta_1\,(\lambda_2+p)}{\alpha_2\,\beta_1 - \alpha_1\,\beta_2}\,\widetilde{\theta}_2(p) - \frac{\beta_2\,(\lambda_1+p)}{\alpha_2\,\beta_1 - \alpha_1\,\beta_2}\,\widetilde{\theta}_1(p), \tag{19}$$

and

$$\widetilde{\mu}_2(p) = \frac{\alpha_1\,(\lambda_2+p)}{\alpha_1\,\beta_2 - \alpha_2\,\beta_1}\,\widetilde{\theta}_2(p) - \frac{\alpha_2\,(\lambda_1+p)}{\alpha_1\,\beta_2 - \alpha_2\,\beta_1}\,\widetilde{\theta}_1(p). \tag{20}$$

Then, when $a \to 0$ from (19) and (20), we obtain the following equalities

$$\mu_1(t) = \frac{1}{2\pi}\int\limits_{-\infty}^{+\infty}\left(\frac{\beta_1\,(\lambda_2+i\xi)}{\alpha_2\,\beta_1 - \alpha_1\,\beta_2}\,\widetilde{\theta}_2(i\xi) - \frac{\beta_2\,(\lambda_1+i\xi)}{\alpha_2\,\beta_1 - \alpha_1\,\beta_2}\,\widetilde{\theta}_1(i\xi)\right)e^{i\xi t}d\xi, \tag{21}$$

and

$$\mu_2(t) = \frac{1}{2\pi}\int\limits_{-\infty}^{+\infty}\left(\frac{\alpha_1\,(\lambda_2+i\xi)}{\alpha_1\,\beta_2 - \alpha_2\,\beta_1}\,\widetilde{\theta}_2(i\xi) - \frac{\alpha_2\,(\lambda_1+i\xi)}{\alpha_1\,\beta_2 - \alpha_2\,\beta_1}\,\widetilde{\theta}_1(i\xi)\right)e^{i\xi t}d\xi. \tag{22}$$

Lemma 1. Let $\theta(t) \in W(M_0)$. Then for the image of the function $\theta(t)$ the following inequality

$$\int\limits_{-\infty}^{+\infty}|\widetilde{\theta}(i\xi)|\sqrt{1+\xi^2}d\xi \leq C\,\|\theta\|_{W_2^2(R_+)}$$

is valid.

Proof. We calculate the Laplace transform of a function $\theta(t)$ as follows

$$\widetilde{\theta}(a+i\xi) = \int\limits_0^\infty e^{-(a+i\xi)t}\theta(t)\,dt = -\theta(t)\,\frac{e^{-(a+i\xi)t}}{a+i\xi}\bigg|_{t=0}^{t=\infty} + \frac{1}{a+i\xi}\int\limits_0^\infty e^{-(a+i\xi)t}\,\theta'(t)\,dt,$$

then we get

$$(a+i\xi)\,\widetilde{\theta}(a+i\xi) = \int\limits_0^\infty e^{-(a+i\xi)t}\,\theta'(t)\,dt,$$

and for $a \to 0$ we have

$$i\xi\,\widetilde{\theta}(i\xi) = \int\limits_0^{\infty} e^{-i\xi t}\,\theta'(t)\,dt.$$

Also, we can write the following equality

$$(i\xi)^2\,\widetilde{\theta}(i\xi) = \int\limits_0^{\infty} e^{-i\xi t}\,\theta''(t)\,dt.$$

Then we have

$$\int\limits_{-\infty}^{+\infty} |\widetilde{\theta}(i\xi)|^2 (1+\xi^2)^2 d\xi \le C_1\,\|\theta\|^2_{W_2^2(R_+)}. \tag{23}$$

Consequently, according to (23) we get the following estimate

$$\int\limits_{-\infty}^{+\infty} |\widetilde{\theta}(i\xi)|\sqrt{1+\xi^2}d\xi = \int\limits_{-\infty}^{+\infty} \frac{|\widetilde{\theta}(i\xi)|(1+\xi^2)}{\sqrt{1+\xi^2}} \le$$

$$\le \Big(\int\limits_{-\infty}^{+\infty} |\widetilde{\theta}(i\xi)|^2(1+\xi^2)^2 d\xi\Big)^{1/2}\Big(\int\limits_{-\infty}^{+\infty}\frac{1}{1+\xi^2}d\xi\Big)^{1/2} \le C\,\|\theta\|_{W_2^2(R_+)}.$$

Lemma 1 is proved.

Proof of Theorem 1. Note that

$$|\lambda_j + i\,\xi| = \sqrt{\lambda_j^2+\xi^2} \le (1+\lambda_j)\,\sqrt{1+\xi^2}.$$

According to (21), (22) and Lemma 1, we obtain the estimates

$$|\mu_1(t)| \le \frac{1}{2\pi}\int\limits_{-\infty}^{+\infty}\left|\frac{\beta_1}{\alpha_2\,\beta_1-\alpha_1\,\beta_2}\right| |\lambda_2+i\xi|\,|\widetilde{\theta}_2(i\xi)|d\xi +$$

$$+\frac{1}{2\pi}\int\limits_{-\infty}^{+\infty}\left|\frac{\beta_2}{\alpha_2\,\beta_1-\alpha_1\,\beta_2}\right| |\lambda_1+i\xi|\,|\widetilde{\theta}_1(i\xi)|d\xi \le$$

$$\le \frac{C_1\,(1+\lambda_2)}{2\pi}\int\limits_{-\infty}^{+\infty}\sqrt{1+\xi^2}\,|\widetilde{\theta}_2(i\xi)|\,d\xi + \frac{C_2\,(1+\lambda_1)}{2\pi}\int\limits_{-\infty}^{+\infty}\sqrt{1+\xi^2}\,|\widetilde{\theta}_1(i\xi)|d\xi \le$$

$$\le \frac{C_1\,C\,(1+\lambda_2)}{2\pi}\,\|\theta_2\|_{W_2^2(R_+)} + \frac{C_2\,C\,(1+\lambda_1)}{2\pi}\,\|\theta_1\|_{W_2^2(R_+)} \le$$

$$\le \frac{C_1\,C\,(1+\lambda_2)}{2\pi}\,M_0 + \frac{C_2\,C\,(1+\lambda_1)}{2\pi}\,M_0 = M_1,$$

and

$$|\mu_2(t)| \le \frac{1}{2\pi}\int\limits_{-\infty}^{+\infty}\left|\frac{\alpha_1}{\alpha_1\,\beta_2-\alpha_2\,\beta_1}\right| |\lambda_2+i\xi|\,|\widetilde{\theta}_2(i\xi)|d\xi +$$

$$+\frac{1}{2\pi}\int\limits_{-\infty}^{+\infty}\left|\frac{\alpha_2}{\alpha_1\,\beta_2-\alpha_2\,\beta_1}\right||\lambda_1+i\xi|\,|\widetilde{\theta}_1(i\xi)|d\xi\le$$

$$\le\frac{C_3\,(1+\lambda_2)}{2\pi}\int\limits_{-\infty}^{+\infty}\sqrt{1+\xi^2}\,|\widetilde{\theta}_2(i\xi)|\,d\xi+\frac{C_4\,(1+\lambda_1)}{2\pi}\int\limits_{-\infty}^{+\infty}\sqrt{1+\xi^2}\,|\widetilde{\theta}_1(i\xi)|d\xi\le$$

$$\le\frac{C_3\,C\,(1+\lambda_2)}{2\pi}\,\|\theta_2\|_{W_2^2(R_+)}+\frac{C_4\,C\,(1+\lambda_1)}{2\pi}\,\|\theta_1\|_{W_2^2(R_+)}\le$$

$$\le\frac{C_3\,C\,(1+\lambda_2)}{2\pi}\,M_0+\frac{C_4\,C\,(1+\lambda_1)}{2\pi}\,M_0=M_2.$$

Theorem 1 is proved.

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Сырықты қыздыру процесіне байланысты жылуды бөлу теңдеуінің шекаралық мәнін бақылау есебі

Мақалада параболалық теңдеу үшін шекаралық бақылау есебі қарастырылған. Температураның мәні берілген аумақтың шекаралық бөлігінде берілген және температураның орташа мәнін алу үшін басқару элементтерін табу қажет. Берілген басқару есебі бірінші типті Вольтерра интегралдық теңдеулер жүйесіне келтірілді. Математиканың физикалық әдістерін қолдану арқылы белгілі бір салада ұқсас басқару функцияларының бар екендігі дәлелденді және қажетті бағалар алынды.

Кілт сөздер: жылуалмасу теңдеуі, интегралдық теңдеулер жүйесі, бастапқы-шекаралық есеп, Лаплас алмастыру.

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Задача граничного управления для уравнения теплопереноса, связанного с процессом нагрева стержня

В статье рассмотрена задача граничного управления для параболического уравнения на отрезке. В части границы данной области задано значение решения, и требуется найти управление, чтобы получить среднее значение решения. Данная задача управления сведена к системе интегральных уравнений Вольтерра первого рода. Методами математической физики доказано, что подобные функции управления существуют в некоторой области, найдены и получены необходимые оценки.

Ключевые слова: уравнение теплопроводности, система интегральных уравнений, начально-краевая задача, преобразование Лапласа.

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A fractionally loaded boundary value problem two-dimensional in the spatial variable

In the paper, the boundary value problem for the loaded heat equation is solved, and the loaded term is represented as the Riemann-Liouville derivative with respect to the time variable. The domain of the unknown function is the cone. The order of the derivative in the loaded term is less than 1, and the load moves along the lateral surface of the cone, that is in the domain of the desired function. The boundary value problem is studied in the case of the isotropy property in an angular coordinate (case of axial symmetry). The problem is reduced to the Volterra integral equation, which is solved by the method of the Laplace integral transformation. It is also shown by direct verification that the resulting function satisfies the boundary value problem.

Keywords: loaded boundary value problem, heat equation, isotropy, Volterra integral equation, Laplace transformation.

Introduction

It is known [1] that, as a rule, mathematical models of nonlocal physical and biological fractal processes are based on loaded differential equations with fractional order partial derivatives. In monograph [2], A.M. Nakhushev gave a detailed bibliography on loaded equations, including various applications of loaded equations as a method for studying problems in mathematical biology, mathematical physics, mathematical modeling of nonlocal processes and phenomena, and continuum mechanics with memory. In [3, 4], a boundary value problem for a fractionally loaded one-dimensional heat equation is considered. The load moves at a variable velocity. The conditions for the unique solvability of the boundary value problem are established depending on the order of the fractional derivative. In this paper, we study the solvability of a boundary value problem that is two-dimensional in the spatial variable. In [5, 6], a boundary value problem for the heat equation is considered in a cone in Lebesgue and Sobolev spaces. The BVP is reduced to a Volterra type integral equation of the second kind, and the method of successive approximations is not applicable to it [5]. This fact follows from the incompressibility property of the integral operator [7, 8]. As a result, nonzero solutions of the homogeneous equation arise [9, 10]. Singular integral operators defined in a bounded domain of the hodograph plane are considered in [11]. In this paper, we show the unique solvability of the reduced integral equation and the boundary value problem posed in a certain functional class.

The paper is organized as follows: in Section 1 we introduce some necessary definitions and mathematical preliminaries of fractional calculus which will be needed in the forthcoming Section. In Section 2, the statement of a fractionally loaded BVP of heat conduction is given. The loaded term is represented as a fractional Riemann-Liouville derivative with respect to the time variable. Since the boundary value problem is studied in the case of the isotropy property in the angular coordinate (when passing to polar coordinates), the problem statement for this case is also given. In Section 3,

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the BVP is equivalently reduced to the Volterra integral equation, namely, to the generalized Abel equation. Section 4 contains solving the integral equation (homogeneous and nonhomogeneous) using the Laplace transform method. Further, the solution of the BVP in the case of axial symmetry is obtained. Also in this Section it is shown that the obtained solution satisfies the BVP. Finally, Section 5 presents the main results of the paper, namely, theorems on the solvability of the integral equation and the boundary value problem posed in Section 2.

Note that in this paper the order of the derivative in the loaded term is less than the order of the differential part of the equation. In [12], the order of the derivative is greater than two, and the boundary value problem was reduced to an integro-differential equation, which led to the non-uniqueness of the problem's solution.

Summing up the above analysis of studies, we can say that boundary value problems for loaded differential equations are well-posed in a number of cases in natural classes of functions, i.e., in this case, the loaded term is interpreted as a weak perturbation. In the case of violation of the uniqueness of the solution to a boundary value problem, the loaded term can be considered as a strong perturbation [13–15]. Everywhere linear equations are considered. An interesting method for studying semilinear equations in the [16].

1 Preliminaries

Let us first recall some previously known concepts and results. The first one is the definition of the Riemann–Liouville fractional derivative.

Definition 1 ([17]). Let $f(t) \in L_1[a,b]$. Then, the Riemann-Liouville derivative of the order β is defined as follows

$$ {}_rD^{\beta}_{a,t}f(t) = \frac{1}{\Gamma(n-\beta)}\frac{d^n}{dt^n}\int_a^t \frac{f(\tau)}{(t-\tau)^{\beta-n+1}}d\tau,\ \beta, a \in R,\ n-1<\beta<n. \tag{1} $$

From formula (1) it follows that

$$ {}_rD^{0}_{a,t}f(t) = f(t), \quad {}_rD^{n}_{a,t}f(t) = f^{(n)}(t), \quad n \in N. $$

We study a boundary value problem for the loaded heat equation, that is two-dimensional in the spatial coordinate when the loaded term is represented in the form of a fractional derivative. The considered problem is reduced to an integral equation by inverting the integral part.

It's known [18] the function

$$ G(r,\xi,t) = \frac{\xi}{2a^2t}\exp\left\{-\frac{r^2+\xi^2}{4a^2t}\right\} I_0\left(\frac{r\xi}{2a^2t}\right) $$

is a fundamental solution to the equation

$$ \frac{\partial w}{\partial t} = \frac{a^2}{r}\left(r\frac{\partial w}{\partial r}\right), $$

where

$$ I_\nu(z) = \sum_{n=0}^{\infty}\frac{\left(\frac{z}{2}\right)^{2n+\nu}}{n!\Gamma(n+\nu+1)}, \quad -\infty<\nu<\infty $$

is the modified Bessel function.

It's known ([18]; p. 76) that in the domain $\Omega_\infty = \{(r,t)\,|\,0 \le r < +\infty;\ t>0\}$ the solution to the boundary value problem of heat conduction

$$ \frac{\partial w}{\partial t} = \frac{a^2}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right) + F(r,t), $$

$$w|_{t=0} = w_0(r)$$

is defined by the formula

$$w(r,t) = \int_0^{+\infty} G(r,\xi,t)\, w_0(\xi)\, d\xi + \int_0^t \int_0^{+\infty} G(r,\xi,t-\tau)\, F(\xi,\tau)\, d\xi d\tau. \quad (2)$$

The Green function $G\left(x,\xi,t-\tau\right)$ satisfies the relation

$$\int_0^{+\infty} G\left(x,\xi,t-\tau\right) d\xi = 1. \quad (3)$$

Indeed,

$$\int_0^{+\infty} G(r,\xi,t) d\xi = \frac{1}{2a^2 t} \int_0^{+\infty} \xi \exp\left(-\frac{r^2+\xi^2}{4a^2 t}\right) I_0\left(\frac{r\xi}{2a^2 t}\right) d\xi =$$
$$= \frac{1}{2a^2 t} \exp\left(-\frac{r^2}{4a^2 t}\right) \int_0^{+\infty} \xi \exp\left(-\frac{\xi^2}{4a^2 t}\right) I_0\left(\frac{r\xi}{2a^2 t}\right) d\xi.$$

From [19] (formula 2.15.5 (4) when $\alpha = 2;\ \nu = 0,\ c = \dfrac{r}{2a^2 t};\ \rho = \dfrac{1}{4a^2 t}$) we have

$$\int_0^{+\infty} G(r,\xi,t) d\xi = \frac{1}{2a^2 t} \exp\left(-\frac{r^2}{4a^2 t}\right) A_\nu^{\nu+2}.$$

Since $\nu = 0 => A_0^2 = A_\nu^{\nu+2}$. Then we get equality (3).

We assume that the right side of the BVP's equation vanishes at $t < 0$ and belongs to the class

$$\Phi(x,y;t) \in L_\infty\left(A\right) \cap C\left(B\right), \quad (4)$$

where $A = \{(x,y;t)\,|\, x > 0,\ -\infty < y < +\infty,\ t \in [0,T]\}$, $B = \{(x,y;t)\,|\, x > 0,\ -\infty < y < +\infty,\ t \geq 0\}$, $T - const > 0$.

The classes in which the problem is studied are determined from the natural requirement for the existence and convergence of improper integrals that arise in the study.

2 Problem setting

Problem 1. In a domain

$$G = \{(x,y;t) \big| \sqrt{x^2+y^2} \leq t;\ t > 0\} \quad (5)$$

we consider a boundary value problem, two-dimensional in the spatial variable for a fractionally loaded heat equation:

$$u_t = a^2 \Delta u + \lambda \{{}_{RL}D_{0t}^{\beta} u(x,y;t)\}\Big|_{\sqrt{x^2+y^2}=t/2} + \Phi(x,y;t) \quad (6)$$

with the condition of solution's boundedness:

$$\lim_{\sqrt{x^2+y^2}\to+\infty} u(x,y;t) = 0, \quad (7)$$

and with the condition on the lateral surface of the cone:

$$u(x,y;t)\Big|_{\sqrt{x^2+y^2}=t} = g(t), \tag{8}$$

where $\Phi(x,y;t)$ is a given function belonging to the class (4), λ is a complex parameter, ${}_{RL}D_{0t}^{\beta}u(x,y;t)$ is the Riemann-Liouville derivative of the order β, $0<\beta<1$, i.e.

$${}_{RL}D_{0t}^{\beta}u(x,y;t) = \frac{1}{\Gamma(1-\beta)}\frac{d}{dt}\int_0^t \frac{u(x,y;\tau)}{(t-\tau)^{\beta}}d\tau. \tag{9}$$

Let's move on to polar coordinates:

$$x = r\,\cos\phi;\ y = r\,\sin\phi;\ 0\le\phi<2\pi;\ r\ge 0.$$

Since the problem (6)–(8) is considered in the case of the isotropy property in the angular coordinate ϕ (case of axial symmetry), we obtain the following problem.

Problem 2. In a domain $\Omega_\infty = \{(r,t)\mid r>0;\ t>0\}$ find a solution to the equation

$$\frac{\partial w}{\partial t} = \frac{a^2}{r}\frac{\partial}{\partial r}\Big(r\frac{\partial w(r,t)}{\partial r}\Big) + \lambda\left\{{}_{RL}D_{0t}^{\beta}w(r;t)\right\}\Big|_{r=\frac{t}{2}} + F(r,t), \tag{10}$$

that satisfies the conditions

$$\lim_{r\to\infty} w(r,t) = 0, \tag{11}$$

$$w(r,t)|_{r=t} = g(t). \tag{12}$$

Hear $w(r,t) = u(r\,\cos\phi;\ r\,\sin\phi;\ t)$ is unknown function, $F(r,t) = \Phi(r\,\cos\phi;\ r\,\sin\phi;\ t)$.

The temperature field is assumed to be axisymmetric, i.e., it is approximated by the functional dependence of the temperature only on the value of r. Note that due to the axisymmetric nature of the problem under consideration and the degeneracy of the definition domain (5) to a point at the initial time, conditions (8) and (12) implies the matching condition at the cone top $w|_{r=0} = w|_{t=0} = g(0)$.

Now we have the following boundary value problem.

Problem 3. In a domain $\Omega_\infty = \{(r,t)\mid r>0;\ t>0\}$ find a solution to the equation (10) that satisfies condition (11) and the initial condition

$$w(r,t)|_{t=0} = g(0). \tag{13}$$

3 Reducing the boundary value problem to an integral equation

We invert the differential part of problem (10), (11), (13) by formula (2):

$$w(r,t) = \int_0^{+\infty} G(r,\xi,t)g(0)\,d\xi + \lambda\int_0^t\int_0^{+\infty} G(r,\xi,t-\tau)\mu(\tau)\,d\xi d\tau + f(r,t), \tag{14}$$

where

$$\mu(t) = \left\{{}_{RL}D_{0t}^{\beta}w(r;t)\right\}\Big|_{r=\frac{t}{2}}, \tag{15}$$

$$f(r,t) = \int_0^t\int_0^{+\infty} G(\xi,r,t-\tau)\,F(\xi,\tau)\,d\xi d\tau. \tag{16}$$

Taking into account equality (3), representation (14) can be rewritten in the form

$$w(r,t) = g(0) + \lambda \int_0^t \mu(\tau)d\tau + f(r,t). \tag{17}$$

Applying to (17) the operator of fractional differentiation according to formula (9), substituting $r = \frac{t}{2}$ into the resulting expression, by virtue of notation (15) on the left in (17) we obtain the function $\mu(t)$.

Since

$$\Gamma(1-\beta)_{RL}D_{0t}^{\beta}\left\{\int_0^t \mu(\tau)d\tau\right\} = \frac{d}{dt}\int_0^t \frac{1}{(t-\tau)^{\beta}}\int_0^{\tau}\mu(\theta)d\theta d\tau = \frac{d}{dt}\int_0^t \mu(\theta)\int_{\theta}^t \frac{d\tau}{(t-\tau)^{\beta}}d\theta =$$
$$= \frac{d}{dt}\int_0^t \frac{\mu(\theta)(t-\theta)^{1-\beta}}{1-\beta}d\theta = \int_0^t \frac{\mu(\theta)}{(t-\theta)^{\beta}}\,d\theta$$

then from (17) after the above procedure we obtain an integral equation

$$\mu(t) = \frac{g(0)}{\Gamma(1-\beta)}t^{-\beta} + \frac{\lambda}{\Gamma(1-\beta)}\int_0^t \frac{\mu(\tau)}{(t-\tau)^{\beta}}d\tau + f_1(t), \quad 0<\beta<1,$$

where

$$f_1(t) = \left\{{}_{RL}D_{0t}^{\beta} f(r,t)\right\}\Big|_{r=\frac{t}{2}}. \tag{18}$$

Thus, problem (10), (11), (13) is reduced to solving the Volterra integral equation of the second kind, namely the generalized Abel equation:

$$\mu(t) - \frac{\lambda}{\Gamma(1-\beta)}\int_0^t \frac{\mu(\tau)}{(t-\tau)^{\beta}}d\tau = \frac{g(0)}{\Gamma(1-\beta)}t^{-\beta} + f_1(t), \quad 0<\beta<1, \tag{19}$$

where $f_1(t)$ is defined by formulas (18), (16).

4 Solving the integral equation

Solving the integral equation in the case of the homogeneous equation in BVP (6)–(8). Consider the corresponding problem for $\Phi(x,y,t) \equiv 0$ in equation (6), i.e. $F(r,t) = 0$ in equation (10). Then integral equation (10) will take the form:

$$\mu(t) - \frac{\lambda}{\Gamma(1-\beta)}\int_0^t \frac{\mu(\tau)}{(t-\tau)^{\beta}}d\tau = \frac{g(0)}{\Gamma(1-\beta)}t^{-\beta}, \tag{20}$$

where $f_1(t)$ is defined by formulas (18), (16).

Let $\Phi(s) = L[\mu(t)]$ be the Laplace image of the function $\mu(t)$. Applying the integral Laplace transform to equation (20) we obtain:

$$\Phi(s) - \frac{\lambda\Phi(s)}{s^{1-\beta}} = \frac{g(0)}{s^{1-\beta}}, \quad Re\, s > |\lambda|^{\frac{1}{1-\beta}}.$$

From here

$$\Phi(s) = \frac{g(0)}{s^{1-\beta}-\lambda}, \quad Re\, s > |\lambda|^{\frac{1}{1-\beta}}. \tag{21}$$

Applying the inverse Laplace transform, taking into account formula 1.80 [20]

$$L\left[t^{\alpha k+\beta-1}E^{(k)}_{\alpha,\beta}(\pm at^{\alpha})\right]=\frac{k!s^{\alpha-\beta}}{(s^{\alpha}\mp a)^{k+1}};\ Re\, s>|a|^{\frac{1}{\alpha}},$$

where $E_{a,b}(z)$ is the Mittag-Leffler function, i.e.

$$E_{a,b}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(ak+b)},$$

from (21) we get

$$\mu(t)=g(0)t^{-\beta}E_{1-\beta;1-\beta}\left(\lambda t^{1-\beta}\right). \tag{22}$$

Due to the representation (17) of the solution to problem (10), (13) for $F(r,t)=0$ in the domain Ω_∞, taking into account (22) we get

$$w(r,t)=g(0)+\lambda g(0)\int_0^t \tau^{-\beta}E_{1-\beta;1-\beta}\left(\lambda\tau^{1-\beta}\right)d\tau.$$

Since [20] (formula 1.99)

$$\int_0^z E_{a,b}(\lambda t^a)t^{b-1}dt=z^bE_{a,b+1}(\lambda z^a);\ (b>0),$$

then

$$w(r,t)=g(0)+\lambda g(0)\,t^{-\beta}E_{1-\beta;2-\beta}\left(\lambda t^{1-\beta}\right). \tag{23}$$

(23) is the solution to problem (10), (13) in the domain Ω_∞, since condition (12) takes the form (13). Thus, the solution to problem (6)–(8) for $\Phi(x,y,t)=0$ in the case of axial symmetry has the form:

$$u(x,y;t)=g(0)+\lambda g(0)\,t^{-\beta}E_{1-\beta;2-\beta}\left(\lambda t^{1-\beta}\right), \tag{24}$$

where $0<\beta<1$.

Due to the formula

$$E_{a;\,b}(z)=z\,E_{a;\,a+b}(z)+\frac{1}{\Gamma(b)}$$

we have at $b=1$ and $z=\lambda t^{1-\beta}$

$$\lambda t^{1-\beta}E_{1-\beta;2-\beta}\left(\lambda t^{1-\beta}\right)=E_{1-\beta;1}\left(\lambda t^{1-\beta}\right)-1.$$

Then (24) will take the form:

$$u(x,y,t)=g(0)E_{1-\beta}\left(\lambda t^{1-\beta}\right), \tag{25}$$

since $E_{a,1}(z)=E_a(z)$.

It can be shown by direct verification that function (25) satisfies homogeneous equation (6) in the case of axial symmetry.

The case of BVP (6)–(8) at $\beta=1/2$ *when* $\Phi(x,y,t)=0$.

If $\beta = \dfrac{1}{2}$ in BVP (6)–(8) then expression (9) can be rewritten as

$$_{RL}D_{0t}^{\frac{1}{2}}u(x,y,t) = \frac{1}{\sqrt{\pi}}\frac{d}{dt}\int_0^t \frac{u(x,t,\tau)}{\sqrt{t-\tau}}d\tau.$$

Let

$$u\Big|_{t=0} = g(0),$$

where

$$g(t) = u(x,y,t)\Big|_{\sqrt{x^2+y^2}=t}$$

and $\Phi(x,y,t) = 0$.

Then the solution to BVP (6)–(8) has the form (see (24))

$$u(x,y;t) = g(0)E_{\frac{1}{2}}\left(\lambda\sqrt{t}\right),$$

when $\Phi(x,y,t) = 0$. Since [20] (formula 1.65)

$$E_{\frac{1}{2}}\left(\pm z^{\frac{1}{2}}\right) = e^{z}\operatorname{erfc}\left(\mp z^{\frac{1}{2}}\right),$$

then

$$u(x,y;t) = g(0)e^{\lambda^2 t}\operatorname{erfc}\left(-\lambda^2 t\right),$$

where

$$\operatorname{erfc} z = \frac{2}{\sqrt{\pi}}\int_0^z e^{-\xi^2}d\xi$$

is the complementary error function.

Solving the integral equation (19). Consider now equation (19). Let $L[f_1(t)] = F_1(s)$. Then, in the space of Laplace images, equation (19) takes the form:

$$\Phi(s) - \frac{\lambda\Phi(s)}{s^{1-\beta}} = \frac{g(0)}{s^{1-\beta}} + F_1(s).$$

From hear

$$\Phi(s) = \frac{g(0)}{s^{1-\beta}} + F_1(s) + \lambda\frac{F_1(s)}{s^{1-\beta}-\lambda}.$$

Applying the inverse Laplace transform, we get:

$$\mu(t) = g(0)t^{-\beta}E_{1-\beta,1-\beta}\left(\lambda t^{1-\beta}\right) + f_1(t) + \lambda f_1(t)\,t^{-\beta}E_{1-\beta,1-\beta}\left(\lambda t^{1-\beta}\right). \quad (26)$$

Then, taking into account function (26), representation (17) has the form:

$$w(r,t) = g(0) + \lambda\int_0^t \left(g(0)\tau^{-\beta}E_{1-\beta,1-\beta}\left(\lambda t^{1-\beta}\right)d\tau + f_1(\tau)\right)d\tau +$$
$$+\lambda^2\int_0^t\int_0^\tau f_1(\theta)(\tau-\theta)^{-\beta}E_{1-\beta,1-\beta}\left(\lambda(\tau-\theta)^{1-\beta}\right)d\theta d\tau + f(r,t) =$$
$$= g(0) + \lambda g(0)t^{1-\beta}E_{1-\beta,2-\beta}\left(\lambda t^{1-\beta}\right) +$$
$$+\lambda\int_0^t f_1(\tau)d\tau + \lambda^2\int_0^t f_1(\theta)d\theta\int_\theta^t (\tau-\theta)^{-\beta}E_{1-\beta,1-\beta}\left(\lambda(\tau-\theta)^{1-\beta}\right)d\tau + f(r,t)$$

that is

$$w(r,t) = g(0) + \lambda g(0)t^{1-\beta}E_{1-\beta,2-\beta}\left(\lambda t^{1-\beta}\right) + \lambda\int_0^t f_1(\tau)d\tau + \lambda^2\int_0^t f_1(\theta)I(\theta;t)d\theta + f(r,t), \quad (27)$$

where

$$I(\theta;t) = \int_\theta^t (\tau-\theta)^{-\beta}E_{1-\beta,1-\beta}\left(\lambda(\tau-\theta)^{1-\beta}\right)d\tau = (t-\theta)^{1-\beta}E_{1-\beta,2-\beta}\left(\lambda(t-\theta)^{1-\beta}\right).$$

Then function (27) can be rewritten as:

$$w(r,t) = g(0) + \lambda g(0)t^{1-\beta}E_{1-\beta,2-\beta}\left(\lambda t^{1-\beta}\right) + \lambda\int_0^t f_1(\tau)d\tau +$$
$$+\lambda^2\int_0^t (t-\tau)^{1-\beta}E_{1-\beta,2-\beta}\left(\lambda(t-\tau)^{1-\beta}\right)f_1(\tau)\,d\tau + f(r,t). \quad (28)$$

Due to the formula

$$E_{a,\,b}(z) = z\,E_{a;\,a+b}(z) + \frac{1}{\Gamma(\beta)}$$

we have at $b=1$ and $z=\lambda t^{1-\beta}$

$$\lambda t^{1-\beta}E_{1-\beta;\,2-\beta}\left(\lambda t^{1-\beta}\right) = E_{1-\beta}\left(\lambda t^{1-\beta}\right) - 1.$$

Then function (28) takes the form:

$$w(r,t) = g(0)E_{1-\beta}\left(\lambda t^{1-\beta}\right) + \lambda\int_0^t E_{1-\beta}\left(\lambda(t-\tau)^{1-\beta}\right)f_1(\tau)d\tau + f(r,t), \quad (29)$$

where $f_1(\tau)$ and $f(r,t)$ are defined by formulas (18) and (16), respectively. (29) is a solution to BVP (10), (11), (13).

So, in the case of axial symmetry in the domain G, the function

$$u(x,y,t) = g(0)E_{1-\beta}\left(\lambda t^{1-\beta}\right) + \lambda\int_0^t E_{1-\beta}\left(\lambda(t-\tau)^{1-\beta}\right)f_1(\tau)\,d\tau + f\left(\sqrt{x^2+y^2},t\right)$$

is a solution to BVP (6)-(8), where $f_1(\tau)$ and $f(r,t)$ are defined by formulas (18) and (16), respectively, and $F(r,t) = \Phi(r\cos\phi,\ r\sin\phi;\ t)$.

Checking that function (29) is a solution to BVP (10), (11), (13).

We first rewrite function (29) in the form (28). Since

$$\frac{d}{dt}\Big[t^{1-\beta}E_{1-\beta;2-\beta}\left(\lambda t^{1-\beta}\right)\Big] = t^{-\beta}E_{1-\beta;1-\beta}\left(\lambda t^{1-\beta}\right),$$

then

$$\frac{d}{dt}\int_0^t (t-\tau)^{1-\beta}E_{1-\beta;2-\beta}\left(\lambda(t-\tau)^{1-\beta}\right)f_1(\tau)\,d\tau =$$
$$= \int_0^t \frac{d}{dt}\Big[(t-\tau)^{1-\beta}E_{1-\beta;2-\beta}\left(\lambda(t-\tau)^{1-\beta}\right)\Big]f_1(\tau)\,d\tau =$$
$$= \int_0^t (t-\tau)^{-\beta}E_{1-\beta;1-\beta}\left(\lambda(t-\tau)^{1-\beta}\right)f_1(\tau)\,d\tau.$$

Then from (28) we have

$$\frac{\partial w}{\partial t} = \lambda g(0)t^{-\beta}E_{1-\beta;1-\beta}\left(\lambda t^{1-\beta}\right) + \lambda f_1(t) + \\ + \lambda^2 \int_0^t (t-\tau)^{-\beta}E_{1-\beta;1-\beta}\left(\lambda (t-\tau)^{1-\beta}\right) f_1(\tau)\, d\tau + \frac{\partial f(r,t)}{\partial t}. \quad (30)$$

$$\frac{a^2}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right) = \frac{a^2}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f(r,t)}{\partial t}\right). \quad (31)$$

By virtue of notation (15) and equality (26), we have

$${}_{RL}D^{\beta}_{0t}w(r,t)\Big|_{r=\frac{t}{2}} = \mu(t) = g(0)t^{-\beta}E_{1-\beta,1-\beta}\left(\lambda t^{1-\beta}\right) + f_1(t) + \\ + \lambda \int_0^t (t-\tau)^{-\beta}E_{1-\beta;1-\beta}\left(\lambda (t-\tau)^{1-\beta}\right) f_1(\tau)\, d\tau. \quad (32)$$

Substituting (30)-(32) into equation (10) we get:

$$\frac{\partial f(r,t)}{\partial t} = \frac{a^2}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f(r,t)}{\partial r}\right) + F(r,t). \quad (33)$$

By notation (16), we have

$$\frac{\partial f(r,t)}{\partial t} = \frac{\partial}{\partial t}\int_0^t \int_0^{+\infty} G(\xi,r,t-\tau)F(\xi,\tau)\, d\xi d\tau = \\ = \int_0^t \int_0^{+\infty} \frac{\partial G(\xi,r,t-\tau)}{\partial t}F(\xi,\tau) d\xi d\tau + \int_0^{+\infty} G(\xi,r;0)F(\xi,0)\, d\xi;$$

$$\frac{a^2}{r}\frac{\partial\left(rf_r(r,t)\right)}{\partial r} = \frac{a^2}{r}\frac{\partial}{\partial r}\left(\int_0^t \int_0^{+\infty} rG(\xi,r,t-\tau)F(\xi,\tau)\, d\xi d\tau\right).$$

It is known [21] that

$$e^{-z}I_\nu(z) \sim \frac{1}{\sqrt{2\pi z}}\left(1 + O\left(\frac{1}{z}\right)\right)$$

when $|\arg z| < \frac{\pi}{2}$ and $|z| \to \infty$. Then $\lim_{t\to 0} G(\xi,r,t) = 0$. Therefore, equality (33) takes the form:

$$\int_0^t \int_0^{+\infty} \frac{\partial G(\xi,r,t-\tau)}{\partial t}F(\xi,\tau)\, d\xi d\tau = \int_0^t \int_0^{+\infty} \frac{a^2}{r}\frac{\partial\left(rG(\xi,r,t-\tau)\right)}{\partial r}F(\xi,\tau) d\xi d\tau + F(r,t)$$

or

$$\int_0^t \int_0^{+\infty} \left[\frac{\partial G(\xi,r,t-\tau)}{\partial t} - \frac{a^2}{r}\frac{\partial(rG(\xi,r,t-\tau))}{\partial r}\right]F(\xi,\tau) d\xi d\tau = F(r,t). \quad (34)$$

Since $G(\xi,r,t)$ is the fundamental solution of the heat equation in polar coordinates, then

$$\frac{\partial G}{\partial t} - \frac{a^2}{r}\frac{\partial(rG)}{\partial r} = \delta(\xi - r)\delta(t),$$

where δ is the Dirac function. Then equality (34) takes the form:

$$\int_0^{+\infty} \delta(\xi - r)\delta(t) * F(\xi, t) d\xi = F(r, t)$$

or

$$\int_0^{+\infty} \delta(\xi - r) F(\xi, t) d\xi = F(r, t).$$

Hence, function (29) satisfies equation (10). Function (29) obviously satisfies condition (11) due to the choice of classes for $F(r, t)$. Let us now show that function (29) satisfies condition (13). We have

$$w(r, t)|_{t=0} = g(0) + \lim_{t\to 0} f(r, t) = g(0)$$

due to equality (16).

So, function (29) is a solution to BVP (10), (11), (13).

5 *Main results*

Theorem 1. Equation (19) is uniquely solvable in the class $\mu(t) \in C([0; T])$, for any function side $f_1(t) \in AC([0; T])$, and the solution to equation (19) is determined by formula (26).

Theorem 2. Let conditions (4) and $F(r, t) = \Phi(r \cos\phi,\ r \sin\phi;\ t) \in L_1(t \in [0; T])$ be satisfied for the function $\Phi(x, y; t)$, the function $\mu(t)$ is defined by formula (26). Then in the class $L_1(t \in [0; T])$ the boundary value problem (6)–(8) for the case of axial symmetry has a unique solution defined by formula

$$u(x, y, t) = g(0)E_{1-\beta}\left(\lambda t^{1-\beta}\right) + \lambda \int_0^t E_{1-\beta}\left(\lambda (t-\tau)^{1-\beta}\right) f_1(\tau)\, d\tau + f\left(\sqrt{x^2 + y^2}, t\right),$$

where $f_1(\tau)$ and $f(r, t)$ are defined by formulas (18) and (16), respectively.

Remark. Since equation (19) is a generalised Abel equation, its solution can be written as [22]

$$\mu(t) = f_1(t) + \int_0^t R(t - \tau) f_1(\tau) d\tau,$$

where

$$R(t - \tau) = \frac{d}{dt} \sum_{k=0}^{\infty} \frac{\left(\lambda (t-\tau)^{1-\beta}\right)^n}{\Gamma(1 + (1-\beta)n)}$$

or

$$R(t) = \frac{d}{dt} E_{1-\beta}\left(\lambda t^{1-\beta}\right).$$

After simple transformations, we get formula (26).

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Кеңістіктік айнымалыдағы екі өлшемді бөлшектік жүктемелі шеттік есеп

Жұмыста жүктемелі жылуөткізгіштік теңдеуі үшін шеттік есеп қарастырылды, жүктелген мүше уақыт айнымалысына қатысты Риман–Лиувилл туындысы ретінде берілген. Белгісіз функцияның анықталу облысы конус болып табылады. Жүктелген мүшедегі туындының реті 1-ден кіші, ал жүк конустың бүйір беті бойымен қозғалады және ізделінді функцияның анықталу облысына жатады. Шеттік есеп бұрыштық координаттағы изотропия қасиеті (осьтік симметрия жағдайы) жағдайында зерттелді. Есеп Вольтерра интегралдық теңдеуіне келтірілді және Лаплас интегралды түрлендіру әдісімен шешілді. Алынған функцияның шеттік есептерді қанағаттандыратыны тікелей тексеру арқылы көрсетілді.

Кілт сөздер: жүктелген шеттік есеп, жылуөткізгіштік теңдеуі, изотропия, Вольтерра интегралдық теңдеуі, Лаплас түрлендіруі.

