ISSN 2518-7929 (Print)
ISSN 2663-5011 (Online)

BULLETIN

OF THE KARAGANDA UNIVERSITY

MATHEMATICS

Series

№ 1(105)/2022

ISSN 2518-7929 (Print)
ISSN 2663–5011(Online)
Индексі 74618
Индекс 74618

ҚАРАҒАНДЫ УНИВЕРСИТЕТІНІҢ **ХАБАРШЫСЫ**

ВЕСТНИК КАРАГАНДИНСКОГО УНИВЕРСИТЕТА

BULLETIN OF THE KARAGANDA UNIVERSITY

МАТЕМАТИКА сериясы

Серия МАТЕМАТИКА

MATHEMATICS Series

№ 1(105)/2022

Қаңтар–ақпан–наурыз
30 наурыз 2022 ж.

Январь–февраль–март
30 марта 2022 г.

January–February–March
March 30th, 2022

1996 жылдан бастап шығады
Издается с 1996 года
Founded in 1996

Жылына 4 рет шығады
Выходит 4 раза в год
Published 4 times a year

Қарағанды, 2022
Караганда, 2022
Karaganda 2022

Postal address: 28, University Str., Karaganda, 100024, Kazakhstan
Tel.: (7212) 77-04-38 (add. 1026); fax: (7212) 35-63-98.
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Editors
Zh.T. Nurmukhanova, S.S. Balkeyeva, Z.E. Ramazanova
Computer layout
G.K. Kalel

Bulletin of the Karaganda University. Mathematics series.
ISSN 2518-7929 (Print). ISSN 2663–5011 (Online).
Proprietary: NLC «Karagandy University of the name of academician E.A. Buketov».
Registered by the Ministry of Information and Social Development of the Republic of Kazakhstan.
Rediscount certificate No. KZ43VPY00027385 dated 30.09.2020.

Signed in print 29.03.2022. Format 60×84 1/8. Offset paper. Volume 19,9 p.sh. Circulation 200 copies.
Price upon request. Order № 11.
Printed in the Publishing house of NLC «Karagandy University of the name of acad. E.A. Buketov».
28, University Str., Karaganda, 100024, Kazakhstan. Tel.: (7212) 35-63-16. E-mail: izd_kargu@mail.ru

CONTENT

MATHEMATICS

ANNIVERSARY

DOI 10.31489/2022M1/4-12

UDC 517.9

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Solvability of an initial-boundary value problem for a nonlinear pseudoparabolic equation with degeneration

This article is devoted to the solvability of degenerate nonlinear equations of pseudoparabolic type. Such problems appear naturally in physical and biological models. The article aims to study the solvability in the classes of regular solutions of (all derivatives generalized in the sense of S.L. Sobolev included in the equation) initial-boundary value problems for differential equations. For the problems under consideration, We have found conditions on parameters ensuring the existence of solutions and we have proved existence and uniqueness theorems. The main method for proving the solvability of boundary value problems is the regularization method.

Keywords: pseudo parabolic equations, degenerate equations, boundary value problems, nonlinear equations, solvability, uniqueness.

Introduction

In the modern theory of partial differential equations, an important place is occupied by the study of degenerate hyperbolic and elliptic equations, as well as equations of mixed type. The increased interest in this class of equations is explained both by the great theoretical significance of the obtained results and by their numerous applications in gas dynamics, hydrodynamics, in the theory of infinitesimal bending of the surface, in the momentless theory of shells, in various branches of mechanics of continuous media, acoustics, and in the theory of electron scattering and many other areas of expertise. Degenerate equations are a good model for physical and biological processes. Such equations have become an actual formulation and solution of various boundary value problems. Consequently, degenerate equations are currently the subject of fundamental research by many mathematicians.

Boundary value problems for pseudo parabolic equations were investigated in the works of D. Colton [1], A.M. Nakhushev [2], A.I. Kozhanov [3], M.S. Salakhitdinov [4], T.D. Dzhuraev [5], and others.

One of the important sections of the theory of partial differential equations is the formulation and study of well-posed boundary value problems for degenerate parabolic equations of the second, third, and higher orders.

Boundary value problems for second-order degenerate parabolic equations are considered in the works of M. Gevrey [6], O.A. Ladyzhenskaya, V.A. Solonnikov, N.N. Uralczeva [7], C.D. Pagani, G. Talenti [8], Yu.P. Gorkov [9].

In this article, we consider boundary value problems for differential equations of the following type:

$$\varphi(t)u_t - \nu\Delta u - \chi\Delta u_t + |u|^{p-2}u + c(x,t)u = f(x,t) \ \ (x \in \Omega \subset R^n,\, n \geq 3,\ 0 < t < T) \tag{1}$$

where $\nu = const > 0$ and $f(x,t)$ is the external force. In these equations, the function $\varphi(t)$ and $\chi(t)$ can arbitrarily change sign on the segment $[0,T]$, and it can vanish on subsets of the segment $[0,T]$ of positive measure.

In article [3] I.A. Kozhanov and E.E. Maczievskaya in a cylindrical domain $Q = \Omega \times (0,T)$ $(0 < T < +\infty$, $\Omega \subset R^n$ – bounded area with smooth border $\Gamma)$ considered the solvability of a boundary value problem for differential equations:

$$\varphi(t)\, u_t + \psi(t)\, \Delta u + c(x,t)\, u = f(x,t)\, (x \in \Omega \subset R^n, 0 < t < T)\,. \tag{2}$$

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In articles [10], [11] for equation (2), a statement of the first boundary value problem is proposed, and the existence of its generalized solutions is proved.

G. Fichera [10] considered an equation of the following type:

$$Lu = a^{ij} u_{x_i x_j} + b^i u_{x_i} + cu = f(x,t)$$

and for the first boundary value problem the existence of its generalized solutions is proved.

O.A. Olejinik and E.V. Radkevich [11] proved the existence of generalized solutions to the first boundary value problem for the following type of equation:

$$Lu = a^{kj} u_{x_k x_j} + b^k u_{x_k} + c(x)\,u = f(x).$$

I.A. Kozhanov [12] proved the uniqueness of solutions to the first boundary value problem for the following type of equation:

$$u_{tt} + \alpha(t) \frac{\partial}{\partial t}(\Delta u) + Bu = f(x,t).$$

The purpose of this work is to study the solvability of the first boundary value problem for doubly degenerate differential equations (1) in classes of regular solutions — solutions that have all derivatives generalized in the sense of Sobolev entering the equation.

In [13], A. Benaissa and Ch. Aichi considered a one-dimensional degenerate wave equation with a boundary control condition of fractional derivative type

$$u_{tt}(x,t) - (a(x)\,u_x(x,t))_x = 0 \text{ in } (0,1)\times(0,\infty), \tag{3}$$

where the coefficient a is a positive function on $[0,1]$ but vanishes at zero. The degeneracy of (3) at $x = 0$ is measured by the parameter μ_a defined by

$$\mu_a = \sup_{0<x\le 1} \frac{x\,|a'(x)|}{a(x)}.$$

The researchers pointed out that the problem is not uniformly stable by a spectrum method and they studied the polynomial stability using the semigroup theory of linear operators.

In [14], the authors considered the following modelization of a flexible torque arm controlled by two feedbacks depending only on the boundary velocities:

$$\begin{cases} u_{tt}(x,t) - (a(x)\,u_x(x,t))_x + \alpha u_t(x,t) + \beta y(x,t) = 0,\ 0<x<1,\ t>0, \\ (a(x)\,u_x)(0) = k_1 u_t(0,t),\ t>0, \\ (a(x)\,u_x)(1) = -k_2 u_t(1,t),\ t>0, \end{cases}$$

where

$$\begin{cases} \alpha \ge 0,\ \beta > 0,\ k_1,\ k_2 \ge 0,\ k_1 + k_2 \ne 0, \\ a \in W^{1,\infty}(0,1),\ a(x) \ge a_0,\ \forall x \in [0,1]. \end{cases}$$

They proved the exponential decay of the solutions.

In [15], the authors considered a biharmonic regularization to the following nonlinear degenerate elliptic equation:

$$\begin{aligned} Qu &= \textstyle\sum_{i=1}^{d} \left[\partial_{x_i}\left(\sum_{j=1}^{d} a_{ij}(x,Du)\,\partial_{x_j} u + b_i(x,u,Du)\right) + c_i(x,u,Du)\,\partial_{x_i} u\right] + d(x)\,u = \\ &= f + \textstyle\sum_{i=1}^{d} \partial_{x_i} g^i,\ x \in \Omega \subset R^d,\ Du = \nabla u = (\partial_{x_1},...,\partial_{x_d}), \end{aligned}$$

where the coefficients will be specified later. By degenerate ellipticity, we imply that the coefficients a_{ij}, $i,j = 1,...,d$, satisfy degenerate ellipticity conditions

$$0 \le \lambda(x,p)\,|\xi|^2 \le a_{ij}(x,p)\,\xi_i \xi_j,\ x \in \Omega,\ p \in R^d,$$

for all $\xi = (\xi_1,...,\xi_d) \in R^d \backslash 0$. Under appropriate assumption on the coeffcients, we prove that a sequence of biharmonic regularization to a nonlinear degenerate elliptic equation with possibly rough coeffcients preserves certain regularity as the approximation parameter tends to zero. In order to obtain the result, they introduced a generalization of the Chebyshev inequality. They also presented numerical example.

In [16], the author considered degenerate quasilinear pseudoparabolic equations with memory terms and variational inequalities:

$$\begin{cases} \partial_t b^j(u) - \nabla \cdot (a(x) \nabla u_t)^j - \nabla \cdot d^j(t, x, u, \nabla u) + M^j(u) = f^j(u), \\ u^j = 0 \text{ on } (0, T) \times \partial\Omega, \\ b^j(u(0, x)) = b^j(u_0(x)) \text{ in } \Omega, \end{cases}$$

where the memory operator M is defined by

$$\langle M^j(t)(u), v^j \rangle = \int_\Omega \int_0^t K^j(t, s) g^j(s, x, \nabla u(s, x)) ds \nabla v^j(t, x) dx$$

for all functions u, $v \in L^p\left(0, T; H_0^{1,p}(\Omega)^l\right)$, for almost all $t \in (0, T)$.

The existence of solutions of degenerate quasilinear pseudoparabolic equations, where the term $\partial_t u$ is replaced by $\partial_t b(u)$, with memory terms and quasilinear variational inequalities is shown. The existence of solutions of equations is proved under the assumption that the nonlinear function b is monotone and a gradient of a convex, continuously differentiable function. The uniqueness is proved for Lipschitz continuous elliptic parts. The existence of solutions of quasilinear variational inequalities is proved under stronger assumptions, namely, the nonlinear function defining the elliptic part is assumed to be a gradient and the function b to be Lipschitz continuous.