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Дробно-нагруженная краевая задача, двумерная по пространственной переменной

В работе найдено решение краевой задачи для нагруженного уравнения теплопроводности, в котором нагруженное слагаемое представлено в виде производной Римана–Лиувилля по временной переменной. Область определения неизвестной функции — конус. Порядок производной в нагруженном члене меньше 1, и нагрузка движется по боковой поверхности конуса, который находится в области определения искомой функции. Краевая задача исследована в случае свойства изотропности по угловой координате (случай осевой симметрии). Задача сведена к интегральному уравнению Вольтерра, которое решается методом интегрального преобразования Лапласа. Непосредственной проверкой также показано, что полученная функция удовлетворяет поставленной задаче.

Ключевые слова: нагруженная краевая задача, уравнение теплопроводности, изотропность, интегральное уравнение Вольтерра, преобразование Лапласа.

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On one approximate solution of a nonlocal boundary value problem for the Benjamin-Bona-Mahony equation

The paper investigates a non-local boundary value problem for the Benjamin-Bona-Mahony equation. This equation is a nonlinear pseudoparabolic equation of the third order with a mixed derivative. To find a solution to this problem, an algorithm for finding an approximate solution is proposed. Sufficient conditions for the feasibility and convergence of the proposed algorithm are established, as well as the existence of an isolated solution of a non-local boundary value problem for a nonlinear equation. Estimates are obtained between the exact and approximate solution of this problem.

Keywords: partial differential equation, Benjamin-Bona-Mahony equation, algorithm, approximate solution.

Introduction

The paper considers a nonlocal boundary value problem for a third-order nonlinear partial differential equation or the Benjamin–Bona–Mahony equation. The Benjamin–Bona–Mahony equation or the regularized long-wavelength equation was studied in [1–4]. Modern studies on this topic can be found in [5–13]. In this article, introducing a new function, a non-local boundary value problem for a third-order nonlinear differential equation is reduced to a non-local boundary value problem for a hyperbolic equation. The resulting problem with different conditions was investigated in [14–18]. Similarly to the linear case [19–22], sufficient conditions for the unique solvability of the problem under consideration are established and an algorithm for finding an approximate solution is proposed.

1 Statement of the initial boundary problem

On $\Omega = [0, X] \times [0, Y]$ we consider a nonlocal boundary value problem for the nonlinear equation

$$\frac{\partial^3 w}{\partial x^2 \partial y} = \frac{\partial w}{\partial y} + w\frac{\partial w}{\partial x} + \frac{\partial w}{\partial x}, \quad (x, y) \in \Omega, \tag{1}$$

$$w(x, 0) = \varphi(x), \quad x \in [0, X], \tag{2}$$

$$\frac{\partial w(0, y)}{\partial y} = \alpha(y)\frac{\partial w(X, y)}{\partial y} + \psi(y), \quad y \in [0, Y], \tag{3}$$

$$\frac{\partial w(0, y)}{\partial x} = \theta(y), \quad y \in [0, Y], \tag{4}$$

where the functions $\psi(y), \theta(y)$ are continuously differentiable on $[0, Y]$, the function $\varphi(x)$ is continuously differentiable on $[0, X]$, $\alpha(y) \neq 1$.

Let $C(\Omega, R)$ be the set of functions $w : \Omega \to R$ continuous on Ω.

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A function $u(x,y)\in C(\Omega,R)$, with partial derivatives $\frac{\partial u(x,y)}{\partial x}\in C(\Omega,R)$, $\frac{\partial u(x,y)}{\partial y}\in C(\Omega,R)$, $\frac{\partial^2 u(x,y)}{\partial x\partial y}\in C(\Omega,R)$, $\frac{\partial^2 u(x,y)}{\partial x^2}\in C(\Omega,R)$, $\frac{\partial^3 u(x,y)}{\partial x^2\partial y}\in C(\Omega,R)$ is called a solution to problem (1)–(4) if it satisfies equation (1), for all $(x,y)\in\Omega$, and boundary conditions (2)–(4).

To find a solution to problem (1)–(4), we introduce the functions

$$w(0,y)=\lambda(y),\quad \widetilde{w}(x,y)=w(x,y)-\lambda(y),$$

the original problem can be written as

$$\frac{\partial^3\widetilde{w}(x,y)}{\partial x^2\partial y}=\frac{\partial\widetilde{w}(x,y)}{\partial y}+\frac{\partial\lambda(y)}{\partial y}+[\widetilde{w}(x,y)+\lambda(y)]\frac{\partial\widetilde{w}(x,y)}{\partial x}+\frac{\partial\widetilde{w}(x,y)}{\partial x},$$

$$\widetilde{w}(0,y)=0,\quad x\in[0,X],$$

$$\widetilde{w}(x,0)+\lambda(0)=\varphi(x),\quad \lambda(0)=\varphi(0),$$

$$\frac{\partial\lambda(y)}{\partial y}=\alpha(y)\frac{\partial\widetilde{w}(X,y)}{\partial y}+\alpha(y)\frac{\partial\lambda(y)}{\partial y}+\psi(y),\quad y\in[0,Y],$$

$$\frac{\partial\widetilde{w}(0,y)}{\partial x}=\theta(y),\quad y\in[0,Y].$$

We introduce a new function $v(x,y)=\frac{\partial\widetilde{w}(x,y)}{\partial x}$, then $\widetilde{w}(x,y)=\int\limits_0^x v(\xi,y)d\xi$ and the problem goes to the equivalent problem

$$\frac{\partial^2 v(x,y)}{\partial x\partial y}=\int\limits_0^x\frac{\partial v(\xi,y)}{\partial y}d\xi+\frac{\partial\lambda(y)}{\partial y}+\left[\int\limits_0^x v(\xi,y)d\xi+\lambda(y)\right]v(x,y)+v(x,y),\tag{5}$$

$$v(x,0)=\varphi'(x),\quad x\in[0,X],\tag{6}$$

$$\frac{\partial\lambda(y)}{\partial y}=\frac{\alpha(y)}{1-\alpha(y)}\int\limits_0^X v(x,y)dx+\frac{\psi(y)}{1-\alpha(y)},\quad \lambda(0)=\varphi(0),\tag{7}$$

$$v(0,y)=\theta(y),\quad y\in[0,Y].\tag{8}$$

Integrating both parts of equation (5) with respect to the variable x and taking into account conditions (8) we get

$$\frac{\partial v(x,y)}{\partial y}=\theta'(y)+\int\limits_0^x\left(\int\limits_0^\xi\frac{\partial v(\xi_1,y)}{\partial y}d\xi_1+\frac{\partial\lambda(y)}{\partial y}+\left[\int\limits_0^\xi v(\xi_1,y)d\xi_1+\lambda(y)\right]v(\xi,y)+v(\xi,y)\right)d\xi.\tag{9}$$

Once again integrating over the variable y and using condition (6), we have [23]

$$v(x,y)=\varphi'(x)+\int\limits_0^y\left(\theta'(\eta)+\int\limits_0^x\int\limits_0^\xi\frac{\partial v(\xi_1,\eta)}{\partial\eta}d\xi_1 d\xi+\int\limits_0^x\frac{\partial\lambda(\eta)}{\partial\eta}d\xi+\right.$$

$$\left.+\int\limits_0^x\left(\left[\int\limits_0^\xi v(\xi_1,\eta)d\xi_1+\lambda(\eta)\right]v(\xi,\eta)+v(\xi,\eta)\right)d\xi\right)d\eta.\tag{10}$$

In addition, from (12) it follows

$$\lambda(y)=\varphi(0)+\int\limits_0^y\left(\frac{\alpha(\eta)}{1-\alpha(\eta)}\int\limits_0^X v(x,\eta)dx+\frac{\psi(\eta)}{1-\alpha(\eta)}\right)d\eta. \tag{11}$$

2 Main result

Setting $v(x,y)=\varphi'(x)$, from (7) and (11) we define

$$\frac{\partial\lambda^{(0)}(y)}{\partial y}=\frac{\alpha(y)}{1-\alpha(y)}\int\limits_0^X\varphi'(x)dx+\frac{\psi(y)}{1-\alpha(y)}=\frac{\alpha(y)}{1-\alpha(y)}[\varphi(X)-\varphi(0)]+\frac{\psi(y)}{1-\alpha(y)},$$

$$\lambda^{(0)}(y)=\varphi(0)+\int\limits_0^y\left(\frac{\alpha(\eta)}{1-\alpha(\eta)}\int\limits_0^X\varphi'(x)dx+\frac{\psi(\eta)}{1-\alpha(\eta)}\right)d\eta=$$

$$=\varphi(0)+\int\limits_0^y\left(\frac{\alpha(\eta)}{1-\alpha(\eta)}[\varphi(X)-\varphi(0)]+\frac{\psi(\eta)}{1-\alpha(\eta)}\right)d\eta.$$

Using equation (9), with $\lambda(y)=\lambda^{(0)}(y)$, we find

$$\frac{\partial v^{(0)}(x,y)}{\partial y}=\theta'(y)+\int\limits_0^x\left(\frac{\partial\lambda^{(0)}(y)}{\partial y}+\left[\int\limits_0^\xi\varphi'(\xi_1)d\xi_1+\lambda(y)\right]\varphi'(\xi)+\varphi'(\xi)\right)d\xi.$$

$$v^{(0)}(x,y)=\varphi'(x)+\int\limits_0^y\frac{\partial v^{(0)}(x,\eta)}{\partial\eta}d\eta.$$

Taking the functions $\lambda^{(0)}(y), v^{(0)}(x,y)$, numbers $\rho_1>0,\rho_2>0$ we construct the sets

$$S\Big(\lambda^{(0)}(y),\rho_1\Big)=\Big\{\lambda(y)\in C([0,Y],R):\|\lambda(y)-\lambda^{(0)}(y)\|<\rho_1\Big\},$$

$$S(v^{(0)}(x,y),\rho_2)=\Big\{v(x,y)\in C(\Omega,R):\|v(x,y)-v^{(0)}(x,y)\|<\rho_2,(x,y)\in\Omega\Big\},$$

$$G^0(\rho_1,\rho_2)=\Big\{(x,y,w,v):(x,y)\in\Omega,\quad\Big\|w(x,y)-\int\limits_0^x v^{(0)}(\xi,y)d\xi-\lambda^{(0)}(y)\Big\|<\rho_1+\rho_2,$$

$$\|v(x,y)-v^{(0)}(x,y)\|<\rho_2\Big\}.$$

Denote by $U(L_1,L_2,x,y)$ the collection $\Big(\lambda^{(0)}(y),v^{(0)}(x,y),\rho_1,\rho_2\Big)$, or which the function $f(x,y,w,v)$ in $G^0(\rho_1,\rho_2)$ has continuous partial derivatives $f'_w(x,y,w,v)$, $f'_v(x,y,w,v)$ and

$$\|f'_w(x,y,w,v)\|\le L_1,\quad \|f'_v(x,y,w,v)\|\le L_2,\quad L_1,L_2-const.$$

Based on the system $\{\lambda(y), \mu(y), v(x,y)\}$ we compose the triple $\{\lambda^{(0)}(y), \mu^{(0)}(y), v^{(0)}(x,y)\}$, which we take as the initial approximation of problem (5)–(8) and build successive approximations according to the following algorithm:

Step 1. Setting $v(x,y) = v^{(0)}(x,y)$, from (7) and (11) we determine $\frac{\partial\lambda^{(1)}(y)}{\partial y}$ and $\lambda^{(1)}(y)$. Using equation (9), with $\lambda(y) = \lambda^{(1)}(y)$, we find $\frac{\partial v^{(1)}(x,y)}{\partial y}$. Next, we find $v^{(1)}(x,y) = \varphi'(x) + \int\limits_0^y \frac{\partial v^{(1)}(x,\eta)}{\partial \eta} d\eta$.

Step 2. Taking $v(x,y) = v^{(1)}(x,y)$, from (7) and (11) we determine $\frac{\partial\lambda^{(2)}(y)}{\partial y}$ and $\lambda^{(2)}(y)$, respectively. Using equation (9), with $\lambda(y) = \lambda^{(2)}(y)$, we find $\frac{\partial v^{(2)}(x,y)}{\partial y}$. Let us find $v^{(2)}(x,y) = \varphi'(x) + \int\limits_0^y \frac{\partial v^{(2)}(x,\eta)}{\partial \eta} d\eta$.

Continuing the process, at the k-th step we obtain the system $\left\{\frac{\partial\lambda^{(2)}(y)}{\partial y}, \lambda_r^{(k)}(x), \frac{\partial v^{(k)}(x,y)}{\partial y}, v_r^{(k)}(x,t)\right\}$.

The conditions of the following statement ensure the feasibility and convergence of the proposed algorithm, as well as the solvability of problem (5)–(8).

Theorem 1. Let there exist $(\lambda^{(0)}(y), v^{(0)}(x,y), \rho_1, \rho_2) \in U(L_1, L_2, x, y)$, where $(x,y) \in \Omega$, $(\lambda(y), v(x,y)) \in S(\lambda^{(0)}(y), \rho_1) \times S(v^{(0)}(x,y), \rho_2)$ and following conditions are satisfied:

1) the function $\varphi(x)$ is continuously differentiable on $[0, X]$,

2) the functions $a(y), \psi(y), \theta(y)$ are continuously differentiable on $[0, Y]$, $\alpha(y) \neq 1$,

3) $q = X\left(\frac{X}{2} + \frac{X\alpha}{1-\alpha} + \frac{XYL_1}{2} + \frac{XY^2}{2}\frac{L_1\alpha}{1-\alpha} + (L_2+1)Y\right) < 1$,

4) $\frac{\sigma}{1-q}\frac{\alpha}{1-\alpha}\frac{XY^2}{2} < \rho_1, \quad \frac{qY\sigma}{1-q} < \rho_2$,

where $\sigma = \theta' + \frac{X\psi}{1-\alpha}\left(1 + \max\limits_{x\in[0,X]} \|\varphi'(x)\| Y\right) + X \max\limits_{x\in[0,X]} \|\varphi'(x)\| \left(1 + \max\limits_{x\in[0,X]} \|\varphi(x)\|\right)$,
$\alpha = \max\limits_{y\in[0,Y]} \|\alpha(y)\|$, $\psi = \max\limits_{y\in[0,Y]} \|\psi(y)\|$, $\theta = \max\limits_{y\in[0,Y]} \|\theta(y)\|$, then the nonlocal problem (5)–(8) for the nonlinear Benjamin-Bona-Mahony equation has a unique solution belonging to $S(\lambda^{(0)}(y), \rho_1) \times S(v^{(0)}(x,y), \rho_2)$ and estimates are made:

$$a)\|\lambda^*(y) - \lambda^{(k)}(y)\| \leq \frac{\alpha}{1-\alpha}\frac{XY^2}{2}\sum_{i=k}^{\infty} q^i\sigma, \quad b)\|v^*(x,y) - v^{(k)}(x,y)\| \leq Y\sum_{i=k+1}^{\infty} q^i\sigma.$$

Proof. From the zero step of the algorithm, the following estimates hold:

$$\|\lambda^{(0)}(y)\| \leq \varphi(0) + \frac{\psi Y}{1-\alpha}, \quad \left\|\frac{\partial\lambda(y)}{\partial y}\right\| \leq \frac{\psi}{1-\alpha},$$

$$\left\|\frac{\partial v^{(0)}(x,y)}{\partial y}\right\| \leq \theta' + X\frac{\psi}{1-\alpha} + X\max_{x\in[0,X]}\|\varphi(x)\| \max_{x\in[0,X]}\|\varphi'(x)\| +$$

$$+XY\frac{\psi}{1-\alpha}\max_{x\in[0,X]}\|\varphi'(x)\| + X\max_{x\in[0,X]}\|\varphi'(x)\| \leq$$

$$\leq \theta' + \frac{X\psi}{1-\alpha}\left(1 + \max_{x\in[0,X]}\|\varphi'(x)\|Y\right) + X\max_{x\in[0,X]}\|\varphi'(x)\|\left(1 + \max_{x\in[0,X]}\|\varphi(x)\|\right) = \sigma,$$

$$\|v^{(0)}(x,y) - \varphi'(x)\| \leq \int\limits_0^y \left\|\frac{\partial v^{(0)}(x,\eta)}{\partial\eta}\right\| d\eta < Y\sigma.$$

From the first step of the algorithm, for $v(x,y)=v^{(0)}(x,y)$, the following inequalities follow

$$\|\lambda^{(1)}(y)-\lambda^{(0)}(y)\|\leq\frac{\alpha}{1-\alpha}\int\limits_0^y\int\limits_0^X\|v^{(0)}(x,\eta)-\varphi'(x)\|dxd\eta<\frac{\alpha}{1-\alpha}\frac{XY^2}{2}\sigma<\rho_1,$$

$$\left\|\frac{\partial\lambda^{(1)}(y)}{\partial y}-\frac{\partial\lambda^{(0)}(y)}{\partial y}\right\|\leq\frac{\alpha}{1-\alpha}\int\limits_0^X\left\|\frac{\partial v^{(0)}(x,y)}{\partial y}\right\|dx,$$

$$\left\|\frac{\partial v^{(1)}(x,y)}{\partial y}-\frac{\partial v^{(0)}(x,y)}{\partial y}\right\|\leq$$

$$\leq\int\limits_0^x\int\limits_0^\xi\left\|\frac{\partial v^{(0)}(\xi_1,y)}{\partial y}\right\|d\xi_1d\xi+\int\limits_0^x\left\|\frac{\partial\lambda^{(1)}(y)}{\partial y}-\frac{\partial\lambda^{(0)}(y)}{\partial y}\right\|d\xi+$$

$$+L_1\int\limits_0^x\int\limits_0^\xi\|v^{(0)}(\xi_1,y)-\varphi'(\xi_1)\|d\xi_1d\xi+L_1\|\lambda^{(1)}(y)-\lambda^{(0)}(y)\|\int\limits_0^x d\xi+$$

$$+L_2\int\limits_0^x\|v^{(0)}(\xi,y)-\varphi'(\xi)\|d\xi+\int\limits_0^x\|v^{(0)}(\xi,y)-\varphi'(\xi)\|d\xi\leq$$

$$\leq q\max_{(x,y)\in\Omega}\left\|\frac{\partial v^{(0)}(x,y)}{\partial y}\right\|\leq q\sigma.$$

$$\|v^{(1)}(x,y)-v^{(0)}(x,y)\|\leq\int\limits_0^y\left\|\frac{\partial v^{(1)}(x,\eta)}{\partial\eta}-\frac{\partial v^{(0)}(x,\eta)}{\partial\eta}\right\|d\eta\leq\int\limits_0^y q\sigma d\eta<\rho_2.$$

At the second step of the algorithm, for $v(x,y)=v^{(1)}(x,y)$, the following estimates hold:

$$\|\lambda^{(2)}(y)-\lambda^{(1)}(y)\|\leq\frac{\alpha}{1-\alpha}\int\limits_0^y\int\limits_0^X\|v^{(1)}(x,\eta)-v^{(0)}(x,\eta)\|dxd\eta\leq\frac{\alpha}{1-\alpha}\frac{XY^2}{2}q\sigma,$$

$$\left\|\frac{\partial v^{(2)}(x,y)}{\partial y}-\frac{\partial v^{(1)}(x,y)}{\partial y}\right\|\leq q\max_{(x,y)\in\Omega}\left\|\frac{\partial v^{(1)}(x,y)}{\partial y}-\frac{\partial v^{(0)}(x,y)}{\partial y}\right\|\leq q^2\max_{(x,y)\in\Omega}\left\|\frac{\partial v^{(0)}(x,y)}{\partial y}\right\|\leq q^2\sigma,$$

$$\|v^{(2)}(x,y)-v^{(1)}(x,y)\|\leq\int\limits_0^y\left\|\frac{\partial v^{(2)}(x,\eta)}{\partial\eta}-\frac{\partial v^{(1)}(x,\eta)}{\partial\eta}\right\|d\eta\leq Yq^2\sigma.$$

$$\|\lambda^{(2)}(y)-\lambda^{(0)}(y)\|\leq\|\lambda^{(2)}(y)-\lambda^{(1)}(y)\|+\|\lambda^{(1)}(y)-\lambda^{(0)}(y)\|\leq$$

$$\leq\frac{\alpha}{1-\alpha}\frac{XY^2}{2}q\sigma+\frac{\alpha}{1-\alpha}\frac{XY^2}{2}\sigma\leq\frac{\alpha}{1-\alpha}\frac{XY^2}{2}(1+q)\sigma<\rho_1,$$

$$\left\|\frac{\partial v^{(2)}(x,y)}{\partial y}-\frac{\partial v^{(0)}(x,y)}{\partial y}\right\|\leq(1+q)\max_{(x,y)\in\Omega}\left\|\frac{\partial v^{(1)}(x,y)}{\partial y}-\frac{\partial v^{(0)}(x,y)}{\partial y}\right\|\leq$$

$$\leq (q+q^2)\max_{(x,y)\in\Omega}\left\|\frac{\partial v^{(0)}(x,y)}{\partial y}\right\| \leq (q+q^2)\sigma.$$

$$\|v^{(2)}(x,y)-v^{(0)}(x,y)\| \leq \|v^{(2)}(x,y)-v^{(1)}(x,y)\| + \|v^{(1)}(x,y)-v^{(0)}(x,y)\| \leq Y(q^2+q)\sigma < \rho_2.$$

At the $(k+1)-$th step of the algorithm, for $v(x,y)=v^{(k)}(x,y)$, the following estimates hold:

$$\|\lambda^{(k+1)}(y)-\lambda^{(k)}(y)\| \leq \frac{\alpha}{1-\alpha}\int\limits_0^y\int\limits_0^X \|v^{(k)}(x,\eta)-v^{(k-1)}(x,\eta)\|dxd\eta. \tag{12}$$

$$\left\|\frac{\partial\lambda^{(k+1)}(y)}{\partial y}-\frac{\partial\lambda^{(k)}(y)}{\partial y}\right\| \leq \frac{\alpha}{1-\alpha}\int\limits_0^X\left\|\frac{\partial v^{(k)}(x,y)}{\partial y}-\frac{\partial v^{(k-1)}(x,y)}{\partial y}\right\|dx, \tag{13}$$

$$\left\|\frac{\partial v^{(k+1)}(x,y)}{\partial y}-\frac{\partial v^{(k)}(x,y)}{\partial y}\right\| \leq q\max_{(x,y)\in\Omega}\left\|\frac{\partial v^{(k)}(x,y)}{\partial y}-\frac{\partial v^{(k-1)}(x,y)}{\partial y}\right\|, \tag{14}$$

$$\|v^{(k+1)}(x,y)-v^{(k)}(x,y)\| \leq \int\limits_0^y\left\|\frac{\partial v^{(k+1)}(x,\eta)}{\partial\eta}-\frac{\partial v^{(k)}(x,\eta)}{\partial\eta}\right\|d\eta. \tag{15}$$

$$\|\lambda^{(k+1)}(y)-\lambda^{(0)}(y)\| \leq \frac{\alpha}{1-\alpha}\int\limits_0^y\int\limits_0^X \|v^{(k)}(x,\eta)-\varphi'(x)\|dxd\eta \leq \frac{\alpha}{1-\alpha}\frac{XY^2}{2}\sum_{i=0}^{k}q^i\sigma < \rho_1,$$

$$\|v^{(k+1)}(x,y)-v^{(0)}(x,y)\| \leq Y\sum_{i=1}^{k+1}q^i\sigma < \rho_2.$$

Thus, it follows from inequalities (12)–(15) and $q<1$ that the sequence $\{\lambda^{(k)}(y), v^{(k)}(x,y)\}$ as $k\to\infty$, converges to $\{\lambda^*(y), v^*(x,y)\}$ to the solution of problem (5)–(8) in $S(\lambda^{(0)}(y),\rho_1)\times S(v^{(0)}(x,y),\rho_2)$.

Let's establish the inequalities

$$\|\lambda^{(k+p)}(y)-\lambda^{(k)}(y)\| \leq \frac{\alpha}{1-\alpha}\frac{XY^2}{2}\sum_{i=k}^{k+p-1}q^i\sigma, \qquad \|v^{(k+p)}(x,y)-v^{(k)}(x,y)\| \leq Y\sum_{i=k+1}^{k+p}q^i\sigma,$$

as $p\to\infty$ we obtain estimates a), b) of Theorem 1.

Let's prove uniqueness.

Let there be two solutions $(\lambda^*(x), v^*(x,y))$, $(\lambda^{**}(x), v^{**}(x,y))$ in $S(\lambda^{(0)}(x),\rho_1)\times S(v^{(0)}(x,y),\rho_2)$ of problem (10)–(14). Similar to relations (12)–(15) for the differences $\lambda^{**}(y)-\lambda^*(y)$, $\frac{\partial\lambda^{**}(y)}{\partial y}-\frac{\partial\lambda^*(y)}{\partial y}$, $\frac{\partial v^{**}(x,y)}{\partial y}-\frac{\partial v^*(x,y)}{\partial y}$, $v^{**}(x,y)-v^*(x,y)$, for all $(x,y)\in\Omega$, we get:

$$\|\lambda^{**}(y)-\lambda^*(y)\| \leq \frac{\alpha}{1-\alpha}\int\limits_0^y\int\limits_0^X \|v^{**}(x,\eta)-v^*(x,\eta)\|dxd\eta,$$

$$\left\|\frac{\partial\lambda^{**}(y)}{\partial y}-\frac{\partial\lambda^*(y)}{\partial y}\right\| \leq \frac{\alpha}{1-\alpha}\int\limits_0^X \|v^{**}(x,y)-v^*(x,y)\|dx,$$

$$\left\| \frac{\partial v^{**}(x,y)}{\partial y} - \frac{\partial v^{*}(x,y)}{\partial y} \right\| \le q \max_{(x,y)\in\Omega} \left\| \frac{\partial v^{**}(x,y)}{\partial y} - \frac{\partial v^{*}(x,y)}{\partial y} \right\|,$$

$$\|v^{**}(x,y) - v^{*}(x,y)\| \le \int_0^y \left\| \frac{\partial v^{**}(x,\eta)}{\partial \eta} - \frac{\partial v^{*}(x,\eta)}{\partial \eta} \right\| d\eta.$$

Whence it follows that $\lambda^{**}(x) = \lambda^{*}(x)$, $v^{**}(x,y) = v^{*}(x,y)$. Theorem 1 is proved.

The function $w^{(k)}(x,y), k = 1, 2, ...,$ is defined by the equality

$$w^{(k)}(x,y) = \int_0^x v^{(k)}(\xi,y)d\xi + \lambda^{(k)}(y)$$

and denote by $S(w^{(0)}(x,y), \rho_1 + \rho_2)$ the set of continuously differentiable with respect to y and twice with respect to x functions $w : \Omega \to R$ satisfying the inequalities

$$\|w(x,y) - \int_0^x v^{(0)}(\xi,y)d\xi - \lambda^{(0)}(y)\| < \rho_1 + \rho_2.$$

In view of the equivalence of problems (1)–(4) and (5)–(8), Theorem 1 implies.

Theorem 2. If the conditions of Theorem 1 are satisfied, then the sequence of functions $w^{(k)}(x,y)$, $k = 1, 2, ...,$ is contained in $S(w^{(0)}(x,y), \rho_1 + \rho_2)$ converges to the unique solution $w^{*}(x,y)$ of problem (1)–(4) in $S(w^{(0)}(x,y), \rho_1 + \rho_2)$ and the inequality

$$\|w^{*}(x,y) - w^{(k)}(x,y)\| \le \frac{\alpha}{1-\alpha} \frac{X^2Y^2}{2} \sum_{i=k+1}^{\infty} q^i\sigma + Y \sum_{i=k}^{\infty} q^i\sigma.$$

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Бенджамин-Бон-Махони теңдеуі үшін бейлокал шеттік есептің бір жуық шешімі жайында

Жұмыста Бенджамин-Бона-Махони теңдеуі үшін бейлокал шеттік есеп зерттелді. Қарастырылып отырған теңдеу аралас туындылы үшінші ретті сызықтық емес псевдопараболалық теңдеу болып табылады. Осы есептің шешімін табу үшін жуық шешімін табу алгоритмі ұсынылған. Ұсынылған алгоритмнің орындалуы мен жинақтылығының жеткілікті шарттары алынды, сондай-ақ, сызықтық емес теңдеу үшін бейлокал шеттік есебінің оқшауланған шешімінің бар болуы табылған. Қарастырылған есептің дәл және жуық шешімі арасындағы бағалаулары алынды.

Кілт сөздер: дербес туындылы диффереинциалдық теңдеу, Бенджамин-Бона-Махони теңдеуі, алгоритм, жуық шешім.

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Об одном приближенном решении нелокальной краевой задачи для уравнения Бенджамина-Бона-Махони

В работе исследована нелокальная краевая задача для уравнения Бенджамина-Бона-Махони. Рассматриваемое уравнение является нелинейным псевдопараболическим уравнением третьего порядка со смешанной производной. Для нахождения решения данной задачи предложен алгоритм поиска приближенного решения. Установлены достаточные условия осуществимости и сходимости предложенного алгоритма, а также существование изолированного решения нелокальной краевой задачи для нелинейного уравнения. Получены оценки между точным и приближенным решениями рассматриваемой задачи.

Ключевые слова: дифференциальные уравнения в частных производных, уравнение Бенджамина-Бона-Махони, алгоритм, приближенное решение.

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Compactness of Commutators for Riesz Potential on Local Morrey-type spaces

The paper considers Morrey-type local spaces from $LM^{w}_{p\theta}$ The main work is the proof of the commutator compactness theorem for the Riesz potential $[b, I_\alpha]$ in local Morrey-type spaces from $LM^{w_1}_{p\theta}$ to $LM^{w_2}_{q\theta}$. We also give new sufficient conditions for the commutator to be bounded for the Riesz potential $[b, I_\alpha]$ in local Morrey-type spaces from $LM^{w_1}_{p\theta}$ to $LM^{w_2}_{q\theta}$. In the proof of the commutator compactness theorem for the Riesz potential, we essentially use the boundedness condition for the commutator for the Riesz potential $[b, I_\alpha]$ in local Morrey-type spaces $LM^{w}_{p\theta}$, and use the sufficient conditions from the theorem of precompactness of sets in local spaces of Morrey type $LM^{w}_{p\theta}$. In the course of proving the commutator compactness theorem for the Riesz potential, we prove lemmas for the commutator ball for the Riesz potential $[b, I_\alpha]$. Similar results were obtained for global Morrey-type spaces $GM^{w}_{p\theta}$ and for generalized Morrey spaces M^{w}_{p}.

Keywords: Compactness, Commutators, Riesz Potential, Local Morrey-type spaces.

Introduction

First we give some definitions.

By $\mathfrak{M}(I)$ we denote the set of all measurable functions on I . The symbol $\mathfrak{M}^+(I)$ stands for the collection of all $f \in \mathfrak{M}(I)$ which are non-negative on I , while $\mathfrak{M}^+(I;\downarrow)$ and $\mathfrak{M}^+(I;\uparrow)$ are used to denote the subset of those functions which are non-increasing and non-decreasing on I , respectively. When $I = (0,\infty)$, we write simply $\mathfrak{M}^+$, $\mathfrak{M}^{\downarrow}$ and $\mathfrak{M}^{\uparrow}$ instead of $\mathfrak{M}^+(I)$, $\mathfrak{M}^+(I;\downarrow)$ and $\mathfrak{M}^+(I;\uparrow)$, accordingly. The family of all weight functions (also called just weights) on I , that is, locally integrable non-negative functions on $(0,\infty)$, is given by $\mathcal{W}(I)$.

For $p \in (0,\infty)$ and $w \in \mathfrak{M}^+(I)$, we define the functional $\|\cdot\|_{p,w,I}$ on $\mathfrak{M}(I)$, by

$$\|f\|_{p,w,I} := \begin{cases} (\int_I |f(x)|^p w(x)dx)^{\frac{1}{p}}, & \text{if} \quad p < \infty; \\ ess \quad sup_I |f(x)| w(x), & \text{if} \quad p = \infty. \end{cases}$$

If, in addition, $w \in \mathcal{W}(I)$, then the weighted Lebesgue space $L^p(w, I)$ is given by

$$L^p(w, I) = f \in \mathfrak{M}(I) : \|f\|_{p,w,I} < \infty,$$

and it is equipped with the quasi-norm $\|\cdot\|_{p,w,I}$. When $w \equiv 1$ on I , we write simply $L^p(I)$ and $\|\cdot\|_{p,I}$ instead of $L^p(w, I)$ and $\|\cdot\|_{p,w,I}$, respectively.

Let $1 \leq p, \theta \leq \infty$, w be a measurable non-negative function on $(0,\infty)$. The Local Morrey-type space $LM^{w}_{p\theta} \equiv LM^{w}_{p\theta}(\mathbb{R}^n)$ is defined as the set of all functions $f \in L^{loc}_p(\mathbb{R}^n)$ with finite quasi-norm

$$\|f\|_{LM^{w}_{p\theta}} \equiv \left\| w(r) \, \|f\|_{L_p(B(0,r))} \right\|_{L_\theta(0,\infty)},$$

where $B(t, r)$ the ball with center at the point t and of radius r.

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The space $LM^w_{p\theta}$ coincides with the known Morrey space M^λ_p at $w(r) = r^{-\lambda}, \theta = \infty$, where $0 \leq \lambda \leq \frac{n}{p}$, which, in turn, for $\lambda = 0$ coincides with the space $L_p(\mathbb{R}^n)$.

Following the notation of $[1, 2]$, we denote by Ω_θ the set of all functions which are non-negative, measurable on $(0, \infty)$, not equivalent to 0 and such that for some $t > 0$

$$\|w(r)\|_{L_\theta(t,\infty)} < \infty.$$

Note that the space $LM^w_{p\theta}$ is non-trivial, that is consists not only of functions equivalent to 0 on $\mathbb{R}^n$, if and only if $w \in \Omega_\theta$.

In this paper we consider the Riesz Potential in the following form

$$I_\alpha f(x) = \int\limits_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} dy.$$

The Riesz Potential I_α plays an important role in the harmonic analysis and theory of operators.

For a function $b \in L_{loc}(\mathbb{R}^n)$ by M_b denote multiplier operator $M_b f = bf$, where f is measurable function. Then the commutator between I_α and M_b is defined by

$$[b, I_\alpha] = M_b I_\alpha - I_\alpha M_b = \int\limits_{\mathbb{R}^n} \frac{[b(x) - b(y)]\, f(y)}{|x-y|^{n-\alpha}} dy.$$

The commutators for Riesz Potential were investigated [3–9].

It is said that the function $b(x) \in L_\infty(\mathbb{R}^n)$ belongs to the space $BMO(\mathbb{R}^n)$, if

$$\|b\|_* = \sup_{Q \subset \mathbb{R}^n} \frac{1}{|Q|} \int\limits_Q |b(x) - b_Q|\, dx = \sup_{Q \in \mathbb{R}^n} M(b, Q) < \infty,$$

where Q - cube $\mathbb{R}^n$ and $b_Q = \frac{1}{|Q|} \int\limits_{\mathbb{R}^n} f(y)dy$.

By $VMO(\mathbb{R}^n)$ we denote the BMO-closure $C^\infty_0(\mathbb{R}^n)$, where $C^\infty_0(\mathbb{R}^n)$ the set of all functions from $C^\infty(\mathbb{R}^n)$ with compact support. Through the $\chi(A)$ denotes the characteristic function of the set $B \subset \mathbb{R}^n$, and ${}^cA$ denotes the complement of A.

The main purpose of this work is to find sufficient conditions for the compactness of commutators operators $[b, I_\alpha]$ on the Local Morrey-type space $LM^w_{p\theta}(\mathbb{R}^n)$.

We note that in the case of the Morrey space this question was investigated in [4]. The following well-known theorem gives necessary and sufficient conditions for the boundedness and compactness for $[b, I_\alpha]$ on the Local Morrey-type spaces $LM^w_{p\theta}(\mathbb{R}^n)$.

1 *Formulas and theorems*

To formulate the following theorem on the boundedness of the Hardy operator in weighted Lebesgue spaces, we introduce the notation.

Denote by

$$H^* g(t) := \int_t^\infty g(s) ds, \quad g \in \mathfrak{M}^+,$$

the Hardy operator.

$$W(t) := \int_0^t w(t) dw,$$

$$U_*(t) := \int_t^\infty u(t)du,$$

$$V_*(t) := \int_t^\infty v(t)dv.$$

Theorem 1. Let $0 < q, p \le \infty$. Assume that $u, v, w \in \mathcal{W}(0,\infty)$. Then inequality

$$\|H_u^*(f)\|_{q,w,(0;\infty)} \le c\|f_u^*\|_{p,w,(0;\infty)}, f \in \mathfrak{M}^{\uparrow}$$

with the best constant c holds if and only if the following holds:

$$A_0^* := sup_{t>0}(\int_t^\infty U_*^q(\tau)w(\tau)d\tau)^{\frac{1}{q}} V_*^{-\frac{1}{p}}(t),$$

$$A_1^* := sup_{t>0} W^{\frac{1}{q}}(t)(\int_t^\infty (\frac{U_*(\tau)}{V_*(\tau)})^{p'} v(\tau)d\tau)^{\frac{1}{p'}},$$

and in this case $c \approx A_0^* + A_1^*$.

Theorem 2. (see. [2]) Let $1 < p < q < \infty$, $0 < \alpha = n(\frac{1}{p} - \frac{1}{q})$, $0 < \theta < \infty$, (w_1, w_2) satisfy the following condition

$$\left\| w_2(r)\left(\frac{r}{t+r}\right)^{\frac{n}{p}} \right\|_{L_\theta(0,\infty)} \le \|w_1(r)\|_{L_\theta(t,\infty)}. \tag{1}$$

Then the operator I_α is bounded from $LM_{p\theta}^{w_1}(\mathbb{R}^n)$ to $LM_{q\theta}^{w_2}(\mathbb{R}^n)$.

It is well known that the boundedness of such operators on Morrey space $LM_{p\theta}^{\lambda}(\mathbb{R})$ was considered in [1, 2].

The following theorem on sufficient conditions for the precompactness of sets on Local Morrey-type and other spaces was proved in [10–14].

Theorem 3. (see. [13]) Suppose that $1 \le p \le \theta \le \infty$ and $w \in \Omega_{p\theta}$. Suppose that a subset S of $LM_{p\theta}^w$ satisfies the following conditions:

$$\sup_{f\in S} \|f\|_{LM_{p\theta}^w} < \infty, \tag{2}$$

$$\lim_{u\to 0} \sup_{f\in S} \|f(\cdot + u) - f(\cdot)\|_{LM_{p\theta}^w} = 0, \tag{3}$$

$$\lim_{r\to\infty} \sup_{f\in S} \left\| f\chi_{^cB(0,r)} \right\|_{LM_{p\theta}^w} = 0. \tag{4}$$

Then S is a pre-compact set in $LM_{p\theta}^w(\mathbb{R})$.

Theorem 4. Let $1 < p \le q < \infty$, $0 < \alpha < n$ and $b \in BMO(R^n)$. $1 < p < \frac{n}{\alpha}$ $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$, $w_1, w_2 \in \Omega_\theta$. Then the condition

$$A_0^* := sup_{t>0}(\int_t^\infty \int_\tau^\infty (1 + ln\frac{\tau}{r})drw(\tau)d\tau)^{\frac{1}{q}} [\int_t^\infty v(t)dv]^{-\frac{1}{p}}, \tag{5}$$

$$A_1^* := sup_{t>0} W^{\frac{1}{q}}(t)(\int_t^\infty (\frac{U_*(\tau)}{V_*(\tau)})^{p'} v(\tau)d\tau)^{\frac{1}{p'}} < \infty. \tag{6}$$

Then the commutator $[b, I_\alpha]$ is the boundedness operator from $LM_{p\theta}^{w_1}$ to $LM_{q\theta}^{w_2}$.

Note that for the case of Morrey space $LM^{\lambda}_{p\theta}(0<\lambda<1)$ (i.e., if $w(r)=r^{-\lambda}$) this assertion was proved earlier in [4], and in the case of $\lambda=0$ is - known Frechet-Kolmogorov theorem [15]. We note that the pre-compactness some sets in Banach function spaces were investigated in [16]. Theorem 4 is proved using theorem 5.4 from [17] and theorem 3.4 from [5].

Now we give theorem about the compactness of the operators $[b, I_\alpha]$ on Local Morrey-type space $LM^{w}_{p\theta}(\mathbb{R}^n)$.