Statement of the problem

Let $\Omega \subset R^n$, $n \geq 3$ is a bounded domain with the smooth border $\partial\Omega$, Q_T is a cylinder $\Omega \times (0, T)$ of finite height T, $S = \partial\Omega \otimes (0, T)$ is a side boundary. Further, let $\nu > 0$, χ, p are constants, $\varphi(t)$, $c(x, t)$ and $f(x, t)$ be the given functions defined at $t \in [0, T]$, $x \in \bar{\Omega}$, L is a differential operator whose action on a given $w(x, t)$ is determined by the equality

$$Lw = \varphi(t) w_t - \nu \Delta w - \chi \Delta w_t + |w|^{p-2} w + c(x, t) w$$

where $2 < p < 4$, Δ is the Laplace operator in the variables $x_1, x_2, ..., x_n$.

Boundary problem I. Find a function $u(x, t)$ that is a solution of the equation:

$$Lu = f(x, t) \tag{4}$$

in the $Q_T = \Omega \times (0, T)$ and such that condition:

$$u|_S = 0, \tag{5}$$

$$u(x, 0) = 0, \; x \in \Omega.$$

Boundary problem II. Find a function $u(x, t)$ that is a solution of equation (4) in the $Q_T = \Omega \times (0, T)$ and such that conditions (5) and

$$u(x, 0) = u(x, T) = 0, \; x \in \Omega.$$

Solvability of boundary value problems I-II

Theorem 1. Let the conditions

$$\varphi(t) \in C^1[0, T], c(x, t) \in C^2(\bar{Q}_T); \tag{6}$$

$$2c(x, t) - \varphi'(t) \geq c_1 > 0, \;\; 2c(x, t) + \varphi'(t) \geq c_2 > 0 \text{ at } (x, t) \in \bar{Q}_T; \tag{7}$$

$$\varphi(0) \leq 0, \;\; \varphi(T) > 0; \tag{8}$$

$$f(x, t) \in W_2^{1,1}(Q_T), \quad f(x, t) = 0 \text{ at } (x, t) \in S, \quad f(x, 0) = 0. \tag{9}$$

Then there is a unique solution $u \in L_2(0,T;\, W_2^2(\Omega)) \cap \overset{0}{W}{}_2^1(\Omega)$, $u_t \in L_2(0,T;\, W_2^2(\Omega)) \cap \overset{0}{W}{}_2^1(\Omega)$, $u_t \in L_2(Q_t)$ of the boundary value problem I.

Proof. For the proof, we use the regularization method. Let ε be a positive number. Let L_ε denote the differential operator whose action on a given function $w\,(x,t)$ is determined by the equality

$$L_\varepsilon w = \varepsilon \Delta w_{tt} + Lw.$$

Consider a boundary value problem: Find a function $w\,(x,t)$ that is a solution of the equation

$$L_\varepsilon w = f\,(x,t) \tag{10}$$

in the $Q_T = \Omega \times (0,T)$ and with conditions (5) and

$$u(x,0) = u_t(x,T) = 0,\ x \in \Omega. \tag{11}$$

Note that the first priori estimate is valid

$$\varepsilon \left\|\nabla u_t\right\|_{2,Q_T}^2 + \left\|u\right\|_{2,Q_T}^2 + \left\|\nabla u\right\|_{2,Q_T}^2 + \left\|u\right\|_{p,Q_T}^p \le c_3. \tag{12}$$

To prove this estimate, it suffices to analyze the equality

$$\int_{Q_T} L_\varepsilon u u dxdt = \int_{Q_T} f u dxdt$$

using the conditions of the theorem (6), (7), (8), (9), and Young's inequality.

Consider the following equation:

$$-\int_Q L_\varepsilon u \Delta u dxdt = -\int_Q f \Delta u dxdt.$$

Let us write this equation by integrating by parts:

$$\begin{array}{l} \varepsilon \left\|\Delta u_t\right\|_{2,Q_T}^2 + \frac{1}{2}\int_\Omega \varphi(T)\left|\nabla u(x,T)\right|^2 dx- \\ -\frac{1}{2}\int_{Q_T} \varphi'(t)\left|\nabla u\right|^2 dxdt + \frac{\chi}{2}\int_\Omega \left|\Delta u(x,T)\right|^2 dx+ \\ +\nu \int_{Q_T}\left|\Delta u\right|^2 dxdt + (p-1)\int_{Q_T}\left|u\right|^{p-2}\left|\nabla u\right|^2 dxdt + \int_{Q_T} c(x,t)\left|\nabla u\right|^2 dxdt = \\ -\int_{Q_T}\nabla c(x,t) u \nabla u dxdt - \int_{Q_T} f\Delta u dxdt. \end{array} \tag{13}$$

Let us estimate right-hand side of the equation (13):

$$\left|-\int_{Q_T}\nabla c(x,t) u\nabla u dxdt\right| \le c_4 \left\|\nabla u\right\|_{2,Q_T}\left\|u\right\|_{2,Q_T} \le \left\|\nabla u\right\|_{2,Q_T}^2 + \frac{c_4}{4}\left\|u\right\|_{2,Q_T}^2.$$

$$\left|\int_Q f\Delta u dxdt\right| \le \frac{\nu}{2}\left\|\Delta u\right\|_{2,Q_T}^2 + \frac{1}{2\nu}\left\|f\right\|_{2,Q_T}^2.$$

Substituting the obtained inequalities into equation (13) and taking into account the conditions of the theorem (6)–(9), we obtain the second priori estimate

$$\varepsilon \left\|\Delta u_t\right\|_{2,Q_T}^2 + \int_{Q_T}\left|\Delta u\right|^2 dxdt + \int_{Q_T}\left|u\right|^{p-2}\left|\nabla u\right|^2 dxdt \le C_5. \tag{14}$$

Now, let us show that at conditions (9), the solutions $u \in L_2(0,T;\, W_2^2(\Omega)) \cap \overset{0}{W}{}_2^1(\Omega)$, $u_t \in L_2(0,T;\, W_2^2(\Omega)) \cap$ $\cap \overset{0}{W}{}_2^1(\Omega)$, $u_t \in L_2(Q_t)$ of the boundary value problem (10), (5), (11) will satisfy the estimates uniform by ε.

In the next step, consider equality

$$\int_Q L_\varepsilon u \Delta u_{tt} dxdt = -\int_Q f \Delta u_{tt} dxdt.$$

This equality is easily transformed to form

$$\begin{array}{l}\varepsilon\left\|\Delta u_{tt}\right\|_{L^2(Q_T)}^2+\frac{1}{2}\int_\Omega\varphi(0)\left|\nabla u_t(x,0)\right|^2dx+\\+\frac{\chi}{2}\int_\Omega\left|\Delta u_t(x,0)\right|^2dxdt+\nu\int_{Q_T}\left|\Delta u_t\right|^2dxdt+\frac{1}{2}\int_{Q_T}(2c(x,t)+\varphi'(t))\left|\nabla u_t\right|^2dxdt=\\=-\int_{Q_T}(c_t(x,t)\nabla u\nabla u_t+\nabla c(x,t)u_t\nabla u_t+\nabla c_t(x,t)u\nabla u_t)dxdt-\\-(p-1)\int_{Q_T}\left|u\right|^{p-2}u_t\Delta u_tdxdt+\int_{Q_T}\nabla f_t\nabla u_tdxdt.\end{array}.$$

Let us estimate to $\int_{Q_T}\left|u\right|^{p-2}u_t\Delta u_tdxdt$:

$$\begin{array}{l}\left|\int_{Q_T}\left|u\right|^{p-2}u_t\Delta u_tdxdt\right|\leq\left\|\Delta u_t\right\|_{2,Q_T}\left\|u_t\right\|_{\frac{2n}{n-2},Q_T}\left\|u\right\|_{(p-2)n,Q_T}^{p-2}\leq\\\leq\frac{\nu}{2}\left\|\Delta u_t\right\|_{2,Q_T}^2+\frac{c_1}{8}\left\|\nabla u_t\right\|_{2,Q_T}^2+C_6.\end{array}$$

Using the conditions of the theorem (6)–(9), we obtain from this that solution $u\,(x,t)$ of the boundary value problem (10), (5), (11) satisfies the estimate

$$\varepsilon\left\|\Delta u_{tt}\right\|_{L^2(Q_T)}^2+\int_{Q_T}\left|\Delta u_t\right|^2dxdt+\int_{Q_T}\left|\nabla u_t\right|^2dxdt\leq C_7. \tag{15}$$

Estimates (12), (14) and (15) are already enough for choosing a sequence converging to the solution of boundary value problem I.

Let $\{\varepsilon_l\}_{l=1}^{\infty}$ be a sequence of positive numbers converging to 0. We denote by $u_l\,(x,t)$ the solution to boundary value problem (10), (5), (11) for $\varepsilon=\varepsilon_l$. For the sequence $\{u_l\,(x,t)\}_{l=1}^{\infty}$ for $\varepsilon=\varepsilon_l$, the priori estimates (12), (14) and (15) hold. It follows from these estimates and the reflexive property of the Hilbert space that there exists a subsequence $\{u_{l_k}\,(x,t)\}_{k=1}^{\infty}$ and a function $u\,(x,t)$ such that

$$\varepsilon_{l_k}\to 0,$$

$$u_{l_k}\,(x,t)\to u\,(x,t)\ \text{ in } L_2\,(Q_T)\ \text{ weakly},$$

$$\nabla u_{l_k t}\,(x,t)\to\nabla u_t\,(x,t)\ \text{ in } L_2\,(Q_T)\ \text{ weakly},$$

$$\Delta u_{l_k}\,(x,t)\to\Delta u\,(x,t)\ \text{ in } L_2\,(Q_T)\ \text{ weakly},$$

$$\Delta u_{l_k t}\,(x,t)\to\Delta u_t\,(x,t)\ \text{ in } L_2\,(Q_T)\ \text{ weakly},$$

$$\varepsilon_{l_k}\Delta u_{l_k tt}\,(x,t)\to 0 \text{ in } L_2\,(Q_T)\ \text{ weakly},$$

converges for $k\to\infty$. Obviously, the limit function $u\,(x,t)$ will belong to the space $u\in L_2(0,T;\,W_2^2(\Omega))\cap\overset{0}{W}{}_2^1(\Omega)$, $u_t\in L_2(0,T;\,W_2^2(\Omega))\cap\overset{0}{W}{}_2^1(\Omega)$, $u_t\in L_2(Q_t)$, and that it is a solution to the problem I. □

The study of the solvability of boundary value problem II in classes $u\in L_2(0,T;\,W_2^2(\Omega))\cap\overset{0}{W}{}_2^1(\Omega)$, $u_t\in L_2(0,T;\,W_2^2(\Omega))\cap\overset{0}{W}{}_2^1(\Omega)$, $u_t\in L_2(Q_t)$ is carried out in the whole similarly to the study of the solvability of boundary value problem I. The regularization method is used again, the operator L_ε is again used as a regularizing operator. The difference is that in the regularizing problem at $t=0$ and $t=T$ there are conditions

$$u(x,0)=u(x,T)=0,x\in\Omega.$$

Theorem 2. Let conditions (6), (7) and (9), and conditions

$$\varphi(0)>0,\,\varphi(T)<0$$

also hold. Then there is a unique solution $u\in L_2(0,T;\,W_2^2(\Omega))\cap\overset{0}{W}{}_2^1(\Omega)$, $u_t\in L_2(0,T;\,W_2^2(\Omega))\cap\overset{0}{W}{}_2^1(\Omega)$, $u_t\in L_2(Q_t)$ of the boundary value problem II.

The uniqueness of a solution

Theorem 3. Let conditions (6)–(9) be satisfied. Then $u(x,t)$ the solution of the boundary value problem I is a unique.

Proof. To prove the uniqueness of the equation suppose that problem has two solutions: $u_1(x,t)$ and $u_2(x,t)$. Then their difference $\vartheta(x,t) = u_1(x,t) - u_2(x,t)$ satisfies condition

$$\vartheta(x,0) = \vartheta_t(x,T) = 0,\ x \in \Omega.$$

Then, (10) is written in a form

$$\varepsilon \Delta \vartheta_{tt} + \varphi(t)\vartheta_t + |u_1|^{p-2} u_1 - |u_2|^{p-2} u_2 - \nu\Delta\vartheta - \chi\Delta\vartheta_t + c(x,t)\vartheta = 0.$$

Consider the following equation:

$$\int_{Q_T} \Big(\varepsilon \Delta \vartheta_{tt} + \varphi(t)\vartheta_t + |u_1|^{p-2} u_1 - |u_2|^{p-2} u_2 - \nu\Delta\vartheta - \\ -\chi\Delta\vartheta_t + c(x,t)\vartheta\Big)\vartheta dxdt = 0.$$

For any $p > 0$, the inequality holds

$$|(|u_1|^p u_1 - |u_2|^p u_2)(u_1 - u_2)| \geq c\,|u_1 - u_2|^{p+2}.$$

Let us write this equation by integrating by parts

$$\varepsilon \|\nabla\vartheta_t\|^2_{2,Q_T} + \frac{c_1}{2}\int_{Q_T}\vartheta^2 dxdt + \\ +\frac{\nu}{2}\int_{Q_T}|\nabla\vartheta|^2 dxdt + c\int_{Q_T}|\vartheta|^p \leq 0$$

In this case, we come to an equality

$$\vartheta = 0 \Rightarrow u_1 - u_2 = 0 \Rightarrow u_1 = u_2.$$

So, we proved the uniqueness of a solution.

Statement of the second problem

Let $\Omega \subset R^n$, $n \geq 3$ is a bounded domain with the smooth border $\partial\Omega$, Q_T is a cylinder $\Omega \times (0,T)$ with a finite height T, $S = \partial\Omega \otimes (0,T)$ is a side boundary. Further, let $\nu > 0$, χ, q be constants, $\varphi(t), c(x,t)$ and $f(x,t)$ be the given functions defined at $t \in [0,T]$, $x \in \bar{\Omega}$, L is a differential operator whose action on a given $w(x,t)$ is determined by the equality

$$Lw = \varphi(t)w_t - \nu\Delta w - \chi\Delta w_t - |\nabla w|^q + c(x,t)w$$

where $0 < q < 1$, Δ is the Laplace operator in the variables $x_1, x_2, ..., x_n$.

Boundary problem III. Find a function $u(x,t)$ that is a solution of the equation

$$Lu = f(x,t) \tag{16}$$

in the $Q_T = \Omega \times (0,T)$ and with the conditions

$$u|_S = 0, \tag{17}$$

$$u(x,0) = 0,\ x \in \Omega.$$

Boundary problem IV. Find a function $u(x,t)$ that is a solution of the equation (16) in the $Q_T = \Omega \times (0,T)$ and with condition (17), and condition

$$u(x,0) = u(x,T) = 0,\ x \in \Omega.$$

Solvability of boundary value problems I-II

Theorem 4. Let conditions (6), (7) and (9), and conditions

$$\varphi(0) \leq 0,\ \varphi(T) > 0$$

also hold. Then there is a unique solution $u \in L_2(0,T;\ W_2^2(\Omega)) \cap \overset{0}{W}{}_2^1(\Omega)$, $u_t \in L_2(0,T;\ W_2^2(\Omega)) \cap \overset{0}{W}{}_2^1(\Omega)$, $u_t \in L_2(Q_t)$ of the boundary value problem III.

Theorem 5. Let conditions (6), (7) and (9), and conditions

$$\varphi(0) > 0,\ \varphi(T) < 0$$

also hold. Then there is a unique solution $u \in L_2(0,T;\ W_2^2(\Omega)) \cap \overset{0}{W}{}_2^1(\Omega)$, $u_t \in L_2(0,T;\ W_2^2(\Omega)) \cap \overset{0}{W}{}_2^1(\Omega)$, $u_t \in L_2(Q_t)$ of the boundary value problem III.

Acknowledgments

The first author was supported by the Ministry of Education and Science of the Republic of Kazakhstan (Grant No. AP08857604).

References

1 Colton D. Pseudoparabolic equation in one space variable / D. Colton // J. Diff. Eq. — 1972. — Vol. 12. — №3. — P. 559–565.

2 Нахушев А.М. Об одном классе линейных краевых задач для гиперболического и смешанного типов уравнений второго порядка / А.М. Нахушев. — Нальчик: Эльбрус, 1992. — 155 с.

3 Кожанов А.И. Вырождающиеся параболические уравнения с переменным направлением эволюции / А.И. Кожанов, Е.Е. Мациевская // Сиб. электрон. мат. изв. — 2019. — 16. — С. 718–731.

4 Салахитдинов М.С. Уравнения смешанно-составного типа / М.С. Салахитдинов. — Ташкент: Фан, 1974. — 156 с.

5 Джураев Т.Д. Краевые задачи для уравнений смешанного и смешанно-составного типов / Т.Д. Джураев. — Ташкент: Фан, 1979. — 236 с.

6 Gevrey M. Sur Les Equations Aux Derivees Partielles Du Type Parabolique / M. Gevrey // J. de Math. — 1914. — Vol. 10. — №6. — P. 105–148.

7 Ладыженская О.А. Линейные и квазилинейные уравнения параболического типа / О.А. Ладыженская, В.А. Солонников, Н.Н. Уральцева. — М.: Наука, 1967. — 736 с.

8 Pagani C.D. On A Forward-Backward Parabolic Equation / C.D. Pagani, G. Talenti // Ann. Mat. Pura ed Appl. — 1971. — 90. — P. 1–58.

9 Горьков Ю.П. О поведении решения краевой задачи для квазилинейного параболического уравнения второго порядка при T→∞ / Ю.П. Горьков // Дифференц. урав. — 1970. — 6. — 6. — С. 1064–1072.

10 Fichera G. On a unified theory of boundary-value problems for elliptic-parabolic equations of second order, in: Boundary Problems in Differential Equations / G. Fichera. — Univ. of Wisconsin Press, Madison, 1960. — P. 97–120.

11 Олейник О.А. Уравнения второго порядка с неотрицательной характеристической формой / О.А. Олейник, Е.В. Радкевич // Итоги науки. Сер. Математика. Мат. анализ. 1969. — М.: ВИНИТИ, 1971. — С. 7–252.

12 Кожанов А.И. Краевые задачи для псевдоэллиптических уравнений третьего порядка с вырождением / А.И. Кожанов // Математические заметки СВФУ. — 2020. — 27. — №3. — С. 16–26.

13 Benaissa A. Energy decay for a degenerate wave equation under fractional derivative controls / A. Benaissa, Ch. Aichi // Filomat. — 2018. — Vol. 32. — Iss. 17. — P. 6045–6072

14 Chentouf B. On the stabilization of a vibrating equation / B. Chentouf, C.Z. Xu, G. Sallet // Nonlinear Anal.Ser. A: Theory Methods. — 2000. — Vol. 39 — Iss. 5. — P. 537–558.

15 Novaka A. On a Regularity of Biharmonic Approximations to a Nonlinear Degenerate Elliptic PDE / A. Novaka, J. Susicb // Filomat. — 2017. — Vol. 31. — Iss. 7. — P. 1835–1842.

16 Ptashnyk M. Degenerate quasilinear pseudoparabolic equations with memory terms and variational inequalities / M. Ptashnyk // Nonlinear Analysis. — 2007. — Vol. 66. — Iss. 12. — P. 2653–2675.

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Азғындалған сызықты емес псевдопараболалық теңдеу үшін бастапқы-шеттік есептің шешілімділігі

Мақала псевдопараболалық типтегі азғындалған сызықты емес теңдеулердің шешілімділігіне арналған. Мұндай проблемалар физиканың және биологияның әр түрлі модельдерінде туындайды. Мақаланың мақсаты — дифференциалдық теңдеулер үшін шекті есептердің (С.Л.Соболев мағынасында жалпыланған барлық туындыларды қоса алғанда) регулярлы шешімдер класындағы шешімділікті зерттеу. Қарастырылып отырған есептің шешілуіне кепілдік беретін, параметрлерге шарттар табылған және қарастырылған есептер үшін шешімнің бар және жалғыздық теоремалары дәлелденген. Шектік есептердің шешімділігін дәлелдеудің негізгі әдісі регуляризация әдісі болады.

Кілт сөздер: псевдопараболық теңдеулер, азғындалған теңдеулер, шеттік есептер, сызықты емес теңдеулер, шешімділік, жалғыздық.

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Разрешимость начально-краевой задачи для нелинейного псевдопараболического уравнения с вырождением

Статья посвящена разрешимости вырожденных нелинейных уравнений псевдопараболического типа. Такие задачи естественно возникают в физических и биологических моделях. Целью статьи является исследование разрешимости в классах регулярных решений (включающих в уравнение все производные, обобщенные по С.Л.Соболеву) краевых задач для дифференциальных уравнений. Для рассматриваемых задач авторами найдены условия на параметры, гарантирующие, что задача имеет решение. Кроме того, доказаны теоремы существования и единственности. В качестве основного метода доказательства разрешимости краевых задач выбран метод регуляризации.