Theorem 5. Let $1<p\le q<\infty$, $0<\alpha<n$ and $b\in VMO(\mathbb{R}^n)$. $1<p<\frac{n}{\alpha}$, $\frac{1}{q}=\frac{1}{p}-\frac{\alpha}{n}$, $w_1, w_2\in\Omega_\theta$ satisfy the conditions (1), (5), (6). Then the commutator $[b, I_\alpha]$ is a compact operator from $LM^{w_1}_{p\theta}$ to $LM^{w_2}_{p\theta}$.

To prove this theorem we need the following auxiliary assertions.

Lemma 1. Let $n\in\mathbb{N}$, $1\le p\le\infty$, $1\le q\le\infty$, $0<\alpha<n\left(1-\frac{1}{q}\right)$, $\beta>0$. Then there exists $C>0$, depending only on n,p,q,α, such that for some $f\in L_p(B(0,\beta))$ satisfying the condition $suppf\subset\overline{B(0,\beta)}$, and for some $\gamma\ge 2\beta$, $t\in\mathbb{R}^n$, $r>0$

$$\left\|(I_\alpha f)\chi_{^cB(0,\gamma)}\right\|_{L_q(B(t,r))}\le C\gamma^{\alpha-n}\left(min\{\gamma,r\}\right)^{\frac{n}{q}}\|f\|_{L_pB(0,\beta)}. \tag{7}$$

Proof. From the definition of the operator I_α, we have

$$I=\left\|\left(I_\alpha f\right)\chi^c_{B(0,\gamma)}\right\|_{L_q(B(t,r))}=$$

$$=\left(\int\limits_{B(t,r)\cap{}^cB(0,\gamma)}\left|\int\limits_{\mathbb{R}^n}\frac{f(y)}{|x-y|^{n-\alpha}}dy\right|^q dx\right)^{\frac{1}{q}}\le$$

$$\le\left(\int\limits_{B(t,r)\cap{}^cB(0,\gamma)}\left|\int\limits_{B(0,\beta)}\frac{f(y)}{|x-y|^{n-\alpha}}dy\right|^q dx\right)^{\frac{1}{q}}.$$

It is clear that $\beta\le\frac{\gamma}{2}$ for $x\in{}^c B(0,\gamma), y\in B(0,\beta)$ we have

$$|x-y|\ge|x|-|y|\ge|x|-\beta=\frac{|x|}{2}+\frac{|x|}{2}-\beta\ge\frac{|x|}{2}. \tag{8}$$

From this it follows that

$$I\le 2^{n-\alpha}\left(\int\limits_{^cB(0,\gamma)}\frac{dx}{|x|^{(n-\alpha)q}}\right)^{\frac{1}{q}}\int\limits_{B(0,\beta)}|f(y)|dy\le$$

$$\le 2^{n-\alpha}\left(\delta_n\int\limits_{\gamma}^{\infty}\rho^{(n-\alpha)q+n-1}d\rho\right)^{\frac{1}{q}}\left(\upsilon_n\beta^n\right)^{1-\frac{1}{p}}\|f\|_{L_p(B(0,\beta))}=$$

$$=2^{n-\alpha}\left(\frac{\delta_n}{(n-\alpha)q-n}\right)^{\frac{1}{q}}\upsilon_n^{1-\frac{1}{p}}\beta^{n(1-\frac{1}{p})}\gamma^{\alpha-n(1-\frac{1}{p})}\|f\|_{L_p(B(0,\beta))}\equiv$$

$$\equiv C_1\gamma^{\alpha-n(1-\frac{1}{p})}\|f\|_{L_p(B(0,\beta))}. \tag{9}$$

$\beta=\frac{\gamma}{2}$ for $x\in{}^c B(0,\gamma), y\in B(0,\beta)$, then using (8) we get $|x-y|\ge\frac{|x|}{2}$.

Next, we consider

$$I\le 2^{n-\alpha}\gamma^{\alpha-n}\left(\int\limits_{B(t,r)}dx\right)^{\frac{1}{q}}\int\limits_{B(0,\beta)}|f(y)|dy\le$$

$$\le 2^{n-\alpha}\gamma^{\alpha-n}\left(\upsilon_n r^n\right)^{\frac{1}{q}}\left(\upsilon_n\beta^n\right)^{1-\frac{1}{p}}\|f\|_{L_p(B(0,\beta))}=$$

$$=C_2\gamma^{\alpha-n}r^{\frac{n}{q}}\|f\|_{L_p(B(0,\beta))}. \tag{10}$$

From inequality (9) and (10) it follows (7), where $C=\max\{C_1,C_2\}$.

Lemma 2. Let $n\in\mathbb{N},\ 1\le p\le\infty,\ 1\le q\le\infty,\ 0<\alpha<n\left(1-\frac{1}{q}\right),\ \beta>0$. Then there exists $C>0$, depending only on n,p,q,α such that for some $f\in L_p(B(0,\beta))$, $b\in L_\infty(\mathbb{R}^n)$, satisfying the condition *supp* $b\subset\overline{B(0,\beta)}$, and for some $\gamma\ge 2\beta$, $t\in\mathbb{R}^n$, $r>0$

$$\left\|([b,I_\alpha]f)\chi_{{}^cB(0,\gamma)}\right\|_{L_q(B(t,r))}\le C\gamma^{\alpha-n}\left(min\{\gamma,r\}\right)^{\frac{n}{q}}\|b\|_{L_\infty(\mathbb{R}^n)}\|f\|_{L_pB(0,\beta)}. \tag{11}$$

Proof. Let $\gamma>\beta$,*supp* $b\subset B(0,\beta)$, for $x\in^c B(0,\gamma), b(x)=0$. Then

$$\left\|[b,I_\alpha]\,f\chi^c_{B(0,\gamma)}\right\|_{L_q(B(t,r))}=$$

$$=\left(\int\limits_{B(t,r)\cap^c B(0,\gamma)}\left|\int\limits_{\mathbb{R}^n}\frac{(b(x)-b(y))f(y)}{|x-y|^{n-\alpha}}dy\right|^q dx\right)^{\frac{1}{q}}\le$$

$$\le\left(\int\limits_{B(t,r)\cap^c B(0,\gamma)}\left|\int\limits_{\mathbb{R}^n}\frac{b(y)f(y)}{|x-y|^{n-\alpha}}dy\right|^q dx\right)^{\frac{1}{q}}\le\left(\int\limits_{B(t,r)\cap^c B(0,\gamma)}\left|\int\limits_{B(0,\beta)}\frac{|b(y)||f(y)|}{|x-y|^{n-\alpha}}dy\right|^q dx\right)^{\frac{1}{q}}\le$$

$$\le\left(\int\limits_{B(t,r)\cap^c B(0,\gamma)}\left|\int\limits_{B(0,\beta)}\frac{|f(y)|}{|x-y|^{n-\alpha}}dy\right|^q dx\right)^{\frac{1}{q}}\|b\|_{L_\theta(\mathbb{R}^n)}\le$$

$$\le\left(\int\limits_{B(t,r)\cap^c B(0,\gamma)}\left|\int\limits_{\mathbb{R}^n}\frac{|f(y)|}{|x-y|^{n-\alpha}}dy\right|^q dx\right)^{\frac{1}{q}}\|b\|_{L_\theta(\mathbb{R}^n)}=\left\|(I_\alpha f)\,\chi^c_{B(0,\gamma)}\right\|_{L_q(B(t,r))}\|b\|_{L_\theta(\mathbb{R}^n)}.$$

From this and from Lemma 1 we obtain the inequality (11).

Proof of Theorem 5. To the proof of Theorem 5 it is sufficient to show that the conditions (2)–(4) of Theorem 3 are hold.

Let F be an arbitrary bounded subset of $LM^{w_1}_{p\theta}$. Since $C^\infty_c(\mathbb{R}^n)$ is dense in $VMO(\mathbb{R}^n)$ we only need to prove that the set $G=\{[b,I_\alpha]f: f\in F, b\in C^\infty_c\}$ is pre-compact in the $GM^{w_2}_p$. By Theorem 3, we only need to verify the conditions (2), (3) and (4) hold uniformly F for $b\in C^\infty_c\}$.

Suppose that

$$\|f\|_{LM^{w_1}_{p\theta}}\le D.$$

Applying condition (1), we have

$$\|[b,I_\alpha]f\|_{LM^{w_2}_{p\theta}}\le C\cdot\|b\|*\sup_{f\in F}\|f\|_{M^{w_1}_p}\le C\cdot D\|b\|*<\infty.$$

This implies that the condition (2) of Theorem 3 is hold.

Now we prove that condition (4) of Theorem 3 also is hold, i. e.

$$\lim_{\gamma\to\infty}\left\|([b,I_\alpha]\,f)\chi^c_{B(0,\gamma)}\right\|_{LM^{w_2}_{p\theta}}=0.$$

It follows from Lemma 2. Indeed

$$\left\|([b,I_\alpha]\,f)\chi^c_{B(0,\gamma)}\right\|_{LM^{w_2}_{p\theta}}=$$

$$=\left\|w(r)\left\|([b,I_\alpha]\,f)\chi^c_{B(0,\gamma)}\right\|_{L_p(B(0,r))}\right\|_{L_\theta(0,\infty)}\leq$$

$$\leq C\gamma^{-n}\left\|b\right\|_{L_\theta(\mathbb{R}^n)}\left\|f\right\|_{L_pB(0,\beta)}\sup_{\substack{r>0,\\ x\in\mathbb{R}^n}}\left\|w_2(r)\,(min\{\gamma,r\})^{\frac{n}{p}}\right\|_{L_\theta(0,\infty)}.$$

When $r<l<\gamma$ we have $(min\{\gamma,r\})^{\frac{n}{p}}=r^{\frac{n}{p}}$. By condition $\left\|w_2(r)r^{\frac{n}{p}}\right\|_{L_\theta(l,\infty)}<\infty$.

When $\gamma<t<r$ we have $(min\{\gamma,r\})^{\frac{n}{p}}=\gamma^{\frac{n}{p}}$. By condition $\left\|w_2(r)\right\|_{L_\theta(0,t)}<\infty$.

Therefore

$$\lim_{\gamma\to\infty}\left\|([b,I_\alpha]\,f)\chi^c_{B(0,\gamma)}\right\|_{LM^{w_2}_{p\theta}}=0.$$

This implies the required condition (4).

Now we prove that condition (3) of Theorem 3 for the set $[b,I_\alpha](f)$, $f\in F$, is hold i.e. we show that for any $0<\varepsilon<\frac{1}{2}$ and if $|z|$ is sufficiently small depending only on ε, then for every $f\in F$.

$$\left\|[(b,I_\alpha f)(\cdot+z)]-[b,I_\alpha]\,f(\cdot)\right\|_{LM^{w_2}_{p\theta}}\leq C\cdot\varepsilon.$$

Let ε arbitrary number such that $0<\varepsilon<\frac{1}{2}$. For $|z|\in\mathbb{R}^n$ we have, that

$$[f,I_\alpha]f(x+z)-[b,I_\alpha]f(x)=\int\limits_{\mathbb{R}^n}\frac{[b(x+z)-b(y)]f(y)}{|x+z-y|^{n-\alpha}}dy-\int\limits_{\mathbb{R}^n}\frac{[b(x)-b(y)]f(y)}{|x-y|^{n-\alpha}}dy=$$

$$=\int\limits_{\mathbb{R}^n}\frac{[b(x+z)-b(y)]f(y)}{|x+z-y|^{n-\alpha}}dy-\int\limits_{\mathbb{R}^n}\frac{[b(x)+b(x+z)-b(x+z)-b(y)]f(y)}{|x-y|^{n-\alpha}}dy=$$

$$=\int\limits_{\mathbb{R}^n}\frac{[b(x+z)-b(y)]f(y)}{|x+z-y|^{n-\alpha}}dy+\int\limits_{\mathbb{R}^n}\frac{[b(y)-b(x+z)]f(y)}{|x-y|^{n-\alpha}}dy+$$

$$+\int\limits_{\mathbb{R}^n}\frac{[b(x+z)-b(x)]f(y)}{|x-y|^{n-\alpha}}dy=$$

$$=\int\limits_{\mathbb{R}^n}[b(y)-b(x+z)]\left(\frac{f(y)}{|x-y|^{n-\alpha}}-\frac{f(y)}{|x+z-y|^{n-\alpha}}\right)dy+\int\limits_{\mathbb{R}^n}\frac{[b(x+z)-b(x)]f(y)}{|x-y|^{n-\alpha}}dy=$$

$$=\int\limits_{|x-y|>|z|e^{\frac{1}{\varepsilon}}}\frac{[b(x+z)-b(x)]f(y)}{|x-y|^{n-\alpha}}dy+$$

$$+\int\limits_{|x-y|>|z|e^{\frac{1}{\varepsilon}}}\left(\frac{f(y)}{|x-y|^{n-\alpha}}-\frac{f(y)}{|x+z-y|^{n-\alpha}}\right)[b(x+z)-b(y)]dy+$$

$$+\int\limits_{|x-y|\leq|z|e^{\frac{1}{\varepsilon}}}\frac{[b(y)-b(x)]f(y)}{|x-y|^{n-\alpha}}dy-\int\limits_{|x-y|\leq|z|e^{\frac{1}{\varepsilon}}}\frac{[b(y)-b(x+z)]f(y)}{|x+z-y|^{n-\alpha}}dy=$$
$$=J_1+J_2+J_3-J_4. \qquad (12)$$

Since $b\in C_0^n(\mathbb{R}^n)$, we have
$$|b(x)-b(x+z)|\leq|\nabla f(x)|\cdot|z|\leq C|z|.$$

Then
$$|J_1|\leq C|z|I_\alpha(|f|)(x).$$

By Theorem 5
$$||J_1||_{LM^{w_2}_{p\theta}}\leq C|z|\parallel I_\alpha(f)\parallel_{LM^{w_2}_{p\theta}}\leq C|z|\parallel f\parallel_{LM^{w}_{p\theta}}\leq CD|z|. \qquad (13)$$

For J_2 we have that
$$(b(x+z)-b(y))\leq 2\parallel b\parallel_\infty\leq C.$$

Therefore
$$|J_2|\leq C|z|\int\limits_{|x-y|>|z|e^{\frac{1}{\varepsilon}}}\frac{f(y)}{|x-y|^n}dy\leq C_\varepsilon I_\alpha(|f|)(x).$$

Again by the of Theorem 1 we get
$$\parallel J_2\parallel_{LM^{w_2}_{p\theta}}\leq c\varepsilon\parallel I_\alpha(f)\parallel_{LM^{w_1}_{p\theta}}\leq c\varepsilon\parallel C\cdot D\cdot\varepsilon.$$

Consequently,
$$|J_3|\leq C\int\limits_{|x-y|\leq|z|e^{\frac{1}{\varepsilon}}}\frac{f(y)}{|x-y|^{n-\alpha}}dy\leq C\varepsilon^{-1}|z|\int\limits_{|x-y|\leq|z|e^{\frac{1}{\varepsilon}}}\frac{f(y)}{|x-y|^{n-\alpha}}dy\leq C\cdot\varepsilon^{-1}|z|I_\alpha(|f|)(x).$$

Thus, we have
$$\parallel J_3\parallel_{LM^{w_2}_{p\theta}}\leq C\cdot\varepsilon^{-1}|z|\parallel I_\alpha(f)\parallel_{LM^{w_2}_{p\theta}}\leq C\cdot\varepsilon^{-1}|z|\parallel f\parallel_{LM^{w_1}_{p\theta}}\leq\varepsilon^{-1}|z|. \qquad (14)$$

Similarly, using the estimate finally by
$$|b(x+z)-b(y)|\leq C|x+z-y|,$$
we have
$$|J_4|\leq C\int\limits_{|x-y|\leq e^{\varepsilon^{-1}}(y)}|x+z-y|^{-n+1+\alpha}|b(y)|dy\leq C(\varepsilon^{-1}|z|+|z|)I_\alpha(|f|)(x+z).$$

Therefore
$$\parallel J_4\parallel_{LM^{w_2}_{p\theta}}\leq C\cdot(\varepsilon^{-1}|z|+|z|)\parallel f\parallel_{LM^{w_1}_{p\theta}}\leq C\cdot(\varepsilon^{-1}|z|+|z|). \qquad (15)$$

Here C does not depend on z and ε. Finally from (12)–(15) taking a $|z|$ small enough we have
$$\|[b,I_\alpha(f)(\cdot+z)]-[b,I_\alpha]\,f(\cdot)\|_{LM^{w_2}_{p\theta}}\leq\|J_1\|_{LM^{w_2}_{p\theta}}+\|J_2\|_{LM^{w_2}_{p\theta}}+\|J_3\|_{LM^{w_2}_{p\theta}}+\|J_4\|_{LM^{w_2}_{p\theta}}\leq C\cdot D\cdot\varepsilon$$
i.e. the set $[b,I_\alpha](f),\ f\in F$ satisfies the condition (3) of Theorem 3. Then by Theorem 3, the set $[b,I_\alpha](f),\ f\in F$ is precompact in the $LM^{w_2}_{p\theta}$. Which completes the proof of the theorem.

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Локальдi Морри типтес кеңiстiгiнде Рисс потенциал коммутаторының компактылығы

Мақалада $LM^{w}_{p\theta}$ локальдi Морри типтi кеңiстiктерi қарастырылған. Негiзгi жұмыс $LM^{w_1}_{p\theta}$-дан $LM^{w_2}_{q\theta}$-ге дейiнгi локальдi Морри типтi кеңiстiктердегi $[b, I_\alpha]$ Рисс потенциалы үшiн коммутатордың компактылық теоремасын дәлелдеу. Сондай-ақ, Рисс потенциалы үшiн коммутатордың $[b, I_\alpha]$ локальдi Морри типтi кеңiстiктердегi $LM^{w_1}_{p\theta}$-дан $LM^{w_2}_{q\theta}$ шенелгендiгi үшiн жаңа жеткiлiктi шарттар берiлген. Рисс потенциалы үшiн коммутатордың компактылық теоремасын дәлелдеуде негiзiнен, $[b, I_\alpha]$ Рисс потенциалы үшiн коммутатордың $LM^{w}_{p\theta}$ локальдi Морри типтi кеңiстiктерiнде шектелген шарты, сонымен қатар $LM^{w}_{p\theta}$ локальдi Морри типтi кеңiстiктерiндегi жиындардың компактылық теоремасының жеткiлiктi шарттары пайдаланылған. Рисс потенциалы үшiн коммутатордың жинақылық теоремасын дәлелдеу барысында $[b, I_\alpha]$ Рисс потенциалы үшiн коммутатор шарының шенелген леммалары анықталған. Осыған ұқсас нәтижелер $GM^{w}_{p\theta}$ глобальды Морри типтi кеңiстiктер үшiн және M^{w}_{p} жалпыланған Морри кеңiстiгi үшiн де алынған.

Кiлт сөздер: компактылы, коммутаторлар, Рисс потенциалы, локальдi Морри типтi кеңiстiктер.

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Компактность коммутаторов для потенциала Рисса в локальных пространствах типа Морри

В статье рассмотрены локальные пространства типа Морри из $LM^{w}_{p\theta}$. Основной работой является доказательство теоремы компактности коммутатора для потенциала Рисса $[b, I_\alpha]$ в локальных пространствах типа Морри из $LM^{w_1}_{p\theta}$ в $LM^{w_2}_{q\theta}$. Приведены также новые достаточные условия ограниченности коммутатора для потенциала Рисса $[b, I_\alpha]$ в локальных пространствах типа Морри из $LM^{w_1}_{p\theta}$ в $LM^{w_2}_{q\theta}$. В доказательстве теоремы компактности коммутатора для потенциала Рисса существенно использованы условие ограниченности коммутатора для потенциала Рисса $[b, I_\alpha]$ в локальных пространствах типа Морри $LM^{w}_{p\theta}$, а также достаточные условия из теоремы предкомпактности множеств в локальных пространствах типа Морри $LM^{w}_{p\theta}$. В ходе доказательства теоремы компактности коммутатора для потенциала Рисса подтверждены леммы оценки по шару коммутатора для потенциала Рисса $[b, I_\alpha]$. Аналогичные результаты были получены для глобальных пространств типа Морри $GM^{w}_{p\theta}$ и для обобщенных пространств Морри M^{w}_{p}.

Ключевые слова: компактность, коммутаторы, потенциал Рисса, локальные пространства типа Морри.

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Numerical solution of differential – difference equations having an interior layer using nonstandard finite differences

This paper addresses the solution of a differential-difference type equation having an interior layer behaviour. A difference scheme is suggested to solve this equation using a non-standard finite difference method. Finite differences are derived from the first and second order derivatives. Using these approximations, the given equation is discretized. The discretized equation is solved using the algorithm for the tridiagonal system. The method is examined for convergence. Numerical examples are illustrated to validate the method. Maximum errors in the solution, in contrast to the other methods are organized to justify the method. The layer behaviour in the solution of the examples is depicted in graphs.

Keywords: Differential-difference equation, Boundary layer, Nonstandard finite difference, Convergence.

Introduction

Differential equations are ones in which the time evolution of a state variable is inconsistently dependent on a particular past. This means that the rate of change of a physical system is dependent not only on its current state but also on its previous history. The layer behavior differential difference equations have been extensively used in control theory for a number of years.

Subsequently, these equations play an important part in predator-prey models [1], population dynamics [2] and models of the red blood cell system [3] and models of neuronal variability [4]. Bender and Orszag [5], Doolan et al. [6] just are a few of the authors who have produced papers and books in recent years explaining various methods for solving differential-difference equations with singular perturbations, El'sgol'ts and Norkin [7], Mickens [8], Driver [9], Kokotovic et al. [10], Miller et al [11], O'Malley [12] are the authors who have produced books explaining various methods for solving delay differential equations and singularly perturbed differential-difference equations. In [13], authors developed an asymptotic analysis for a class of singularly perturbed problems with negative and positive shifts. In [14], the authors concentrate on problems with solutions that display layer behaviour at either one of the boundaries or both of the boundaries. The Laplace transforms used to the analysis of the layer equations produce new and interesting findings. The authors in [15] designed non-standard fitted finite difference methods based on the methods given in El-Mistikawy–Werle exponential finite difference scheme for differential–difference equations with negative and positive shifts. Rai and Sharma [16] developed numerical schemes using some modifications in El-Mistikawy–Werle exponential finite difference scheme. Sirisha et al. [17] devised a mixed difference scheme to solve the same problem. Salama and Al-Amery [18] constructed a mixed asymptotic solution for SPDDE using the composite expansion method. This work deals with constant shift arguments, which are independent of perturbation parameter. Swamy et al. [19] constructed a computational method of order four to solve SPDDE with mixed arguments. Bestehorn and Grigorieva [20] solved coupled nonlinear partial differential equations and single diffusion equation with an additional nonlinear delay term. Kadalbajoo and Sharma [21] solved a mathematical model arising from a model of neuronal variability and mathematical modelling

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for the determination of the expected time for generation of action potentials in nerve cells by random synaptic inputs in dendrites. Kadalbajoo and Sharma [22] exponentially fitted method based on finite difference to solve boundary-value problem for a singularly perturbed differential-difference equation with small shifts of mixed type.

The rest of the paper is organized as: In Section 1, problem description is given. In Section 2, a maximum principle and some important properties of the exact solution and its derivatives are established. The proposed numerical scheme is described in Section 3. Error estimate is derived in Section 4. Section 5 presents numerical examples to support the theoretical findings. Finally, Section 6 concludes with a summary and discussion.

1 Description of the problem

Consider a differential-difference equation with layer behaviour consisting a small delay and advanced terms of the form:

$$\varepsilon z''(u) + P(u)z'(u) + Q(u)z(u-\delta) + R(u)z(u) + V(u)z(u+\eta) = F(u), \tag{1}$$

on $(-1, 1)$, with the boundary conditions

$$z(u) = \phi(u), \quad -1-\delta \le u \le -1, z(u) = \gamma(u) \quad 1 \le u \le 1+\eta, \tag{2}$$

where $0 \le \varepsilon \ll 1$ is a perturbation parameter, $P(u), Q(u), R(u), V(u), F(u), \phi(u)$ and $\gamma(u)$ are smooth functions and $0 < \delta = o(\varepsilon)$ is the delay or negative shift and $0 < \eta = o(\varepsilon)$ is the advance or positive shift parameter. If $(P(u) - \delta Q(u) + \eta V(u)) > 0$, the solution of problem (1) with conditions (2) exposes layer at the left end of the interval and if $(P(u) - \delta Q(u) + \eta V(u)) < 0$ then the layer at the right-end of the interval. If $P(u) = 0$, the problem has either an oscillatory solution or two layers, depending on whether $Q(u) + R(u) + V(u)$ is positive or negative.

Since the solution $z(u)$ of problem (1) is sufficiently differentiable, the terms $z(u-\delta)$ and $z(u-\eta)$ can be expanded using Taylor series, then we have

$$z(u-\delta) = z(u) - \delta z'(u) + \frac{\delta^2}{2} z''(u), \tag{3}$$

$$z(u+\eta) = z(u) + \eta z'(u) + \frac{\eta^2}{2} z''(u). \tag{4}$$

Using formula (3) and formula (4) in problem (1), we get

$$C_\varepsilon z''(u) + a(u)z'(u) + b(u)z(u) = F(u), -1 < u < 1. \tag{5}$$

Problem (5) is a convection-diffusion problem. Here $C_\varepsilon = (\varepsilon + Q\frac{\delta^2}{2} + V\frac{\eta^2}{2}), \quad a(u) = P(u) - \delta(u) + {}$ $+ \eta V(u), \quad b(u) = Q(u) + R(u) + V(u)$. We solve problem (5) subject to the boundary constraints

$$z(-1) = \phi(-1), \quad z(1) = \gamma(1), \tag{6}$$

where the solution of the problem (5) with conditions (6) is taken as the approximation to the solution of the problem (1) with conditions (2). Let $a(u)$ vanishes at some $l_i \in (-1, 1)$. Let $N_i = [l_i - \xi, l_i - \xi]$ be a neighborhood of the turning point l_i such that it does not contain any other turning point. Also, it is assumed that

$$|a'(u)| \ge |\frac{a'(l_i)}{2}| \quad \text{for} \quad u \in N_i.$$

The transformation $u = \xi^{-1}(u - l_i)$ reduces the study of the behaviour of $z(u)$ near a given turning point l_i to the case when $a(u)$ has only one zero located at $u = 0$. Thus, we consider problems (5) with conditions (6) under the following hypothesis:

(i) $a(u) \in C^2[-1, 1]$, $F(u)$ and $b(u) \in C^1[-1, 1]$,

(ii) $b(u) \geq b_0 \geq 0$ on $[-1, 1]$, where b_0 is a positive constant,

(iii) $a(u)$ has simple zero at $u = 0$ and no other zeros in [-1,1],

(iv) $|a'(u)| \geq |\frac{(a'(0))}{2}|$ for $-1 \leq u \leq 1$,

(v) $\beta = \frac{b(0)}{a'(0)}$, and β_l,β_s be positive constants such that $\beta_l \leq 1 \leq \beta_s$ and $\beta_l \leq |\beta| \leq \beta_s$.

For a given function $g(u) \in C^k[-1, 1]$, let $||g||_k$ denote $\sum_{i=0}^{k} \max\limits_{-1 \leq u \leq 1} |g^i|$, where g^i denote i^{ih} derivative of $g(u)$, $C_\varepsilon(u) = \left(\varepsilon + Q\frac{\delta^2}{2} + V\frac{\eta^2}{2}\right)$. C_ε is taken as constant part of $C_\varepsilon(u)$ when $a(u)$ depends on u.

2 Analytical results

Lemma 1. (Continuous maximum principle): Let $\psi(u)$ be any sufficiently smooth function satisfying $\psi(-1) \geq 0$ and $\psi(1) \geq 0$. Then, $L\psi(u) \geq 0 \quad \forall\ u \in (-1, 1)$ implies that $\psi(u) \in 0\ \forall\ \ u \in [-1, 1]$.

Proof. Let u^* be such that $\psi(u^*) = \min\limits_{u \in [-1,1]} \psi(u)$. Let us assure that $\psi(u^*) \leq 0$.
Clearly $u^* \notin (-1, 1)$. Since u^* is the point of minima therefore $\psi'(u^*) = 0$ and $\psi''(u^*) > 0$.

Now

$$
\begin{aligned}
L\psi(u^*) &= C_\varepsilon z''(u^*) + a(u^*)z'(u^*) + b(u^*)z(u^*) \\
&= C_\varepsilon z''(u^*) + b(u^*)z(u^*) < 0,
\end{aligned}
$$

which is contradiction. This follows $\psi(u^*) \geq 0$ and since u^* is chosen arbitrarily therefore $\psi(u) \geq 0$, for $u \in [-1, 1]$.

Lemma 2. Let $z(u)$ be the solution of the problem (1) with the conditions (2) then

$$
||z||_0 \leq \frac{||f||_0}{b_0} + \max(|\phi(-1)|, |\gamma(1)|).
$$

Proof. Let us define

$$
\psi^{\pm} \leq \frac{||f||_0}{b_0} + \max(|\phi(-1)|, |\gamma(1)|) + z(u),
$$

then we have

$$
\begin{aligned}
\psi^{\pm}(-1) &= \frac{||f||_0}{b_0} + \max(|\phi(-1)|, |\gamma(1)|) + z(-1) = \\
&= \frac{||f||_0}{b_0} + \max(|\phi(-1)|, |\gamma(1)|) \pm \phi(-1) \geq 0. \\
\psi^{\pm}(1) &= \frac{||f||_0}{b_0} + \max(|\phi(-1)|, |\gamma(1)|) + z(1) = \\
&= \frac{||f||_0}{b_0} + \max(|\phi(-1)|, |\gamma(1)|) \pm \gamma(1) \geq 0. \\
L\psi^{\pm}(u) &= C_\varepsilon(\psi^{\pm}(u))'' + a(u)(\psi^{\pm}(u))' + b(u)(\psi^{\pm}(u)) = \\
&= b(u)\left(\frac{||f||_0}{b_0} + \max(|\phi(-1)|, |\gamma(1)|) + Lz(u)\right) = \\
&= b(u)\left(\frac{||f||_0}{b_0} + \max(|\phi(-1)|, |\gamma(1)|) + f(u)\right) =
\end{aligned}
$$

$$= (||f||_0 \pm f(u) + b(u)\max(|\phi(-1)|, |\gamma(1)|)) \geq 0 \quad (\text{since} \quad b(u) \geq b_0 \geq 0).$$

Therefore, using maximum principle, we obtain $\psi^{\pm}(u) \geq 0$ for $u \in [-1, 1]$, which is the required bound on the solution of the problem (1) with conditions (2). Lemma 3 provides bound on the solution of the problem (5) with conditions (6). We now derive bounds on $z(u)$ and its derivatives on a subinterval $[p, q]$ of $[-1, 1]$ which does not contain the turning point.

Lemma 3. Let $z(u)$ be the solution to the problem (5) with conditions (6) and $a(u), b(u), f(u) \in C^j[-1, 0],\ j \geq 0$, are sufficiently smooth functions in $[-1, 1]$. Then, there exit positive constant C and η such that $|D^i z| \geq C$ for $\quad u \in [-1, 1]$.

Proof. See in [23].

Theorem 1. Let $z(u)$ be the solution to the problem (5) with conditions (6) and $a(u), b(u), f(u) \in C^j[-1, 0),\ j > 0, |a(u)| \geq v$ (v is a positive constant) are sufficiently smooth functions in $[-1, 1]$. Then, there exit positive constant C and η such that

$$|D^i z| \leq C\left(1 + C_\varepsilon^{-i} e^{\left(\frac{vu}{C_\varepsilon}\right)}\right) \quad \text{for} \quad i = 1, 2, ...j+1, \quad u \in [-1, 0),$$

and

$$|D^i z| \leq C\left(1 + C_\varepsilon^{-s} e^{\left(\frac{-vu}{C_\varepsilon}\right)}\right) \quad \text{for} \quad i = 1, 2, ...j+1, \quad u \in [0, 1].$$

Proof. See in [23].

3 Numerical scheme

In this section, we construct a numerical scheme based on EI-Mistikawy and Werle exponentially fitted operator scheme to approximate the solution $z(u)$ of problem (5). Let uniform partition of the interval $[-1, 1]$ be given by $u_i = -1 + ih$ for $i = 0, 1, 2, ..., n$ where $h = \frac{2}{n}$. We construct the numerical scheme as:

$$Lz_j = \begin{cases} C_\varepsilon D^+ D^- z_j + a(u) D^- z_j + b(u) z(u) = F(u), & j = 1, 2, 3 \ldots .n/2 - 1, \\ C_\varepsilon D^+ D^- z_j + a(u) D^+ z_j + b(u) z(u) = F(u), & j = n/2, n/2+1, n/2+2, \ldots .n-1, \end{cases} \tag{7}$$

where

$$D^- z_j = \frac{z_j - z_{j-1}}{h}, \quad D^+ z_j = \frac{z_{j+1} - z_j}{h}, \quad D^+ D^- z_j = \frac{z_{j+1} - 2z_j + z_{j-1}}{\phi_j^2},$$

and

$$\phi_j^2 = \begin{cases} \frac{h\varepsilon}{a_j}\left[e^{\frac{(a_j h)}{\varepsilon}} - 1\right], & j = 1, 2, 3 \ldots .n/2 - 1, \\ \frac{h\varepsilon}{a_j}\left[1 - e^{\frac{(-a_j h)}{\varepsilon}}\right], & j = n/2, n/2+1, n/2+2, \ldots .n-1. \end{cases}$$

The system of equations (7) can be written in tridiagonal form as:

$$\left(\frac{\varepsilon}{\phi_j^2} - \frac{a_j}{h}\right) z_{j-1} + \left(\frac{-2\varepsilon}{\phi_j^2} + \frac{a_j}{h} + b_j\right) z_j + \left(\frac{\varepsilon}{\phi_j^2}\right) z_{j+1} = f_j \quad \text{for} \quad j = 1, 2, 3, ..., \frac{n}{2} - 1,$$

$$\left(\frac{\varepsilon}{\phi_j^2}\right) z_{j-1} + \left(\frac{-2\varepsilon}{\phi_j^2} - \frac{a_j}{h} + b_j\right) z_j + \left(\frac{\varepsilon}{\phi_j^2} + \frac{a_j}{h}\right) z_{j+1} = f_j \quad \text{for} \quad j = \frac{n}{2}, \frac{n}{2} + 1, \frac{n}{2} + 2, ..., n-1.$$

The above system of equations can be written as:

$$\begin{cases} A_j z_{j-1} + B_j z_j + C_j z_{j+1} = f_j, & j = 1, 2, 3, ..., \frac{n}{2} - 1, \\ A_j z_{j-1} + B_j z_j + C_j z_{j+1} = f_j, & j = \frac{n}{2}, \frac{n}{2} + 1, \frac{n}{2} + 2, ..., n - 1. \end{cases} \tag{8}$$

Here

$$\begin{cases} A_j = \left(\frac{\varepsilon}{\phi_j^2} - \frac{a_j}{h}\right), \ B_j = \left(\frac{-2\varepsilon}{\phi_j^2} + \frac{a_j}{h} + b_j\right), \ C_j = \left(\frac{\varepsilon}{\phi_j^2}\right), \ f_j = f_j, & j = 1, 2, 3, ..., \frac{n}{2} - 1, \\ A_j = \left(\frac{\varepsilon}{\phi_j^2}\right), \ B_j = \left(\frac{-2\varepsilon}{\phi_j^2} - \frac{a_j}{h} + b_j\right), \ C_j = \left(\frac{\varepsilon}{\phi_j^2} + \frac{a_j}{h}\right), \ f_j = f_j, & j = \frac{n}{2}, \frac{n}{2} + 1, \frac{n}{2} + 2, ..., n - 1. \end{cases}$$

4 Convergence Analysis

In this section, we analyze the convergence of the difference scheme (8). The analysis will be done on $u \in [-1, 0]$ and similarly the same will be done on $u \in (0, 1]$.
Let us define operator L as:

$$Lz_j = \begin{cases} C_\varepsilon \frac{d^2 z(u)}{du^2} + a(u)\frac{dz(u)}{du} + b(u)z(u) = F(u), & j = 1, 2, 3, ..., \frac{n}{2} - 1, \\ C_\varepsilon \frac{d^2 z(u)}{du^2} + a(u)\frac{dz(u)}{du} + b(u)z(u) = F(u), & j = \frac{n}{2}, \frac{n}{2} + 1, \frac{n}{2} + 2, ..., n - 1. \end{cases}$$

The local truncation error of the discretization on $[-1, 0]$ can be given as:

$$L\left(U_j - z_j\right) = \varepsilon z_j'' + a_j z_j - \left[\frac{\varepsilon\left(z_{j+1} - 2z_j + z_{j-1}\right)}{\left(\phi_j^2\right)} + a_j \frac{z_j - z_{j-1}}{h}\right] =$$

$$= \varepsilon U_j'' - \frac{\varepsilon}{\phi_j^2}\left[h^2 U_j'' + \frac{h^4}{24}(z^4)(\xi_1) + \frac{h^4}{24}(z^4)(\xi_2)\right] + \frac{(a_j h)}{2} z_j'' - \frac{a_j h^2}{6} z_j''' + \frac{(a_j h^3)}{24}(\xi_3).$$

Using the truncated Taylor expansion of $\frac{1}{(\phi_j^2)} = \frac{1}{h^2} + \frac{a_j}{2\varepsilon h}$, it follows that

$$L(U_j - z_j) = h^2\left\{\frac{-2\varepsilon}{12}(z^4)(\xi_1) - \frac{a_j}{6} z_j'''(\xi_2)\right\} + \frac{h^3}{24}\left\{(z^4)(\xi_1 - \xi_2)\right\} +$$

$$+ h^4\left\{\frac{-\varepsilon}{360}(z^6)(\xi_1) - \frac{a_j}{120} z_j^{vi}(\xi_3)\right\} + h^5\left\{\frac{a_j}{720} z_j^6(\xi_1 - \xi_2)\right\}, \tag{9}$$

where $\xi_i \in (u_j, u_{j+1}), i \in 1, 3$ and $\xi_2 \in (u_{j-1}, u_j)$. Using bounds on derivatives of z, for small h, we have

$$L_1(U_j - z_j) \leq Mh^2 \quad \forall j = 1(1)\frac{n}{2} - 1.$$

In a similar way, we can prove that

$$L_2(U_j - z_j) \leq Mh^2 \quad \forall j = \frac{n}{2}(1)n + 1.$$

Theorem 2. Let U_j be the numerical result of the difference scheme (8) along with the condition (9) and z_j is the solution to the problem (1) with condition (2), then a constant M is an independent of ε, h such that

$$\max_{1 \leq j \leq n+1} |U_j - z_j| \leq Mh^2.$$

Proof. Using th triangular inequality $|U_j - u_j| \leq |U_j - z_j| + |z_j - u_j|$, along with the truncation error, the fundamental outcome was generated by a global error.
Therefore,

$$\max_{1 \leq j \leq n+1} |U_j - z_j| \leq Mh^2.$$

5 Numerical examples

Example 1.

$$-\varepsilon z''(u) + 2(1-2u)z'(u) + 4z(u) + 2z(u-\delta) + z(u+\eta) = 0,$$

$$\text{on } u \in (0,1) \;\text{ with } z(u) = 1 \text{ on } -\delta \le u \le 0,\; z(1) = 1 \text{ on } 1 \le u \le 1+\eta.$$

Example 2.

$$-\varepsilon z''(u) + 2(1-2u)z'(u) + 4z(u) + 2z(u-\delta) + z(u+\eta) = 4(1-4u),$$

$$\text{on } u \in (0,1) \;\text{ with }\; z(u) = 1 \text{ on } -\delta \le u \le 0,\; z(1) = 1 \text{ on } 1 \le u \le 1+\eta.$$

6 Conclusion and discussions

We have discussed a numerical scheme with nonstandard finite differences for the solution of singularly perturbed differential–difference equations with delay and advance shifts. The domain is divided into two subintervals since the problem under consideration involves internal layer behavior. We constructed numerical scheme in each subinterval to get the solution. The proposed numerical method is analyzed for convergence.

In order to discuss the efficiency of the suggested scheme, some numerical experiments are carried out. The maximum absolute error in the solution of examples is tabulated in the form of Tables 1–4 in comparison to the method given in [16]. The effect of small delay and advance on the interior layer solution is shown by plotting the graphs (Figures 1–4). It is observed that when η is increasing for a fixed delay the width of the interior layer decreases, whereas it increases when δ increases for a fixed η.