Ключевые слова: псевдопараболические уравнения, вырожденные уравнения, краевые задачи, нелинейные уравнения, разрешимость, единственность.

References

1 Colton, D. (1972). Pseudoparabolic equation in one space variable. *J. Diff. Eq., 12*(3), 559–565.

2 Nakhushev, A.M. (1992). *Ob odnom klasse lineinykh kraevykh zadach dlia giperbolicheskogo i smeshannogo tipov uravnenii vtorogo poriadka [On a class of linear boundary value problems for hyperbolic and mixed types of second-order equations]*. Nalchik: Elbrus [in Russian].

3 Kozhanov, A.I., & Maczievskaya, E.E. (2019). Vyrozhdaiushchiesia parabolicheskie uravneniia s peremennym napravleniem evoliutsii [Degenerate parabolic equations with a variable direction of evolution]. *Sibirskie elektronnye matematicheskie izvestiia – Siberian Electronic Mathematical Reports, 16*, 718–731 [in Russian].

4 Salakhitdinov, M.S. (1974). *Uravneniia smeshanno-sostavnogo tipa [Mixed-compound equations]*. Tashkent: Fan [in Russian].

5 Dzhuraev, T.D. (1979). *Kraevye zadachi dlia uravnenii smeshannogo i smeshanno-sostavnogo tipov [Boundary value problems for equations of mixed and mixed-composite types]*. Tashkent: Fan [in Russian].

6 Gevrey, M. (1914). Sur Les Equations Aux Derivees Partielles Du Type Parabolique. *J. de Math., 10*(6), 105–148 [in French].

7 Ladyzhenskaya, O.A., Solonnikov, V.A., & Uralczeva, N.N. (1967). *Lineinye i kvazilineinye uravneniia parabolicheskogo tipa [Linear and quasi-linear equations of parabolic type]*. Moscow: Nauka [in Russian].

8 Pagani, C.D., & Talenti, G. (1971). On A Forward-Backward Parabolic Equation. *Ann. Mat. Pura ed Appl., 90*, 1–58.

9 Gorkov, Yu.P. (1970). O povedenii resheniia kraevoi zadachi dlia kvazilineinogo parabolicheskogo uravneniia vtorogo poriadka pri T→∞ [On the behavior of the solution of a boundary value problem for a quasilinear parabolic equation of the second order at T→∞]. *Differentsialnye uravneniia, 6*(6), 1064–1072 [in Russian].

10 Fichera, G. (1960). *On a unified theory of boundary-value problems for elliptic-parabolic equations of second order, in: Boundary Problems in Differential Equations*. University of Wisconsin Press, Madison, 97–120.

11 Olejnik, O.A., & Radkevich, E.V. (1971). *Uravneniia vtorogo poriadka s neotritsatelnoi kharakteristicheskoi formoi [Second-order equations with a non-negative characteristic form]*. Itogi nauki. Seriia. Matematika. Matematicheskii analiz, 1969 — Science Results. Mathematical Series. Mathematical analysis. Moscow: VINITI [in Russian].

12 Kozhanov, A.I. (2020). Kraevye zadachi dlia psevdoellipticheskikh uravnenii tretego poriadka s vyrozhdeniem [Boundary value problems for third-order pseudoelliptic equations with degeneration]. *Matematicheskie zametki SVFU – Mathematical notes of NEFU, 27*(3), 16–26 [in Russian].

13 Benaissa, A., & Aichi, Ch. (2018). Energy decay for a degenerate wave equation under fractional derivative controls. *Filomat, 32*(17), 6045–6072.

14 Chentouf, B., Xu, C.Z., & Sallet, G. (2000). On the stabilization of a vibrating equation. *Nonlinear Anal. Ser. A: Theory Methods, 39*(5), 537–558.

15 Novaka, A., & Susicb, A. (2017). On a Regularity of Biharmonic Approximations to a Nonlinear Degenerate Elliptic PDE *Filomat, 31*(7), 1835–1842.

16 Ptashnyk, M. (2007). Degenerate quasilinear pseudoparabolic equations with memory terms and variational inequalities. *Nonlinear Analysis, 66*(12), 2653–2675.

DOI 10.31489/2022M1/13-24
UDC 519.63

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Mathematical modeling of the energy consumption problem

The importance of energy-saving and correct design is obvious for energy efficiency. Correct design means that before construction considerable things, such as orientation or isolation decisions, need to be made. This study gives a mathematical model of the nonstationary energy consumption calculation problems. The model is well-posedness in Holder spaces of the mixed one-dimensional parabolic problem with Robin boundary conditions. In this study, an effective numerical method is also developed for energy consumption calculation which is related to this mathematical model. The three case problems are taken to test this numerical method. The dynamic model results have been compared with the previous finite-difference or steady-state solutions. The study also aims to develop a mathematical model in which the result can be found at any time.

Keywords: mathematical modeling, heat diffusion equation, difference scheme, stability.

Introduction

An important part of energy consumption occurs in buildings. Energy Efficient Building Design (EEBD) is a design that reduces energy usage and pollution controlling the criteria. Architectural building design rules are functionality, stability, and aesthetics. Today, efficiency and healthiness also are added. An efficient design means not only doing things during operating but also doing correct design before the construction. There are numerous studies on EEBD all over the world (see [1–7]). A national software, that calculates the energy consumption of buildings according to the Turkish Standards Institute (TS EN 13790), exists in Turkey. Note that the problem is complicated because the energy consumption calculation depends on many variables, such as nonstationary external temperature and solar radiation, building materials, heat losses and gains and energy consumption change with time. Energy consumption numerical calculations take a lot of time because of the stability criterion. It is not easy to check hour by hour for the whole year. For these reasons, the mathematical model and theoretical solution are valuable. In this article, the mathematical model of a building's outer wall consisting of an opaque wall is obtained by taking as a boundary value problem for the annual energy consumption calculation. The heat conduction differential equation and the boundary equations of the one-dimensional nonstationary boundary value problem are given. This study also gives a one-dimensional nonstationary general solution for some energy consumption calculation problems. Finally, the dynamic model results were compared with the numerical results.

Theoretical background

In this section, we consider the theoretical background of the mathematical model of energy-saving problems. The well-posedness of differential and difference heat problems with third boundary conditions in Hölder spaces is established. Numerical results are provided.

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Stability and coercive stability of differential problem

We study the initial-boundary value problem

$$\begin{cases} \frac{\partial u(t,x)}{\partial t} - \frac{\partial}{\partial x}\left(a\left(x\right)\frac{\partial u(t,x)}{\partial x}\right) + \delta u(t,x) = f\left(t,x\right),\ t\in\left(0,T\right),\ x\in\left(0,l\right), \\ u\left(0,x\right) = \varphi\left(x\right),\ x\in\left[0,l\right], \\ u\left(t,0\right) - \psi(t) = bu_x\left(t,0\right), -u\left(t,l\right) - \omega(t) = cu_x\left(t,l\right),\ t\in[0,T], \end{cases} \tag{1}$$

for the one-dimensional heat equation with Robin boundary conditions. Here $0 < a \leq a\left(x\right)$ and b, c, δ are positive constants. Under compatibility conditions problem, (1) has a unique solution $u(t,x)$ for smooth functions $a\left(x\right)$, $x\in(0,l)$, $\varphi(x)$, $x\in[0,l]$, $\psi(t)$, $\omega(t)$, $t\in[0,T]$, $f(t,x)$, $(t,x)\in(0,1)\times(0,l)$.

Assume that H be a Hilbert space and A be the self-adjoint positive-definite operator defined by the formula

$$Az = -\frac{d}{dx}\left(a(x)\frac{dz(x)}{dx}\right) + \delta z(x) \tag{2}$$

with domain

$$D(A) = \{z : z, z^{''} \in L_2(0,l),\ z(0) = bz^{'}(0),\ -z(l) = cz^{'}(l)\}.$$

Here and in the rest of this paper, $C_0^{\alpha}\left(\left[0,T\right],H\right)$ $(0<\alpha<1)$ stands for Banach spaces of all abstract continuous functions $\varphi(t)$ defined on $[0,T]$ with values in H satisfying a Hölder condition with weight t^{α} for which the following norm is finite

$$\|\varphi\|_{C_0^{\alpha}([0,T],H)} = \|\varphi\|_{C([0,T],H)} + \sup_{0\leq t<t+\tau\leq T}\frac{(t+\tau)^{\alpha}\|\varphi(t+\tau)-\varphi(t)\|_H}{\tau^{\alpha}}.$$

Here, $C\left(\left[0,T\right],H\right)$ stands for the Banach space of all abstract continuous functions $\varphi(t)$ defined on $[0,T]$ with values in H equipped with the norm

$$\|\varphi\|_{C([0,T],H)} = \max_{0\leq t\leq T}\|\varphi(t)\|_H.$$

Let the Sobolev space $W_2^2(0,l)$ be defined as the set of all functions $v(x)$ defined on $(0,l)$ such that both $v(x)$ and $v''(x)$ are locally integrable in $L_2(0,l)$, equipped with the norm

$$\|v\|_{W_2^2(0,l)} = \left(\int_0^{\ell}|v(x)|^2\,dx\right)^{1/2} + \left(\int_0^{\ell}|v''(x)|^2\,dx\right)^{1/2}.$$

Theorem 1. Assume that $f(t,x)$ and $\psi(t)$, $\omega(t)$ are continuous functions and satisfying a Hölder condition with weight t^{α}. Then the problem (1) has a unique solution $u\in C_0^{\alpha}\left(L_2(0,l)\right)$ and for the solution of problem (1) the following stability estimates

$$\|u\|_{C_0^{\alpha}([0,T],L_2(0,l))} \leq M\left(q,\delta\right)\left[\|\varphi\|_{L_2(0,l)} + \|f\|_{C_0^{\alpha}([0,T],L_2(0,l))} + \|\psi\|_{C_0^{\alpha}[0,T]} + \|\omega\|_{C_0^{\alpha}[0,T]}\right]$$

and coercive stability estimates

$$\|u_t\|_{C_0^{\alpha}([0,T],L_2(0,l))} + \|u\|_{C_0^{\alpha}\left([0,T],W_2^2(0,l)\right)} \leq M\left(q,\delta\right)\Big[\|\varphi\|_{W_2^2(0,l)}$$
$$+\frac{1}{\alpha(1-\alpha)}\|f\|_{C_0^{\alpha}([0,T],L_2(0,l))} + \|\psi\|_{C_0^{\alpha}[0,T]} + \|\omega\|_{C_0^{\alpha}[0,T]}\Big]$$

are satisfied.

Proof. Denote by

$$u\left(t,x\right) = w\left(t,x\right) + \left(1-\frac{x^2}{l^2+2lc}\right)\psi(t) - \frac{x^2}{l^2+2lc}\omega\left(t\right), \tag{3}$$

where $w\left(t,x\right)$ is the solution of the following initial-boundary value problem:

$$\begin{cases} w_t(t,x) - (a(x)\, w_x(t,x))_x + \delta w(t,x) \\ = f(t,x) + \psi_t(t)\left(1 - \frac{x^2}{l^2+2lc}\right) - \frac{x^2}{l^2+2lc}\omega_t(t) \\ +\delta\left(\psi(t)\left(1 - \frac{x^2}{l^2+2lc}\right) - \frac{x^2}{l^2+2lc}\omega(t)\right) + \frac{2}{l^2+2lc}\psi(t)\,(a(x)\,x)_x \\ +\frac{2}{l^2+2lc}\omega(t)\,(a(x)\,x)_x\,,\ t\in(0,T)\,,\ x\in(0,l)\,, \\ w(0,x) = \varphi(x) + \psi(0)\left(1 - \frac{x^2}{l^2+2lc}\right) - \frac{x^2}{l^2+2lc}\omega(0)\,,\ x\in[-l,l]\,, \\ w(t,0) = bw_x(t,0)\,,\ -w(t,l) - cw_x(t,l) = 0,\ t\in[0,T]\,. \end{cases} \qquad (4)$$

Applying (3), we get

$$\|u\|_{C_0^\alpha([0,T],L_2(0,l))} \le \|w\|_{C_0^\alpha([0,T],L_2(0,l))} + K_1(l)\left[\|\psi\|_{C_0^\alpha[0,T]} + \|\omega\|_{C_0^\alpha[0,T]}\right],$$

$$\|u\|_{C_0^\alpha\left([0,T],W_2^2(0,l)\right)} \le \|w\|_{C_0^\alpha([0,T],L_2(0,l)])} + K_2(l)\left[\|\psi\|_{C_0^\alpha[0,T]} + \|\omega\|_{C_0^\alpha[0,T]}\right].$$

Therefore, the following theorem will be complete the proof of Theorem 1.

Theorem 2. Under assumptions of Theorem 1, the problem (4) has a unique solution in $C([0,T],L_2(0,l))$ and the following stability estimate:

$$\|w\|_{C_0^\alpha([0,T],L_2(0,l))} \le M(q,\delta)\left[\|\varphi\|_{L_2[0,l]} + \|f\|_{C_0^\alpha([0,T],L_2[0,l])} + \|\psi\|_{C_0^\alpha[0,T]} + \|\omega\|_{C_0^\alpha[0,T]}\right]$$

and coercive stability estimate

$$\|w_t\|_{C_0^\alpha([0,T],L_2(0,l))} + \|w\|_{C_0^\alpha\left([0,T],W_2^2(0,l)\right)} \le M(q,\delta)\left[\|\varphi\|_{W_2^2(0,l)} + \frac{1}{\alpha(1-\alpha)}\|f\|_{C_0^\alpha([0,T],L_2(0,l))} \right.$$
$$\left. + \|\psi\|_{C_0^\alpha[0,T]} + \|\omega\|_{C_0^\alpha[0,T]}\right]$$

are satisfied.

Proof. Problem (4) can be written in the following abstract form

$$\begin{cases} w'(t) + Aw(t) = f(t) + \psi_t(t)q_1 + \psi_t(t)q_2 + \psi(t)q_3 + \omega(t)q_4,\ 0<t<T, \\ w(0) = \varphi + \psi(0)q_1 + \omega(0)\,q_2 \end{cases} \qquad (5)$$

in a Hilbert space $H = L_2(0,l)$ with the space operator $A = A^x$ defined by the formula (2). Here, $f(t) = f(t,x)$ is the given abstract function, $w(t) = w(t,x)$ is unknown function and

$$q_1 = q_1(x) = 1 - \frac{x^2}{l^2+2lc}, q_2 = q_2(x) = -\frac{x^2}{l^2+2lc}, q_3 = q_3(x) = \delta\left(1 - \frac{x^2}{l^2+2lc}\right) + \frac{2}{l^2+2lc}(a(x)\,x)_x\,,$$
$$q_4 = q_4(x) = -\delta\frac{x^2}{l^2+2lc} + \frac{2}{l^2+2lc}(a(x)\,x)_x$$

are known elements of $L_2(0,l)$. The proof of Theorem 2 is based on theorems on stability and coercive stability of the abstract problem (5) (see, [1,2]), the self-adjointness and positive definiteness of the space operator A^x defined by formula (2).

Stability and coercive stability of difference problem

Let $\alpha\in(0,1)$ is a given number and $C_\tau^\alpha(H) = C_0^\alpha([0,T]_\tau, H]),\, C_\tau(H) = C([0,T]_\tau, H)$ be Banach spaces of all H-valued mesh functions $w_\tau = \{w_k\}_{k=0}^N$ defined on

$$[0,T]_\tau = \{t_k = k\tau, 0\le k\le N, N\tau = T\}$$

with the corresponding norms

$$\|w_\tau\|_{C_\tau(H)} = \max_{0\le k\le N}\|w_k\|_H\,,$$

$$\|w_\tau\|_{C_\tau^\alpha(H)} = \sup_{1\le k<k+n\le N}(N-n)^{-\alpha}(k)^\alpha\|w_{k+n} - w_k\|_H + \|w_\tau\|_{C_\tau(H)}\,.$$

Moreover, let $L_{2h}=L_2\left[0,l\right]_h$ and $W_{2h}^2=W_2^2(0,l)_h$ be normed spaces of all mesh functions $\gamma^h\left(x\right)=\left\{\gamma_n\right\}_{n=0}^M$ defined on

$$\left[0,l\right]_h=\left\{x_n=nh,0\leq n\leq M,Mh=l\right\}$$

equipped with norms

$$\left\|\gamma^h\right\|_{L_{2h}}=\left(\sum_{x\in[0,l]_h}\left|\gamma^h(x)\right|^2h\right)^{1/2}$$

and

$$\left\|\gamma^h\right\|_{W_{2h}^2}=\left\|\gamma^h\right\|_{L_{2h}}+\left(\sum_{x\in(0,l)_h}\left|\left(\gamma^h\right)_{x\overline{x},j}\right|^2h\right)^{1/2},$$

respectively. Furthermore, we introduce the difference operator A_h^x defined by the formula

$$A_h^xu^h(x)=\left\{-\frac{1}{h}\left(a_{n+1}\frac{u_{n+1}-u_n}{h}-a_n\frac{u_n-u_{n-1}}{h}\right)+\delta u_n\right\}_1^{M-1}, \tag{6}$$

acting in the space of mesh functions $u^h\left(x\right)=\left\{u_n\right\}_{n=0}^M$ defined on $\left[0,l\right]_h$ satisfying the conditions $\left(h+b\right)u_0-$ $-bu_1=0,-cu_{M-1}+\left(h+c\right)u_M=0$. For the numerical solution $\left\{u_k^h\left(x\right)\right\}_{k=0}^N$ of problem (1), we present DS of the first order of approximation

$$\begin{cases}\frac{u_n^k-u_n^{k-1}}{\tau}-\frac{1}{h}\left(a_{n+1}\frac{u_{n+1}^k-u_n^k}{h}-a_n\frac{u_n^k-u_{n-1}^k}{h}\right)+\delta u_n^k\\=f_n^k,\ f_n^k=f\left(t_k,x_n\right),\ t_k\in k\tau,\ x_n=nh,\ k\in\overline{1,N},\ n\in\overline{1,M-1},\\u_n^0=\varphi_n,\ \varphi_n=\varphi\left(x_n\right),\ n\in\overline{0,M},\\\left(h+b\right)u_0^k-bu_1^k=h\psi_k,\ cu_{M-1}^k-\left(h+c\right)u_M^k=h\omega_k,\\\psi_k=\psi(t_k),\ \omega_k=\omega(t_k),\ k\in\overline{0,N}\end{cases} \tag{7}$$

and of the second order of approximation

$$\begin{cases}\frac{u_n^k-u_n^{k-1}}{\tau}-\frac{1}{2h}\left(a_{n+1}\frac{u_{n+1}^k-u_n^k}{h}-a_n\frac{u_n^k-u_{n-1}^k}{h}\right)-\frac{1}{2h}\left(a_{n+1}\frac{u_{n+1}^{k-1}-u_n^{k-1}}{h}-a_n\frac{u_n^{k-1}-u_{n-1}^{k-1}}{h}\right)\\+\delta\frac{u_n^k+u_n^{k-1}}{2}=f_n^k,\ f_n^k=f\left(t_k-\frac{\tau}{2},x_n\right),\ t_k\in k\tau,\ x_n=nh,\ k\in\overline{1,N},\ n\in\overline{1,M-1},\\u_n^0=\varphi_n,\ \varphi_n=\varphi\left(x_n\right),\ n\in\overline{0,M},\\\frac{u_0^k+u_0^{k-1}}{2}-\psi_k=b\left(\frac{u_1^k-u_0^k}{2h}+\frac{u_1^{k-1}-u_0^{k-1}}{2h}\right),\\-\frac{u_M^k+u_M^{k-1}}{2}-\omega_k=c\left(\frac{u_M^k-u_{M-1}^k}{2h}+\frac{u_M^{k-1}-u_{M-1}^{k-1}}{2h}\right)\\\psi_k=\psi(t_k),\ \omega_k=\omega(t_k),\ k\in\overline{0,N}.\end{cases} \tag{8}$$

Let us give the following results on the stability and coercive stability of DSs (7) and (8).