Table 1

Maximum absolute errors in the solution of Example 1 for $\eta = 0.8 * \varepsilon$, $\delta = 0.6 * \varepsilon$.

$\varepsilon \setminus N$	**32**	**64**	**128**	**256**	**512**	**1024**
Present method:						
10^{-1}	1.8346e-03	5.6862e-04	1.4517e-04	3.4370e-05	8.0119e-06	1.8466e-06
10^{-2}	7.3498e-04	2.1050e-04	5.8093e-05	1.6472e-05	4.1595e-06	1.0125e-06
10^{-3}	1.1163e-03	2.3226e-04	6.4850e-05	1.7021e-05	4.1640e-06	9.5162e-07
10^{-4}	1.0462e-03	2.3228e-04	6.4875e-05	1.7163e-05	4.4164e-06	1.1201e-06
10^{-5}	1.0384e-03	2.3228e-04	6.4876e-05	1.7164e-05	4.4165e-06	1.1202e-06
10^{-6}	1.0376e-03	2.3228e-04	6.4876e-05	1.7164e-05	4.4165e-06	1.1202e-06
10^{-7}	1.0376e-03	2.3228e-04	6.4876e-05	1.7164e-05	4.4165e-06	1.1202e-06
$10-8$	1.0376e-03	2.3228e-04	6.4876e-05	1.7164e-05	4.4165e-06	1.1202e-06
Results in [16]						
10^{-1}	1.405e-03	4.13e-04	1.124e-04	2.936e-05	7.505e-06	1.897e-06
10^{-2}	3.763e-03	1.706e-03	6.567e-04	2.135e-04	6.170e-05	1.664e-05
10^{-3}	3.985e-03	1.966e-03	9.772e-04	4.849e-04	2.284e-04	9.256e-05
10^{-4}	4.005e-03	1.974e-03	9.813e-04	4.893e-04	2.443e-04	1.221e-04
10^{-5}	4.005e-03	1.974e-03	9.813e-04	4.893e-04	2.443e-04	1.221e-04
10^{-6}	4.007e-03	1.975e-03	9.817e-04	4.895e-04	2.444e-04	1.221e-04
10^{-7}	4.007e-03	1.975e-03	9.817e-04	4.895e-04	2.444e-04	1.221e-04
$10-8$	4.007e-03	1.975e-03	9.817e-04	4.895e-04	2.444e-04	1.221e-04

Table 2

Maximum absolute errors in the solution of Example 1 for $\eta = 0.8 * \varepsilon$, $\delta = 0.6 * \varepsilon$.

$\varepsilon \setminus N$	**32**	**64**	**128**	**256**	**512**	**1024**
Present method:						
10^{-1}	4.3167e-03	1.0336e-04	2.4771e-05	6.3523e-06	1.6076e-06	4.0430e-07
10^{-2}	4.5589e-04	9.2210e-05	2.4224e-05	6.2238e-06	1.5747e-06	3.9598e-07
10^{-3}	3.2398e-04	8.9201e-05	1.5556e-05	2.0509e-06	1.5043e-06	3.7861e-07
10^{-4}	3.2274e-04	8.8905e-05	2.3206e-05	5.9299e-06	1.6799e-06	3.7677e-07
10^{-5}	3.2274e-04	8.8905e-05	2.3206e-05	5.9299e-06	1.6799e-06	3.7677e-07
10^{-6}	3.2261e-04	8.8872e-05	2.3197e-05	5.9276e-06	1.4987e-06	3.7676e-07
10^{-7}	3.2261e-04	8.8872e-05	2.3197e-05	5.9276e-06	1.4987e-06	3.7676e-07
$10-8$	3.2261e-04	8.8872e-05	2.3197e-05	5.9276e-06	1.4987e-06	3.7676e-07
Results in [16]						
10^{-1}	1.445e-03	4.246e-04	1.156e-04	3.019e-05	7.716e-06	1.951e-06
10^{-2}	3.779e-03	1.712e-03	6.589e-04	2.142e-04	6.188e-05	1.669e-05
10^{-3}	3.987e-03	1.967e-03	9.776e-04	4.852e-04	2.285e-04	9.259e-05
10^{-4}	4.005e-03	1.974e-03	9.813e-04	4.893e-04	2.443e-04	1.221e-04
10^{-5}	4.007e-03	1.975e-03	9.817e-04	4.895e-04	2.444e-04	1.221e-04
10^{-6}	4.007e-03	1.975e-03	9.817e-04	4.895e-04	2.444e-04	1.221e-04
10^{-7}	4.007e-03	1.975e-03	9.817e-04	4.895e-04	2.444e-04	1.221e-04
$10-8$	4.007e-03	1.975e-03	9.817e-04	4.895e-04	2.444e-04	1.221e-04

Table 3

Maximum absolute errors in the solution of Example 2 for $\eta = 0.8 * \varepsilon$, $\delta = 0.6 * \varepsilon$.

$\varepsilon \setminus N$	**32**	**64**	**128**	**256**	**512**	**1024**
Present method:						
10^{-1}	2.3261e-03	1.2337e-03	4.3756e-04	1.3150e-04	3.6255e-05	9.6086e-06
10^{-2}	3.1095e-03	8.9149e-04	2.2287e-04	5.3497e-05	1.2954e-05	3.5879e-06
10^{-3}	3.1095e-03	8.9149e-04	2.2287e-04	5.3497e-05	1.2954e-05	3.5879e-06
10^{-4}	3.1571e-03	9.8441e-04	2.7495e-04	7.2740e-05	1.8717e-05	4.7473e-06
10^{-5}	3.1569e-03	9.8441e-04	2.7495e-04	7.2741e-05	1.8717e-05	4.7475e-06
10^{-6}	3.1569e-03	9.8441e-04	2.7495e-04	7.2741e-05	1.8717e-05	4.7475e-06
10^{-7}	3.1569e-03	9.8441e-04	2.7495e-04	7.2741e-05	1.8717e-05	4.7475e-06
$10-8$	3.1569e-03	9.8441e-04	2.7495e-04	7.2741e-05	1.8717e-05	4.7475e-06
Results in [16]						
10^{-1}	1.817e-02	5.352e-03	1.459e-03	3.814e-04	9.752e-05	2.466e-05
10^{-2}	4.874e-02	2.212e-02	8.512e-03	2.773e-03	8.017e-04	2.163e-04
10^{-3}	5.148e-02	2.551e-02	1.27e-02	6.302e-03	2.968e-03	1.203e-03
10^{-4}	5.171e-02	2.562e-02	1.275e-02	6.360e-03	3.176e-03	1.587e-03
10^{-5}	5.174e-02	2.563e-02	1.276e-02	6.363e-03	3.178e-03	1.588e-03
10^{-6}	5.174e-02	2.563e-02	1.276e-02	6.363e-03	3.178e-03	1.588e-03
10^{-7}	5.174e-02	2.563e-02	1.276e-02	6.363e-03	3.178e-03	1.588e-03
$10-8$	5.174e-02	2.563e-02	1.276e-02	6.363e-03	3.178e-03	1.588e-03

Table 4

Maximum absolute errors in the solution of Example 2 for $\eta = 0.8 * \varepsilon$, $\delta = 0.6 * \varepsilon$.

$\varepsilon \setminus N$	**32**	**64**	**128**	**256**	**512**	**1024**
Present method:						
10^{-1}	3.1747e-03	1.5835e-03	5.6153e-04	1.7120e-04	7.0667e-05	1.3083e-05
10^{-2}	3.1047e-03	8.9098e-04	2.2250e-04	5.3391e-05	1.3766e-05	3.9585e-06
10^{-3}	3.1584e-03	9.8424e-04	2.7479e-04	7.2123e-05	1.7645e-05	4.0325e-06
10^{-4}	3.1571e-03	9.8440e-04	2.7494e-04	7.2738e-05	1.8717e-05	4.7472e-06
10^{-5}	3.1569e-03	9.8441e-04	2.7495e-04	7.2741e-05	1.8717e-05	4.7475e-06
10^{-6}	3.1569e-03	9.8441e-04	2.7495e-04	7.2741e-05	1.8717e-05	4.7475e-06
10^{-7}	3.1569e-03	9.8441e-04	2.7495e-04	7.2741e-05	1.8717e-05	4.7475e-06
$10-8$	3.1569e-03	9.8441e-04	2.7495e-04	7.2741e-05	1.8717e-05	4.7475e-06
Results in [16]						
10^{-1}	1.843e-02	5.462e-03	1.494e-03	3.913e-04	1.002e-04	4.202e-05
10^{-2}	4.855e-02	2.206e-02	8.515e-03	2.775e-03	8.036e-04	2.168e-04
10^{-3}	5.145e-02	2.556e-02	1.269e-02	6.298e-03	2.967e-03	1.203e-03
10^{-4}	5.171e-02	2.562e-02	1.275e-02	6.360e-03	3.176e-03	1.587e-03
10^{-5}	5.171e-02	2.562e-02	1.275e-02	6.360e-03	3.178e-03	1.588e-03
10^{-6}	5.171e-02	2.562e-02	1.275e-02	6.360e-03	3.178e-03	1.588e-03
10^{-7}	5.171e-02	2.562e-02	1.275e-02	6.360e-03	3.178e-03	1.588e-03
$10-8$	5.171e-02	2.562e-02	1.275e-02	6.360e-03	3.178e-03	1.588e-03

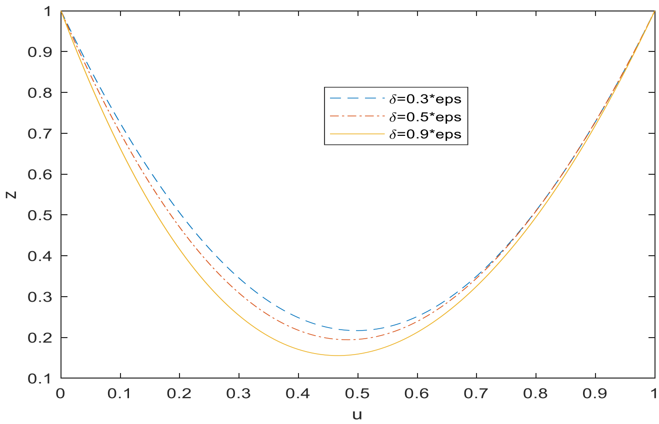

Fig1. Layer profile in Example 1 $\varepsilon = 2^{-2}$, $\eta = 0.5\varepsilon$.

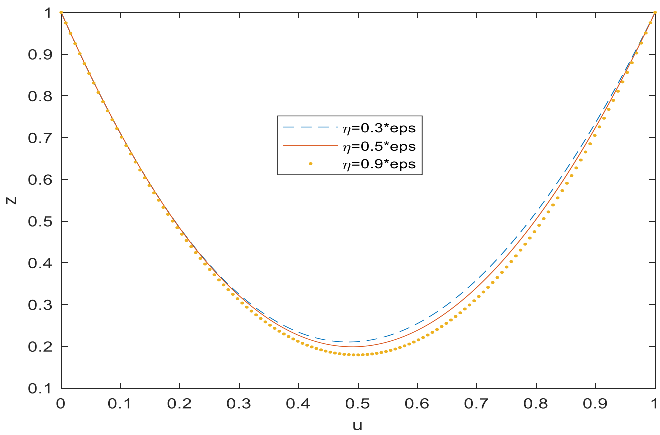

Fig 2. Layer profile in Example 1 $\varepsilon = 2^{-2}$, $\delta = 0.5\varepsilon$.

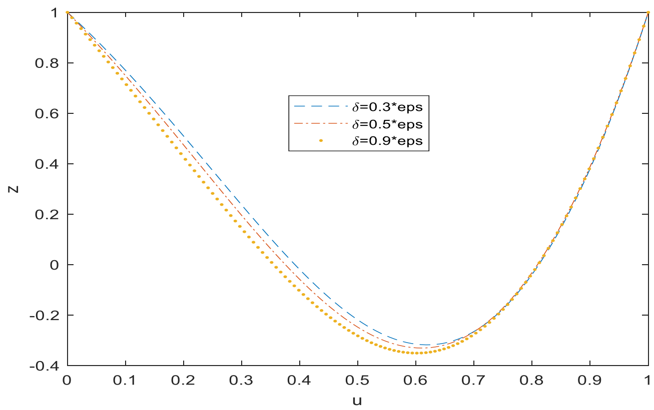

Fig 3. Layer profile in Example 2 $\varepsilon = 2^{-2}$, $\eta = 0.5\varepsilon$.

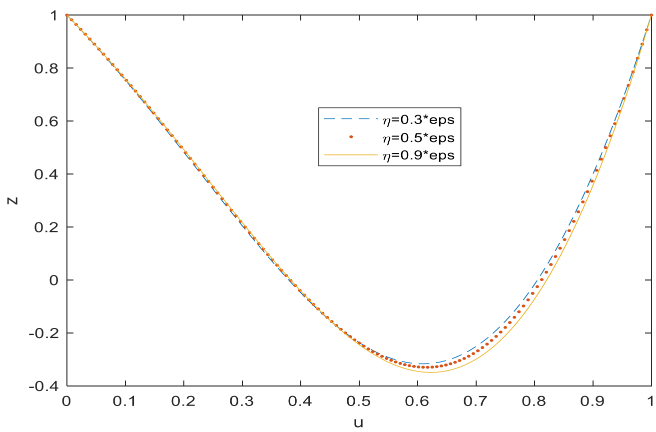

Fig 4. Layer profile in Example 2 with $\varepsilon = 2^{-2}$, $\delta = 0.5\varepsilon$.

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Ішкі қабаты бар дифференциалдық-айырымдық теңдеулерді стандартты емес шекті айырымдарды қолданып сандық шешу

Мақалада ішкі қабаттың әрекеті бар дифференциалдық-айырымдық типті теңдеудің шешімі қарастырылған. Бұл теңдеуде стандартты емес шекті айырымдық әдісі арқылы шешуге арналған айырымдар схемасы ұсынылған. Шекті айырымдар бірінші және екінші ретті туындылардан алынған. Осы жуықтауларды пайдалана отырып, бұл теңдеу дискреттелген. Дискреттелген теңдеу үш диагональдық жүйенің алгоритмі арқылы шешілген. Әдіс жинақтылыққа тексеріледі. Әдісті тексеру үшін сандық мысалдар келтірілген. Шешімдегі максималды қателер басқа әдістерге қарағанда әдісті негіздеу үшін ұйымдастырылған. Мысалдарды шешудегі қабаттың әрекеті графиктерде көрсетілген.

Кілт сөздер: дифференциалдық-айырымдық теңдеуі, шекаралық қабат, стандартты емес шекті айырым, жинақтылық.

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Численное решение дифференциально-разностных уравнений с внутренним слоем с использованием нестандартных конечных разностей

В статье рассмотрено решение уравнения дифференциально-разностного типа, имеющего поведение внутреннего слоя. Предложена разностная схема решения этого уравнения с использованием нестандартного метода конечных разностей. Конечные разности получены из производных первого и второго порядка. Используя эти приближения, данное уравнение дискретизируется. Дискретизированное уравнение решается с помощью алгоритма для трехдиагональной системы. Метод проверяется на сходимость. Для проверки метода проиллюстрированы численные примеры. Максимальные ошибки в решении, в отличие от других методов, организованы для обоснования метода. Поведение слоя в решении примеров изображено на графиках.

Ключевые слова: дифференциально-разностное уравнение, пограничный слой, нестандартная конечная разность, сходимость.

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A family of definite integrals involving Legendre's polynomials

The main objective of this article is to provide the analytical solutions (not previously found and not available in the literature) of some problems related with definite integrals integrands of which are the products of the derivatives of Legendre's polynomials of first kind having different order, with the help of some derivatives of Legendre's polynomials of first kind $\mathrm{P}_n(x)$, Rodrigues formula, Leibnitz's generalized rule for successive integration by parts and certain values of successive differential coefficients of $(x^2-1)^r$ at $x=\pm 1$.

Keywords: Legendre polynomials; Rodrigues formula; Leibnitz generalized rule for successive integration by parts; Murphy formula for Legendre polynomial.

1 Motivation and objectives

Legendre polynomials are studied in most science and engineering mathematics courses, mainly in those courses focused on differential equations or special functions. Legendre polynomials, also known as spherical harmonics or zonal harmonics, were first introduced in 1782 by Adrien-Marie Legendre. Legendre polynomials are used in several areas in physics and mathematics. For example, Legendre and Associate Legendre polynomials are widely used in the determination of wave functions of electrons in the orbits of an atom [1,2] and in the determination of potential functions in the spherically symmetric geometry [3]. In 1784, the significant of Legendre polynomials is sensed when the attraction of spheroids and ellipsoids was studying by A. Legendre. They may arise from solutions of Legendre ODE, such as the analog ODEs in spherical polar coordinates and the famous Helmholtz equation.

The main aim of this work is to fill up the gap in the existing literature on definite integrals integrands of which are the product of the derivatives of two families of classical Legendre's polynomials of first kind, by adding certain definite integrals in the incomplete list, as shown in the following possible combinations of definite integrals:

First combination of definite integrals:

$$\text{Already Solved} \int_{-1}^{+1} \mathrm{P}_n(x)\mathrm{P}_m(x)\mathrm{d}x \tag{1}$$

$$\text{Unsolved} \int_{-1}^{+1} \mathrm{P}_n(x)\mathrm{P}_m{}'(x)\mathrm{d}x \tag{2}$$

$$\text{Unsolved} \int_{-1}^{+1} \mathrm{P}_n(x)\mathrm{P}_m{}''(x)\mathrm{d}x \tag{3}$$

$$\text{Unsolved} \int_{-1}^{+1} \mathrm{P}_n(x)\mathrm{P}_m{}'''(x)\mathrm{d}x \tag{4}$$

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$$\text{Already Solved} \int_{-1}^{+1} \mathrm{P}_n(x)\mathrm{P}_n(x)\mathrm{d}x \tag{5}$$

$$\text{Unsolved} \int_{-1}^{+1} \mathrm{P}_n(x)\mathrm{P}_n{}'(x)\mathrm{d}x \tag{6}$$

$$\text{Unsolved} \int_{-1}^{+1} \mathrm{P}_n(x)\mathrm{P}_n{}''(x)\mathrm{d}x \tag{7}$$

$$\text{Unsolved} \int_{-1}^{+1} \mathrm{P}_n(x)\mathrm{P}_n{}'''(x)\mathrm{d}x \tag{8}$$

Second combination of definite integrals:

$$\text{Unsolved} \int_{-1}^{+1} \mathrm{P}_n{}'(x)\mathrm{P}_m(x)\mathrm{d}x \tag{9}$$

$$\text{Already Solved} \int_{-1}^{+1} \mathrm{P}_n{}'(x)\mathrm{P}_m{}'(x)\mathrm{d}x \tag{10}$$

$$\text{Unsolved} \int_{-1}^{+1} \mathrm{P}_n{}'(x)\mathrm{P}_m{}''(x)\mathrm{d}x \tag{11}$$

$$\text{Unsolved} \int_{-1}^{+1} \mathrm{P}_n{}'(x)\mathrm{P}_m{}'''(x)\mathrm{d}x \tag{12}$$

$$\text{Repeated with (6)} \int_{-1}^{+1} \mathrm{P}_n{}'(x)\mathrm{P}_n(x)\mathrm{d}x \tag{13}$$

$$\text{Already Solved} \int_{-1}^{+1} \mathrm{P}_n{}'(x)\mathrm{P}_n{}'(x)\mathrm{d}x \tag{14}$$

$$\text{Unsolved} \int_{-1}^{+1} \mathrm{P}_n{}'(x)\mathrm{P}_n{}''(x)\mathrm{d}x \tag{15}$$

$$\text{Unsolved} \int_{-1}^{+1} \mathrm{P}_n{}'(x)\mathrm{P}_n{}'''(x)\mathrm{d}x \tag{16}$$

Third combination of definite integrals:

$$\text{Unsolved} \int_{-1}^{+1} \mathrm{P}_n{}''(x)\mathrm{P}_m(x)\mathrm{d}x \tag{17}$$

$$\text{Unsolved} \int_{-1}^{+1} \mathrm{P}_n{}''(x)\mathrm{P}_m{}'(x)\mathrm{d}x \tag{18}$$

$$\text{Already Solved} \int_{-1}^{+1} \mathrm{P}_n{}''(x)\mathrm{P}_m{}''(x)\mathrm{d}x \tag{19}$$

$$\text{Unsolved} \int_{-1}^{+1} \mathrm{P}_n{}''(x)\mathrm{P}_m{}'''(x)\mathrm{d}x \tag{20}$$

$$\text{Repeated with (7)} \int_{-1}^{+1} \mathrm{P}_n{}''(x)\mathrm{P}_n(x)\mathrm{d}x \tag{21}$$

$$\text{Repeated with (15)} \int_{-1}^{+1} \mathrm{P}_n{}''(x)\mathrm{P}_n{}'(x)\mathrm{d}x \tag{22}$$

$$\text{Already Solved} \int_{-1}^{+1} \mathrm{P}_n{}''(x)\mathrm{P}_n{}''(x)\mathrm{d}x \tag{23}$$

$$\text{Unsolved} \int_{-1}^{+1} \mathrm{P}_n{}''(x)\mathrm{P}_n{}'''(x)\mathrm{d}x \tag{24}$$

Fourth combination of definite integrals:

$$\text{Unsolved} \int_{-1}^{+1} \mathrm{P}_n{}'''(x)\mathrm{P}_m(x)\mathrm{d}x \tag{25}$$

$$\text{Unsolved} \int_{-1}^{+1} \mathrm{P}_n{}'''(x)\mathrm{P}_m{}'(x)\mathrm{d}x \tag{26}$$

$$\text{Unsolved} \int_{-1}^{+1} \mathrm{P}_n{}'''(x)\mathrm{P}_m{}''(x)\mathrm{d}x \tag{27}$$

$$\text{Unsolved} \int_{-1}^{+1} \mathrm{P}_n{}'''(x)\mathrm{P}_m{}'''(x)\mathrm{d}x \tag{28}$$

$$\text{Repeated with (8)} \int_{-1}^{+1} \mathrm{P}_n{}'''(x)\mathrm{P}_n(x)\mathrm{d}x \tag{29}$$

$$\text{Repeated with (16)} \int_{-1}^{+1} \mathrm{P}_n{}'''(x)\mathrm{P}_n{}'(x)\mathrm{d}x \tag{30}$$

$$\text{Repeated with (24)} \int_{-1}^{+1} \mathrm{P}_n{}'''(x)\mathrm{P}_n{}''(x)\mathrm{d}x \tag{31}$$

$$\text{Unsolved} \int_{-1}^{+1} \mathrm{P}_n{}'''(x)\mathrm{P}_n{}'''(x)\mathrm{d}x \tag{32}$$

Now there are twenty-six non-repeated combinations of the product of derivatives of two Legendre's polynomials. Out of twenty-six integrals only six integrals are solved. Now we have to solve remaining twenty integrals solutions of which are not available in the literature of special functions.

For the sake of convenience we shall use the following notations and other results:
Suppose $D^{+1}\{F(x)\} = \frac{d}{dx}\{F(x)\}$, $D^m\{F(x)\} = \frac{d^m}{dx^m}\{F(x)\}$, $D^{-1}\{F(x)\} = \frac{1}{D}\{F(x)\} = \int\{F(x)\}\,dx$,
$D^{-m}\{F(x)\} = \frac{1}{D^m}\{F(x)\} = \underbrace{\int\int\int ...m\text{ - times}... \int\int}\ \{F(x)\}\ \underbrace{\mathrm{d}x\ \mathrm{d}x\ \mathrm{d}x...m\text{ - times }...\mathrm{d}x\ \mathrm{d}x}$.
Some derivatives of Legendre's polynomials of first kind $\mathrm{P}_n(x)$, using Rodrigues formula [4; p.162, Eq.(7)]:

$$\mathrm{P}_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n,$$

$$D\{\mathrm{P}_n(x)\} = \mathrm{P}_n{}'(x) = \frac{1}{2^n n!}\frac{d^{n+1}}{dx^{n+1}}(x^2-1)^n,$$

$$D^2\{\mathrm{P}_n(x)\} = \mathrm{P}_n{}''(x) = \frac{1}{2^n n!}\frac{d^{n+2}}{dx^{n+2}}(x^2-1)^n,$$

$$D^3\{\mathrm{P}_n(x)\} = \mathrm{P}_n{}'''(x) = \frac{1}{2^n n!}\frac{d^{n+3}}{dx^{n+3}}(x^2-1)^n, \tag{33}$$

where n is zero and positive integer.

Leibnitz's (also Leibniz) generalized rule for successive integration by parts:

$$I = \int U(x).T(x)dx = \int U.Tdx$$

$$\begin{aligned} I = (-1)^0 \left\{D^0 U\right\}\left\{D^{-1} T\right\} + (-1)^1 \left\{D^1 U\right\}\left\{D^{-2} T\right\} + \\ +(-1)^2 \left\{D^2 U\right\}\left\{D^{-3} T\right\} + (-1)^3 \left\{D^3 U\right\}\left\{D^{-4} T\right\} + \\ +................................ + (-1)^J \left\{D^J U\right\}\left\{D^{-J-1} T\right\} + \\ +(-1)^{J+1} \int \left\{D^{J+1} U\right\}\left\{D^{-J-1} T\right\} \mathrm{d}x + \text{constant of integration.} \end{aligned} \tag{34}$$

$$\frac{d^{2n}(x^2-1)^n}{dx^{2n}} = D^{2n}(x^2-1)^n = (2n)!$$

$$\{\text{Factorial of any negative integer }\}^{-1} = 0.$$

Our present investigation is motivated by the work collected in beautiful monographs of [5–14]. The article is organized as follows. In Section 2, we present some values of successive differential coefficients of $(x^2-1)^r$ at $x = \pm 1$. In Section 3, we mention six known definite integrals. In Section 4, we establish twenty new definite integrals. In Section 5, we have given the derivation of these new definite integrals.

2 Some successive differential coefficients of $(x^2-1)^r$ at $x = \pm 1$

$$\left[D^p(x^2-1)^r\right]_{x=\pm 1} = \begin{cases} \text{nonzero,} & \text{if } p \geq r \\ \text{zero,} & \text{if } p < r \end{cases} \tag{35}$$

$$\left[D^r(x^2-1)^r\right]_{x=1} = 2^r r! \tag{36}$$

$$\left[D^r(x^2-1)^r\right]_{x=-1} = 2^r r!(-1)^r \tag{37}$$

$$\left[D^{r+1}(x^2-1)^r\right]_{x=1} = \frac{2^r r! r(r+1)}{2} \tag{38}$$

$$\left[D^{r+1}(x^2-1)^r\right]_{x=-1} = \frac{(-1)2^r r! r(r+1)(-1)^r}{2} \tag{39}$$

$$\left[D^{r+2}(x^2-1)^r\right]_{x=1} = \frac{2^r r!(r+1)(r+2)r(r-1)}{8} \tag{40}$$

$$\left[D^{r+2}(x^2-1)^r\right]_{x=-1} = \frac{2^r r!(r+1)(r+2)r(r-1)(-1)^r}{8} \tag{41}$$

$$\left[D^{r+3}(x^2-1)^r\right]_{x=1} = \frac{2^r r!(r+1)(r+2)(r+3)r(r-1)(r-2)}{48} \tag{42}$$

$$\left[D^{r+3}(x^2-1)^r\right]_{x=-1} = \frac{(-1)2^r r!(r+1)(r+2)(r+3)r(r-1)(r-2)(-1)^r}{48} \tag{43}$$

$$\left[D^{r+4}(x^2-1)^r\right]_{x=1} = \frac{2^r r!(r+1)(r+2)(r+3)(r+4)r(r-1)(r-2)(r-3)}{384} \tag{44}$$

$$\left[D^{r+4}(x^2-1)^r\right]_{x=-1} = \frac{2^r r!(r+1)(r+2)(r+3)(r+4)r(r-1)(r-2)(r-3)(-1)^r}{384} \tag{45}$$

$$\left[D^{r+5}(x^2-1)^r\right]_{x=1} = \frac{2^r r!(r+1)(r+2)(r+3)(r+4)(r+5)r(r-1)(r-2)(r-3)(r-4)}{3840} \tag{46}$$

$$\left[D^{r+5}(x^2-1)^r\right]_{x=-1} = \frac{(-1)2^r r!(r+1)(r+2)(r+3)(r+4)(r+5)r(r-1)(r-2)(r-3)(r-4)(-1)^r}{3840} \tag{47}$$

With the help of Rodrigues formula and derivatives of the hypergeometric forms (Murphy formula [4; p.166, Eqs.(2) and (3)]) of Legendre's polynomials $\mathrm{P}_r(x)$, we can derive successive differential coefficients of $(x^2-1)^r$ at $x=\pm1$.

3 Six known definite integrals

Integral(1). $$\int_{-1}^{+1} \mathrm{P}_n(x)\mathrm{P}_m(x)\mathrm{d}x = 0, \qquad \text{if } \ m \neq n. \tag{48}$$

Integral(5). $$\int_{-1}^{+1} \{\mathrm{P}_n(x)\}^2\,\mathrm{d}x = \frac{2}{2n+1}. \tag{49}$$

The integrals (1) or (48) and (5) or (49) were derived by A. M. Legendre in the years 1784 and 1789 respectively ([15; p.281]; see also[16; p.277, Eqn's (13) and (14)]).

Integral(10). When m and n are positive integers and $m \geq n \geq 1$, then

$$\int_{-1}^{+1} \mathrm{P}_n{}'(x)\mathrm{P}_m{}'(x)\mathrm{d}x = \frac{n(n+1)}{2}\left\{1+(-1)^{m+n}\right\}. \tag{50}$$

Special case of the integral (50)

$$\int_{-1}^{+1} \mathrm{P}_n{}'(x)\mathrm{P}_m{}'(x)\mathrm{d}x = \begin{cases} 0 & \text{if } (m+n) \text{is an odd integer and } \ m \geq n \geq 1, \\ n(n+1) & \text{if } (m+n) \ \text{ is an even integer and } \ m \geq n \geq 1. \end{cases}$$

Integral(14). When m and n are positive integers and $m = n$ in equation (50), then

$$\int_{-1}^{+1} \{\mathrm{P}_n{}'(x)\}^2\,\mathrm{d}x = n(n+1). \tag{51}$$

The integrals (10) or (50) and (14) or (51) were asked in examination of Clare College London, Cambridge University (1898)[17; p.170, Q.N.11];[18; p.309, e.g.(3)].

Integral(19). When m and n are positive integers such that $m \geq n \geq 2$, then

$$\int_{-1}^{+1} \mathrm{P}_n{}''(x)\mathrm{P}_m{}''(x)\mathrm{d}x = \frac{(n+2)!}{(n-2)!(48)}\left\{1+(-1)^{m+n}\right\}\left\{3m(m+1)-n(n+1)+6\right\}. \tag{52}$$

Special case of the integral (52)

$$\int_{-1}^{+1} \mathrm{P}_n''(x)\mathrm{P}_m''(x)\mathrm{d}x = \begin{cases} 0 & \text{if } (m+n) \text{ is an odd integer and } m \geq n \geq 2, \\ \frac{n(n+1)(n+2)(n-1)}{24}\{3m(m+1) - n(n+1) + 6\} & \text{if } (m+n) \text{ is an even integer and } m \geq n \geq 2. \end{cases}$$

Integral(23). When m and n are positive integers and $m = n$ in equation (52), then

$$\int_{-1}^{+1} \{\mathrm{P}_n''(x)\}^2 \, \mathrm{d}x = \frac{(n+2)!}{(n-2)!(12)}\{n(n+1)+3\}. \tag{53}$$

The integrals (19) or (52) and (23) or (53) were asked in examination of Mathematical Tripos, Cambridge University (1897) [15; p.308, Q.N.2]; [18; p.309, e.g.(4)]. But the solutions of (52) and (53) are not available in the literature of special functions.

Remark: We have verified the definite integrals of Legendre's polynomials (48), (49), (50), (51), (52) and (53) numerically by using Mathematica software.

4 Twenty unsolved and new definite integrals

Integral(2). When m and n are positive integers such that $m \geq n$, then

$$\int_{-1}^{+1} \mathrm{P}_n(x)\mathrm{P}_m'(x)\mathrm{d}x = \{1 - (-1)^{m+n}\}. \tag{54}$$

Special case of the integral (54)

$$\int_{-1}^{+1} \mathrm{P}_n(x)\mathrm{P}_m'(x)\mathrm{d}x = \begin{cases} 2 & \text{if } (m+n) \text{ is an odd integer and } m \geq n, \\ 0 & \text{if } (m+n) \text{ is an even integer and } m \geq n. \end{cases}$$

Integral(6). When m and n are positive integers such that $m = n$ in equation (54), then

$$\int_{-1}^{+1} \mathrm{P}_n(x)\mathrm{P}_n'(x)\mathrm{d}x = 0.$$

Integral(3). When m and n are positive integers such that $m \geq n$, then

$$\int_{-1}^{+1} \mathrm{P}_n(x)\mathrm{P}_m''(x)\mathrm{d}x = \frac{\{1 + (-1)^{m+n}\}}{2}\{m(m+1) - n(n+1)\}. \tag{55}$$

Special case of the integral (55)

$$\int_{-1}^{+1} \mathrm{P}_n(x)\mathrm{P}_m''(x)\mathrm{d}x = \begin{cases} 0 & \text{if } (m+n) \text{ is an odd integer and } m \geq n, \\ m(m+1) - n(n+1) & \text{if } (m+n) \text{ is an even integer and } m \geq n. \end{cases}$$

Integral(7). When m and n are positive integers such that $m = n$ in equation (55), then

$$\int_{-1}^{+1} \mathrm{P}_n(x)\mathrm{P}_n''(x)\mathrm{d}x = 0.$$

Integral(4). When m and n are positive integers such that $m \geq n$, then

$$\int_{-1}^{+1} \mathrm{P}_n(x)\mathrm{P}_m'''(x)\mathrm{d}x = \frac{\{1-(-1)^{m+n}\}}{8}\times$$

$$\times\{m(m+1)(m+2)(m-1) - 2n(n+1)m(m+1) + n(n+1)(n+2)(n-1)\}. \tag{56}$$

Special case of the integral (56)

$$\int_{-1}^{+1} \mathrm{P}_n(x)\mathrm{P}_m'''(x)\mathrm{d}x$$

$$= \begin{cases} \frac{1}{4}\{m(m+1)(m+2)(m-1)- \\ -2n(n+1)m(m+1)+ \\ +n(n+1)(n+2)(n-1)\} & \text{if } (m+n) \text{ is an odd integer and } m \geq n, \\ 0 & \text{if } (m+n) \text{ is an even integer and } m \geq n. \end{cases}$$

Integral(8). When m and n are positive integers such that $m = n$ in equation (56), then

$$\int_{-1}^{+1} \mathrm{P}_n(x)\mathrm{P}_n'''(x)\mathrm{d}x = 0.$$

Integral(9). When m and n are positive integers such that $m \geq n$, then

$$\int_{-1}^{+1} \mathrm{P}_n'(x)\mathrm{P}_m(x)\mathrm{d}x = 0. \tag{57}$$

Repeated Integral(13). When m and n are positive integers such that $m = n$ in equation (57), then

$$\int_{-1}^{+1} \mathrm{P}_n'(x)\mathrm{P}_n(x)\mathrm{d}x = 0.$$

Integral(11). When m and n are positive integers such that $m \geq n$, then

$$\int_{-1}^{+1} \mathrm{P}_n'(x)\mathrm{P}_m''(x)\mathrm{d}x = \frac{n(n+1)}{8}\left\{1-(-1)^{m+n}\right\}\{2m(m+1)-(n+2)(n-1)\}. \tag{58}$$

Special case of the integral(58)

$$\int_{-1}^{+1} \mathrm{P}_n'(x)\mathrm{P}_m''(x)\mathrm{d}x$$

$$= \begin{cases} \frac{n(n+1)}{4}\{2m(m+1)-(n+2)(n-1)\} & \text{if } (m+n) \text{ is an odd integer and } m \geq n, \\ 0 & \text{if } (m+n) \text{ is an even integer and } m \geq n. \end{cases}$$

Integral(15). When m and n are positive integers such that $m = n$ in equation (58), then

$$\int_{-1}^{+1} \mathrm{P}_n'(x)\mathrm{P}_n''(x)\mathrm{d}x = 0.$$

Integral(12). When m and n are positive integers such that $m \geq n$, then

$$\int_{-1}^{+1} \mathrm{P}_n{}'(x)\mathrm{P}_m{}'''(x)\mathrm{d}x = \frac{n(n+1)}{48}\left\{1+(-1)^{m+n}\right\} \times$$

$$\times \left\{3m(m+1)(m+2)(m-1) - 3(n+2)(n-1)m(m+1) + (n+2)(n+3)(n-1)(n-2)\right\}. \quad (59)$$

Special case of the integral (59)

$$\int_{-1}^{+1} \mathrm{P}_n{}'(x)\mathrm{P}_m{}'''(x)\mathrm{d}x$$
$$= \begin{cases} 0 & \text{if } (m+n) \text{ is an odd integer and } m \geq n, \\ \frac{n(n+1)}{(24)}\left\{3m(m+1)(m+2)(m-1)-\right. & \\ -3(n+2)(n-1)m(m+1)+ & \\ \left.+(n+2)(n+3)(n-1)(n-2)\right\} & \text{if } (m+n) \text{ is an even integer and } m \geq n. \end{cases}$$

Integral(16). When m and n are positive integers such that $m = n$ in equation (59), then

$$\int_{-1}^{+1} \mathrm{P}_n{}'(x)\mathrm{P}_n{}'''(x)\mathrm{d}x = \frac{(n+3)!}{(n-3)!(24)};\ n \geq 3.$$

Integral(17). When m and n are positive integers such that $m \geq n$, then

$$\int_{-1}^{+1} \mathrm{P}_n{}''(x)\mathrm{P}_m(x)\mathrm{d}x = 0. \quad (60)$$

Repeated Integral(21). When m and n are positive integers such that $m = n$ in equation (60), then

$$\int_{-1}^{+1} \mathrm{P}_n{}''(x)\mathrm{P}_n(x)\mathrm{d}x = 0.$$

Integral(18). When m and n are positive integers and $m \geq n \geq 2$, then

$$\int_{-1}^{+1} \mathrm{P}_n{}''(x)\mathrm{P}_m{}'(x)\mathrm{d}x = \frac{(n+2)!}{(n-2)!(8)}\left\{1-(-1)^{m+n}\right\}. \quad (61)$$

Special case of the integral (61)

$$\int_{-1}^{+1} \mathrm{P}_n{}''(x)\mathrm{P}_m{}'(x)\mathrm{d}x = \begin{cases} \frac{(n+2)!}{(n-2)!(4)} & \text{if } (m+n) \text{ is an odd integer and } m \geq n \geq 2, \\ 0 & \text{if } (m+n) \text{ is an even integer and } m \geq n \geq 2. \end{cases}$$

Repeated Integral(22). When m and n are positive integers such that $m = n$ in equation (61), then

$$\int_{-1}^{+1} \mathrm{P}_n{}''(x)\mathrm{P}_n{}'(x)\mathrm{d}x = 0.$$

Integral(20). When m and n are positive integers such that $m \geq n \geq 2$, then

$$\int_{-1}^{+1} \mathrm{P}_n{}''(x)\mathrm{P}_m{}'''(x)\mathrm{d}x = \frac{(n+2)!}{(n-2)!(384)}\left\{1-(-1)^{m+n}\right\} \times$$

$$\times \left\{6m(m+1)(m+2)(m-1) - 4(n+3)(n-2)m(m+1) + (n+3)(n+4)(n-2)(n-3)\right\}. \quad (62)$$

Special case of the integral(62)

$$\int_{-1}^{+1} \mathrm{P}_n{}''(x)\mathrm{P}_m{}'''(x)\mathrm{d}x = \begin{cases} \frac{(n+2)!}{(n-2)!(192)}\{6m(m+1)(m+2)(m-1)- \\ -4(n+3)(n-2)m(m+1)+ \\ +(n+3)(n+4)(n-2)(n-3)\} & \text{if } (m+n) \text{ is an odd integer and } m \geq n \geq 2, \\ 0 & \text{if } (m+n) \text{ is an even integer and } m \geq n \geq 2. \end{cases}$$

Integral(24). When m and n are positive integers such that $m = n$ in equation (62), then

$$\int_{-1}^{+1} \mathrm{P}_n{}''(x)\mathrm{P}_n{}'''(x)\mathrm{d}x = 0.$$

Integral(25). When m and n are positive integers such that $m \geq n$, then

$$\int_{-1}^{+1} \mathrm{P}_n{}'''(x)\mathrm{P}_m(x)\mathrm{d}x = 0. \tag{63}$$

Repeated Integral(29). When m and n are positive integers such that $m = n$ in equation (63), then

$$\int_{-1}^{+1} \mathrm{P}_n{}'''(x)\mathrm{P}_n(x)\mathrm{d}x = 0.$$

Integral(26). When m and n are positive integers and $m \geq n \geq 3$, then

$$\int_{-1}^{+1} \mathrm{P}_n{}'''(x)\mathrm{P}_m{}'(x)\mathrm{d}x = \frac{(n+3)!}{(n-3)!(48)}\left\{1 + (-1)^{m+n}\right\}. \tag{64}$$

Special case of the integral (64)

$$\int_{-1}^{+1} \mathrm{P}_n{}'''(x)\mathrm{P}_m{}'(x)\mathrm{d}x = \begin{cases} 0 & \text{if } (m+n) \text{ is an odd integer and } m \geq n \geq 3, \\ \frac{(n+3)!}{(n-3)!(24)} & \text{if } (m+n) \text{ is an even integer and } m \geq n \geq 3. \end{cases}$$

Repeated Integral(30). When m and n are positive integers such that $m = n$ in equation (64), then

$$\int_{-1}^{+1} \mathrm{P}_n{}'''(x)\mathrm{P}_n{}'(x)\mathrm{d}x = \frac{(n+3)!}{(n-3)!(24)};\ n \geq 3.$$

Integral(27). When m and n are positive integers and $m \geq n \geq 3$, then

$$\int_{-1}^{+1} \mathrm{P}_n{}'''(x)\mathrm{P}_m{}''(x)\mathrm{d}x = \frac{(n+3)!}{(n-3)!(384)}\left\{1 - (-1)^{m+n}\right\}[4m(m+1) - (n+4)(n-3)]. \tag{65}$$

Special case of the integral (65)

$$\int_{-1}^{+1} \mathrm{P}_n{}'''(x)\mathrm{P}_m{}''(x)\mathrm{d}x = \begin{cases} \frac{(n+3)!}{(n-3)!(192)}\times \\ \times\{4m(m+1) - (n+4)(n-3)\} & \text{if } (m+n) \text{ is an odd integer and } m \geq n \geq 3, \\ 0 & \text{if } (m+n) \text{ is an even integer and } m \geq n \geq 3. \end{cases}$$

Repeated Integral(31). When m and n are positive integers such that $m = n$ in equation (65), then

$$\int_{-1}^{+1} \mathrm{P}_n'''(x)\mathrm{P}_n''(x)\mathrm{d}x = 0.$$

Integral(28). When m and n are positive integers such that $m \geq n \geq 3$, then

$$\int_{-1}^{+1} \mathrm{P}_n'''(x)\mathrm{P}_m'''(x)\mathrm{d}x = \frac{(n+3)!}{(n-3)!(3840)} \left\{1 + (-1)^{m+n}\right\} \times$$

$$\times \left\{10m(m+1)(m+2)(m-1) - 5(n+4)(n-3)m(m+1) + (n+4)(n+5)(n-3)(n-4)\right\}. \quad (66)$$

Special case of the integral(66)

$$\int_{-1}^{+1} \mathrm{P}_n'''(x)\mathrm{P}_m'''(x)\mathrm{d}x$$
$$= \begin{cases} 0 & \text{if } (m+n) \text{ is an odd integer and } m \geq n \geq 3, \\ \frac{(n+3)!}{(n-3)!(1920)} \times \{10m(m+1)(m+2)(m-1)- \\ -5(n+4)(n-3)m(m+1)+ \\ +(n+4)(n+5)(n-3)(n-4)\} & \text{if } (m+n) \text{ is an even integer and } m \geq n \geq 3. \end{cases}$$

Integral(32). When m and n are positive integers such that $m = n$ in equation (66), then

$$\int_{-1}^{+1} \{\mathrm{P}_n'''(x)\}^2\mathrm{d}x = \frac{(n+3)!}{(n-3)!(960)}\{3n^4 + 6n^3 + 7n^2 + 4n + 120\};\ n \geq 3. \quad (67)$$

Remark: We have verified the definite integrals from (54) to (67) of Legendre's polynomials numerically by using Mathematica software.

5 Derivation of new definite integrals of section 4

Here, in this section we shall provide the detailed and systematic derivation of any one integral.
Derivation of integral (66):
Consider the integral when $m \geq n \geq 3$

$$I = \left\{2^{m+n}(m!)(n!)\right\} \int_{-1}^{+1} \mathrm{P}_n'''(x)\mathrm{P}_m'''(x)\mathrm{d}x. \quad (68)$$

Using the equation (33), we have

$$I = \int_{-1}^{+1} D^{n+3}(x^2-1)^n D^{m+3}(x^2-1)^m \mathrm{d}x.$$

Taking $U = D^{n+3}(x^2-1)^n$, $T = D^{m+3}(x^2-1)^m$ and using the Leibnitz integration formula (34) (with suitable value of $J = n-4$), we get

$$\begin{aligned}
I = \Big[& (-1)^0 \left\{D^{n+3}(x^2-1)^n\right\}\left\{D^{m+2}(x^2-1)^m\right\} + \\
&+(-1)^1 \left\{D^{n+4}(x^2-1)^n\right\}\left\{D^{m+1}(x^2-1)^m\right\} + \\
&+(-1)^2 \left\{D^{n+5}(x^2-1)^n\right\}\left\{D^{m}(x^2-1)^m\right\} + \\
&+(-1)^3 \left\{D^{n+6}(x^2-1)^n\right\}\left\{D^{m-1}(x^2-1)^m\right\} + \\
&+ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots + \\
&+(-1)^{n-4} \left\{D^{2n-1}(x^2-1)^n\right\}\left\{D^{m-n+6}(x^2-1)^m\right\}\Big]_{-1}^{+1} + \\
&+(-1)^{n-3}\int_{-1}^{+1} \left\{D^{2n}(x^2-1)^n\right\}\left\{D^{m-n+6}(x^2-1)^m\right\}\mathrm{d}x,
\end{aligned}$$

$$\begin{aligned}
I = \Big[& (-1)^0 \left\{D^{n+3}(x^2-1)^n\right\}\left\{D^{m+2}(x^2-1)^m\right\} + \\
&+(-1)^1 \left\{D^{n+4}(x^2-1)^n\right\}\left\{D^{m+1}(x^2-1)^m\right\} + \\
&+(-1)^2 \left\{D^{n+5}(x^2-1)^n\right\}\left\{D^{m}(x^2-1)^m\right\} + \\
&+(-1)^3 \left\{D^{n+6}(x^2-1)^n\right\}\left\{D^{m-1}(x^2-1)^m\right\} + \\
&+ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots + \\
&+(-1)^{n-4} \left\{D^{2n-1}(x^2-1)^n\right\}\left\{D^{m-n+6}(x^2-1)^m\right\}\Big]_{-1}^{+1} + \\
&+(-1)^{n-3}(2n)!\int_{-1}^{+1} \left\{D^{m-n+6}(x^2-1)^m\right\}\mathrm{d}x,
\end{aligned}$$

$$\begin{aligned}
I = \Big[& (-1)^0 \left\{D^{n+3}(x^2-1)^n\right\}\left\{D^{m+2}(x^2-1)^m\right\} + \\
&+(-1)^1 \left\{D^{n+4}(x^2-1)^n\right\}\left\{D^{m+1}(x^2-1)^m\right\} + \\
&+(-1)^2 \left\{D^{n+5}(x^2-1)^n\right\}\left\{D^{m}(x^2-1)^m\right\} + \\
&+(-1)^3 \left\{D^{n+6}(x^2-1)^n\right\}\left\{D^{m-1}(x^2-1)^m\right\} + \\
&+ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots + \\
&+(-1)^{n-4} \left\{D^{2n-1}(x^2-1)^n\right\}\left\{D^{m-n+6}(x^2-1)^m\right\} + \\
&+(-1)^{n-3}(2n)!\left\{D^{m-n+5}(x^2-1)^m\right\}\Big]_{-1}^{+1},
\end{aligned} \tag{69}$$

$$\begin{aligned}
&= [\text{upper limit of right-hand side expression of (69) at } x=1] - \\
&-[\text{lower limit of right-hand side expression of (69) at } x=-1].
\end{aligned}$$

The values of fourth line to last line of right-hand side expression of (69) will be zero at $x = \pm 1$ because $(m-1)$, $(m-2)$,....,$\{m-(n-5)\}$ are less than m, in view of the equation (35).

$$I = \Big[(-1)^0 \left\{D^{n+3}(x^2-1)^n\right\}\left\{D^{m+2}(x^2-1)^m\right\} + (-1)^1 \left\{D^{n+4}(x^2-1)^n\right\}\left\{D^{m+1}(x^2-1)^m\right\} +$$

$$+(-1)^2\left\{D^{n+5}(x^2-1)^n\right\}\left\{D^m(x^2-1)^m\right\}\Bigg]_{-1}^{+1}. \tag{70}$$

Now using the results [see equations (36), (37), (38), (39), (40), (41), (42), (43), (44), (45), (46) and (47)] (with suitable values of p and r) in the equation (70), we get

$$I=\Bigg[\left\{\frac{(10)2^n n!n(n+1)(n+2)(n+3)(n-1)(n-2)}{480}\right\}\left\{\frac{2^m m!m(m+1)(m+2)(m-1)}{8}\right\}-$$

$$-\left\{\frac{(5)2^n n!n(n+1)(n+2)(n+3)(n+4)(n-1)(n-2)(n-3)}{1920}\right\}\left\{\frac{2^m m!m(m+1)}{2}\right\}+$$

$$+\left\{\frac{2^n n!n(n+1)(n+2)(n+3)(n+4)(n+5)(n-1)(n-2)(n-3)(n-4)}{3840}\right\}\{2^m m!\}\Bigg]-$$

$$-\Bigg[\left\{\frac{(10)(-1)2^n n!n(n+1)(n+2)(n+3)(n-1)(n-2)(-1)^n}{480}\right\}\times$$

$$\times\left\{\frac{2^m m!m(m+1)(m+2)(m-1)(-1)^m}{8}\right\}-$$

$$-\left\{\frac{(5)2^n n!n(n+1)(n+2)(n+3)(n+4)(n-1)(n-2)(n-3)(-1)^n}{1920}\right\}\times$$

$$\times\left\{\frac{(-1)2^m m!m(m+1)(-1)^m}{2}\right\}+$$

$$+\left\{\frac{(-1)2^n n!n(n+1)(n+2)(n+3)(n+4)(n+5)(n-1)(n-2)(n-3)(n-4)(-1)^n}{3840}\right\}\times$$

$$\times\{2^m m!(-1)^m\}\Bigg],$$

$$I=\left\{\frac{2^{m+n}m!n!n(n+1)(n+2)(n+3)(n-1)(n-2)}{3840}\right\}\times$$

$$\times[10m(m+1)(m+2)(m-1)+10m(m+1)(m+2)(m-1)(-1)^{m+n}-5(n+4)(n-3)m(m+1)-$$
$$-5(n+4)(n-3)m(m+1)(-1)^{m+n}+(n+4)(n+5)(n-3)(n-4)+(n+4)(n+5)(n-3)(n-4)(-1)^{m+n}],$$

$$I=\left\{\frac{2^{m+n}m!n!n(n+1)(n+2)(n+3)(n-1)(n-2)}{3840}\right\}\times$$

$$\times[10m(m+1)(m+2)(m-1)\{1+(-1)^{m+n}\}-5(n+4)(n-3)m(m+1)\{1+(-1)^{m+n}\}+$$
$$+(n+4)(n+5)(n-3)(n-4)\{1+(-1)^{m+n}\}],$$

$$I=\{2^{m+n}m!n!\}\left\{\frac{(n+3)(n+2)(n+1)n(n-1)(n-2)(n-3)!}{3840\ (n-3)!}\right\}\{1+(-1)^{m+n}\}\times$$

$$\times[10m(m+1)(m+2)(m-1)-5(n+4)(n-3)m(m+1)+(n+4)(n+5)(n-3)(n-4)]. \tag{71}$$

Finally, cancelling the factor $\{2^{m+n}m!n!\}$ in the equations (68) and (71), we obtain the integral (66). Similarly, we can derive the remaining integrals.

Conclusion

Here in this paper, we obtain some definite integrals related with the product of the derivatives of Legendre's polynomials of first kind of different order, by using the derivatives of Legendre's polynomials of first kind $\mathrm{P}_n(x)$, Rodrigues formula, Leibnitz's generalized rule for successive integration by parts and some values of successive differential coefficients of $(x^2-1)^r$ at $x=\pm 1$.

We conclude our present investigation by observing that, we can evaluate the following integral $\int_{-1}^{+1}\left\{\frac{d^r}{dx^r}\mathrm{P}_m(x)\right\}\left\{\frac{d^s}{dx^s}\mathrm{P}_n(x)\right\}dx$ by taking positive integral values of r and s, in analogous manner. The classical Legendre polynomials $\mathrm{P}_n(x)$ form a sequence of orthogonal polynomials with many historical applications. Their use continues in recent times in applications such as beam theory [19], phone segmentation [20], neural networks [21] and signal processing [22]; see also the recent works [23–25] dealing extensively with the methodology and techniques based on Legendre polynomials.

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Құрамында Лежандр көпмүшелері бар анықталған интегралдар үйірі

Мақаланың негізгі мақсаты–анықталған интегралдармен байланысты кейбір есептерді (бұрын табылмаған және әдебиетте жарияланбаған) аналитикалық жолмен шешу. Анықталған интегралдың интеграл астындағы өрнегі әр түрлі ретті бірінші текті Лежандр полиномдарының туындыларының көбейтіндісі болып табылады, бұл ретте $\mathrm{P}_n(x)$ бірінші текті Лежандр полиномдарының кейбір туындылары, Родригес формулалары, Лейбництің тізбектелген дифференциалдық $(x^2-1)^r$, $x=\pm1$ коэффициенттерінің бөліктері мен кейбір мәндері бойынша тізбектеп интегралдауға арналған жалпыланған ережесі қолданылады.

Кілт сөздер: Лежандр көпмүшелері, Родригес формуласы, біртіндеп бөліктеп интегралдау үшін Лейбництің жалпыланған ережесі, Лежандр көпмүшесі үшін Мерфи формуласы.

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Семейство определенных интегралов, содержащих многочлены Лежандра

Основная цель настоящей статьи — дать аналитические решения (ранее не найденные и не опубликованные в литературе) некоторых задач, связанных с определенными интегралами, подынтегральными выражениями которых являются произведения производных полиномов Лежандра первого рода разного порядка, с помощью некоторых производных полиномов Лежандра первого рода $\mathrm{P}_n(x)$, формулы Родригеса, обобщенного правила Лейбница для последовательного интегрирования по частям и некоторых значений последовательных дифференциальных коэффициентов $(x^2-1)^r$ при $x=\pm 1$.

Ключевые слова: полиномы Лежандра, формула Родригеса, обобщенное правило Лейбница для последовательного интегрирования по частям, формула Мерфи для многочлена Лежандра.

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Boundary value problems of integrodifferential equations under boundary conditions taking into account physical nonlinearity

When solving integrodifferential equations under boundary conditions, taking into account physical nonlinearity, a broad class of boundary-value problems of oscillations arises associated with various boundary conditions at the edges of a flat element. When taking into account non-stationary external influences, the main parameters is the frequency of natural vibrations of a flat component, taking into account temperature, prestressing, and other factors. The study of such problems, taking into account complicating factors, reduces to solving rather complex problems. The difficulty of solving these problems is due to both the type of equations and the variety. We analyze the results of previous works on the boundary problems of vibrations of plane elements. Possible boundary conditions at the edges of a flat element and the necessary initial conditions for solving particular problems of self-oscillation and forced vibrations, and other problems are considered. The set of equations, boundaries, and initial conditions make it possible to formulate and solve various boundary value problems of vibrations for a flat element. The oscillation equations for a flat element in the form of a plate given in this paper contain viscoelastic operators that describe the viscous behavior of the materials of a flat component. In studying oscillations and wave processes, it is advisable to take the kernels of viscoelastic operators regularly, since only such operators describe instantaneous elasticity and then viscous flow.

Keywords: physical nonlinearity, plates, oscillations, boundary value problems, wave process, isotropic plates, integrodifferential equation, approximate equation, nonlinear operators.

Introduction

When solving integrodifferential equations under boundary conditions, taking into account physical nonlinearity, a broad class of boundary-value problems of oscillations arises associated with various boundary conditions at the edges of a flat element. When taking into account non-stationary external influences, the main parameters is the frequency of natural vibrations of a flat element, taking into account temperature, prestressing, and other factors. The study of such problems, taking into account complicating factors, reduces to solving rather complex problems. The difficulty of solving these problems is due to both the type of equations and the variety. Let us systematize the results of previous works on boundary value problems of oscillations of flat elements. Possible boundary conditions at the edges of a flat element and initial conditions necessary for solving particular problems of natural and forced vibrations, and other problems are considered. The set of equations, boundary, and initial conditions make it possible to formulate and solve various boundary value problems of vibrations for a flat element. Integrodifferential equations with regular kernels are known to be equivalent to partial differential equations. For other approximate equations of oscillations of a plane element, these equations for regular nuclei can also be reduced to partial differential equations, which will be shown below.

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The assumed mathematical approach allows us to consider problems in a nonlinear setting when the nonlinearity is physical. The necessary theoretical information on the substantiation of the nonlinear dependence law $\sigma_{ij} \sim \varepsilon_{ij}$ for a viscoelastic isotropic body was presented in other papers.

1 *General staging*

For simplicity, we will consider a flat structure in the form of a plate and a base in the plane (x, z) or when external forces do not depend on the y coordinate. In this case, displacements u_l, w_l are non-zero, and displacement $v_l = 0$, i.e. absent. We assume that vibrations of a plate lying on a deformable base can be caused both by external forces on the surface of the plate and by disturbances propagating from the side of the base. In addition, we will assume that along the boundaries of the contact of the plate with the base, these contacts are ideal, i.e. there is no friction. Let us consider the case when the base material is isotropic and the dependence of stresses on strains is linear, i.e. Boltzmann-type relations hold [1–3]:

$$\sigma_{jj}^{(2)} = L_2(\varepsilon_{jj}^{(2)}) + 2M_2(\varepsilon_{jj}^{(2)}),$$

$$\sigma_{ij}^{(2)} = M_2(\varepsilon_{ij}^{(2)}), \quad (i, j = x, z; i \neq j).$$

Let us assume that the dependences of stresses on deformations for a plate are cubic.

$$\sigma_{jj}^{(1)} = 3K_1 R_0^{(1)}\{\varepsilon_0^{(1)}[1 + \alpha\chi_0^{(1)} K_2^{(1)}(\varepsilon_0^{(1)2})]\} + 2G_1 R^{(1)}\{(\varepsilon_{jj}^{(1)} - \varepsilon_0^{(1)}) \cdot [1 + \alpha\gamma_0^{(1)} G_1^{(1)}(\psi_0^{(1)2})]\}, \tag{1}$$

$$\sigma_{ij}^{(1)} = G_1 R^{(1)} \varepsilon_{ij}^{(1)}[1 + \alpha\gamma_0^{(1)} G_1^{(1)}(\psi_0^{(1)2})], (i \neq j; i = x, y; l = 1, 2),$$

where $\varepsilon^{(1)}$ is the average volumetric strain. $(\psi_0^{(1)2})$ is the square strain intensity, i.e.

$$(\psi_0^{(1)2}) = \frac{2}{\sqrt{3}}\left[\frac{2}{3}(\varepsilon_{xx}^{(1)2} + \varepsilon_{zz}^{(1)2} - \varepsilon_{xx}^{(1)2}\varepsilon_{zz}^{(1)2} + \frac{1}{2}\varepsilon_{xz}^{(1)2}\right].$$

$\chi_0^{(1)}, \gamma_0^{(1)}$ are the elongation and shear functions, respectively, which are expressed by the formulas:

$$\chi_0^{(1)} = 1 + F_0^{(1)}(\varepsilon_0^1); \gamma_0^1(\psi_0^{(1)2}) = 1 + F_1^1(\psi_0^{(1)2}); \; F_j^{(1)}(0) = 0.$$

In this case, the functions $F_0^{(1)}$ and $F_1^{(1)}$ are expanded in a power series

$$F_0^{(1)}(\varepsilon_0^{(1)}) = \sum_{n=0}^{\infty} \alpha_n \cdot (\varepsilon_0^{(1)})^{2(n+1)},$$

$$F_1^{(1)}(\psi_0^{(1)2}) = \sum_{n=0}^{\infty} \gamma_n \cdot (\psi_0^{(1)})^{2(n+1)}.$$

$R_0^{(1)}$ and $R^{(1)}$ are linear integral operators of Voltaire type

$$R_0^{(1)}(\zeta) = \zeta(t) - \int_0^t F_{10}(t-\xi)\zeta(\xi)d\xi,$$

$$R^{(1)}(\zeta) = \zeta(t) - \int_0^t F_{20}(t-\xi)\zeta(\xi)d\xi.$$

$K_2^{(1)}$ and $G_1^{(1)}$ are non-linear viscoelastic operators.

$$K_2^{(1)}(\varepsilon_0^{(1)2}) = \varepsilon_0^{(1)2} - \int_0^t \int_0^t F_0^{(1)}[(t-\xi_1)(t-\xi_2)]\varepsilon_0^{(1)}(\xi_2)d\xi_1 d\xi_2,$$

$$G_1^{(1)}(\psi_0^{(1)2}) = \psi_0^{(1)2} - \int_0^t F_1^{(1)}(t-\xi)\psi_0^{(1)2}(\xi)d\xi.$$

Constants K_1 and G_1 are equal

$$K_1 = \lambda_1 + \frac{2}{3}\mu_1; G_1 = \mu_1.$$

The vibration equations for a plate as a viscoelastic layer have the form: [4].

$$\left(K_2^{(1)}R_0^{(1)} + \frac{4}{3}G_1R^{(1)}\right)\frac{\partial^2 u_1}{\partial x^2} + G_1R^{(1)}\frac{\partial^2 u_1}{\partial z^2} + \left(K_1R_0^{(1)} + \frac{1}{3}G_1R^{(1)}\right)\times$$

$$\times\frac{\partial^2 w_1}{\partial x \partial z} + \alpha F_1^{(1)}(u_1, w_1) = \rho_1\frac{\partial^2 u_1}{\partial t^2}, \tag{2}$$

$$\left(K_1R_0^{(1)} + \frac{1}{3}G_1R^{(1)}\right)\frac{\partial^2 u_1}{\partial x \partial z} + G_1R^{(1)}\frac{\partial^2 w_1}{\partial z^2} + \left(K_1R_0^{(1)} + \frac{4}{3}G_1R^{(1)}\right)\times$$

$$\times\frac{\partial^2 w_1}{\partial z^2} + \alpha F_2^{(1)}(u_1, w_1) = \rho_1\frac{\partial^2 w_1}{\partial t^2},$$

where $F_1^{(1)}, F_2^{(1)}$ are non-linear operators.

$$F_1^{(1)}(u_1, w_1) = 3K_1\chi_0^{(1)}R_0^{(1)}\left\{\frac{\partial}{\partial x}\left[\varepsilon_0^{(1)}K_2^{(1)}(\varepsilon_0^{(1)2})\right]\right\} +$$

$$+\gamma_0\left\{G_1R^{(1)}\frac{\partial}{\partial x}[(\varepsilon_{xx}^{(1)} - \varepsilon_0^{(1)})G_1^{(1)}(\psi_0^{(1)2})]\right\} + \gamma_0G_1R^{(1)}\frac{\partial}{\partial z}[\varepsilon_{xz}^{(1)}G_1^{(1)}(\psi_0^{(1)2})] \tag{3}$$

$$F_2^{(1)}(u_1, w_1) = 3K_1\chi_0^{(1)}R_0^{(1)}\left\{\frac{\partial}{\partial z}\left[\varepsilon_0^{(1)}K_2^{(1)}(\varepsilon_0^{(1)2})\right]\right\} +$$

$$+\gamma_0\left\{G_1R_1\frac{\partial}{\partial z}\left[(\varepsilon_{xx}^{(1)} - \varepsilon_0^{(1)})G_1^{(1)}(\psi_0^{(1)2})\right]\right\} + \gamma_0G_1R^{(1)}\frac{\partial}{\partial x}\left[\varepsilon_{xz}^{(1)}G_1^{(1)}(\psi_0^{(1)2})\right]$$

Boundary conditions: at $z = h$.

$$\sigma_{zz}^{(1)} = f_z^{(1)}(x,t), \sigma_{xz}^{(1)} = 0 \tag{4}$$

at $z = -h$.

$$\sigma_{zz}^{(1)} = \sigma_{zz}^{(2)}; \sigma_{xz}^{(1)} = 0; \sigma_{xz}^{(2)} = 0; w_1 = w_2. \tag{5}$$

The initial conditions are zero, i.e. $u_l = \frac{\partial u_l}{\partial t} = w_l = \frac{\partial w_l}{\partial t} = 0$, at $t = 0$.

Thus, the boundary-value problem of the vibration of isotropic plates lying on a deformable foundation, taking into account the physical nonlinearity of stresses from deformation, is reduced to solving integrodifferential equations (2) under boundary and initial conditions (4)–(5).

Let us consider the oscillation equations taking into account the physical nonlinearity of stresses due to deformations [5].

Relations (1) hold for the plate material.

We will look for the displacements of the u and v plates in the form of a series with respect to parameter α.

$$u(x,z,t)=\sum_{n=0}^{\infty}a^n u_n(x,z,t),$$

$$\omega(x,z,t)=\sum_{n=0}^{\infty}a^n w_n(x,z,t). \tag{6}$$

In this case, the parameter α will be considered small, i.e. the nonlinearity is considered weak. We restrict ourselves to the first two terms in the series (6). Then for u_0, w_0 and u_1, w_1 we have the equations:

$$L_1\left(\frac{\partial^2 u_0}{\partial x^2}\right)+M_1\left(\frac{\partial^2 u_0}{\partial z^2}\right)+(L_1+M_1)\left(\frac{\partial^2 w_0}{\partial x\partial z}\right)=\rho_1\frac{\partial^2 u_0}{\partial t_2}, \tag{7}$$

$$(L_1+M_1)\left(\frac{\partial^2 u_0}{\partial x\partial z}\right)+M_1\left(\frac{\partial^2 w_0}{\partial x^2}\right)+L_1\left(\frac{\partial^2 w_0}{\partial x^2}\right)=\rho_1\frac{\partial^2 w_0}{\partial t_2},$$

$$L_1\left(\frac{\partial^2 u_1}{\partial x^2}\right)+M_1\left(\frac{\partial^2 u_1}{\partial z^2}\right)+(L_1+M_1)\left(\frac{\partial^2 w_1}{\partial x\partial z}\right)+F_1(u_0,w_0)=\rho_1\frac{\partial^2 u_1}{\partial t_2},$$

$$(L_1+M_1)\left(\frac{\partial^2 u_1}{\partial x\partial z}\right)+M_1\left(\frac{\partial^2 w_1}{\partial x^2}\right)+L_1\left(\frac{\partial^2 w_1}{\partial z^2}\right)+F_2(u_0,w_0)=\rho_1\frac{\partial^2 w_1}{\partial t_2}, \tag{8}$$

where $L_1=K_1R_0^{(1)}=\frac{4}{3}G_1R^{(1)}; M_1=G_1R^{(1)}$.

That problem was reduced to systems of two linear problems.

Problem (7) under boundary conditions (4) and (5) was solved in the second chapter in a three-dimensional formulation, so we will consider it solved. For example, the exact equations for the longitudinal-transverse oscillation of a plate lying on a deformable base in the first or linear approximations in a flat setting have the form:

$$M_{1(n)}(U^{(1)})+M_{2(n)}(W_1^{(1)})+M_{3(n)}(U_1^{(1)})+M_{4(n)}(W^{(1)})=M_1^{-1}f_z,$$

$$K_{1(n)}(U^{(1)})+K_{2(n)}(W_1^{(1)})+K_{3(n)}(U_1^{(1)})+K_{4(n)}(W^{(1)})=0,$$

$$D_{1(n)}(U^{(1)})+D_{2(n)}(W_1^{(1)})+D_{3(n)}(U_1^{(1)})+D_{4(n)}(W^{(1)})=0,$$

$$-K_{1(n)}(U^{(1)})-K_{2(n)}(W_1^{(1)})+K_{3(n)}(U_1^{(1)})+K_{4(n)}(W^{(1)})=0,$$

where the operators $M_{j(n)}, K_{j(n)}, D_{j(n)}$ are expressed from a system of general equations describing the longitudinal-transverse oscillation of a plate of constant thickness located in a deformable medium under the surface obtained in [5–7].

In particular, for the main part of the transverse displacement $W^{(1)}$ in the classical approximation we have the equation

$$\rho_1\frac{\partial^2 W_0^{(1)}}{\partial t^2}+\frac{h^2}{6}\left[\rho_1^2(N_1^{-1}+3M_1^{-1})-4\rho_1(3-2ML^{-1})\frac{\partial^4 W_0^{(1)}}{\partial t^2\partial x^2}+8(1-M_1L_1^{-1})\frac{\partial^4 W_0^{(1)}}{\partial x^4}\right]+$$

$$+P(W_0^{(1)}=\Phi_1(x,t),$$

where the operator P is equal to

$$P=\frac{s}{2h}\rho_1\left\{\frac{\partial}{\partial t}+\frac{h^2}{2}\left[\rho_1(M_1^{-1}+3L_1^{-1})\frac{\partial^3}{\partial t^3}-4\frac{\partial^3}{\partial t\partial x^2}\right]\right\}.$$

For equations (8), the boundary conditions have the form:

$$\text{at } z = h,\ \sigma_{zz}^{(1)} = 0;\ \sigma_{xz}^{(1)} = 0, \tag{9}$$

$$\text{at } z = -h,\ \sigma_{zz}^{(1)} = R(w_1);\ \sigma_{xz}^{(1)} = 0, \tag{10}$$

where the operator R is found after the invocation of the operator

$$R_0 = \frac{(\beta^2 + k^2 + q^2)^2 - 4\alpha_2\beta_2(k^2 + q^2)}{\alpha^2(\beta_2^2 - k^2 - q^2)}$$

for k, q, ρ (k and q are the parameters of the Fourier transform, ρ is the parameter of the Laplace transform) [8].

It should be noted that from the boundary conditions (9), namely $\sigma_{xz}^{(1)} = 0$ $w_1 = w_2$ at $z = -h$ we have eliminated the base parameters, and R is the base reaction.

Thus, we have the problem (5) of vibrations of an isotropic plate under boundary conditions (9) and (10) taking into account the physical nonlinearity of stress versus strain [9, 10].

With this formulation of the problem, we have a linear problem (8) under boundary conditions (9) and (10), and in the left parts of equation (8), there are nonlinear terms $F_1(u_0, w_0)$ and $F_2(u_0, w_0)$ depending on displacements u_0, w_0 and having the form (3). Representing displacements u_1, w_1 as

$$u_1 = \int_0^\infty \left.\begin{matrix} sinkx \\ -coskx \end{matrix}\right\} dk \int_l u_{10}\, exp\,(pt) dt$$

$$w_1 = \int_0^\infty \left.\begin{matrix} coskx \\ sinkx \end{matrix}\right\} dk \int_l w_{10}\, exp\,(pt) dt$$

for quantities u_{10}, w_{10} from equations (8) we obtain ordinary differential equations

$$M_{10}\frac{d^2 u_{10}}{dz^2} - [\rho_1 p^2 + k^2 L_{10}]u_{10} - k[L_{10} + M_{10}]\frac{dw_{10}}{dz} = F_{10}(u_0, w_0) \tag{11}$$

$$L_{10}\frac{d^2 w_{10}}{dz^2} - [\rho_1 p^2 + k^2 M_{10}]w_{10} - k[L_{10} + M_{10}]\frac{du_{10}}{dz} = F_{20}(u_0, w_0),$$

where F_{10} and F_{20} are Fourier and Laplace-transformed nonlinear functions $F_1(u_0, w_0)$, $F_2(u_0, w_0)$.

$$F_{10} = \int_0^\infty \left.\begin{matrix} sinkx \\ -coskx \end{matrix}\right\} dk \int_l F_1\, exp\,(pt) dp$$

$$F_{20} = \int_0^\infty \begin{matrix} coskx \\ sinkx \end{matrix}\ dk \int_l F_2\, exp\,(pt) dp.$$

General solutions of equations (11) are sought in the form

$$u_{10} = k\,[A_1 ch(\alpha z) + B_1 sh(\alpha z)] + \beta\,[A_2 ch(\beta z) + B_2 sh(\beta z)] - \frac{1}{\alpha(\beta^2 - \alpha^2)}$$

$$\int_0^z F(\xi) sh\,[\alpha(z - \xi)]\, d\xi + \frac{1}{\beta(\beta^2 - \alpha^2)} \int_0^z F(\xi) sh\,[(\beta(z - \xi)]\, d\xi$$

$$w_{10} = -\alpha\left[A_1 sh(\alpha z) + B_1 ch(\alpha z)\right] - k\left[A_2 sh(\beta z) + B_2 ch(\beta z)\right] + \frac{1}{k(\beta^2-\alpha^2)}$$

$$\int_0^z F(\xi) ch\left[\alpha(z-\xi)\right] d\xi - \frac{k}{\beta^2(\beta^2-\alpha^2)}\int_0^z F(\xi) ch\left[(\beta(z-\xi)\right] d\xi$$

where $F(z) = \frac{k(L_{10}+M_{10})}{L_{10}\cdot M_{10}}\frac{dF_{20}}{dz} + \frac{1}{L_{10}}\frac{d^2F_{10}}{dz^2} - \frac{\beta^2}{L_{10}}F_{10}$.

In this case, function $F(z)$ is considered to be given, and the integrals $\int_0^z ch\left[\gamma(z-\xi)\right] d\xi$ and $\int_0^z sh\left[\gamma(z-\xi)\right] d\xi$ can be expanded into power series.

Expanding the expressions for u_{10} and w_{10} into power series in coordinate z and introducing the main parts of the displacement according to the formulas [11]:

$$U_{10} = kA_1 + \beta A_2;\ U_{10}^{(1)} = kB_1\alpha + \beta^2 B_2$$

$$W_{10}^{(1)} = -\alpha^2 A_1 - k\beta A_2; W_0^{(1)} = -\alpha B_1 - kB_2$$

and reversing k and ρ we get:

$$u_1 = \sum_{n=0}^{\infty}\left\{\left[\lambda_1^{(n)} - \lambda_1^{(1)} c_t Q_n\right] U_1 + c_t Q_n \frac{\partial W_1^{(1)}}{\partial x} + F_{2n}^{(1)}\right\}\frac{Z^{2n}}{(2n)!} +$$

$$+\sum_{n=0}^{\infty}\left\{\left[\lambda_2^{(n)} - \frac{\partial^2}{\partial x^2} D_1 Q_n\right] U_1^{(1)} + D_1 Q_n \frac{\partial}{\partial x}\lambda_2^{(1)} W^{(1)} + F_{2n+1}^{(2)}\right\}\frac{Z^{2n+1}}{(2n+1)!},$$

$$w_1 = \sum_{n=0}^{\infty}\left\{\left[c_t \lambda_1^{(1)} Q_n\right]\frac{\partial U_1}{\partial x} + \left[\lambda^{(n)} - c_t\frac{\partial^2}{\partial x^2} Q_n\right] W_1^{(1)} + F_{2n}^{(3)}\right\}\frac{Z^{2n+1}}{(2n+1)!}$$

$$+\sum_{n=0}^{\infty}\left\{-D_1 Q_n \frac{\partial U_1^{(1)}}{\partial x} + \left[\lambda_2^{(n)} - \lambda_2^{(1)} D_1 Q_n\right] W^{(1)} + F_{2n+1}^{(4)}\right\}\frac{Z^{2n}}{(2n)!},$$

where

$$F_{2n}^{(1)} = F\left[\frac{\beta^2-k^2}{k(\beta^2-\alpha^2)} + \cdots + \frac{\alpha^{2n}(\beta^2+k^2) - 2k^2\beta^{2n}}{k(\beta^2-\alpha^2)}\right]$$

$$F_{2n+1}^{(2)} = F\left[\frac{\beta^2-k^2}{\beta^2(\beta^2-\alpha^2)} + \cdots + \frac{2\beta^{2(n+1)} - (\beta^2+k^2)\beta^{2n}}{\beta^2(\beta^2-\alpha^2)}\right]$$

$$F_{2n}^{(3)} = \frac{\partial}{\partial z}F_{2n}^{(1)}/_{Z=0};\ \ F_{2n+1}^{(4)} = \frac{\partial}{\partial z}F_{2n+1}^{(2)}/_{Z=0}.$$

Then from the boundary conditions (9) and (10) we obtain a system of four equations for U_1, $U_1^{(1)}$, $W_1^{(1)}$ and $W^{(1)}$.

$$M'_{1(n)}(U_1) + M'_{2(n)}(W_1^{(1)}) + M'_{3(n)}(U_1^{(1)}) + M_{4(n)}(W^{(1)}) = M_{5(n)}(F_{2n}^{(1,3)})$$

$$K'_{1(n)}(U_1) + K'_{2(n)}(W_1^{(1)}) + K'_{3(n)}(U_1^{(1)}) + K'_{4(n)}(W^{(1)}) = K'_{5(n)}(F_{2n}^{(1,3)})$$

$$D_{1(n)}^{(R)'}(U_1) + D_{2(n)}^{(R)'}(W_1^{(1)}) + D_{3(n)}^{(R)'}(U_1^{(1)}) + D_{4(n)}^{(R)'}(W^{(1)}) = D_{5(n)}(F_{2n}^{(i)}, F_{2n+1}^{(j)})$$

$$-K'_{1(n)}(U_1) - K'_{2(n)}(W_1^{(1)}) + K'_{3(n)}(U_1^{(1)}) + K'_{4(n)}(W^{(1)}) = -K'_{5(n)}(F_{2n+1}^{(2,4)}), \tag{12}$$

where the operators $M'_{j(n)},\ K'_{j(n)}\ D^{(R)'}_{j(n)},\ j=\overline{1,5}$ have the form:

$$M'_{1(n)} = \sum_{n=0}^{\infty}\left\{-\left[c_t\left(\lambda_2^{(1)} - \frac{\partial^2}{\partial x^2}\right)Q_n - (1+c_t)\lambda_2^{(n)}\right]\right\}\frac{h^{2n}}{(2n)!}$$

$$M'_{2(n)} = \sum_{n=0}^{\infty}\left\{\left[c_t\left(\lambda_2^{(1)} - \frac{\partial^2}{\partial x^2}\right)Q_n + (1-c_t)\lambda_2^{(1)}\right]\right\}\frac{h^{2n}}{(2n)!}$$

$$M'_{3(n)} = \sum_{n=0}^{\infty}\left\{\left[2\lambda_2^{(1)}D_1Q_n\psi_n + \lambda_1^{(n)}\right]\right\}\frac{h^{2n+1}}{(2n+1)!}$$

$$M'_{4(n)} = \sum_{n=0}^{\infty}\left\{\lambda_2^{(1)}\left(2\frac{\partial^2}{\partial x^2}F_1Q_n + \lambda_1^{(n)}\right)\right\}\frac{h^{2n+1}}{(2n+1)!}$$

$$M'_{5(n)} = \sum_{n=0}^{\infty}\left\{-F_{2n}^{(1)}\frac{h^{(2n)}}{(2n)!} + F_{2n}^{(3)}\frac{h^{2n+1}}{(2n+1)!}\right\}$$

$$K'_{1(n)} = \sum_{n=0}^{\infty} = \left\{-\left[Q_n\left(\lambda_1^{(1)} + \lambda_2^{(1)}\right) + \lambda_1^{(n)}\right]\right\}\frac{h^{2n+1}}{(2n+1)!}$$

$$K'_{2(n)} = \sum_{n=0}^{\infty}\left\{\left[2\lambda_1^{(1)}Q_1c_t + (1+c_t)\lambda_2^{(n)}\right]\right\}\frac{h^{2n+1}}{(2n+1)!}$$

$$K'_{3(n)} = \sum_{n=0}^{\infty}\left\{\left[\left(\lambda_2^{(1)} + \frac{\partial^2}{\partial x^2}\right)D_1Q_n + \lambda_1^{(n)}\right]\right\}\frac{h^{2n}}{(2n)!}$$

$$K'_{4(n)} = \sum_{n=0}^{\infty}\left\{\left(\lambda_2^{(1)} - \frac{\partial^2}{\partial x^2}\right)D_1Q_n - \lambda_1^{(n)}\right\}\frac{h^{2n}}{(2n)!}$$

$$K'_{5(n)} = \sum_{n=0}^{\infty}\left\{-F_{2n+1}^{(2)}\frac{h^{(2n+1)}}{(2n+1)!} + F_{2n+1}^{(4)}\frac{h^{2n}}{(2n)!}\right\}$$

$$D^{(R)'}_{1(n)} = \sum_{n=0}^{\infty}\left\{\left[(1+c_t)\lambda_2^{(n)} - c_t\left(\lambda_2^{(1)} - \frac{\partial^2}{\partial x^2}\right)Q_n\right]\frac{h^{2n}}{(2n)!} + R\left[c_t\lambda_1^{(1)}Q_n\frac{\partial}{\partial x}\right]\frac{h^{(2n+1)}}{(2n+1)!}\right\}$$

$$D^{(R)'}_{2(n)} = \sum_{n=0}^{\infty}\left\{\left[c_t\left(\lambda_2^{(1)} - \frac{\partial^2}{\partial x^2}\right)Q_n + (1-c_t)\lambda_2^{(n)}\right]\frac{h^{2n}}{(2n)!} + R\left[\lambda_1^{(n)} - c_t\frac{\partial^2}{\partial x^2}Q_n\right]\frac{h^{(2n+1)}}{(2n+1)!}\right\}$$

$$D^{(R)'}_{3(n)} = -\sum_{n=0}^{\infty}\left\{\left[2\lambda_2^{(1)}D_1Q_n + \lambda_1^{(n)}\right]\frac{h^{2n+1}}{(2n+1)!} + R\left[-D_1Q_n\frac{\partial}{\partial x}\right]\frac{h^{(2n)}}{(2n)!}\right\}$$

$$D^{(R)'}_{4(n)} = -\sum_{n=0}^{\infty}\left\{\left[\lambda_2^{(1)}\left(2\frac{\partial}{\partial x^2}D_1Q_n + \lambda_1^{(n)}\right)\right]\frac{h^{(2n+1)}}{(2n+1)!} + R\left[\lambda_2^{(n)} - \lambda_2^{(1)}D_1Q_n\right]\frac{h^{2n}}{(2n)!}\right\}$$

$$D^{(R)'}_{5(n)} = \sum_{n=0}^{\infty}\left\{[RF_{2n+1}^{(4)} - F_{2n}^{(1)}]\frac{h^{2n}}{(2n)!} - (1+R)F_{2n}^{(3)}\frac{h^{2n+1}}{(2n+1)!}\right\}$$

The system of equations (12) are the equations of the longitudinal-transverse oscillation of a plate in a non-linear formulation, lying on a deformable foundation in the first approximation.