Theorem 3. For the solution of DSs (7) and (8) the stability estimates

$$\left\|\left\{u_k^h\right\}_{k=1}^N\right\|_{C_\tau^\alpha(L_{2h})}\leq M\left(q,\delta\right)\left[\left\|\varphi^h\right\|_{L_{2h}}\right.$$

$$\left.+\left\|\left\{f_k^h\right\}_{k=1}^N\right\|_{C_\tau^\alpha(L_{2h})}+\left\|\left\{\psi_k\right\}_1^N\right\|_{C_0^\alpha[0,T]_\tau}+\left\|\left\{\omega_k\right\}_1^N\right\|_{C_0^\alpha[0,T]_\tau}\right]$$

and coercive stability estimates

$$\left\|\left\{\frac{1}{\tau}\left(u_k^h-u_{k-1}^h\right)\right\}_{k=1}^N\right\|_{C_\tau^\alpha(L_{2h})}+\left\|\left\{\widetilde{u_k^h}\right\}_{k=1}^N\right\|_{C_\tau^\alpha\left(W_{2h}^2\right)}\leq M\left(q,\delta\right)\left[\left\|\varphi^h\right\|_{W_{2h}^2}\right.$$

$$\left.+\frac{1}{\alpha\left(1-\alpha\right)}\left\|\left\{f_k^h\right\}_{k=1}^N\right\|_{C_\tau^\alpha(L_{2h})}+\left\|\left\{\psi_k\right\}_1^N\right\|_{C_0^\alpha[0,T]_\tau}+\left\|\left\{\omega_k\right\}_1^N\right\|_{C_0^\alpha[0,T]_\tau}\right]$$

hold. Here,

$$\widetilde{u_k^h}=\begin{cases} u_k^h \quad , & \text{for first order DS,} \\ \frac{u_k^h+u_{k-1}^h}{2}, & \text{for Crank-Nicolson DS.}\end{cases}$$

Proof. We will use

$$u_n^k=w_n^k+\left(1-\frac{(nh)^2-nh^2}{l^2+2lc+(c-l)h}\right)\psi_k-\frac{(nh)^2-nh^2}{l^2+2lc+(c-l)h}\omega_k, \tag{9}$$

where $\left\{w_k^h(x)\right\}_{k=0}^N$ is the solution of the following DSs

$$\begin{cases} \frac{w_n^k-w_n^{k-1}}{\tau}-\frac{1}{h}\left(a_{n+1}\frac{w_{n+1}^k-w_n^k}{h}-a_n\frac{w_n^k-w_{n-1}^k}{h}\right)+\delta w_n^k \\ =f_n^k-\left(1-\frac{(nh)^2-nh^2}{l^2+2lc+(c-l)h}\right)\frac{\psi_k-\psi_{k-1}}{\tau}-\frac{(nh)^2-nh^2}{l^2+2lc+(c-l)h}\frac{\omega_k-\omega_{k-1}}{\tau} \\ +\delta\left[\left(1-\frac{(nh)^2-nh^2}{l^2+2lc+(c-l)h}\right)\psi_k-\frac{(nh)^2-nh^2}{l^2+2lc+(c-l)h}\omega_k\right] \\ +\frac{2}{l^2+2lc+(c-l)h}\left(na_{n+1}-(n-1)a_n\right)\left[\psi_k+\omega_k\right],\ k\in\overline{1,N},\ n\in\overline{1,M-1}, \\ w_n^0=\varphi_n+\left(1-\frac{(nh)^2-nh^2}{l^2+2lc+(c-l)h}\right)\psi_0-\frac{(nh)^2-nh^2}{l^2+2lc+(c-l)h}\omega_0,\ n\in\overline{0,M}, \\ (h+b)\,w_0^k-bw_1^k=0, cw_{M-1}^k-(h+c)\,w_M^k=0,\ 9k\in\overline{0,N}\end{cases} \tag{10}$$

and

$$\begin{cases} \frac{w_n^k-w_n^{k-1}}{\tau}-\frac{1}{2h}\left(a_{n+1}\frac{w_{n+1}^k-w_n^k}{h}-a_n\frac{w_n^k-w_{n-1}^k}{h}\right)-\frac{1}{2h}\left(a_{n+1}\frac{w_{n+1}^{k-1}-w_n^{k-1}}{h}-a_n\frac{w_n^{k-1}-w_{n-1}^{k-1}}{h}\right) \\ +\delta\frac{w_n^k+w_n^{k-1}}{2}=f_n^k-\left(1-\frac{(nh)^2-nh^2}{l^2+2lc+(c-l)h}\right)\frac{\psi_k-\psi_{k-1}}{\tau}-\frac{(nh)^2-nh^2}{l^2+2lc+(c-l)h}\frac{\omega_k-\omega_{k-1}}{\tau} \\ +\delta\left[\left(1-\frac{(nh)^2-nh^2}{l^2+2lc+(c-l)h}\right)\frac{\psi_k+\psi_{k-1}}{2}_k-\frac{(nh)^2-nh^2}{l^2+2lc+(c-l)h}\frac{\omega_k+\omega_{k-1}}{2}\right] \\ +\frac{2}{l^2+2lc+(c-l)h}\left(na_{n+1}-(n-1)a_n\right)\left[\psi_k+\omega_k\right],\ k\in\overline{1,N},\ n\in\overline{1,M-1}, \\ w_n^0=\varphi_n+\left(1-\frac{(nh)^2-nh^2}{l^2+2lc+(c-l)h}\right)\psi_0-\frac{(nh)^2-nh^2}{l^2+2lc+(c-l)h}\omega_0,\ n\in\overline{0,M}, \\ (h+b)\,w_0^k-bw_1^k=0, cw_{M-1}^k-(h+c)\,w_M^k=0,\ k\in\overline{0,N}\end{cases} \tag{11}$$

for (7) and (8), respectively. Applying (9), we obtain

$$\left\|\left\{u_k^h\right\}_{k=1}^N\right\|_{C_\tau^\alpha(L_{2h})}\le\left\|\left\{w_k^h\right\}_{k=1}^N\right\|_{C_\tau^\alpha(L_{2h})}+K\left[\left\|\left\{\psi_k\right\}_{k=1}^N\right\|_{C_0^\alpha[0,T]_\tau}+\left\|\left\{\omega_k\right\}_{k=1}^N\right\|_{C_0^\alpha[0,T]_\tau}\right]$$

and

$$\left\|\left\{\widetilde{u_k^h}\right\}_{k=1}^N\right\|_{C_\tau^\alpha\left(W_{2h}^2\right)}\le\left\|\left\{w_k^h\right\}_{k=1}^N\right\|_{C_\tau^\alpha\left(W_{2h}^2\right)}+K\left[\left\|\left\{\psi_k\right\}_{k=1}^N\right\|_{C_0^\alpha[0,T]_\tau}+\left\|\left\{\omega_k\right\}_{k=1}^N\right\|_{C_0^\alpha[0,T]_\tau}\right].$$

Therefore, the following theorem will be complete the proof of Theorem 3.

Theorem 4. For the solution of DSs (10) and (11) the stability estimates

$$\left\|\left\{w_k^h\right\}_{k=1}^N\right\|_{C_\tau^\alpha(L_{2h})}\le K_3(a)\left[\left\|\varphi^h\right\|_{L_{2h}}\right.$$

$$\left.+\left\|\left\{f_k^h\right\}_{k=1}^N\right\|_{C_\tau^\alpha}+\left\|\left\{\psi_k\right\}_{k=1}^N\right\|_{C_0^\alpha[0,T]_\tau}+\left\|\left\{\omega_k\right\}_{k=1}^N\right\|_{C_0^\alpha[0,T]_\tau}\right]$$

and coercive stability estimates

$$\left\|\left\{\frac{1}{\tau}\left(w_k^h-w_{k-1}^h\right)\right\}_{k=1}^N\right\|_{C_\tau^\alpha(L_{2h})}\le K_3(q)\left[\left\|\varphi^h\right\|_{W_{2h}^2}\right.$$

$$\left.+\frac{1}{\alpha(1-\alpha)}\left\|\left\{f_k^h\right\}_{k=1}^N\right\|_{C_\tau^\alpha}+\left\|\left\{\psi_k\right\}_{k=1}^N\right\|_{C_0^\alpha[0,T]_\tau}+\left\|\left\{\omega_k\right\}_{k=1}^N\right\|_{C_0^\alpha[0,T]_\tau}\right]$$

hold.

Proof. Problems (10) and (11) can be written in the following abstract forms

$$\begin{cases} \frac{w_k^h - w_{k-1}^h}{\tau} + A^h w_k^h = f_k^h + q_1^h \frac{\psi_k - \psi_{k-1}}{\tau} + q_2^h \frac{\omega_k - \omega_{k-1}}{\tau} + q_3^h \psi_k + q_4^h \omega_k, \ 1 \le k \le N, \\ w_0^h = \varphi^h + q_1^h \psi_0 + q_2^h \omega_0 \end{cases} \tag{12}$$

and Crank-Nicolson

$$\begin{cases} \frac{w_k^h - w_{k-1}^h}{\tau} + A^h \frac{w_k^h + w_{k-1}^h}{2} = f_k^h + q_1^h \frac{\psi_k - \psi_{k-1}}{\tau} + q_2^h \frac{\omega_k - \omega_{k-1}}{\tau} + q_3^h \frac{\psi_k + \psi_{k-1}}{2} + q_4^h \frac{\omega_k + \omega_{k-1}}{2}, \ 1 \le k \le N, \\ w_0^h = \varphi^h + q_1^h \psi_0 + q_2^h \omega_0, \end{cases}$$

in a Hilbert space $H = L_{2h}$ with the space operator $A^h = A_h^x$ defined by the formula (6). Here, $f_k^h = f_k^h(x)$ is given abstract mesh function, $w_k^h = w_k^h(x)$ is unknown mesh function and
$q_1^h = q_1^h(x) = \left(1 - \frac{(nh)^2 - nh^2}{l^2 + 2lc + (c-l)h}\right)$, $q_2^h = q_2^h(x) = -\frac{(nh)^2 - nh^2}{l^2 + 2lc + (c-l)h}$, $q_3^h = q_3^h(x) = \delta\left(1 - \frac{(nh)^2 - nh^2}{l^2 + 2lc + (c-l)h}\right)$ $+ \frac{2}{l^2 + 2lc + (c-l)h}\left(na_{n+1} - (n-1)a_n\right)$, $q_4^h = q_4^h(x) = -\delta\frac{(nh)^2 - nh^2}{l^2 + 2lc + (c-l)h} + \frac{2}{l^2 + 2lc + (c-l)h}\left(na_{n+1} - (n-1)a_n\right)$ are known elements of L_{2h}. The proof of Theorem 4 is based on theorems on stability and coercive stability of the abstract problem (12) (see [1, 2]), the self-adjointness and positive definiteness of the difference operator A_h^x defined by the formula (6).

Numerical results

Now, the numerical results for the solution of the initial boundary value problem

$$\begin{cases} u_t(t,x) - u_{xx}(t,x) = -\frac{3}{4}e^{-t}\cos\frac{x}{2}, \\ 0 < t < 1, \ 0 < x < \pi, \\ u(0,x) = \cos\frac{x}{2}, \ 0 \le x \le \pi, \\ u(t,0) - e^{-t} = u_x(t,0), \\ -u(t,\pi) - \frac{1}{2}e^{-t} = u_x(t,\pi), \ 0 \le t \le 1 \end{cases} \tag{13}$$

for the parabolic equation with Robin conditions are presented. The exact solution of this problem is

$$u(t,x) = e^{-t}\cos\frac{x}{2}.$$

For the approximate solution of problem (13), the set $[0,1]_\tau \times [0,\pi]_h$ of a family of grid points depending on the small parameters τ and h

$$[0,1]_\tau \times [0,\pi]_h$$
$$= \{(t_k, x_n) : t_k = k\tau, \ 0 \le k \le N, \ N\tau = 1, \ x_n = nh, \ 0 \le n \le M, \ Mh = \pi\}$$

is defined. For the numerical solution of problem (13), we present the first order of accuracy Rothe DS:

$$\begin{cases} \frac{u_n^k - u_n^{k-1}}{\tau} - \frac{u_{n+1}^k - 2u_n^k + u_{n-1}^k}{h^2} = f_n^k, f_n^k = -\frac{3}{4}e^{-t_k}\cos\frac{x_n}{2}, \\ 1 \le k \le N, \ 1 \le n \le M-1, \\ u_n^0 = \cos\frac{x_n}{2}, \ 0 \le n \le M, \\ u_0^k - e^{-t_k} = \frac{u_1^k - u_0^k}{h}, \\ u_M^k + \frac{1}{2}e^{-t_k} = -\frac{u_M^k - u_{M-1}^k}{h} = 0, \ 0 \le k \le N \end{cases} \tag{14}$$

and second order of accuracy Crank-Nicolson DS

$$\begin{cases} \frac{u_n^k - u_n^{k-1}}{\tau} - \frac{u_{n+1}^k - 2u_n^k + u_{n-1}^k}{2h^2} - \frac{u_{n+1}^{k-1} - 2u_n^{k-1} + u_{n-1}^{k-1}}{2h^2} = f_n^k, \\ f_n^k = -\frac{3}{4}e^{-t_k + \frac{\tau}{2}}\cos\frac{x_n}{2}, \ 1 \le k \le N, \ 1 \le n \le M-1, \\ u_n^0 = \cos\frac{x_n}{2}, \ 0 \le n \le M, \\ \frac{u_0^k + u_0^{k-1}}{2} - e^{-t_k + \frac{\tau}{2}} = \frac{u_1^k - u_0^k}{2h} + \frac{u_1^{k-1} - u_0^{k-1}}{2h}, \\ \frac{u_M^k + u_M^{k-1}}{2} + \frac{1}{2}e^{-t_k + \frac{\tau}{2}} = -\frac{u_M^k - u_{M-1}^k}{2h} - \frac{u_M^{k-1} - u_{M-1}^{k-1}}{2h}, \\ 1 \le k \le N. \end{cases} \tag{15}$$

For the computer implementation of DS (14), we can apply two approaches.

First, for obtaining the solution of difference scheme (14), we rewrite it as the initial value problem for the first order difference equation with respect to k and matrix coefficients

$$Au^{k+1} + Bu^k = If^k,\ 1 \le k \le N,\ u^0 = \varphi \tag{16}$$

where A, B are $(M+1)\times(M+1)$ square matrices and f^k is $(M+1)\times 1$ column matrix. Here,

$$A = \begin{bmatrix} 1+\frac{1}{h} & -\frac{1}{h} & 0 & \cdot & 0 & 0 & 0 \\ a & b & a & \cdot & 0 & 0 & 0 \\ 0 & a & b & \cdot & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & b & b & 0 \\ 0 & 0 & 0 & \cdot & a & b & a \\ 0 & 0 & 0 & \cdot & 0 & \frac{1}{h} & -1-\frac{1}{h} \end{bmatrix}_{(M+1)\times(M+1)},$$

$$B = \begin{bmatrix} 0 & 0 & 0 & \cdot & 0 & 0 & 0 \\ 0 & c & 0 & \cdot & 0 & 0 & 0 \\ 0 & 0 & c & \cdot & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & c & 0 & 0 \\ 0 & 0 & 0 & \cdot & 0 & c & 0 \\ 0 & 0 & 0 & \cdot & 0 & 0 & 0 \end{bmatrix}_{(M+1)\times(M+1)}$$

here and in future

$$a = -\frac{1}{h^2},\ b = \frac{1}{\tau} + \frac{2}{h^2},\ c = -\frac{1}{\tau}$$

and

$$f^k = \begin{bmatrix} f_0^k \\ f_1^k \\ \cdot \\ f_{M-1}^k \\ f_M^k \end{bmatrix}_{(M+1)\times 1} = \begin{bmatrix} e^{-t_k} \\ -0.75e^{-t_k}\cos\frac{x_1}{2} \\ \cdot \\ -0.75e^{-t_k}\cos\frac{x_{M-1}}{2} \\ \frac{1}{2}e^{-t_k} \end{bmatrix}_{(M+1)\times 1}.$$

From (16) it follows that

$$u^k = -inv(A)Bu^{k-1} + inv(A)If^k,\quad k = 1,\cdots,N,\quad u^0 = \varphi.$$

for DS (14) and

$$A = \begin{bmatrix} \frac{1}{2}+\frac{1}{2h} & -\frac{1}{2h} & 0 & \cdot & 0 & 0 & 0 \\ a & b & a & \cdot & 0 & 0 & 0 \\ 0 & a & b & \cdot & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & b & a & 0 \\ 0 & 0 & 0 & \cdot & a & b & a \\ 0 & 0 & 0 & \cdot & 0 & \frac{1}{2h} & -\frac{1}{2}-\frac{1}{2h} \end{bmatrix}_{(M+1)\times(M+1)},$$

$$B = \begin{bmatrix} \frac{1}{2}+\frac{1}{2h} & -\frac{1}{2h} & 0 & \cdot & 0 & 0 & 0 \\ a & c & a & \cdot & 0 & 0 & 0 \\ 0 & a & c & \cdot & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & c & a & 0 \\ 0 & 0 & 0 & \cdot & a & c & a \\ 0 & 0 & 0 & \cdot & 0 & \frac{1}{2h} & -\frac{1}{2}-\frac{1}{2h} \end{bmatrix}_{(M+1)\times(M+1)}$$

and

$$a = -\frac{1}{2h^2},\ b = \frac{1}{\tau} + \frac{1}{h^2},\ c = -\frac{1}{\tau} + \frac{1}{h^2},$$

and

$$f^k = \begin{bmatrix} f_0^k \\ f_1^k \\ \cdot \\ f_{M-1}^k \\ f_M^k \end{bmatrix}_{(M+1)\times 1} = \begin{bmatrix} e^{-t_k+\frac{\tau}{2}} \\ -0.75e^{-t_k+\frac{\tau}{2}}\cos\frac{x_1}{2} \\ \cdot \\ -0.75e^{-t_k+\frac{\tau}{2}}\cos\frac{x_{M-1}}{2} \\ \frac{1}{2}e^{-t_k+\frac{\tau}{2}} \end{bmatrix}_{(M+1)\times 1}$$

for DS (14). From (16) it follows that

$$u^k = -inv(A)Bu^{k-1} + inv(A)If^k, \quad k = 1, \cdots, N, \quad u^0 = \varphi.$$

Now, we will give the results of the numerical analysis. We recorded the numerical solutions u_n^k of these difference schemes at (t_k, x_n) for different N and M values. For their comparison, here and future errors are computed by

$$E_u = \max_{\substack{1 \le k \le N \\ 0 \le n \le M}} \left| u(t_k, x_n) - u_n^k \right|.$$

Table 1 demonstrates the error analysis between the exact solution and the solutions derived by the difference scheme. The error of Crank-Nicolson DS is $E_u = O(\tau^2 + h)$. It is constructed for $N = M = 20$, 40 and 80.

Table 1

Error analysis of first order Rothe DS (14)

Error	$N = M = 20$	$N = M = 40$	$N = M = 80$
E_u	0,0076	0,0038	0,0019

Table 2 illustrates the error analysis between the exact solution and the solutions derived by Crank-Nicolson. It is constructed for $N^2 = M = 100$, 400 and 1600.

Table 2

Error analysis of Crank-Nicolson DS (15)

Error	$N = 10$	$N = 20$	$N = 40$
E_u	0,0020	0,00046	0,00011

As it is seen in Tables 1 and 2, if N is multiplied by 2, the value of errors decreases approximately 1/2 for the DS (14) and 1/4 for the Crank-Nicolson DS (15). This shows that DS (15) has the second order of accuracy in time.

Mathematical modeling of the energy consumption problem

The importance of energy-saving and correct design is obvious for energy efficiency. Correct design means that before construction of something as orientation or isolation decisions needs to be made. The energy-saving means things to do during operation as automatic control. An important part of energy consumption occurs in buildings. For decision making, there are numerous studies on this subject all over the world. A national software calculates the energy consumption of buildings according to the TS EN 13790 standard.

The problem is complicated because the energy consumption calculation depends on many variables, such as the external temperature and the heat losses and gains, including the sun radiation change over time. Energy consumption numerical calculations given by the standard is time-consuming. Thus, the mathematical model and theoretical solution are valuable.

In this article, the annual energy consumption mathematical model of a house's room assumed heat loss and gain through the opaque outer wall. The heat conduction differential equation and boundary equations of the one-dimensional, nonstationary boundary value problem are obtained for the outer wall. This study aims at a dynamic model to compare the results of the numerical calculations ([7]). The study also aims to develop a mathematical model in which the result can be found at any time.

In this study, an effective numerical method is developed for energy consumption calculation. The three case problems are taken to test this method.

Case 1. Outer wall with different convection boundary problems; outer wall of a building which is initially 20^oC, is suddenly subjected to the convection boundary condition from the outer surface with air at 0^oC and the convection coefficient $25\,W/m^2K$ while inner temperature 20^oC and inner convection resistance $0.13\,m^2K/W$ are constant. Time-dependent temperature distribution and how long it will take to reach steady-state conditions are needed to be determined. Thermo-physical properties of the wall; $\rho = 2000\,kg/m^3$, $k = 1\,W/mK$, $c = 1000\,J/kgK$.

$$\begin{cases} u_t(t,x) - 5.10^{-7}u_{xx}(t,x) = 0, \ 0 < t < 3600, \ 0 < x < 0.2, \\ u\,(0,x) = 0, \ 0 \le x \le 0.2, \\ u_x\,(t,0) = 25u\,(t,0)\,, \ 0 \le t \le 3600, \\ u_x\,(t,0.2) = 140 - 7u\,(t,0.2)\,, \ 0 \le t \le 3600. \end{cases}$$

Case 2. Time-dependent on outer temperature problem; outer wall of a building which is initially 20^oC, is suddenly subjected to the convection boundary condition from the outer surface with time-dependent air temperature with the convection coefficient $25\,W/m^2K$ while inner temperature 20^oC and inner convection resistance $0.13\,m^2K/W$ are constant. Time-dependent temperature distribution and energy consumption are needed to be determined. Thermo-physical properties of the wall; $\rho = 2000\ kg/m^3$, $k = 1\,W/mK$, $c = 1000\,J/kgK$.