2 Conclusions

Thus, the boundary-value problem of plate oscillations, taking into account the physical nonlinearity of stresses, is reduced to solving integrodifferential equations, under given boundary and initial conditions.

A general formulation of the boundary value problem of vibrations of isotropic plates in a nonlinear formulation, lying on a deformable foundation, is given. To solve specific problems, instead of exact equations, it is advisable to use approximate ones, which include one or another finite order in derivatives: such approximate equations can be easily obtained from exact equations, limited to a finite number of first terms. If the nonlinear dependence on the stress intensity does not depend, i.e. parameter $\gamma_0 = 0$, the obtained results are greatly simplified. Of theoretical and applied interest is the problem of the effects of a normal load on the surface of an elastic plate lying on an absolutely rigid half-space with ideal contact between them. As above, it is assumed that the dependencies of stresses on strains are non-linear (physical non-linearity).

Due to the ideality of the contact, the desired displacements of the points of the plate are symmetrical with respect to displacement u and antisymmetric with respect to displacement v.

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Физикалық бейсызықты негізіндегі шекаралық шарттардағы интегралдық-дифференциалдық теңдеулердің шеттік есептері

Физикалық бейсызықты негізінде, шекаралық шарттарда интегралдық-дифференциалдық теңдеулерді шешу кезінде жазық элементтің шеттеріндегі әртүрлі шекаралық шарттармен байланысты тербелістердің шеттік есептерінің кең класы туындайды. Стационарлы емес сыртқы әсерлерді есепке алғанда, температураны, алдын ала кернеуді және басқа факторларды ескере отырып, жазық элементтің табиғи тербелістерінің жиілігі негізгі параметрлердің негізгісі болып табылады. Күрделі факторларды ескере отырып, мұндай проблемаларды зерттеу өте күрделі мәселелерді шешу жолына әкеледі. Бұл есептерді шешудің қиындығы теңдеулердің түріне және әртүрлілігіне байланысты. Жазық элементтер тербелістерінің шекаралық есептері бойынша алдыңғы жасалған жұмыстардың нәтижелері талданған. Жазық элементтің шеттеріндегі мүмкін болатын шекаралық шарттар мен меншікті және мәжбүрлі тербелістердің дербес есептерін шешуге қажетті бастапқы шарттар және басқа да есептер қарастырылады. Бұл теңдеулер жиыны, шекаралық және бастапқы шарттар жазық элемент үшін тербелістердің әртүрлі шекаралық есептерін құрастыруға және шешуге мүмкіндік береді. Осы жұмыста берілген пластина түріндегі жазық элементтің тербелістерінің теңдеулері жалпақ элемент материалдарының тұтқырлық әрекетін сипаттайтын тұтқыр серпімді операторларды қамтиды. Тербелістер мен толқындық процестерді зерттеуде тұтқыр серпімді операторлардың ядроларын жүйелі түрде қабылдаған жөн, өйткені тек осындай операторлар лездік серпімділікті, содан кейін тұтқыр ағынды сипаттайды.

Кілт сөздер: физикалық бейсызық, пластиналар, тербеліс, шекаралық есептер, толқындық процесс, изотропты қалақшалар, интегралдық-дифференциалдық теңдеу, жуықтық теңдеу, сызықтық емес операторлар.

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Краевые задачи интегро–дифференциальных уравнений при граничных условиях с учетом физической нелинейности

При решении интегро-дифференциальных уравнений при граничных условиях с учетом физической нелинейности возникает широкий класс краевых задач колебаний, связанных с различными граничными условиями на краях плоского элемента. При учете нестационарных внешних воздействий основным из главных параметров является частота собственных колебаний плоского элемента с учетом температуры, предварительной напряженности и других факторов. Исследование таких задач, с учетом усложняющих факторов, сводится к решению достаточно сложных задач. Трудность решения данных задач обусловлена как типом уравнений, так и разнообразием. Проанализированы результаты предыдущих работ по краевым задачам колебания плоских элементов. Рассмотрены возможные граничные условия на краях плоского элемента и необходимые начальные условия для решения частных задач собственных и вынужденных колебаний и другие задачи. Совокупность уравнений, граничных и начальных условий позволяют формулировать и решать различные краевые задачи колебания для плоского элемента. Приведенные в данной работе уравнения колебания плоского элемента в виде пластинки содержат вязкоупругие операторы, описывающие вязкое поведение материалов плоского элемента. При исследовании колебания и волновых процессов ядра вязкоупругих операторов целесообразно брать регулярными, так как только такие операторы описывают мгновенную упругость, а затем вязкое течение.

Ключевые слова: физическая нелинейность, пластинки, колебания, краевые задачи, волновой процесс, изотропные пластинки, интегро-дифференциальное уравнение, приближенные уравнения, нелинейные операторы.

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Generalized Hankel shifts and exact Jackson–Stechkin inequalities in L_2

In this paper, we have solved several extremal problems of the best mean-square approximation of functions f on the semiaxis with a power-law weight. In the Hilbert space L^2 with a power-law weight $t^{2\alpha+1}$ we obtain Jackson–Stechkin type inequalities between the value of the $E_\sigma(f)$-best approximation of a function $f(t)$ by partial Hankel integrals of an order not higher than σ over the Bessel functions of the first kind and the k-th order generalized modulus of smoothnes $\omega_k(B^r f, t)$, where B is a second–order differential operator.

Keywords: best approximation, generalized modulus of smoothness of m-th order, Hilbert space.

Introduction

At present, there is a number of meaningful papers [1–3] devoted to the theory of approximation of a function from $L_2[0, 2\pi]$. Let $\alpha > -\frac{1}{2}$. For $p = 2$ by L_{2,μ_α} we denote the space consisting of measurable functions f on $[0, \infty)$, for which the norm is finite

$$\|f\|_{2,\mu_\alpha} = \left(\int_0^\infty |f(x)|^2 d\mu_\alpha(x)\right)^{\frac{1}{p}},$$

where

$$d\mu_\alpha(x) = \frac{x^{2\alpha+1}}{2^\alpha\Gamma(\alpha+1)}dx.$$

Consider the Hankel transform defined for the function f:

$$h_\alpha(f)(\lambda) = \int_0^\infty x^{2\alpha+1}(x\lambda)^{-\alpha}J_\alpha(x\lambda)f(x)dx, \quad \lambda \in (0, \infty),$$

where $J_\alpha(z)$ is the Bessel function of the first kind of an order $\alpha \geq -\frac{1}{2}$, $\Gamma(x)$ is the gamma-function.

In particular, for $\alpha = \frac{1}{2}$ and $\alpha = -\frac{1}{2}$ the Hankel transforms turn into the sine transform and the cosine Fourier transform, respectively:

$$F_s(\lambda) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin(\lambda x)dx,$$

$$F_c(\lambda) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos(\lambda x)dx,$$

since the formulas $J_{\frac{1}{2}} = \sqrt{\frac{2}{\pi}}\sin x$ and $J_{-\frac{1}{2}} = \sqrt{\frac{2}{\pi}}\cos x$ hold.

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For a function $f \in L_{2,\mu_\alpha}$ the expansion into the Hankel integral [4], is valid:

$$\hat{H}_\alpha(f)(\lambda) = \int_0^\infty f(x) j_\alpha(\lambda x) d\mu_\alpha(x),$$

and

$$f(x) = \int_0^\infty \hat{H}_\alpha(f)(\lambda) j_\alpha(\lambda x) d\mu_\alpha(\lambda).$$

Let $T > 0$ and we denote by $S_T(f, x)$ the partial Hankel integral of a function $f \in L_{2,\mu_\alpha}$ i.e.

$$S_T(f, x) = \int_0^T \hat{H}_\alpha(f)(\lambda) j_\alpha(\lambda x) d\mu_\alpha(\lambda), \quad x \in (0, \infty).$$

For functions $f, g \in L_{2,\mu_\alpha}$, the generalized Plancherel's theorem holds [5]

$$(f, g) = (\hat{f}, \hat{g}),$$

where $(f, g) = \int_0^\infty f(x)\overline{g(x)} d\mu_\alpha$ is the scalar product of f and g.

In the space $L_{p,\alpha}$ consider the generalized shift operator of functions $f(x)$ [6]

$$(T^h f)(x) = \frac{\Gamma(\alpha + 1)}{\sqrt{\pi}\Gamma(\alpha + 1/2)} \int_0^\pi f(\sqrt{x^2 + h^2 - 2xh\cos\varphi})(\sin\varphi)^{2\alpha} d\varphi.$$

For a function $f \in L_{2,\mu_\alpha}$, $\Delta_h^k f(x)$ finite differences of the k-th order with a step $h > 0$ are defined as follows (see [7]):

$$\Delta_h^1 f(x) = (I - T^h)(x), \Delta_h^k f(x) = (I - T^h)^k f(x), k > 1.$$

The value

$$\omega_k(f, \delta)_{2,\mu_\alpha} = \sup_{0 \le h \le \delta} \|\Delta_h^k f(x)\|_{2,\mu_\alpha} = \sup_{0 \le h \le \delta} \left\{ \int_0^\infty (1 - j_\alpha(\lambda h))^{2k} |\hat{H}_\alpha(f)(\lambda)|^2 d\mu_\alpha(\lambda) \right\}^{\frac{1}{2}} \tag{01}$$

will be called the generalized modulus of smoothness of the k-th order of a function$f \in L_{2,\mu_\alpha}$. We denote by $M(\nu, 2, \alpha), \nu > 0$ the set of all functions $Q_\nu(x)$ satisfying the following conditions (see [7]):

1. $Q_\nu(x)$ is an even entire function of exponential type ν;
2. $Q_\nu(x)$ belongs to the class L_{2,μ_α}.

The best approximation of a function $f \in L_{2,\mu_\alpha}$ from the class $M(\sigma, 2, \alpha), \nu > 0$ is defined as follows:

$$E_\sigma(f)_{2,\mu_\alpha} = \inf\{\|f - Q_\sigma\|_{2,\mu_\alpha} : Q_\sigma \in M(\sigma, 2, \alpha)\} = \left\{ \int_\sigma^\infty |\hat{H}_\alpha(f)(\lambda)|^2 d\mu_\alpha(\lambda) \right\}^{\frac{1}{2}}. \tag{02}$$

Let

$$B = B_t = \frac{d^2}{dt^2} + \frac{2\alpha + 1}{t}\frac{d}{dt}$$

be a differential Bessel operator. We denote by $j_\alpha(\lambda t)$ the normalized Bessel function

$$j_\alpha(\lambda t) = \frac{2^\alpha \Gamma(\alpha + 1) J_\alpha(\lambda t)}{(\lambda t)^\alpha}.$$

The function $j_\alpha(\sqrt{\lambda} t)$ is a solution to the problem

$$\frac{d^2 y}{dt^2} + \frac{2\alpha + 1}{t}\frac{dy}{dt} + \lambda y = 0,$$

$$y(0) = 1, y'(0) = 0.$$

In [8], when solving problems of the theory approximations in the space L_{2,μ_α} associated with finding the exact constants in the Jackson–Stechkin inequality

$$E_\sigma(f) \le \omega_r(f, \frac{\tau}{\sigma})$$

it is considered the following extreme characteristic:

$$K_{\sigma,r,m,\tau} = sup\left\{\frac{E_\sigma(f)}{\omega_r(f, \frac{\tau}{\sigma})} : f \in L_2(R^m)\right\}.$$

In this article, we want to get the exact constant in Jackson's inequality

$$E_\sigma(f) \le K\sigma^{-2r}\omega_r(B^r f, \frac{\tau}{\sigma})$$

for the functions $f \in W^r_{2,\mu_\alpha}(B)$. For the goal, we introduce an extremal approximate characteristic of the following form

$$\Xi_{\sigma,r,m,p,s}(\varphi, h) = \sup_{f\in W^r_{2,\mu_\alpha}(B)} \frac{E_\sigma(f)}{\left(\int_0^h \omega^p_m(B^r f, t)\varphi(t)dt\right)^s}, \tag{03}$$

where $r, m \in \mathbb{N},\ 0 < p < 2,\ h > 0,\ \sigma > 0,\ \varphi(t) \ge 0$ is an arbitrary integrable, not equivalent to zero on the segment $[0, h]$, weight function and $W^r_{2,\mu_\alpha}(B), r = 1, 2, ...$ is a Sobolev space, constructed by the differential operator B, i.e.

$$W^r_{2,\mu_\alpha}(B) = \left\{f \in L_{2,\mu_\alpha} : B^j f \in L_{2,\mu_\alpha}, j = 1, 2, ..., r\right\}.$$

Note that values $\Xi_{\sigma,r,m,p,s}(\varphi, h)$ for different values of the parameters therein and specific weight functions were examined by Chernykh, Taykov, Yudin, Esmaganbetov, Ivanov, Babenko, Shalaev, Vakarchuk, Shabozov, Tukhliev and many others (see., e.g., [6-11] and the literature cited therein).

In the case of approximation of 2π-periodic function from L_2 by the subspace of trigonometric polynomials of an order $(n-1)$ in the metric L_2, similar problems were solved in [9] by Taikov, in [10] by M. Esmaganbetov, and in [11] by Sh.Shabozov and K. Tukhliev.

The extension of this question to the case of the best mean-square approximation by entire functions of exponential $\sigma > 0$ type in space L_2 with a power-law weight was carried out in [8] by A.G. Babenko and in [12] by D. V. Gorbachev, in [5] by V.I. Ivanov.

1 Auxiliary results

Lemma 1. Let $q_{\alpha+1,1}$ be the smallest positive zero of the function $j_{\alpha+1}(t)$. Let $\sigma > 0$ and $t \in (0, \frac{q_{\alpha+1,1}}{\sigma}],\ \alpha \ge -\frac{1}{2}$. Then

$$\sup_{0<h\le t}(1 - j_\alpha(\sigma h)) = 1 - j_\alpha(\sigma t).$$

Proof of Lemma 1. Since

$$j'_\alpha(t) = -\frac{t}{2(\alpha+1)}j_{\alpha+1}(t), 0 \le t \le \infty$$

(see [5]), then from $j_{\alpha+1}(0) > 0$ and $j_{\alpha+1}(q_{\alpha+1,1}) = 0$ we obtain for all $t \in [0, q_{\alpha+1,1}]$ values $(1 - j_\alpha(t))' = \frac{t}{2(\alpha+1)}j_{\alpha+1}(t) > 0$. It follows that the function $1 - j_\alpha(t)$ increases on $[0, q_{\alpha+1,1}]$. Hence, for all $t \in (0, q_{\alpha+1,1}]$ we have

$$\sup_{0<h\le t}(1 - j_\alpha(h)) = 1 - j_\alpha(t).$$

Therefore, for all $t \in (0, \frac{q_{\alpha+1,1}}{\sigma}]$ we get

$$\sup_{0<h\leq t}(1 - j_\alpha(\sigma h)) = 1 - j_\alpha(\sigma t). \tag{1}$$

Lemma 1 is proved.

Lemma 2. Let $q_{\alpha+1,1}$ be the first positive zero of the function $j_{\alpha+1}(t),\ h \in (0, \frac{q_{\alpha+1,1}}{\sigma}],\ \alpha \geq -\frac{1}{2}$ and $\sigma > 0$. Let

$$\Psi(y) = y^{4r}\int_0^h (1 - j_\alpha(yt))^{2k}dt, \quad y \in G, \quad \text{where} \quad G = \{y : \sigma \leq y < \infty\}.$$

Then

$$\min\{\Psi(y) : y \in G\} = \sigma^{4r}\int_0^h (1 - j_\alpha(\sigma t))^{2k}dt.$$

Proof of Lemma 2. Since $j'_\alpha(t) = -\frac{t}{2(\alpha+1)}j_{\alpha+1}(t),\ 0 \leq x \leq \infty$, then for $y \in G$ we have

$$\Psi'(y) = 4ry^{4r-1}\int_0^h (1 - j_\alpha(yt))^{2k}dt + y^{4r}\int_0^h \frac{\partial}{\partial y}\left((1 - j_\alpha(yt))^{2k}\right)dt. \tag{2}$$

Since it is not difficult to verify by direct verification that the equality is true

$$\frac{1}{y}\frac{\partial}{\partial t}\left((1 - j_\alpha(yt))^{2k}\right) = \frac{1}{t}\frac{\partial}{\partial y}\left((1 - j_\alpha(yt))^{2k}\right), \tag{3}$$

where t, y are non-zero, then from (2) by virtue of equality (3) we have

$$\Psi'(y) = y^{4r-1}\left[4r\int_0^h (1 - j_\alpha(yt))^{2k}dt + \int_0^h t\frac{\partial}{\partial t}\left((1 - j_\alpha(yt))^{2k}\right)dt\right]. \tag{4}$$

Applying the method of integration by parts to calculate the second integral in the right-hand side of (4), we conclude

$$\Psi'(y) = y^{4r-1}\left[(4r - 1)\int_0^h (1 - j_\alpha(yt))^{2k}dt + h(1 - j_\alpha(yh))^{2k}\right]. \tag{5}$$

Since $|j_\alpha(u)| \leq 1, \forall u \geq 0$ (see [8], formula (21)) and (1), then by virtue of (5), we have $\Psi'(y) > 0$ for all $y \geq \sigma$. Lemma is proved.

2 Main results

The main results of this work are the following theorems.

Theorem 1. For any function $f \in W^r_{2,\mu_\alpha}(B)$ for any $h > 0$, the following estimate holds:

$$E_\sigma(f)_{2,\mu_\alpha} \leq \frac{\left(\int_0^h \omega_k^2(B^r f, t)_{2,\mu_\alpha}dt\right)^{\frac{1}{2}}}{\sigma^{2r}\left(\int_0^h (1 - j_\alpha(\sigma t))^{2k}dt\right)^{\frac{1}{2}}}.$$

Proof of Theorem 1. Let $f \in W^r_{2,\mu_\alpha}(B)$. Then from Parseval's equality, we have

$$\omega_k^2(B^r f, t)_{2,\mu_\alpha} \geq \int_\sigma^\infty (1 - j_\alpha(\lambda t))^{2k}\lambda^{4r}|\hat{H}_\alpha(f)(\lambda)|^2 d\mu_\alpha(\lambda).$$

Integrating both sides of this inequality variable t over the range $t = 0$ and $t = h$, we obtain

$$\int_0^h \omega_k^2(B^r f, t)_{2,\mu_\alpha} dt \geq \int_0^h \left(\int_\sigma^\infty (1 - j_\alpha(\lambda t))^{2k} \lambda^{4r} |\hat{H}_\alpha(f)(\lambda)|^2 d\mu_\alpha(\lambda) \right) dt =$$

$$= \int_\sigma^\infty |\hat{H}_\alpha(f)(\lambda)|^2 \left(\int_0^h \lambda^{4r} (1 - j_\alpha(\lambda t))^{2k} \right) dt d\mu_\alpha(\lambda). \tag{6}$$

From (6) by virtue of lemma 2, we have

$$\int_0^h \omega_k^2(B^r f, t)_{2,\mu_\alpha} dt \geq \sigma^{4r} \int_0^h (1 - j_\alpha(\sigma t))^{2k} dt \int_\sigma^\infty |\hat{H}_\alpha(f)(\lambda)|^2 d\mu_\alpha(\lambda).$$

It follows that

$$\int_\sigma^\infty |\hat{H}_\alpha(f)(\lambda)|^2 d\mu_\alpha(\lambda) \leq \frac{\int_0^h \omega_k^2(B^r f, t)_{2,\mu_\alpha} dt}{\sigma^{4r} \int_0^h (1 - j_\alpha(\sigma t))^{2k} dt}. \tag{7}$$

Further, given the following equality

$$\|f - S_\sigma(f, x)\|_{2,\mu_\alpha} = E_\sigma(f)_{2,\mu_\alpha}$$

in view of the inequality (7) we get

$$E_\sigma^2(f)_{2,\mu_\alpha} \leq \frac{\int_0^h \omega_k^2(B^r f, t)_{2,\mu_\alpha} dt}{\sigma^{4r} \int_0^h (1 - j_\alpha(\sigma t))^{2k} dt}.$$

Theorem 1 is proved.

Theorem 2. For any function $f \in W_{2,\mu_\alpha}^r(B)$ for any $h > 0$, the following estimate holds:

$$\sup_{f \in W_{2,\mu_\alpha}^r(B)} \frac{\sigma^{2r} E_\sigma(f)_{2,\mu_\alpha}}{\left(\int_0^h \omega_k^2(B^r f, t)_{2,\mu_\alpha} dt \right)^{\frac{1}{2}}} = \frac{1}{\left(\int_0^h (1 - j_\alpha(\sigma t))^{2k} dt \right)^{\frac{1}{2}}}. \tag{8}$$

Proof of Theorem 2. Let $f \in W_{2,\mu_\alpha}^r(B)$. Arguing in the same way as in Theorem 1, for $f \in W_{2,\mu_\alpha}^r(B)$ we have

$$\frac{\sigma^{2r} E_\sigma(f)_{2,\mu_\alpha}}{\left(\int_0^h \omega_k^2(B^r f, t)_{2,\mu_\alpha} dt \right)^{\frac{1}{2}}} \leq \frac{1}{\left(\int_0^h (1 - j_\alpha(\sigma t))^{2k} dt \right)^{\frac{1}{2}}}.$$

Hence we get

$$\sup_{f \in W_{2,\mu_\alpha}^r(B)} \frac{\sigma^{2r} E_\sigma(f)_{2,\mu_\alpha}}{\left(\int_0^h \omega_k^2(B^r f, t)_{2,\mu_\alpha} dt \right)^{\frac{1}{2}}} \leq \frac{1}{\left(\int_0^h (1 - j_\alpha(\sigma t))^{2k} dt \right)^{\frac{1}{2}}}. \tag{9}$$

To obtain a lower estimate, we construct the function $f_\epsilon \in W_{2,\mu_\alpha}^r(B)$ so that it satisfies the inequality:

$$\sup_{f \in W_{2,\mu_\alpha}^r(B)} \frac{\sigma^{2r} E_\sigma(f)_{2,\mu_\alpha}}{\left(\int_0^h \omega_k^2(B^r f, t)_{2,\mu_\alpha} dt \right)^{\frac{1}{2}}} \geq \frac{\sigma^{2r} E_\sigma(f_\epsilon)_{2,\mu_\alpha}}{\left(\int_0^h \omega_k^2(B^r f_\epsilon, t)_{2,\mu_\alpha} dt \right)^{\frac{1}{2}}}.$$

To do this, we use function $f_\epsilon \in W_{2,\mu_\alpha}^r(B)$ constructed by Babenko in [9] and such that

$$\hat{H}_\alpha(f_\epsilon)(\lambda) = \begin{cases} |\lambda|^{-\alpha - \frac{1}{2}} & \text{if} \quad \sigma < |\lambda| < \sigma + \epsilon, \\ 0, & \text{otherwise.} \end{cases}$$

Relations (2) and using the properties of the Hankel transform (see [7]) imply the equality

$$E_\sigma^2(f_\epsilon)_{2,\mu_\alpha} = \int_\sigma^\infty |\hat{H}_\alpha(f_\epsilon)(\lambda)|^2 d\mu_\alpha(\lambda) = \int_\sigma^{\sigma+\epsilon} |\hat{H}_\alpha(f_\epsilon)(\lambda)|^2 d\mu_\alpha(\lambda) = \frac{\epsilon}{2^\alpha\Gamma(\alpha+1)}.$$

Therefore

$$E_\sigma(f_\epsilon)_{2,\mu_\alpha} = \sqrt{\frac{\epsilon}{2^\alpha\Gamma(\alpha+1)}}. \tag{10}$$

In virtue of the equality (01) and using the properties of the Hankel transform (see [7], [4])

$$\hat{H}_\alpha(B^r f_\epsilon)(\lambda) = \lambda^{2r}\hat{H}_\alpha(f_\epsilon)(\lambda)$$

we write:

$$\omega_k^2(B^r f_\epsilon, t)_{2,\mu_\alpha} = \int_\sigma^{\sigma+\epsilon} \lambda^{4r}|\hat{H}_\alpha(f_\epsilon)(\lambda)|^2(1-j_\alpha(\lambda t))^{2k} d\mu_\alpha(\lambda) \leq$$

$$\leq (\sigma+\epsilon)^{4r}(1-j_\alpha((\sigma+\epsilon)t))^{2k}\frac{\epsilon}{2^\alpha\Gamma(\alpha+1)}. \tag{11}$$

Integrating both parts of the inequality (11), we have

$$\left\{\int_0^h \omega_k^2(B^r f_\epsilon, t)_{2,\mu_\alpha} dt\right\}^{\frac{1}{2}} \leq (\sigma+\epsilon)^{2r}\sqrt{\frac{\epsilon}{2^\alpha\Gamma(\alpha+1)}}\left\{\int_0^h (1-j_\alpha((\sigma+\epsilon)t)^{2k} dt\right\}^{\frac{1}{2}}. \tag{12}$$

Using (10), (12) we write

$$\frac{\sigma^{2r}E_\sigma(f_\epsilon)_{2,\mu_\alpha}}{\left(\int_0^h \omega_k^2(B^r f_\epsilon, t)_{2,\mu_\alpha} dt\right)^{\frac{1}{p}}} \geq \frac{\sigma^{2r}}{(\sigma+\epsilon)^{2r}\{\int_0^h (1-j_\alpha((\sigma+\epsilon)t))^{2k} dt\}^{\frac{1}{p}}}. \tag{13}$$

Since $f_\epsilon \in W^r_{2,\mu_\alpha}(B)$, then from (13) and from left side of equality (8) we obtain

$$\sup_{f\in W^r_{2,\mu_\alpha}(B)} \frac{\sigma^{2r}E_\sigma(f)_{2,\mu_\alpha}}{\left(\int_0^h \omega_k^2(B^r f, t)_{2,\mu_\alpha} dt\right)^{\frac{1}{2}}} \geq \frac{\sigma^{2r}}{(\sigma+\epsilon)^{2r}\{\int_0^h (1-j_\alpha((\sigma+\epsilon)t))^{2k} dt\}^{\frac{1}{2}}}. \tag{14}$$

Obviously, the left side of inequality (14) does not depend on ϵ, and the expression located on its right side is the function of ϵ (with fixed values of other parameters). Since the left side of inequality (14) does not depend on ϵ, then calculating the supremum with respect to ϵ from its right side, we write

$$\sup_{f\in W^r_{2,\mu_\alpha}(B)} \frac{\sigma^{2r}E_\sigma(f)_{2,\mu_\alpha}}{\left(\int_0^h \omega_k^2(B^r f, t)_{2,\mu_\alpha} dt\right)^{\frac{1}{2}}} \geq \frac{\sigma^{2r}E_\sigma(f_\epsilon)_{2,\mu_\alpha}}{\left(\int_0^h \omega_k^2(B^r f_\epsilon, t)_{2,\mu_\alpha} dt\right)^{\frac{1}{2}}} = \frac{1}{\left(\int_0^h (1-j_\alpha(\sigma t))^{2k} dt\right)^{\frac{1}{2}}}. \tag{15}$$

Comparing the upper estimate (9) and the lower estimate (15), we obtain the required equality. Theorem 2 is proved.

Theorem 3. Let $m, n \in \mathbb{N},\ r \in \mathbb{Z}_+,\ 0 < p \leq 2,\ h > 0,\ \alpha \geq -\frac{1}{2}$. Then the following estimate is valid

$$\sup_{f\in W^r_{2,\mu_\alpha}(B)} \frac{\sigma^{2r}E_\sigma(f)_{2,\mu_\alpha}}{\left(\int_0^h \omega_k^p(B^r f, t)_{2,\mu_\alpha} dt\right)^{\frac{1}{p}}} = \frac{1}{\{\int_0^h (1-j_\alpha(\sigma t))^{kp} dt\}^{\frac{1}{p}}}. \tag{16}$$

Proof of Theorem 3. Let $0 < p \leq 2$, then, arguing as in the previous theorem, we have

$$\omega_k^2(B^r f, t)_{2,\mu_\alpha} \geq \int_\sigma^\infty \lambda^{4r}(1 - j_\alpha(\lambda t))^{2k} |\hat{H}_\alpha(f)(\lambda)|^2 d\mu_\alpha(\lambda).$$

Raising both sides of this inequality by the power $p/2$, integrating the variable t over the range $t = 0$ and $t = h$ we obtain

$$\left(\int_0^h \omega_k^p(B^r f, t)_{2,\mu_\alpha} dt\right)^{\frac{1}{p}} \geq \left\{\int_0^h \left(\int_\sigma^\infty \lambda^{4r}(1 - j_\alpha(\lambda t))^{2k} |\hat{H}_\alpha(f)(\lambda)|^2 d\mu_\alpha(\lambda)\right)^{\frac{p}{2}} dt\right\}^{\frac{1}{p}} = I.$$

Applying the inverse Minkowski inequality for $\frac{p}{2} \leq 1$, we have

$$I \geq \left\{\int_\sigma^\infty |\hat{H}_\alpha(f)(\lambda)|^2 \left(\int_0^h \lambda^{2rp}(1 - j_\alpha(\lambda t))^{kp} dt\right)^{\frac{2}{p}} d\mu_\alpha(\lambda)\right\}^{\frac{1}{2}}. \tag{17}$$

Then from inequality (17) and in view of Lemma 2, we obtain

$$I \geq \sigma^{2r} \left\{\int_\sigma^\infty |\hat{H}_\alpha(f)(\lambda)|^2 d\mu_\alpha(\lambda)\right\}^{\frac{1}{2}} \left(\int_0^h (1 - j_\alpha(\sigma t))^{kp} dt\right)^{\frac{1}{p}} =$$

$$= \sigma^{2r} E_\sigma(f)_{2,\mu_\alpha} \left(\int_0^h (1 - j_\alpha(\sigma t))^{kp} dt\right)^{\frac{1}{p}}. \tag{18}$$

So combining (17) and (18), we get

$$\left(\int_0^h \left(\omega_k^2(B^r f, t)_{2,\mu_\alpha}\right)^{\frac{p}{2}} dt\right)^{\frac{1}{p}} \geq \sigma^{2r} E_\sigma(f)_{2,\mu_\alpha} \left(\int_0^h (1 - j_\alpha(\sigma t))^{kp} dt\right)^{\frac{1}{p}}.$$

Hence it follows that for all $f \in W^r_{2,\mu_\alpha}(B)$ the inequality holds

$$\frac{\sigma^{2r} E_\sigma(f)_{2,\mu_\alpha}}{\left(\int_0^h \omega_k^p(B^r f, t)_{2,\mu_\alpha} dt\right)^{\frac{1}{p}}} \leq \frac{1}{\left(\int_0^h (1 - j_\alpha(\sigma t))^{kp} dt\right)^{\frac{1}{p}}}.$$

For all $f \in W^r_{2,\mu_\alpha}(B)$, we have

$$\sup_{f \in W^r_{2,\mu_\alpha}(B)} \frac{\sigma^{2r} E_\sigma(f)_{2,\mu_\alpha}}{\left\{\int_0^h \omega_k^p(B^r f, t)_{2,\mu_\alpha} dt\right\}^{\frac{1}{p}}} \leq \frac{1}{\left\{\int_0^h (1 - j_\alpha(\sigma t))^{kp} dt\right\}^{\frac{1}{p}}}. \tag{19}$$

Thus, the upper estimate is proved.

To obtain a lower estimate, we construct a function $f_\epsilon \in W^r_{2,\mu_\alpha}(B)$ so that the inequality is fulfilled:

$$\sup_{f \in W^r_{2,\mu_\alpha}(B)} \frac{\sigma^{2r} E_\sigma(f)_{2,\mu\alpha}}{\left\{\int_0^h \omega_k^p(B^r f, t)_{2,\mu_\alpha} dt\right\}^{\frac{1}{p}}} \geq \frac{\sigma^{2r} E_\sigma(f_\epsilon)_{2,\mu\alpha}}{\left\{\int_0^h \omega_k^p(B^r f_\epsilon, t)_{2,\mu_\alpha} dt\right\}^{\frac{1}{p}}}. \tag{20}$$

To do this, we use function $f_\epsilon \in W^r_{2,\mu_\alpha}(B)$ constructed by Babenko in [8] and such that

$$\hat{H}_\alpha(f_\epsilon)(\lambda) = \begin{cases} |\lambda|^{-\alpha - \frac{1}{2}} & \text{if} \quad \sigma < |\lambda| < \sigma + \epsilon, \\ 0, & \text{otherwise.} \end{cases}$$

Raising both sides of the inequality (11) by the power $\frac{p}{2}$ and integrating the variable t over the range $t=0$ to $t=h$, we have

$$\left\{\int_0^h \omega_k^p(B^r f_\epsilon, t)_{2,\mu_\alpha} dt\right\}^{\frac{1}{p}} \leq (\sigma+\epsilon)^{2r}\sqrt{\frac{\epsilon}{2^\alpha\Gamma(\alpha+1)}}\left\{\int_0^h (1-j_\alpha((\sigma+\epsilon)t)^{kp}dt\right\}^{\frac{1}{p}}. \tag{21}$$

Using (21), (10) we write

$$\frac{\sigma^{2r}E_\sigma(f_\epsilon)_{2,\mu_\alpha}}{\left(\int_0^h \omega_k^p(B^r f_\epsilon, t)_{2,\mu_\alpha} dt\right)^{\frac{1}{p}}} \geq \frac{\sigma^{2r}}{(\sigma+\epsilon)^{2r}\{\int_0^h (1-j_\alpha((\sigma+\epsilon)t))^{kp}dt\}^{\frac{1}{p}}}. \tag{22}$$

In view of the fact that the function f_ϵ belongs to the class $W^r_{2,\mu_\alpha}(B)$ and from the right-hand side of equality (16) and by virtue of the inequality (22), (20) we obtain

$$\sup_{f\in W^r_{2,\mu_\alpha}(B)} \frac{\sigma^{2r}E_\sigma(f)_{2,\mu_\alpha}}{\left(\int_0^h \omega_k^p(B^r f, t)_{2,\mu_\alpha} dt\right)^{\frac{1}{p}}} \geq \frac{\sigma^{2r}}{(\sigma+\epsilon)^{2r}\{\int_0^h (1-j_\alpha((\sigma+\epsilon)t))^{kp}dt\}^{\frac{1}{p}}}. \tag{23}$$

Obviously, the left side of inequality (23) does not depend on ϵ, and the expression located on its right side is the function of ϵ (with fixed values of other parameters). Since the left side of inequality (23) does not depend on ϵ, then calculating the supremum with respect to ϵ from its right side, we write

$$\sup_{f\in W^r_{2,\mu_\alpha}(B)} \frac{\sigma^{2r}E_\sigma(f)_{2,\mu\alpha}}{\left(\int_0^h \omega_k^p(B^r f, t)_{2,\mu_\alpha} dt\right)^{\frac{1}{p}}} \geq \frac{1}{\{\int_0^h (1-j_\alpha(\sigma t))^{kp}dt\}^{\frac{1}{p}}}. \tag{24}$$

Comparing the upper estimate (19) and the lower estimate (24), we obtain the required equality. The theorem 3 is proved.

Theorem 4. Let $m,n\in\mathbb{N},\ r\in\mathbb{Z}_+,\ 0<p\leq 2,\ h>0,\ \ \alpha\geq -\frac{1}{2}$ and $\varphi(t)\geq 0$ be a measurable function on the interval (0, h). Then the inequality

$$\left\{\gamma_{\sigma,r,m,p,\frac{1}{p}}(\varphi,h)\right\}^{-1} \leq \Xi_{\sigma,r,m,p,\frac{1}{p}}(\varphi,h) \leq \left\{\inf_{\sigma\leq\lambda<\infty}\gamma_{\lambda,r,m,p,\frac{1}{p}}(\varphi,h)\right\}^{-1}$$

holds, where

$$\gamma_{\lambda,r,m,p,\frac{1}{p}}(\varphi,h) = \left(\lambda^{2rp}\int_0^h (1-j_\alpha(\sigma t))^{kp}\varphi(t)dt\right)^{\frac{1}{p}}, \quad \lambda\geq\sigma.$$

Proof of Theorem 4. Let $0<p\leq 2$ then, arguing as in the previous theorem, we have

$$\omega_k^2(B^r f, t)_{2,\mu_\alpha} \geq \int_\sigma^\infty \lambda^{4r}(1-j_\alpha(\lambda t))^{2k}|\hat{H}_\alpha(f)(\lambda)|^2 d\mu_\alpha(\lambda).$$

Raising both sides of this inequality by the power $p/2$ and multiplying them by a function $\varphi(t)$ and integrating the variable t over the range $t=0$ to $t=h$ we get

$$\left(\int_0^h \omega_k^p(B^r, t)_{2,\mu_\alpha}\varphi(t)dt\right)^{\frac{1}{p}} \geq \left\{\int_0^h \left(\int_\sigma^\infty \lambda^{4r}(1-j_\alpha(\lambda t))^{2k}|\hat{H}_\alpha(f)(\lambda)|^2 d\mu_\alpha(\lambda)\right)^{\frac{p}{2}}\varphi(t)dt\right\}^{\frac{1}{p}} = I. \tag{25}$$

Applying the inverse Minkowski inequality for $\frac{p}{2} \leq 1$ and by virtue of Lemma 2 we obtain

$$I \geq \left\{ \int_\sigma^\infty |\hat{H}_\alpha(f)(\lambda)|^2 d\mu_\alpha(\lambda) \left(\int_0^h \lambda^{2rp}(1-j_\alpha(\lambda t))^{kp}\varphi(t)dt \right)^{\frac{2}{p}} \right\}^{\frac{1}{2}} =$$

$$= \left\{ \int_\sigma^\infty |\hat{H}_\alpha(f)(\lambda)|^2 d\mu_\alpha(\lambda) \left\{ \gamma_{\lambda,r,k,p,\frac{1}{p}}(\varphi,h) \right\}^2 \right\}^{\frac{1}{2}} \geq$$

$$\geq E_\sigma(f)_{2,\mu_\alpha} \inf_{\sigma \leq \lambda < \infty} \gamma_{\lambda,r,k,p,\frac{1}{p}}(\varphi,h). \tag{26}$$

So combining (25) and (26) we get

$$\left(\int_0^h \omega_k^p(B^r,t)_{2,\mu_\alpha}\varphi(t)dt \right)^{\frac{1}{p}} \geq E_\sigma(f)_{2,\mu_\alpha} \inf_{\sigma \leq \lambda < \infty} \gamma_{\lambda,r,k,p,\frac{1}{p}}(\varphi,h).$$

Therefore, according the definition of quantity (03), by previous inequality we obtain an upper bound for the extremal characteristics $\Xi_{\sigma,r,k,p,\frac{1}{p}}(\varphi,h)$, namely

$$\Xi_{\sigma,r,k,p,\frac{1}{p}}(\varphi,h) = \sup_{f \in W^r_{2,\mu_\alpha}(B)} \frac{E_\sigma(f)_{2,\mu_\alpha}}{\left\{ \int_0^h \omega_k^p(B^r f,t)_{2,\mu_\alpha}\varphi(t)dt \right\}^{\frac{1}{p}}} \leq \frac{1}{\inf\limits_{\sigma \leq \lambda < \infty} \gamma_{\lambda,r,k,p,\frac{1}{p}}(\varphi,h)}. \tag{27}$$

To obtain a lower estimate, we construct the function $f_\epsilon \in W^r_{2,\mu_\alpha}(B)$ so that the inequality would be fulfilled:

$$\Xi_{\sigma,r,k,p,\frac{1}{p}}(\varphi,h) = \sup_{f \in W^r_{2,\mu_\alpha}(B)} \frac{E_\sigma(f)_{2,\mu_\alpha}}{\left\{ \int_0^h \omega_k^p(B^r f,t)_{2,\mu_\alpha}\varphi(t)dt \right\}^{\frac{1}{p}}} \geq \frac{E_\sigma(f_\epsilon)_{2,\mu_\alpha}}{\left\{ \int_0^h \omega_k^p(B^r f_\epsilon,t)_{2,\mu_\alpha}\varphi(t)dt \right\}^{\frac{1}{p}}}. \tag{28}$$

To do this, we use function $f_\epsilon \in W^r_{2,\mu_\alpha}(B)$ constructed by Babenko in [9] and such that

$$\hat{H}_\alpha(f_\epsilon)(\lambda) = \begin{cases} |\lambda|^{-\alpha-\frac{1}{2}} & \text{if} \quad \sigma < |\lambda| < \sigma+\epsilon, \\ 0, & \text{otherwise.} \end{cases}$$

Raising both sides of the inequality (11) by the power $\frac{p}{2}$, multiplying them by the weight function $\varphi(t)$, and integrating the variable t over the range $t=0$ to $t=h$, we have

$$\left\{ \int_0^h \omega_k^p(B^r f_\epsilon,t)_{2,\mu_\alpha}\varphi(t)dt \right\}^{\frac{1}{p}} \leq (\sigma+\epsilon)^{2r} \sqrt{\frac{\epsilon}{2^\alpha \Gamma(\alpha+1)}} \left\{ \int_0^h (1-j_\alpha((\sigma+\epsilon)t)^{kp}\varphi(t)dt \right\}^{\frac{1}{p}}. \tag{29}$$

Using (29) and (10) we write

$$\frac{E_\sigma(f_\epsilon)_{2,\mu_\alpha}}{\left(\int_0^h \omega_k^p(B^r f_\epsilon,t)_{2,\mu_\alpha}\varphi(t)dt \right)^{\frac{1}{p}}} \geq \frac{1}{(\sigma+\epsilon)^{2r}\{\int_0^h (1-j_\alpha((\sigma+\epsilon)t))^{kp}\varphi(t)dt\}^{\frac{1}{p}}}. \tag{30}$$

In view of the fact that the function f_ϵ belongs to the class $W^r_{2,\mu_\alpha}(B)$, by virtue of inequality (30) and relation (03), (28) we obtain

$$\sup_{f \in W^r_{2,\mu_\alpha}(B)} \frac{E_\sigma(f)_{2,\mu_\alpha}}{\left(\int_0^h \omega_k^p(B^r f,t)_{2,\mu_\alpha}\varphi(t)dt \right)^{\frac{1}{p}}} \geq \frac{1}{(\sigma+\epsilon)^{2r}\{\int_0^h (1-j_\alpha((\sigma+\epsilon)t))^{kp}\varphi(t)dt\}^{\frac{1}{p}}}. \tag{31}$$

Obviously, the left side of the inequality (31) does not depend on ϵ, and the expression located on its right side is the function of ϵ (with fixed values of other parameters). Since the left side of inequality (31) does not depend on ϵ, then calculating the supremum with respect to ϵ from its right side, we write

$$\sup_{f\in W^r_{2,\mu_\alpha}(B)} \frac{\sigma^{2r}E_\sigma(f)_{2,\mu\alpha}}{\left(\int_0^h \omega_k^p(B^r f,t)_{2,\mu_\alpha}\varphi(t)dt\right)^{\frac{1}{p}}} \geq \frac{1}{\{\int_0^h(1-j_\alpha(\sigma t))^{kp}\varphi(t)dt\}^{\frac{1}{p}}}. \quad (32)$$

Comparing the upper estimate (27) and the lower estimate (32), we obtain the required equality. Theorem 4 is proved.

Let us find: what differential properties the weight function φ must possess in order that the following equality holds

$$\gamma_{\sigma,r,k,p,\frac{1}{p}}(\varphi,h) = \inf_{\sigma\leq\lambda<\infty} \gamma_{\lambda,r,k,p,\frac{1}{p}}(\varphi,h).$$

The following statement gives an answer to this question.

Theorem 5. Let $\varphi(t)$ be a non-negative continuously differentiable function on the interval $[0,h]$. If for some $p \in (0,2]$, $r \in \mathbb{N}$ any $t \in [0,h]$, $\alpha \geq -\frac{1}{2}$, φ satisfies the differential inequality

$$(2rp-1)\varphi(t) - t\varphi'(t) \geq 0,$$

then for all $\sigma \in (0,\infty)$ and $0 < h \leq \frac{q_{\alpha+1,1}}{\sigma}$ we have

$$\inf\left\{\gamma_{\lambda,k,r,p,\frac{1}{p}}(\varphi,h) : \sigma \leq \lambda < \infty\right\} = \gamma_{\sigma,k,r,p,\frac{1}{p}}(\varphi,h)$$

and there is a relation

$$\Xi_{\sigma,k,r,p,\frac{1}{p}}(\varphi,h) = \left(\gamma_{\sigma,k,r,p,\frac{1}{p}}(\varphi,h)\right)^{-1}.$$

Proof of Theorem 5. Since

$$\gamma_{\lambda,r,k,p,\frac{1}{p}}(\varphi,h) = \left\{\lambda^{2rp}\int_0^h (1-j_\alpha(\lambda t))^{kp}\varphi(t)dt\right\}^{\frac{1}{p}}$$

it is sufficient to prove that under the above assumptions on $\varphi(t)$ and the function

$$\eta(y) = y^{2rp}\int_0^h (1-j_\alpha(yt))^{kp}\varphi(t)dt$$

is strictly increasing on the interval $G = \{y : y \geq \sigma\}$. Since

$$\eta'(y) = 2rpy^{2rp-1}\int_0^h (1-j_\alpha(yt))^{kp}\varphi(t)dt + y^{2rp}\int_0^h \frac{d}{dy}(1-j_\alpha(yt))^{kp}\varphi(t)dt, \quad (33)$$

then, using the obvious identity

$$\frac{d}{dy}(1-j_\alpha(yt))^{kp} = \frac{t}{y}\frac{d}{dt}(1-j_\alpha(yt))^{kp} \quad (34)$$

from (33) and taking into account (34) we have

$$\eta'(y) = 2rpy^{2rp-1}\int_0^h (1-j_\alpha(yt))^{kp}\varphi(t)dt + y^{2rp-1}\int_0^h \frac{d}{dt}(1-j_\alpha(yt))^{kp}(t\varphi(t))dt.$$

Applying the method of integration by parts when calculating the second integral, we come to the conclusion

$$\eta'(y) = y^{2rp-1}\left((1-j_\alpha(yh))^{kp}h\varphi(h) + \int_0^h (1-j_\alpha(yt))^{kp}[(2rp-1)\varphi(t)+t\varphi'(t)]dt\right). \qquad (35)$$

Since $|j_\alpha(y)| \leq 1$ for all $y \in [0,\infty)$, then by virtue of the

$$(2rp-1)\varphi(t) - t\varphi'(t) \geq 0,$$

taking into account the conditions $p \in (0,2], r \in \mathbb{N}$ from (35) we have $\eta'(y) \geq 0$, for $y \geq \sigma$. Whence follows $\inf\{\eta(y) : \sigma \leq y < \infty\} = \eta(\sigma)$, which is equivalent to equality

$$\inf\left\{\gamma_{\lambda,k,r,p,\frac{1}{p}}(\varphi,h) : \sigma \leq \lambda < \infty\right\} = \gamma_{\sigma,k,r,p,\frac{1}{p}}(\varphi,h).$$

Then by virtue of the double inequality from Theorem 4, we obtain the required equality. Theorem 5 is proved.

4 Approximation in $L^2(\mathbb{R}^m)$

The exact inequality and its various generalizations have been the subject of study for many specialists in the last 50 years. Some historical information on the Jackson–Stechkin inequalities in $L^2(\mathbb{R}^m)$ can be found in [5, 8, 13–18].

Let $L^2 = L^2(R^m)$ be the Hilbert space of complex functions on R^m with a scalar product and norm

$$(f,g) = \int_{R^m} f(x)g(x)dx, \quad \|f\| = \sqrt{(f,f)}.$$

The Fourier transform of the function $f \in L^2$ is defined by this formula

$$\hat{f}(y) = \frac{1}{(2\pi)^{m/2}}\int_{R^m} f(x)e^{-ix\cdot y}dx,$$

where $x \cdot y = \sum_{l=1}^m x_l \cdot y_l$ is the scalar product of vectors x and y of R^m.

The function f can be decomposed through its Fourier transform $\hat{f}$ as:

$$f(x) = \frac{1}{(2\pi)^{m/2}}\int_{R^m} f(y)e^{ix\cdot y}dy. \qquad (36)$$

For the Fourier transform in L^2 space, the Plancherell formula applies

$$(f,g) = (\hat{f},\hat{g}), \quad f,g \in L^2.$$

Let us denote by W_σ the class of exponential spherical integer functions $\sigma > 0$ belonging to the space. The class W_σ of integer functions consists of integer functions $g \in L^2$ such that the support *supp* $\hat{g}$ of Fourier transform lies in a Euclidean ball $B_{\sigma^m} = \left\{x \in R^m : |x| = \sqrt{(x,x)} \leq \sigma\right\}$ of a radius $\sigma > 0$ and with a center at the origin of the space R^m. The best approximation of the function f of L^2 by the class W_σ is

$$A_\sigma f = \inf\left\{\|f-g\| : g \in W_\sigma\right\}.$$

The spherical shift with a step h is the operator S_h acting according to the rule

$$S_h f(x) = \frac{1}{|\mathbb{S}^{m-1}|}\int_{\mathbb{S}^{m-1}} f(x+h\xi)d\xi,$$

where $\mathbb{S}^{m-1}$ is a unit Euclidean sphere in R^m, $|\mathbb{S}^{m-1}|$ is its surface area. Let I be an identical operator, k is a positive number. Following H.P. Rustamov's operator $(I - S_h f)^{\frac{k}{2}}$ (see [17]), will be called a difference operator of order k with step h and will be denoted by Δ_h^k:

$$\Delta_h^k = \sum_{l=0}^{\infty} (-1)^l \binom{\frac{k}{2}}{l} S_h^l,$$

and the k-order continuity module of the function $f \in L^2(S^{m-1})$ will be the function of the variable $\tau > 0$:

$$\omega_k(f, \tau) = \sup \left\{ \|\Delta_h^k f\| : 0 < h \le \tau \right\}.$$

Denote by $K_n(\tau, k, m), \tau > 0, k \ge 1, m = 2, 3, ...$ the exact constant K the Jackson–Stechkin inequality in $L^2(S^{m-1})$

$$A_\sigma(f) \le K\omega_k(f, \tau), f \in L^2(S^{m-1}),$$

let's put

$$K_\sigma(\tau, k, m) = \sup \left\{ \frac{A_\sigma(f)}{\omega_k(f, \tau)} : f \in L^2(S^{m-1}) \right\}.$$

Using the Plancherell formula, it is easy to see that the value of the best approximation for the function $f \in L^2(S^{m-1})$ is expressed by

$$A_\sigma^2 f = \int_{|y|>\sigma} |\hat{f}(y)|^2 dy.$$

It is known ([19], [13; 176]) that the S_h spherical shift operator with step $h > 0$ acts on the function $e_y(x) = e^{ix\cdot y}$ as follows:

$$S_h e_y(x) = \frac{1}{|\mathbb{S}^{m-1}|} \int_{\mathbb{S}^{m-1}} e^{i(x+h\xi)\cdot y} d\xi =$$

$$= \frac{e^{ix\cdot y}}{|S^{m-1}|} \int_{S^{m-1}} e^{ih\xi\cdot y} d\xi = j_{\frac{m-2}{2}}(h|y|) e_y(x). \tag{37}$$

Applying k times to both parts of equality (36) the spherical shift operator and using relation (37) we have

$$S_h^k f(x) = \frac{1}{(2\pi)^{m/2}} \int_{\mathbb{S}^{m-1}} (j_{\frac{m-2}{2}}(h|y|))^k \hat{f}(y) e^{ix\cdot y} dy. \tag{38}$$

From the definition of the difference operator by virtue of (38) we obtain

$$\Delta_h^k f(x) = \int_{\mathbb{R}^m} (1 - j_{\frac{m-2}{2}}(h|y|))^{\frac{k}{2}} \hat{f}(y) e^{ix\cdot y} dy. \tag{39}$$

Hence, by virtue of the Plancherell formula from (39) we have

$$\|\Delta_h^k f\|^2 = \int_{\mathbb{R}^m} (1 - j_{\frac{m-2}{2}}(h|y|))^k |\hat{f}(y)|^2 dy.$$

5 The Jackson–Stechkin Theorem in $L^2(\mathbb{R}^m)$

Theorem 6. Let $k \ge 1, \sigma > 0$. Then for any function $f \in L^2(\mathbb{S}^{m-1})$ it holds:

$$A_\sigma(f) \le \frac{\left(\int_0^{\frac{2q_{\alpha,1}}{\sigma}} \omega_k^{\frac{2}{k}}(B^r f, t) j_{\frac{m-2}{2}}(\sigma t) t^{m-1} dt \right)^{\frac{k}{2}}}{\sigma^{2r} \left(\int_0^{\frac{2q_{\alpha,1}}{\sigma}} j_{\frac{m-2}{2}}(\sigma t) t^{m-1} dt \right)^{\frac{k}{2}}},$$

where $q_{\frac{m-2}{2},1}$ is the first positive zero of the function $j_{\frac{m-2}{2}}(t)$.

Proof of Theorem 6. For any function $f \in L^2(\mathbb{R}^m)$ and by the equality

$$A_\sigma(f) = \left\{ \int_{|y|>\sigma} |\hat{f}(y)|^2 dy \right\}^{\frac{1}{2}}$$

and applying the Hölder inequality we have

$$A_\sigma^2(f) - \int_\sigma^\infty |\hat{f}(y)|^2 j_{\frac{m-2}{2}}(t|y|)dy = \int_\sigma^\infty |\hat{f}(y)|^2 (1 - j_{\frac{m-2}{2}}(t|y|))dy =$$

$$= \int_\sigma^\infty |\hat{f}(y)|^{2-\frac{2}{k}} |\hat{f}(y)|^{\frac{2}{k}} (1 - j_{\frac{m-2}{2}}(t|y|))dy =$$

$$\leq A_\sigma^{2-\frac{2}{k}}(f) \left(\sigma^{-4r} \int_\sigma^\infty y^{4r} |\hat{f}(y)|^2 (1 - j_{\frac{m-2}{2}}(t|y|))^k dy \right)^{\frac{1}{k}}. \tag{40}$$

Since the equality holds

$$\omega_k^2(B^r f, t) = \int_0^\infty y^{4r} |\hat{f}(y)|^2 (1 - j_{\frac{m-2}{2}}(t|y|))^k dy$$

then from (40) we have

$$A_\sigma^2(f) - \int_\sigma^\infty |\hat{f}(y)|^2 j_{\frac{m-2}{2}}(t|y|)dy \leq A_\sigma^{2-\frac{2}{k}}(f) \sigma^{-\frac{4r}{k}} \omega_k^{\frac{2}{k}}(B^r f, t). \tag{41}$$

By multiplying both parts of the inequality (41) by the Babenko weight function (see [8]) $v(t) = t^{2\alpha+1} T_{\tau_{\alpha,1}} V(t),\ t \in R_+,\ \alpha > \frac{1}{2},\ \alpha = \frac{m-2}{2}$, where

$$V(t) = \begin{cases} j_{\frac{m-2}{2}}(\sigma t), & 0 < t < \frac{q_{\alpha,1}}{\sigma} \\ 0, & t > \frac{q_{\alpha,1}}{\sigma}, \end{cases}$$

$$T_h f(x) = \frac{\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+1/2)} \int_0^\pi f(\sqrt{x^2 + h^2 - 2xh\cos\varphi})(sin\varphi)^{2\alpha} d\varphi$$

and integrating them over t to zero to $q_{\alpha,1} = q_{\frac{m-2}{2},1}$ we obtain

$$\int_0^{\frac{2q_{\alpha,1}}{\sigma}} A_\sigma^2(f) v(t)dt - \int_0^{\frac{2q_{\alpha,1}}{\sigma}} \int_\sigma^\infty |\hat{f}(y)|^2 j_{\frac{m-2}{2}}(t|y|)dy v(t)dt \leq$$

$$\leq \sigma^{-\frac{4r}{k}} \int_0^{\frac{2q_{\alpha,1}}{\sigma}} A_\sigma^{2-\frac{2}{k}}(f) \omega_k^{\frac{2}{k}}(B^r f, t) v(t)dt, \tag{42}$$

where $q_{\frac{m-2}{2},1}$ is the smallest root of the function $j_{\frac{m-2}{2}}(t)$. Since in [8] the inequality

$$\int_0^{\frac{2q_{\alpha,1}}{\sigma}} j_{\frac{m-2}{2}}(t|y|)v(t)dt < 0, \text{ for all } |y| > 1 \tag{43}$$

has been proved, so from (42) and (43), we obtain

$$A_\sigma^2(f) \int_0^{\frac{2q_{\alpha,1}}{\sigma}} v(t)dt \leq \sigma^{-\frac{4r}{k}} A_\sigma^{2-\frac{2}{k}}(f) \int_0^{\frac{2q_{\alpha,1}}{\sigma}} \omega_k^{\frac{2}{k}}(B^r f, t) v(t)dt.$$

Then, applying the properties of the generalized shift operator $T_h f$ (see [6–8]) we have

$$A_\sigma^{\frac{2}{k}}(f) \leq \frac{\int_0^{\frac{2q_{\alpha,1}}{\sigma}} \omega_k^{\frac{2}{k}}(B^r f, t) j_{\frac{m-2}{2}}(\sigma t) t^{m-1} dt}{\sigma^{\frac{4r}{k}} \int_0^{\frac{2q_{\alpha,1}}{\sigma}} j_{\frac{m-2}{2}}(\sigma t) t^{m-1} dt}.$$

It follows that

$$A_\sigma(f) \leq \frac{\left(\int_0^{\frac{2q_{\alpha,1}}{\sigma}} \omega_k^{\frac{2}{k}}(B^r f, t) j_{\frac{m-2}{2}}(\sigma t) t^{m-1} dt\right)^{\frac{k}{2}}}{\sigma^{2r} \left(\int_0^{\frac{2q_{\alpha,1}}{\sigma}} j_{\frac{m-2}{2}}(\sigma t) t^{m-1} dt\right)^{\frac{k}{2}}}.$$

Theorem 6 is proved.

Corollary 1. Let $k \in \mathbb{R}_+, k \geq 1,\ q_{\alpha,1} > 0, \sigma > 0, \alpha = \frac{m-2}{2}$. Then for any function $f \in L^2(\mathbb{R}^m)$ the inequality holds

$$A_\sigma(f) \leq \sigma^{-2r} \omega_k(B^r f, \frac{2q_{\alpha,1}}{\sigma}),$$

where $q_{\alpha,1}$ is the smallest root of the function $j_\alpha(t)$.

Proof of Corollary 1. Let's first show that the functionality of the

$$J_k(f, q_{\alpha,1}) = \frac{\left(\int_0^{\frac{2q_{\alpha,1}}{\sigma}} \omega_k^{\frac{2}{k}}(B^r f, t) j_{\frac{m-2}{2}}(\sigma t) t^{m-1} dt\right)^{\frac{k}{2}}}{\sigma^{2r} \left(\int_0^{\frac{2q_{\alpha,1}}{\sigma}} j_{\frac{m-2}{2}}(\sigma t) t^{m-1} dt\right)^{\frac{k}{2}}}$$

is smaller than $\omega_k(f, \frac{2q_{\alpha,1}}{\sigma})$. Indeed, it follows from the monotonicity of $\omega_k(f,t)$ that

$$J_k(f, \frac{2q_{\alpha,1}}{\sigma}) = \frac{\left(\int_0^{\frac{2q_{\alpha,1}}{\sigma}} \omega_k^{\frac{2}{k}}(B^r f, t) j_{\frac{m-2}{2}}(\sigma t) t^{m-1} dt\right)^{\frac{k}{2}}}{\sigma^{2r} \left(\int_0^{\frac{2q_{\alpha,1}}{\sigma}} j_{\frac{m-2}{2}}(\sigma t) t^{m-1} dt\right)^{\frac{k}{2}}} \leq \sigma^{-2r} \omega_k(B^r f, \frac{2q_{\alpha,1}}{\sigma}). \tag{44}$$

From Theorem 4 and by virtue of (44) we have

$$A_\sigma(f) \leq \frac{\left(\int_0^{\frac{2q_{\alpha,1}}{\sigma}} \omega_k^{\frac{2}{k}}(B^r f, t) j_{\frac{m-2}{2}}(\sigma t) t^{m-1} dt\right)^{\frac{k}{2}}}{\sigma^{2r} \left(\int_0^{\frac{2q_{\alpha,1}}{\sigma}} j_{\frac{m-2}{2}}(\sigma t) t^{m-1} dt\right)^{\frac{k}{2}}} = J_k(f, \frac{2q_{\alpha,1}}{\sigma}) \leq \sigma^{-2r} \omega_k(B^r f, \frac{2q_{\alpha,1}}{\sigma}).$$

Remark. Earlier in [5, 8, 12] similar results were obtained. The proof of Corollary 1 of Theorem 6 given here is new, i.e. it differs from the proofs of the theorems of A.G. Babenko [8], D.V. Gorbachev [12] and V.I. Ivanov [5]. The obtained result, which is a consequence of Theorem 6, coincides with the exact result of A.G. Babenko [8] at $k \geq 1$. In the works [20–22] , direct theorems of the theory of approximation were proved without refining the coefficients

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L_2 метрикасындағы дәл Джексон–Стечкин теңсіздіктері және жалпыланған Ганкелдің ығыстыруы

Жұмыста f функциясының ең жақсы орташа квадраттық жуықтауы бойынша, дәрежелі салмағы бар жарты осьте бірнеше экстремалды есептер шешілген. Гильберт кеңістігінде L_2 салмағы $t^{2\alpha+1}$ дәрежесі болатын, f функциясының Бессельдің бірінші текті функциялары бойынша құрылған σ-ретті дербес Ганкел интегралдарымен ең жақсы жуықтауы $E_\sigma(f)$ және k-ретті үздіксіздіктің жалпыланған модулі $\omega_k f(B^r)f, t)$ арасындағы Джексон–Стечкин типті теңсіздіктер алынған, мұндағы B-екінші ретті дифференциалдық оператор.

Кілт сөздер: ең жақсы жуықтау, үзіліссіздік модулі, m-ретті жалпыланған, тегістік модулі, гильберт кеңістігі.

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Обобщенные сдвиги Ганкеля и точные неравенства Джексона–Стечкина в L_2

В работе решено несколько экстремальных задач о наилучшем среднеквадратическом приближении функции f на полуоси о степенным весом. В гильбертовом пространстве L_2 со степенным весом $t^{2\alpha+1}$ получены неравенства типа Джексона–Стечкина между величиной $E_\sigma(f)$ — наилучшего приближения функции f частичными интегралами Ганкеля порядка не выше σ по функциям Бесселя первого рода и обобщенным модулем непрерывности k-го порядка $\omega_k f(B^r f, t)$, где B — дифференциальный оператор второго порядка.

Ключевые слова: наилучшее приближение, обобщенный модуль гладкости m-го порядка, гильбертово пространство.

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On weighted integrability of the sum of series with monotone coefficients with respect to multiplicative systems

In this paper, we consider the questions about the weighted integrability of the sum of series with respect to multiplicative systems with monotone coefficients. Conditions are obtained for weight functions that ensure that the sum of such series belongs to the weighted Lebesgue space. The main theorems are proved without the condition that the generator sequence is bounded; in particular, it can be unbounded. In the case of boundedness of the generator sequence, the proved theorems imply an analogue of the well-known Hardy-Littlewood theorem on trigonometric series with monotone coefficients.

Keywords: multiplicative systems, decomposition, weighted integrability, sum of series, generator sequence, monotone coefficients, Hardy-Littlewood theorem, Lebesgue space.

Introduction

In the theory of trigonometric series, the Hardy-Littlewood theorem on series with monotone coefficients is known [1, 2]: in order to the series $\sum_{n=0}^{\infty} a_n \cos nx$, where $a_n \downarrow 0$ at $n \to \infty$, was the Fourier series of some function $f(x) \in L_p[0, 2\pi],\ 1 < p < \infty$, is necessary and sufficient to $\sum_{n=1}^{\infty} a_n^p n^{p-2} < \infty$.

An analogue of this theorem for the Walsh system was proved by Moricz F. [3], for multiplicative systems with bounded generating sequences p $(1 \le \sup_n p_n < c)$ was proved by Timan M.F., Tukhliev K. [4].

The weighted integrability of the trigonometric series' sum with generalized monotone coefficients was studied in the works of Tikhonov S.Yu., Dyachenko M.I. [5, 6] and others. Weighted integrability for the sum of series with respect to multiplicative systems is considered in the works of Volosivecs S.S., Fadeev R.N. [7, 8], Bokayev N.A., Mukanov Zh.B. [9].

In this paper, we consider weight functions with other conditions.

1 *Notation and Preliminaries*

In this paper we consider series with monotone coefficients on multiplicative systems. We investigate the problem: under what conditions imposed on the weight function and the coefficients of the series, the sum of this series will belong to the space L_p with weight. Let us give a definition of the multiplicative systems.

Definition 1. Let $\{p_k\}_{k=1}^{\infty}$ is a sequence of natural numbers $p_k \ge 2,\ k \in \mathbb{N}$, $\sup\limits_k p_k = N < \infty$. By definition let us put

$$m_0 = 1,\ m_n = p_1 p_2 \cdots p_n,\ n \in \mathbb{N}.$$

Then every point $x \in [0, 1)$ has a decomposition

$$x = \sum_{k=1}^{\infty} \frac{x_k}{m_k}, \quad x_k \in \mathbb{Z} \cap [0, p_k), \tag{1}$$

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where $\mathbb{Z}$ is the set of integers. Decomposition (1) is uniquely defined if for $x = n/m_k$ take a decomposition with a finite number of nonzero x_k. If $n \in \mathbb{Z}_+ := \{0, 1, 2, \dots\}$ is represented as

$$n = \sum_{j=1}^{\infty} \alpha_j m_{j-1}, \quad \alpha_j \in \mathbb{Z} \cap [0, p_j),$$

then for the numbers $x \in [0, 1)$ we put by definition

$$\psi_n(x) = exp\left(2\pi i \sum_{j=1}^{\infty} \frac{\alpha_j x_j}{p_j}\right), \quad n \in \mathbb{Z}_+.$$

It is known that the system $\{\psi_n\}_{n=0}^{\infty}$, called the Price system, is an orthonormalized system that is complete in $L^1(0, 1)$ (see [10] or [11]). If all $p_k = 2$, then $\{\psi_n\}_{n=0}^{\infty}$ coincides with Walsh system in the Paley numbering.

Let $L^p(G)$, $G := [0, 1)$, $1 \le p < \infty$, be a Lebesgue space with a norm

$$\|f\|_p = \left(\int_G |f(x)|^p dx\right)^{\frac{1}{p}}, \quad \|f\|_\infty = ess \sup_{x \in G} |f(x)|.$$

Definition 2. Let $\varphi(x)$ be a non-negative measurable function on $[1, \infty)$. We say that $\varphi(x)$ satisfies condition B_1, if for all $x \ge 1$

$$\int_x^{\infty} \frac{\varphi(t)}{t^2} dt \le C \frac{\varphi(x)}{x},$$

where C is a positive number independent of x.

For example, the function $\varphi(t) = t^\alpha$ $(\alpha < 1)$ satisfies condition B_1.

To prove the main results, we need the following auxiliary assertions.

Lemma A. (Potapov M.K. [12]). If $a_n,\ b_n \ge 0$ $(n = 1,\ 2,\ \dots)$, $1 \le p < \infty$ and $\sum_{m=n}^{\infty} b_m = \gamma_n b_n$, then

$$\sum_{m=1}^{\infty} b_m \left(\sum_{n=1}^{m} a_n\right)^p \le C_p \sum_{m=1}^{\infty} b_m \left(a_m \gamma_m\right)^p.$$

Lemma B. (Simonyan A.S. [13, 14]) Let $1 < p < \infty$, $\frac{1}{p} + \frac{1}{p'} = 1$, $f(x) \in L[0, 1]$, $f(x) \ge 0$ and function $[\varphi(x)]^{-p'}$ satisfies condition B_1,

$$F(x) = \int_0^x f(t) dt.$$

Then

$$\int_0^1 \varphi^p \left(\frac{1}{x}\right) \left(\frac{F(x)}{x}\right)^p dx \le C_p \int_0^1 \varphi^p \left(\frac{1}{x}\right) f^p(x)\, dx.$$

By

$$D_n(x) = \sum_{k=0}^{n-1} \psi_k(x), \quad n = 1, 2, ...,$$

denote the Dirichlet kernel of the system $\{\psi_n(x)\}$.

Lemma C. (see [10] or [11]) For any $k \in \mathbb{N}$ and $x \in [0,1)$ the Dirichlet kernels have the following properties:

$$D_{m_k} = \begin{cases} m_k, & \text{if } x \in \left[0, \frac{1}{m_k}\right), \\ 0, & \text{if } x \notin \left[0, \frac{1}{m_k}\right). \end{cases} \tag{2}$$

The Dirichlet kernel $D_n(x)$ satisfies the estimate

$$\frac{q(x)}{2} \leq \sup_{j \leq p_{n+1}} |D_{j_{m_k}}(x)| \leq 2q(x), \tag{3}$$

where $q(x)$ is the function introduced in [15]:

$$q(x) = \frac{m_n(x)}{\sin \frac{\pi l(x)}{p_{n(x)+1}}}, \quad x \in [0,1], \tag{4}$$

where $n(x)$ is the number of the last zero in the initial series of the decomposition of the element $x \in [0,1]$, $l(x)$ is the value of the first nonzero coordinate of this decomposition.

Lemma 1. Let $S_n(x) = \sum_{k=0}^{n-1} a_k \psi_k(x)$, $(n = 0,\ 1,\ 2, \ldots)$ $a_k \downarrow 0$ at $k \to \infty$. Then for any $x \in \left[\frac{1}{m_{\nu+1}},\ \frac{1}{m_\nu}\right]$,

$$|S_n(x)| \leq \sum_{k=0}^{m_\nu - 1} a_k + a_{m_\nu} \cdot m_{\nu+1}.$$

Proof of Lemma 1. Let $x \in \left[\frac{1}{m_{\nu+1}},\ \frac{1}{m_\nu}\right]$, $0 \leq \nu < \infty$. Considering, $|\psi_k(x)| = 1$, we have

$$|S_n(x)| \leq \sum_{k=0}^{\nu-1} a_k + \sum_{k=\nu}^{n-1} a_k \psi_k(x).$$

Applying the Abel transformation for the second sum, by inequality (3) we have

$$\left|\sum_{k=m_\nu}^{n-1} a_k \psi_k(x)\right| = \left|\sum_{k=m_\nu}^{n-2} \Delta a_k D_{k+1}(x) + a_{n-1} D_n(x) - a_{m_\nu} D_{m_\nu}(x)\right| \leq$$

$$\leq q(x) \left|\sum_{k=m_\nu}^{n-2} \Delta a_k + a_{n-1} + a_{m_\nu}\right| \leq 2 a_{m_\nu} q(x),$$

but for the function $q(x)$ at $x \in \left[\frac{1}{m_{\nu+1}},\ \frac{1}{m_\nu}\right]$, an estimate

$$q(x) \leq \frac{m_{\nu+1}}{2}$$

holds (see (4)).

Consequently,

$$|S_n(x)| \leq \sum_{k=0}^{m_\nu - 1} a_k + a_{m_\nu} \cdot m_{\nu+1}.$$

Lemma 1 is proved.

2 Main Results

The following theorems about integrability with weight of the series' sum with monotone coefficients are valid.

Theorem 1. Let $1 < p < \infty,\ \frac{1}{p} + \frac{1}{p'} = 1$

$$f(x) = \sum_{k=0}^{\infty} a_k \psi_k(x), \quad a_k \downarrow 0 \text{ at } k \to \infty$$

and let $\varphi(x)$ is a non-negative measurable function on $[1,\ \infty)$. Then

1^0. If $\varphi\left(\frac{1}{x}\right) \in L_p(0,\ 1)$ and

$$\sum_{n=1}^{\infty} \left(\sum_{k=0}^{m_n - 1} a_k + a_{m_n} m_{n+1} \right)^p \int_{1/m_{n+1}}^{1/m_n} \varphi^p \left(\frac{1}{x} \right) dx < \infty, \tag{5}$$

then $\varphi\left(\frac{1}{x}\right) f(x) \in L_p(0,\ 1)$.

2^0. If the function $\varphi^{-p'}(x)$ satisfies the condition B_1 and $\varphi\left(\frac{1}{x}\right) f(x) \in L_p(0,\ 1)$, $\sup\limits_n p_n = K < \infty$, then

$$\sum_{n=1}^{\infty} \left(\sum_{k=0}^{m_{n+1} - 1} a_k \right)^p \int_{1/m_{n+1}}^{1/m_n} \varphi^p \left(\frac{1}{x} \right) dx < \infty.$$

Theorem 2. Let $1 < p < \infty,\ \frac{1}{p} + \frac{1}{p'} = 1$

$$f(x) = \sum_{k=0}^{\infty} a_k \psi_k(x), \quad a_k \downarrow 0 \text{ at } k \to \infty$$

and let $\varphi(x)$ be a non-negative measurable function on $[1,\ \infty)$. Then

1^0. If function $\varphi^p(x)$ satisfies condition B_1 and

$$\sum_{n=0}^{\infty} a_{m_n}^p \cdot m_{n+1}^p \int_{1/m_{n+1}}^{1/m_n} \varphi^p \left(\frac{1}{x} \right) dx < \infty, \tag{6}$$

then $\varphi\left(\frac{1}{x}\right) f(x) \in L_p(0,\ 1)$.

2^0. If $\varphi^{-p'}(x)$ satisfies condition B_1 and $\varphi\left(\frac{1}{x}\right) f(x) \in L_p(0,\ 1)$, then

$$\sum_{n=0}^{\infty} a_{m_{n+1}}^p \cdot m_{n+1}^p \int_{1/m_{n+1}}^{1/m_n} \varphi^p \left(\frac{1}{x} \right) dx < \infty. \tag{7}$$

In the case $\sup p_n = k < \infty$ theorem 1 is equivalent to the following theorem:

Theorem 3. Let $1 < p < \infty,\ \frac{1}{p} + \frac{1}{p'} = 1,$

$$f(x) = \sum_{k=0}^{\infty} a_k \psi_k(x), \quad a_k \downarrow 0 \text{ at } k \to \infty$$

and let $\varphi(x)$ be a non-negative measurable function on $[1,\ \infty)$, $\sup\limits_n p_n = N < \infty$. Then

1^0. If the function $\varphi^p(x)$ satisfies condition B_1 and

$$\sum_{n=1}^{\infty} a_n^p \cdot n^p \int_n^{n+1} \frac{\varphi^p(x)}{x^2} dx < \infty,$$

then $\varphi\left(\frac{1}{x}\right) f(x) \in L_p(0,\ 1)$.

2^0. If $\varphi^{-p'}(x)$ it satisfies the condition B_1 and $\varphi\left(\frac{1}{x}\right) f(x) \in L_p(0,\ 1)$, then it takes place (6).

From this theorem in the case of the Walsh system follow the corresponding results of A.S. Simonyan [13].

Remark. If the weight function $\varphi(x)$ has the form $\varphi(x) = x^{\alpha}$, then in this case $\varphi^{p}(x)$ and $\varphi^{-p'}(x)$ satisfy condition B_1 at $-\frac{1}{p'} < \alpha < \frac{1}{p}$ and condition (7) has the form

$$\sum_{n=1}^{\infty} a_n^p \cdot n^{p(\alpha+1)-1} < \infty.$$

Proof of Theorem 1. 1^0. By Lemma 1 and condition (5) we have

$$\int_0^1 \varphi^p\left(\frac{1}{x}\right) |f(x)|^p\,dx = \sum_{n=0}^{\infty} \int_{1/m_{n+1}}^{1/m_n} \varphi^p\left(\frac{1}{x}\right) |f(x)|^p\,dx \leq$$

$$\leq \sum_{n=0}^{\infty} \left(\sum_{k=0}^{m_n-1} a_k + a_{m_n}\cdot m_{n+1}\right)^p \int_{1/m_{n+1}}^{1/m_n} \varphi^p\left(\frac{1}{x}\right) dx < \infty,$$

that is

$$f(x)\ \varphi\left(\frac{1}{x}\right) \in L_p[0,\ 1].$$

2^0. Let

$$\varphi\left(\frac{1}{x}\right) f(x) \in L_p(0,\ 1) \quad and \quad \varphi^{-p'}\left(\frac{1}{x}\right) \in L(0,\ 1).$$

By Gelder's inequality

$$\int_0^1 |f(x)|\ dx \leq \left(\int_0^1 \varphi^p\left(\frac{1}{x}\right) |f(x)|^p\,dx\right)^{1/p} \left(\int_0^1 \varphi^{-p'}\left(\frac{1}{x}\right) dx\right)^{1/p'} < \infty.$$

Consequently, $f(x) \in L(0,1)$ and $a_k = a_k(f)$.

Let $F(x) = \int_0^x |f(t)|\,dt$. By (2) from Lemma C we get

$$\sum_{k=0}^{m_n-1} a_k(f) = \sum_{k=0}^{m_n-1} \int_0^1 f(x)\overline{\psi_x(x)}dx = \int_0^1 f(x) \sum_{k=0}^{m_n-1} \overline{\psi_k(x)}dx =$$

$$= \int_0^1 f(x)\, D_{m_n}(x)\,dx = m_n \int_0^{1/m_n} f(x)\,dx \leq m_n F\left(\frac{1}{m_n}\right),$$

where $F(x) = \int_0^x |f(x)|\,dx$.

From here, using the monotonicity of the sequence $\{a_k\}$ and Lemma B, we obtain

$$\sum_{n=0}^{\infty} \left(\sum_{k=0}^{m_{n+1}-1} a_k\right)^p \int_{1/m_{n=1}}^{1/m_n} \varphi^p\left(\frac{1}{x}\right) dx \leq \sum_{n=0}^{\infty} \left[m_{n+1} F\left(\frac{1}{m_{n+1}}\right)\right]^p \int_{1/m_{n+1}}^{1/m_n} \varphi^p\left(\frac{1}{x}\right) dx \leq$$

$$\leq C_p \sum_{n=0}^{\infty} \int_{1/m_{n+1}}^{1/m_n} \varphi^p\left(\frac{1}{x}\right) \left(\frac{F(x)}{x}\right)^p dx \leq C_p \int_0^1 \varphi^p\left(\frac{1}{x}\right) |f(x)|^p\,dx < \infty.$$

Theorem 1 is proved.

Proof of Theorem 2. 1^0. We denote

$$b_n = \int_{m_n}^{m_{n+1}} \frac{\varphi^p(t)}{t^2} dt.$$

Then

$$\sum_{n=\nu}^{\infty} b_n = \left(\int_{m_\nu}^{\infty} \frac{\varphi^p(t)}{t^2} dt \right) \left(\int_{m_\nu}^{m_{\nu+1}} \frac{\varphi^p(t)}{t^2} dt \right)^{-1} b_\nu = \gamma_\nu \cdot b_\nu,$$

where

$$\gamma_\nu = \left(\int_{m_\nu}^{\infty} \frac{\varphi^p(t)}{t^2} dt \right) \cdot \left(\int_{m_\nu}^{m_{\nu+1}} \frac{\varphi^p(t)}{t^2} dt \right)^{-1}.$$

The function $\varphi^p(x)$ satisfies the condition B_1, therefore

$$\gamma_\nu = 1 + \left(\int_{m_\nu}^{m_{\nu+1}} \frac{\varphi^p(t)}{t^2} dt \right)^{-1} \cdot \int_{m_\nu}^{m_{\nu+1}} \left(\int_{x}^{\infty} \frac{\varphi^p(t)}{t^2} dt \right) dx \leq$$

$$\leq 1 + C_1 \left(\int_{m_\nu}^{m_{\nu+1}} \frac{\varphi^p(t)}{t^2} dt \right)^{-1} \cdot \int_{m_\nu}^{m_{\nu+1}} \frac{\varphi^p(x)}{x} dx \leq C_2.$$

Using the Lemma A we have

$$\sum_{n=1}^{\infty} b_n \left(\sum_{k=0}^{m_n-1} a_k \right)^p = \sum_{n=1}^{\infty} b_n \left[\sum_{k=0}^{n-1} \left(\sum_{j=m_k}^{m_{k+1}-1} a_j \right) \right]^p \leq C \sum_{n=1}^{\infty} a_{m_n}^p \cdot m_n^p \cdot b_n;$$

consequently,

$$\sum_{n=0}^{\infty} \left(\sum_{k=0}^{m_n-1} a_k + a_{m_n} \cdot m_{n+1} \right)^p \cdot b_n \leq C \sum_{n=0}^{\infty} a_{m_n}^p \cdot m_{n+1}^p \cdot b_n < \infty.$$

Hence, on the basis of the first point of Theorem 1 follows the first point of Theorem 2.

2^0. Due to the monotonicity of the sequence a_n

$$\sum_{n=1}^{\infty} \left(\sum_{k=0}^{m_{n+1}-1} a_n \right)^p \int_{m_n}^{m_{n+1}} \frac{\varphi^p(x)}{x^2} dx \geq \sum_{n=1}^{\infty} a_{m_{n+1}} \cdot m_{n+1}^p \int_{m_n}^{m_{n+1}} \frac{\varphi^p(x)}{x^2} dx.$$

Therefore, the statement of Theorem 2 follows from Theorem 1.

Theorem 2 is proved.

Proof of Theorem 3. Sufficiency. By the monotonicity of the sequence $\{a_n\}$ and by the condition $\sup p_n = N < \infty$, we have

$$\sum_{n=1}^{\infty} a_n^p n^p \int_{n}^{n+1} \frac{\varphi^p(x)}{x^2} dx = \sum_{n=0}^{\infty} \sum_{k=m_n}^{m_{n+1}-1} a_k^p k^p \int_{k}^{k+1} \frac{\varphi^p(x)}{x^2} dx \geq$$

$$\geq \sum_{n=0}^{\infty} a_{m_{n+1}}^p \cdot m_n^p \sum_{k=m_n}^{m_{n+1}-1} \int_{k}^{k+1} \frac{\varphi^p(x)}{x^2} dx \geq C_p \sum_{n=0}^{\infty} a_{m_{n+1}}^p \cdot m_{n+2}^p \int_{m_n}^{m_{n+1}} \frac{\varphi^p(x)}{x^2} dx \geq$$

$$\geq C_p \sum_{n=1}^{\infty} a_{m_n}^p \cdot m_{n+1}^p \int_{1/m_{n+1}}^{1/m_n} \varphi^p \left(\frac{1}{t} \right) dx.$$

Therefore, from the condition of point 1^0 of Theorem 3 it follows the condition of point 1^0 of the Theorem 2. Therefore $f(x)\, \varphi\left(\frac{1}{x}\right) \in L_p(0,\ 1)$.

On the other hand, also due to monotonicity of the sequence $\{a_n\}$ and boundedness of the sequence $\{p_n\}$, we have

$$\sum_{n=1}^{\infty} a_n^p n^p \int_{n}^{n+1} \frac{\varphi^p(x)}{x^2} dx = \sum_{n=0}^{\infty} \sum_{k=m_n}^{m_{n+1}-1} a_k^p k^p \int_{k}^{k+1} \frac{\varphi^p(x)}{x^2} dx \leq$$

$$\leq \sum_{n=0}^{\infty} a_{m_n}^p \cdot m_{n+1}^p \sum_{k=m_n}^{m_{n+1}-1} \int_k^{k+1} \frac{\varphi^p(x)}{x^2} dx = C_p \sum_{n=0}^{\infty} a_{m_{n+1}}^p \cdot m_{n+1}^p \int_{1/m_{n+1}}^{1/m_n} \varphi^p\left(\frac{1}{t}\right) dt.$$

Therefore, from the point 2^0 of Theorem 3 follows the condition of point 2^0 of Theorem 2. From this follows the necessity of the Theorem 3.

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Коэффициенттері монотонды мультипликативтік жүйелер бойынша қатарлардың қосындысының салмақты интегралдануы туралы

Жұмыста коэффициенттері монотонды мультипликативті жүйелер бойынша құрылған қатарлар қосындысының салмақты интегралдануы туралы сұрақтар қарастырылған. Осындай қатарлардың қосындысы салмақты Лебег кеңістігінде жататынын қамтамасыз ететін салмақты функцияларға шарттар алынған. Негізгі теоремалар жасаушы тізбегіне шенелгендік шарт қойылмағанда дәлелденеді; атап айтқанда, ол шенелмеген болуы мүмкін. Жасаушы тізбегі шенелгендігі жағдайында дәлелденген теоремалар монотонды коэффициенттері бар тригонометриялық қатарлар бойынша белгілі Харди-Литлвуд теоремасының аналогын білдіреді.

Кілт сөздер: мультипликативті жүйелер, жіктеу, қатарлар қосындысы, салмақты интегралдану, жасаушы тізбек, монотонды коэффициенттер, Харди-Литлвуд теоремасы, Лебег кеңістігі.

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Об интегрируемости с весом суммы рядов с монотонными коэффициентами по мультипликативным системам

В работе рассмотрены вопросы о весовой интегрируемости суммы рядов по мультипликативным системам с монотонными коэффициентами. Получены условия на весовые функции, обеспечивающие принадлежность суммы таких рядов весовому пространству Лебега. Основные теоремы доказаны без условия ограниченности образующей последовательности, в частности, она может быть неограниченной. В случае ограниченности образующей последовательности из доказанных теорем следует аналог известной теоремы Харди–Литлвуда о тригонометрических рядах с монотонными коэффициентами.

Ключевые слова: мультипликативные системы, разложение, весовая интегрируемость, сумма рядов, образующая последовательность, монотонные коэффициенты, теорема Харди-Литлвуда, пространство Лебега.

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On Robinson spectrum of the semantic Jonsson quasivariety of unars

Given article is devoted to the study of semantic Jonsson quasivariety of universal unars of signature containing only unary functional symbol. The first section of the article consists of basic necessary concepts. There were defined new notions of semantic Jonsson quasivariety of Robinson unars $J\mathbb{C}_U$, its elementary theory and semantic model. In order to prove the main result of the article, there were considered Robinson spectrum $RSp(J\mathbb{C}_U)$ and its partition onto equivalence classes $[\Delta]$ by cosemanticness relation. The characteristic features of such equivalence classes $[\Delta] \in RSp(J\mathbb{C}_U)$ were analysed. The main result is the following theorem of the existence of: characteristic for every class $[\Delta]$ the meaning of which is Robinson theories of unars; class $[\Delta]$ for any arbitrary characteristic; criteria of equivalence of two classes $[\Delta]_1, [\Delta]_2$. The obtained results can be useful for continuation of the various Jonsson algebras' research, particularly semantic Jonsson quasivariety of S-acts over cyclic monoid.

Keywords: Jonsson theory, unars, universal theory, Robinson theory, quasivariety, semantic Jonsson quasivariety, Jonsson spectrum, Robinson spectrum, equivalence class, cosemanticness.

Introduction

The study of model-theoretic relations of classical algebras and their syntactic properties from the Jonsson theories consideration, which are, generally speaking, incomplete, allows one to describe quite broad classes of theories. The article is a continuation of the work [1]. The authors of this article aimed at deepening of the universal unar's semantic model's characteristic study and strengthening the existing result by considering new and more general notion of semantic Jonsson quasivariety, and also by defining the notion of Robinson spectrum and its equivalence classes for unars.

The first section of the article gives the required notions of Jonsson theories, particularly Jonsson spectrum and its related notions. The second is devoted to the definitions connected with Jonsson universal unars and their semantic model's characteristic. The main section contains the definition of arbitrary characteristic and the main theorem on cosemanticness classes of factor-set $RSp(J\mathbb{C})_{/\bowtie}$, obtained during research conduction. All necessary base definitions can be found in [2], definitions and notions concerning Jonsson theories in [3–18].

All definitions that were not given in the current article can be extracted from [3].

1 Semantic Jonsson quasivariety

One of the important definitions, used by the authors of given article, is the definition of Jonsson theory. Let us recall the conditions, that should be satisfied in order for a theory to be Jonsson.

Definition 1. [3; 80] A theory T is said to be Jonsson, if:
1) T has at least one infinite model;
2) T is $\forall\exists$-axiomatising;
3) T has JEP property;
4) T has AP property.

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$\forall$-axiomatizing Jonsson theory is called the Robinson theory.

Let us recall some necessary notions from Jonsson model theory.

Theorem 1. [3; 155] T is Jonsson iff it has a semantic model $\mathfrak{C}_T$.

The definition of Jonsson theory's semantic model.

Definition 2. [3; 155] Let T be a Jonsson theory. A model $\mathfrak{C}_T$ of power $2^{|T|}$ is called to be a semantic model of the theory T if $\mathfrak{C}_T$ is a $|T|^+$-homogeneous $|T|^+$-universal model of the theory T.

The next definition was introduced by T.G. Mustafin.

Definition 3. [3; 161] The elementary theory of a semantic model of the Jonsson theory T is called the center of this theory. The center is denoted by T^*, i.e. $Th(C) = T^*$.

Since the current research is connected with consideration of Robinson spectrum for classes of algebras, let us give the following conditions of Jonsson theories' cosemanticness.

Definition 4. [3; 40] Let T_1 and T_2 be Jonsson theories, T_1^* and T_2^* be their centres, respectively. T_1 and T_2 are said to be cosemantic Jonsson theories (denoted by $T_1 \bowtie T_2$), if $T_1^* = T_2^*$.

Theorem 2. [3; 176] Let T_1 and T_2 be Jonsson theories, $\mathfrak{C}_{T_1}$ and $\mathfrak{C}_{T_2}$ be their semantic models, respectively. Then the next conditions are equivalent:
1) $\mathfrak{C}_{T_1} \bowtie \mathfrak{C}_{T_2}$;
2) $\mathfrak{C}_{T_1} \equiv_J \mathfrak{C}_{T_2}$;
3) $\mathfrak{C}_{T_1} = \mathfrak{C}_{T_2}$.

Let K be a class of models of fixed signature σ. Then we can consider Jonsson spectrum for K, which can be defined as follows.

Definition 5. [5] A set $JSp(K)$ of Jonsson theories of signature σ, where

$$JSp(K) = \{T \mid T \text{ is Jonsson theory and } K \subseteq Mod(T)\}$$

is called the Jonsson spectrum for class K.

Hence, in the particular case, when the Jonsson theory is $\forall$-axiomatising we get the concept of the Robinson theory, respectively, the notion of the Jonsson spectrum allows us to consider the Robinson spectrum.

Definition 6. A set $RSp(K)$ of Robinson theories of signature σ, where

$$RSp(K) = \{T \mid T \text{ is Robinson theory and } \forall A \in K,\ A \models T\}$$

is called the Robinson spectrum for class K.

Definition 4 states, that two Jonsson theories are cosemantic ($T_1 \bowtie T_2$), if their centres are equal. It is easy to check, that such cosematicness relation, given on a set of Jonsson theories, will be an equivalence relation. The proof of this fact one can find in detail in [4]. Hence, based on theorem 2, we can consider the cosemanticity relation on Jonsson spectrum $JSp(K)$ and obtain a partition of $JSp(K)$ onto equivalence classes. We get a factor-set, denoted as $JSp(K)_{/\bowtie}$. The factor-set $RSp(K)_{/\bowtie}$ will be obtained correspondingly.

According to A.I. Malcev [2], quasivarieties of algebras are the classes of algebras, that can be set by means of collection of quasi-identities (conditional identities). Quasi-identities are $\forall$-formulas, and quasivarieties are presented as particular types of universally axiomatising classes of algebras. A class $\mathfrak{R}$ of algebraic system is called a quasivariety if there is such collection of quasi-identities of signature σ that this algebraic system consists of those and only those systems of signature σ, in which all formulas from σ are true [2].

We want to define semantic Jonsson quasivariety as follows. Let K be a class of quasivariety in the sense of [2] of first-order language L, $L_0 \subset L$, where L_0 is the set of sentences of language L. Let us consider the elementary theory $Th(K)$ of such class K. By adding to $Th(K)$ $\forall\exists$ sentences of language L, that are not contained in the $Th(K)$, we can consider the set of Jonsson theories $J(Th(K))$ defined as follows.

Denotation 1. A set $J(Th(K)) = \{\Delta \mid \Delta-\text{Jonsson theory},\ \Delta = Th(K) \cup \{\varphi^i\}\}$, where $\varphi^i \in \forall\exists(L_0)$ and $\varphi^i \notin Th(K)$ for some $i \in \{0, 1\}$, $Th(K)$ is elementary theory of class of quasivariety K, $\forall\exists(L_0)$ is a set of all $\forall\exists$ sentences of language L.

According to theorem 1 the theory is Jonsson iff it has a semantic model. Hence every Jonsson theory $\Delta \in J(Th(K))$ has its own semantic model $\mathfrak{C}_\Delta$. Let us consider the set of such semantic models and denote it as $J\mathbb{C}$.

Denotation 2. A set $J\mathbb{C} = \{\mathfrak{C}_\Delta \mid \Delta \in J(Th(K)), \mathfrak{C}_\Delta \text{ is semantic model of } \Delta\}$.

We will call the set $J\mathbb{C}$ semantic Jonsson quasivariety of class K if its elementary theory $Th(J\mathbb{C})$ is Jonsson theory.

2 Robinson spectrum of semantic Jonsson quasivariety of Robinson unars

We will consider some basic definitions, denotations, properties of arbitrary Jonsson universals, necessary for proofing the main result of the article.

Denotation 3. [1] 1) If Γ is collection or type of the sentences, then T_Γ is following set of formulas $\{\psi \in T : \{\varphi \in \Gamma : T \vdash \varphi\} \vdash \psi\}$;
2) ∇ is $\Pi_1 \cup \Sigma_1$, that is ∇ is a collection of all universal and existential formulas.

Here, in the second item, Π_1 denotes universal formulas, Σ_1 denotes existential ones.

Definition 7. [1] 1) If $T = T_\forall$, then $T_\forall$ is said to be universal;
2) If $T = T_\nabla$, then the theory T is called primitive.

Thus, by the universal we call a set of all universal conclusions of Jonsson theory T. The next proposition plays an important role in the proof of the obtained main theorem of the article.

Proposition 1. [1] Let T_1, T_2 be Jonsson universals. Then the following conditions are equivalent:
1) $T_1 = T_2$;
2) $\mathfrak{C}_{T_1} \simeq \mathfrak{C}_{T_2}$;
3) $T_1^* = T_2^*$.

$\mathfrak{C}_{T_1}$ and $\mathfrak{C}_{T_2}$ are semantic models of Jonsson theories T_1, T_2 respectively. Each model U of Jonsson theory of unars T is an unar. Consequently, the following fact is true.

Lemma 1. [1] For any unar U the following is satisfied

$$U \models T \Leftrightarrow U \text{ embeds in } \mathfrak{C}.$$

The following definitions are necessary for the construction of semantic model of cosemanticness classes of Robinson spectrum for semantic Jonsson quasivariety of Robinson unars.

Definition 8. [1] 1) If $A \subseteq \mathfrak{C}, a \in \mathfrak{C}$, then $[A, a]$ denotes sub-unar, generated by subset $A \cup \{a\}$.
2) We will write $tp_{at}^{\mathfrak{C}}(a, A) = tp_{at}^{\mathfrak{C}}(b, A)$ if there is such isomorphism $\varphi : [A, a] \simeq [A, b]$, that $\varphi(c) = c, \forall c \in A$, and $\varphi(a) = b$.

Definition 9. [1] 1) If H is sub-unar $\mathfrak{C}, f^n(a) = h \in H, f^k(a) \notin H$ for all $k < n$, then the element h will be called input element from a in H, and number n will be called the distance from a to H. In this case we will use denotation $h = \text{input}(a, H), n = \rho(a, H)$. We will write $\rho(a, H) = \infty$, if $f^n(a) \notin H, \forall n < \omega$.

2) $\chi(a) = \begin{cases} \omega, \text{if } f^n(a) \neq f^k(a), \forall n < k < \omega \\ < n, m >, \text{if } < n, m >= min\{< n, m >: f^n(a) = f^{n+m}(a)\}. \end{cases}$

Definition 10. [1] If $a \in \mathfrak{C}$, then

$$k(a) \ \ |\{b \in \mathfrak{C} : f(b) = a\}|.$$

Definition 11. [1] A set $\{a_1, ..., a_m\}$ of elements $\mathfrak{C}$ will be called m-loop, if $a_i \neq a_j, f(a_i) = a_{i+1}$ for all $1 \leq i < j \leq m$ and $f(a_m) = a_1$.

The next definition determines the characteristic of semantic model of Robinson unar's Jonsson theory.

Definition 12. [1] A fourset $(\Omega, \nu, \mu, \varepsilon)$ will be called a characteristic $\mathfrak{C}$ and denoted as $char(\mathfrak{C})$, if

$\Omega = \{\chi(a) : a \in \mathfrak{C}\}$,

$\nu \ \ \omega \setminus \{0\} \to \omega \cup \{\infty\}$ such that $\forall m > 0$,

$\nu(m) = \begin{cases} k, \text{if the quantity } m - \text{loops in } \mathfrak{C} \text{ is equal to } k < \omega, \\ \infty, \text{otherwise;} \end{cases}$

$\mu : \Omega \to \omega \cup \{\infty\}$ such that if $\alpha \in \Omega$ and $\alpha \in \chi(a)$, then $\mu(\alpha) = k(a)$, if $k(a) < \omega$ and $\mu(\alpha) = \infty$, if $k(a) = |\mathfrak{C}|$;

$\varepsilon = \begin{cases} 0, \text{if } |\{a \in \mathfrak{C} : \chi(a) = \omega\}| = 0, \\ \infty, \text{otherwise.} \end{cases}$

The next lemma gives some useful specification to the definition of above-mentioned fourset.

Lemma 2. [1] If $char(\mathfrak{C}) = (\Omega, \nu, \mu, \varepsilon)$, then

1°. $\varnothing \neq \Omega \subseteq \{\omega\} \cup (\omega \times \omega)$;

2°. $(n, m) \in \Omega \& 0 \leq k < n \Rightarrow (k, m) \in \Omega$;

3°. $\nu(m) > 0 \Leftrightarrow (0, m) \in \Omega$;

4°. $\omega \in \Omega \Leftrightarrow \varepsilon = \infty$;

5°. $|\Omega| = \omega \Rightarrow \omega \in \Omega$;

6°. $(n, m) \in \Omega \Rightarrow ((n + 1, m) \notin \Omega \Leftrightarrow \mu^*(n, m) = 0$;

7°. $\omega \notin \Omega \& |\Omega| < \omega \Rightarrow \exists m < \omega(\nu(m) = \infty) \vee \exists n < \omega, m < \omega((n, m) \in \Omega \& \mu(n, m) = \infty)$;

8°. $|\Omega| = \omega \Rightarrow \begin{cases} \mu(\omega) \geq k, \text{if } \exists k, l < \omega(k = max\{\mu(n, m) \in \Omega, n + m \geq l\}); \\ \mu(\omega) = \infty, \ \text{otherwise.} \end{cases}$

3 Main result

The theory $Th_{\forall}(U)$ of all universal sentences, true in U is the Jonsson theory. This statement was proven in the work [1]. By virtue of $\forall$-axiomatisability of elemantary theory of unars, $Th_{\forall}(U)$ is the Robinson theory of unars.

Thus, we use the denotation 2 of semantic Jonsson quasivariety of class K and consider a set $J\mathbb{C}_U = \{\mathfrak{C}_\Delta \mid \Delta \in J(Th(K)), \mathfrak{C}_\Delta$ is a semantic model $\Delta\}$ of signature $\sigma_U =< f >$, where Δ is a Robinson theory of unars, f is unary functional symbol. Such $J\mathbb{C}_U$ defines semantic Jonsson quasivariety of Robinson unars.

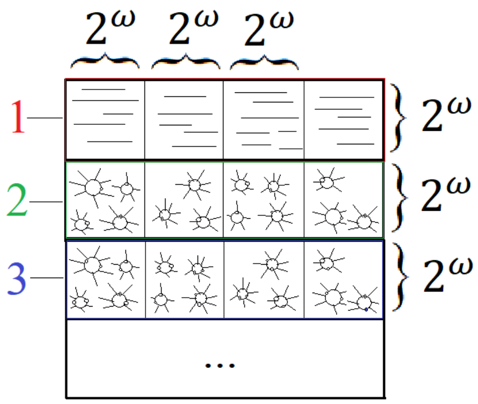

Figure 1. Semantic Jonsson quasivariety of Robinson unars $J\mathbb{C}_U$

We can see on the figure 1, that 1, 2, 3 are $\mathfrak{C}_{\Delta_1}$, $\mathfrak{C}_{\Delta_2}$, $\mathfrak{C}_{\Delta_3}$, which are semantic models of $[\Delta_1]$, $[\Delta_2]$, $[\Delta_3]$ respectively. The semantic models consist of unars of length 0, 1, 2 and so on.

Let us define the Robinson spectrum of the set $J\mathbb{C}_U$ as follows.

Definition 13. A set $RSp(J\mathbb{C}_U)$ of Robinson theories of signature σ_U, where

$$RSp(J\mathbb{C}_U) = \{\Delta \mid \Delta \text{ is Robinson theory of unars and } \forall \mathfrak{C}_\Delta \in J\mathbb{C}_U, \mathfrak{C}_\Delta \models \Delta\}$$

is called the Robinson spectrum for class $J\mathbb{C}_U$, where $J\mathbb{C}_U$ is semantic Jonsson quasivariety of Robinson unars.

Further we can consider the notion of cosemanticness relation on Robinson spectrum $RSp(J\mathbb{C}_U)$ and get the partition $RSp(J\mathbb{C}_U)$ on equivalence classes. As a result we obtain a factor-set, denoted as $RSp(J\mathbb{C}_U)_{/\bowtie}$ and consisted of equivalence classes parted by cosemanticness relation $[\Delta] \in RSp(J\mathbb{C}_U)_{/\bowtie}$.

Remark 1. Everywhere in this section $[\Delta]$ denotes an equivalence class of Robinson theories of unars parted by cosemanticness relation on Robinson spectrum $RSp(J\mathbb{C}_U)$, $\mathfrak{C}_{[\Delta]}$ denotes this class's semantic model.

According to theorem 2 and definition 6 we can deduce the conclusion, that $char(\mathfrak{C}_{[\Delta]})$ defines similarly to $char(\mathfrak{C})$ from definition 12 for every semantic model $\mathfrak{C}_{[\Delta]}$ of every class $[\Delta]$.

Definition 14. As a characteristic $Char([\Delta])$ we will understand $Char(\mathfrak{C}_{[\Delta]})$.

Lemma 3. For classes $[\Delta_1], [\Delta_2]$ of Robinson theories of unars the following conditions are equivalent:
1) $[\Delta_1]$ is equivalent to $[\Delta_2]$;
2) $Char([\Delta_1]) = Char([\Delta_2])$.

Proof. 1) $\Rightarrow$ 2) According to the theorem 2 if two classes are equivalent, then their semantic models will be equal to each other. Therefore, the characteristics of those models will also be equal to each other.

2) $\Rightarrow$ 1) follows from the fact that $Char(\mathfrak{C}_{[\Delta]})$ defines the semantic models $\mathfrak{C}_{[\Delta]}$ up to isomorphism. Hence $\mathfrak{C}_{[\Delta]_1} \simeq \mathfrak{C}_{[\Delta]_2}$, according to proposition 1.

Definition 15. [1] An arbitrary fourset $(\Omega^{**}, \nu^{**}, \mu^{**}, \varepsilon^{**})$ will be called a characteristic if the following conditions are satisfied:
1) $\varnothing \neq \Omega^{**} \subseteq \{\omega\} \cup (\omega \times \omega)$;
2) $\nu^{**} : \omega \setminus \{0\} \to \omega \cup \{\infty\}$;
3) $\mu^{**} : \Omega^{**} \to \omega \cup \{\infty\}$;
4) $\varepsilon^{**} = 0$ or $\varepsilon^{**} = \infty$;

5) $\omega \in \Omega^{**} \Leftrightarrow \varepsilon = \infty$;
6) $(n, m) \in \Omega^{**} \cap (\omega \times \omega) \& 0 \leq k < n \Rightarrow (k, m) \in \Omega^{**}$;
7) $\nu^{**}(m) > 0 \Leftrightarrow (0, m) \in \Omega{**}$;
8) $|\Omega^{**}| = \omega \Rightarrow \omega \in \Omega^{**}$;
9) $(n, m) \in \Omega^{**} \cap (\omega \times \omega) \Rightarrow (\mu(n, m) = 0 \Leftrightarrow (n + 1, m) \neq \Omega^{**})$;
10) $\omega \notin \Omega^{**} \& |\Omega^{**}| < \omega \Rightarrow \exists m < \omega(\nu(m) = \infty) \vee \exists n < \omega, m < \omega \ \ ((n, m) \in \Omega^{**} \& \mu(n, m) = \infty)$;
11) $|\Omega^{**}| = \omega \Rightarrow \begin{cases} \mu(\omega) \geq k, \text{if } \ \exists k, l < \omega(k = max\{\mu(n, m) : (n, m) \in \Omega^{**}, n + m \geq l\}); \\ \mu(\omega) = \infty, \text{otherwise}; \end{cases}$
12) $\mu^{**}(\omega) > 0$.

Lemma 4. $Char(\mathfrak{C}_{[\Delta]})$ is characteristic.

Proof. Follows immediately from lemma 2.

Theorem 3 (Main theorem). 1) Every class $[\Delta]$ has a characteristic.
2) For any characteristic π there is a class $[\Delta]$, that has characteristic π.
3) Two classes $[\Delta_1], [\Delta_2]$ are equivalent iff their characteristics are equal.

Proof. Items 1) and 3) are proven in lemmas 2 and 3 respectively.

2) Let us consider given arbitrary characteristic $\pi = (\Omega^{**}, \nu^{**}, \mu^{**}, \varepsilon^{**})$. We need to define class $[\Delta]_\pi$, that is the equivalence class of Robinson theories of unars parted by cosemanticness relation on Robinson spectrum $RSp(J\mathbb{C}_U)$ of characteristic π. Let us start from the denotation of collection of universal sentences of unars' language

$Q_{k,n,m} = \forall x(f^n(x) = f^{n+m}(x) \wedge (\&_{0 \leq i < j < n+m} f^i(x) \neq f^j(x)) \to \forall y_1, ..., y_{k+1}(\wedge_{i=1}^{k+1} f(y_i) = (x) \to \&_{1 \leq i < j \leq k+1} y_i = y_j)$.

$Q_{k,n,m}$ expresses "$\chi(x) = (n, m) \Rightarrow k(x) \leq k$".

$P_{l,m}$ is $\forall x_1, ..., x_{l+1}([\wedge_{i+1}^{l+1}(f^m(x_i) = x_i \wedge \wedge_{m-1}^{j=1} f^j(x_i)) \neq x_i)] \to \&_{1 \leq i < j < l+1_{0 \leq k, n \leq m-1}} f^k(x_i) = f^m(x_j))$.

$P_{l,m}$ states that the quantity of m-loops is no more then l.

R_m is $\forall x \neg(x = f^m(x) \wedge \wedge_{i=1}^{m-1} x \neq f^i(x))$.

R_m expresses the absence of m-loops.

Φ_m is $\forall x(f^m(x) \neq x)$. No comments needed here.

F_r is $\forall x \forall y_1, ..., y_{r+1}(\wedge_{i=1}^{r+1} f(y_i) = x \to \vee_{1 \leq i < j \leq r+1} y_i = y_j)$.

$F_r \Leftrightarrow \forall \alpha \in \Omega^{**}(\mu^{**}(\alpha) \leq r) \Leftrightarrow \forall x(k(x) \leq r)$.

$E_{r,m}$ is $\forall x(\wedge_{0 \leq i < j \leq m} f^i(x) \neq f^j(x) \to \forall y_1, ..., y_{r+1}(\wedge_{i=1}^{r+1} f(y_i) = x \to \vee_{1 \leq i < j \leq r+1} y_i = y_j)$.

$E_{m,r}$ states that if x is not an elemnt of s-loop for all $s \leq m$, then $K(x) \leq r$.

If $|\Omega^{**}| < \omega$ and $\omega \notin \Omega^{**}$, then D_Ω^{**} is $\forall x \vee_{(n,m) \in \Omega^{**}} (\vee_{0 \leq i < j \leq n+m-1} f^i(x) \neq f^j(x) \wedge f^n(x) = f^{n+m}(x))$.

In this case $D_\Omega^{**} \Leftrightarrow \forall x(\chi(x) \in \Omega^{**})$.

Let us move on to definition of $[\Delta]_\pi$.

Case 1. $\varepsilon^{**} = \infty$.

By the condition 5) of definition 15 it is equivalent to $\omega \in \Omega^{**}$. By the condition 12) $\mu^{**}(\omega) > 0$.

Case 1.1. $\Omega^{**} \setminus \{\omega\} \neq \varnothing$.

Case 1.1.1. $\mu^{**}(\omega) = \infty$.

Let $\theta_{\Omega^{**}, \nu^{**}, \mu^{**}}$ be $\{Q_{k,n,m} : (n, m) \in \Omega^{**} \setminus \{\omega\}, k = \mu^{**}(n, m)\} \cup \{P_{l,m} : 0 < m < \omega, 1 \leq l = \nu^{**}(m) < \omega\} \cup \{R_m : 0 < m < \omega, \nu^{**}(m) = 0\}$.

We suppose $[\Delta]_\pi = \theta_{\Omega^{**}, \nu^{**}, \mu^{**}}$

Case 1.1.2. $\mu^{**}(\omega) = r < \omega$.

Let $[\Delta]_\pi = \theta_{\Omega^{**}, \nu^{**}, \mu^{**}} \cup \{F_r\}$.

Case 1.2. $\Omega^{**} = \{\omega\}$.

Case 1.2.1. $\mu^{**}(\omega) = \infty$

By definition $[\Delta]_\pi = \{\Phi_m : 0 < m < \omega\}$.

Case 1.2.2. $\mu^{**}(\omega) = r$.

By definition $[\Delta]_\pi = \{\Phi_m : 0 < m < \omega\} \cup \{F_r\}$.

Case 2. $\varepsilon^{**} = 0$.

Note, that in this case by conditions 5) and 8) $\omega \notin \Omega^{**}$ and $|\Omega^{**}| < \omega$. Let us suppose $[\Delta]_\pi = \{Q_{k,n,m} : (n,m) \in \Omega^{**}, k = \mu^{**}(n,m)\} \cup \{P_{l,m} : 0 < m < \omega, 1 \leq l = \nu^{**}(m) < \omega\} \cup \{D^{**}_\Omega\}$. It is not hard to check, that in every case $[\Delta]_\pi$ is the equivalence class of Robinson theories of unars parted by cosemanticness relation on Robinson spectrum $RSp(J\mathbb{C}_U)$ and $Char(\mathfrak{C}_{[\Delta]_\pi}) = \pi$. The theorem is proven.

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Унарлардың семантикалық йонсондық квазикөптүрліліктерінің робинсондық спектрі

Мақала сигнатурасы тек бір орынды функционалдық символдан тұратын, универсалды унарлардың семантикалық йонсондық квазикөптүрліліктерін зерттеуге арналған. Мақаланың бірінші бөлімі негізгі қажетті ұғымдардан тұрады. Сонымен қатар $J\mathbb{C}_U$ робинсондық унарлардың семантикалық йонсондық квазикөптүрліліктерінің, оның элементарлы теориясы мен семантикалық моделінің жаңа түсініктері анықталды. Мақаланың негізгі нәтижесін дәлелдеу үшін $RSp(J\mathbb{C}_U)$ робинсондық спектр және оның косемантты қатынас арқылы $[\Delta]$ эквиваленттік кластарға бөлінуі қарастырылған. Мұндай $[\Delta] \in RSp(J\mathbb{C}_U)$ эквиваленттік кластардың сипаттамалық ерекшеліктері талданған. Мәні унарлардың робинсондық теориялары болатын әрбір $[\Delta]$ үшін кездейсоқ сипаттаманың; кез келген кездейсоқ сипаттама үшін $[\Delta]$ класының; екі $[\Delta]_1, [\Delta]_2$ кластарының эквиваленттілік критерийінің бар болу теоремасы негізгі нәтиже болып табылады. Алынған нәтижелер әртүрлі йонсондық алгебраларды, атап айтқанда, циклді моноид арқылы анықталған полигондардың семантикалық йонсондық квазикөптүрліліктерді зерттеуді жалғастыру үшін пайдалы болуы мүмкін.

Кілт сөздер: йонсондық теория, унарлар, универсалды теория, робинсондық теория, квазикөптүрлілік, семантикалық йонсондық квазикөптүрлілік, йонсондық спектр, робинсондық спектр, эквиваленттік класс, косеманттылық.

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Робинсоновский спектр семантического йонсоновского квазимногообразия унаров

Статья посвящена изучению семантического йонсоновского квазимногообразия универсальных унаров сигнатуры, содержащей единственный функциональный символ. Первый раздел статьи состоит из базовых необходимых понятий. Были определены новые понятия семантического йонсоновского квазимногообразия робинсоновских унаров $J\mathbb{C}_U$, его элементарной теории и семантической модели. Для того чтобы доказать главный результат статьи, были рассмотрены робинсоновский спектр $RSp(J\mathbb{C}_U)$ и его разбиение на классы эквивалентности $[\Delta]$ с помощью отношения косемантичности. Проанализированы характерные особенности таких классов эквивалентностей $[\Delta] \in RSp(J\mathbb{C}_U)$. Основным результатом является следующая теорема о существовании: произвольной характеристики для каждого $[\Delta]$, значение которого-робинсоновские теории унаров; класс $[\Delta]$ для любой произвольной характеристики; критерий эквивалентности классов $[\Delta]_1, [\Delta]_2$. Полученные результаты могут быть полезны в продолжении исследования различных йонсоновских алгебр, в частности, семантического йонсоновского квазимногообразия полигонов над циклическим моноидом.

Ключевые слова: йонсоновская теория, унары, универсальная теория, робинсоновская теория, квазимногообразие, семантическое йонсоновское квазимногообразие, йонсоновский спектр, робинсоновский спектр, класс эквивалентности, косемантичность.

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IN MEMORIAM OF SCIENTIST

In memory of the talented mathematician, our comrade and friend Krasnov Yakov Alexandrovich (1951–2023)

Krasnov Yakov Alexandrovich was born on March 28, 1951 in Aktyubinsk town, Kazakh SSR, in a family of doctors – Alexander Grigoryevich (subsequently, a personal pensioner of republican significance) and Lyudmila Samoilovna Krasnova. In 1968 Yakov graduated secondary school No. 2 in Aktyubinsk and in the same year entered the Mechanics and Mathematics Faculty, Mathematics department of the Moscow State University named after M.V. Lomonosov, which graduated in 1973.

In 1973 he started to work as lecturer of the Math department at the Ust-Kamenegorsk Road-Construction Institute. In 1975 he became postgraduate student of the Institute of Mathematics and Mechanics of the Academy of Sciences of the Kazakh SSR. In 1978 he successfully completed his postgraduate studies and in the same year, under the supervising of a Corresponding Member of the Academy of Sciences E.I. Kim he successfully defended his Candidate thesis of the Physics and Mathematics Sciences "Solution of the problems of the theory of heat conduction and potential with nonlinear boundary conditions of a special type and their applications" on the specialization 01.01.02 "Differential equations, dynamical systems and optimal control".

After defending his dissertation, he continued research as a Researcher at the (headed by E.I. Kim) laboratory of Equations of Mathematical Physics of the Institute of Mathematics and Mechanics (IMM). In 1981, he was certified as a Senior Researcher. In this time, he was already the author of 15 published scientific papers.

During his work at the IMM of the Academy of Sciences of the Kazakh SSR, Ya.A. Krasnov carried out a number of important scientific researches in the field of the qualitative theory of partial differential equations, which have practical application in the construction of electrical apparatus and in other areas of science and technology. In particular, he obtained solutions to a number of nonlinear boundary problems of the theory of heat conduction and potential, in a form suitable for engineering calculations.

Under his and E.I. Kim's supervising R.N. Kantaeva prepared and succesfully defended the PhD dissertation "Potential Method in Boundary Value Problems with a Moving Boundary for a System of Equations of a Parabolic Type" (01.01.02).

In addition to scientific work, Ya.A. Krasnov also performed various complimentary assignments and duties: he was the editor of the institute's wall newspaper, worked at the Small Academy of School pupils, took part in the design of thematic stands, posters.

In the collective of the Institute, Yakov Alexandrovich was distinguished by a cheerful and responsive character, and he was a respected person.

In 1991, the family of Yakov Aleksandrovich together with his parents went to Israel and settled in Tel Aviv, in the Ramat Gan region. In Israel, he was hired in a renowned university Bar Ilana to the position of a researcher at the Department of Mathematics, where he continued his scientific activities, and also taught the courses:

– Numerical analysis,
– Calculus of variations,
– Ordinary differential equations.

The themes of his scientific works were:

Elements of a spectral theory in non associative algebras,
Application of the stability theory to homogeneous of ODEs,
The operator analytic functions theory,
Symmetries of the Dirac equation,
Numerical methods for free boundary value problem,
Theory of non-conformal finite element method preserving harmonic moments,
Geometrically optimal space-time motion algorithms.

Unfortunately, in recent years, he was diagnosed with heart problems, which eventually led to his untimely death in May 2023.

Numerous friends and colleagues who worked with him remember him as a talented scientist, a respected and responsible emploeee, a kind and sympathetic person, and a wonderful family man.

Yakov Aleksandrovich Krasnov is the author and co-author of numerous scientific articles, the most significant of which are listed below:

1 The solution of nonlinear moving boundary value problems in the theory of heat conduction and potential and their applications. Ph.D. thesis, 1978.

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