$$\begin{cases} u_t(t,x) - 5.10^{-7}u_{xx}(t,x) = 0, \ 0 < t < 3600, \ 0 < x < 0.2, \\ u\,(0,x) = 20, \ 0 \le x \le 0.2, \\ 25\,(u\,(t,0) - 20|\sin(\pi t/86400)|) = u_x\,(t,0)\,, \ 0 \le t \le 3600, \\ -1.438u\,(t,0.2) + 28.76 = u_x\,(t,0.2)\,, \ 0 \le t \le 3600. \end{cases}$$

Case 3. Time-dependent on outer temperature and solar radiation problems; An outer wall of a building which is initially 20^oC, is suddenly subjected to the convection boundary condition from the outer surface with time-dependent air temperature with the convection coefficient $25\,W/m^2K$ while inner temperature 20^oC and inner convection resistance $0.13\,m^2K/W$ are constant and time-dependent (constant) solar energy gain. Time-dependent temperature distribution and energy consumption are needed to be determined. Thermo-physical properties of the wall; $\rho = 2000\,kg/m^3$, $k = 1\,W/mK$, $c = 1000\,J/kgK$.

$$\begin{cases} u_t(t,x) - 5.10^{-7}u_{xx}(t,x) = f(t), \ 0 < t < 86400, \ 0 < x < 0.2, \\ u\,(0,x) = 20, \ 0 \le x \le 0.2, \\ 25\left(u\,(t,0) - 20\sin^2(\pi t/86400)\right) = u_x\,(t,0)\,, \ 0 \le t \le 3600, \\ 7[20 - u\,(t,0.2)] = u_x\,(t,0.2)\,, \ 0 \le t \le 3600, \end{cases}$$

$$f(t) = \begin{cases} 0, & t \le 21600, \\ 5.10^{-4}\sin^2(\pi t/43200), & 21600 < t < 64800, \\ 0, & 64800 \le t \le 86400. \end{cases}$$

Results

The results are compared with the previous finite-difference or steady-state solutions [7].

Case 1. One layer residence outer wall composed of one material initially is at the homogenously 20^oC. Then suddenly outside air temperature falls 0^oC and stays stable. Wall is 20 cm thick. Wall material properties are wall conduction coefficient $1\,W/mK$ and specific heat $1000\,J/kgK$, density $2000\,kg/m^3$. Heat convection coefficients inner and outer temperatures are 7.69 and $25\,W/m^2K$ respectively. This method's time-dependent results for the wall inner temperature distribution are given in Table 3.

Table 3

Temperature distribution for Case 1

Time(h)		Temperature				$q(W/m^2K)$
	Outside temp.	Outer surface	Mid point	Inner surfce	Inner Temp.	Heat Loss
0	20	20	20	20	20	0
6	0	5.48	14.33	17.53	20	17
12	0	3.50	10.92	15.36	20	35
24	0	2.43	8.26	13.52	20	49
48	0	2.11	7.38	12.55	20	57

The limit of this time-dependent solution for $t \to \infty$ is the steady-state solution, which is shown in Table 4. Steady-state temperature distribution goes to the linear line. Integrating heat loss over time we can get energy consumption rate approximation $2000\,Wh/m^2$.

Table 4

Steady state temperature distribution for Case 1

Time(h)		Temperature				$q(W/m^2K)$
	Outside temp.	Outer surface	Mid point	Inner surfce	Inner Temp.	Heat Loss
0	0	2.16	7.57	12.97	20	54

If we compare Table 3 results with Table 4, steady-state solutions are reasonable.

Case 2. Similar wall with Case 1, subjected this time with variable outer temperature according to $u_{outside}(t) = 20\sin^2(t/24)$ function. The temperature distribution of this wall is found by this method in Table 5 and compared with finite difference solution, Table 6.

Table 5

Temperature distribution for Case 2 variable outside temperature with sin function

Time(h)		Temperature				$q(W/m^2K)$
	Outside temp.	Outer surface	Mid point	Inner surfce	Inner Temp.	Heat Loss
0	20	20	20	20	20	0
6	7	13.93	15.49	17.08	20	22
12	20	18.88	17.05	16.91	20	28
24	0	5.12	13.42	15.87	20	32
48	0	4.91	12.25	14.58	20	41

Table 5 temperatures are over Table 3 temperatures as expected. Energy consumption rate is approximately $1200\,Wh/m^2$.

Table 6

The finite difference temperature distribution for Case 2

Time(h)		Temperature				$q(W/m^2K)$
	Outside temp.	Outer surface	Mid point	Inner surfce	Inner Temp.	Heat Loss
0	20	20	20	20	20	0
6	7	6.05	12.60	16.56	20	26
12	20	17.91	14.80	16.42	20	38
24	0	3.81	11.50	15.89	20	32
48	0	3.77	11.37	15.80	20	32

If the heat losses are integrated over a time period, heat energy consumption can be found.

The finite-difference numerical results of the article [7] for Case 2 are illustrated in Table 6. If we compare this study result of Table 5 with Table 6, then time-dependent solutions are reasonable.

Case 3. Similar wall with Case 1, subjected this time with variable outer temperature according to $u_{outside}(t) = 20\sin^2(t/24)$ function and variable sun radiation with a periodic sin function $6 < t < 18$, $q'' = 20\sin^2(t/24)$ function. The temperature distribution of this wall is found by this method in Table 7 and compared with finite difference solution in Table 8.

Table 7

Temperature distribution for Case 3 variable outside temperature and sun radiation with sin function

Time(h)		Temperature				$q(W/m^2K)$
	Outside temp.	Outer surface	Mid point	Inner surfce	Inner Temp.	Heat Loss
0	20	20	20	20	20	0
6	7	11.65	16.02	18.20	20	13
12	20	19.00	18.11	18.80	20	13
24	0	5.43	11.53	15.65	20	24
48	0	5.34.	11.27	15.44	20	24

Table 7 temperatures exceed Table 3 and Table 5 temperatures as expected. The finite-difference numerical result of the article [7] for Case 3 is pointed out in Table 8. If we compare this study results of Table 7 with Table 8, then time-dependent solutions are reasonable.

Table 8

The finite-difference temperature distribution solution for Case 3 variable outside temperature and sun radiation with sin function for a window and an opaque wall [7]

Time(h)		Temperature				$q(W/m^2K)$
	Outside temp.	Outer surface	Mid point	Inner surfce	Inner Temp.	Heat Loss
0	20	20	20	20	20	0
6	7	6	12.43	16.24	20	32
12	20	18.10	15.65	16.92	20	24
24	0	3.85	11.58	15.74	20	36
48	0	4.13	10.95	13.26	20	36

Conclusions

The energy consumption problem is complicated because the energy consumption calculation depends on many variables, such as the external temperature and the heat losses and gains, including the sun radiation change over time. Energy consumption numerical calculations given by the standard take a lot of time. In the present paper, we have examined the model of the nonstationary energy consumption calculation problems. The theoretical background of this model has been provided. The well-posedness of the mixed problem for parabolic equations with Robin conditions has been studied. The first and second-order accuracy single-step absolute stable difference schemes have been constructed. Well-posedness in Hölder spaces on time of these differential and difference parabolic problems has been established. Finally, these difference schemes have been applied for the energy consumption problems for the heat equations. The developed results are justified.

Acknowledgments

The authors are grateful to Charyyar Ashyralyyev for his support and to reviewers for the useful comments and relevant references.

References

1 Ashyralyev, A., & Sobolevskii, P.E. (1994). *Well-Posedness of Parabolic Difference Equations, Operator Theory Advances, and Applications.* Birkhauser Verlag, Basel, Boston, Berlin.

2 Ashyralyev, A., & Sobolevskii, P.E. (2004). *New Difference Schemes for Partial Differential Equations, Operator Theory Advances, and Applications.* Birkhauser Verlag, Basel, Boston, Berlin.

3 TS EN ISO 13790:2008. (2013). Energy performance of buildings – Calculation of energy use for space heating and cooling.

4 Incropera, F.P., Dewitt, D.P., Bergman, T.L., & Lavine, A.S. (2007). *Fundamentals of Heat and Mass Transfer.* United States, America.

5 Kokogiannakis, G., Strachan, P., & Clarke, J. (2008). Comparison of the simplified methods of the ISO 13790 standard and detailed modeling programs in a regulatory context. *Journal of Building Performance Simulation, 1*(4), 209–219.

6 Michalak, P. (2014). The simple hourly method of EN ISO 13790 standard in Matlab/Simulink: A comparative study for the climatic conditions of Poland. *Energy 75*, 568–578.

7 Koc, I., Sayar, E., & Parmaksizoglu, I.C. (2017). Thermal Modelling of the Building Energy Consumption. *National Thermal System Engineering Congress.*

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Энергияны тұтыну мәселелерін математикалық модельдеу

Энергияны үнемдеу мен дұрыс жобалау энергия тиімділігі үшін маңызды. Дұрыс дизайн дегеніміз — құрылысқа дейін бағдарлау немесе оқшаулау жұмыстарын жасау керек. Бұл зерттеуде бейстационарлық энергияны тұтынуды есептеу есептерінің математикалық моделі ұсынылған, яғни Гельдер кеңістігіндегі Робен шарттары бар аралас бір өлшемді параболалық есептің корректілігі. Авторлар осы математикалық модельге байланысты энергияны тұтынуды есептеудің тиімді сандық әдісін жасаған. Бұл сандық әдісті тексеру үшін үш есеп алынды. Динамикалық модельдің нәтижелері алдыңғы айырымдық немесе стационарлық шешімдермен салыстырылды. Сонымен қатар, зерттеу нәтижені кез келген уақытта табуға болатын математикалық модельді жасауға бағытталған.

Кілт сөздер: математикалық модельдеу, жылуөткізгіштік теңдеуі, айырымдық схемасы, тұрақтылық.

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Математическое моделирование проблемы энергопотребления

Важность энергосбережения и правильного проектирования очевидна для энергоэффективности. Правильный дизайн означает, что перед строительством нужно сделать что-то вроде решения по ориентации или изоляции. В данном исследовании предложена математическая модель задач расчета нестационарного энергопотребления, которая представляет собой корректность в пространствах Гёльдера смешанной одномерной параболической задачи с условиями Робена. Авторами разработан эффективный численный метод расчета энергопотребления, связанный с данной математической моделью. Для проверки этого численного метода взяты три задачи. Результаты динамической модели сравнивались с предыдущими конечно-разностными или стационарными решениями. Кроме того, исследование направлено на разработку математической модели, в которой результат может быть найден в любое время.

Ключевые слова: математическое моделирование, уравнение теплопроводности, разностная схема, устойчивость.

DOI 10.31489/2022M1/25-34

UDC 517.956

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On solvability of boundary value problem for a nonlinear Fredholm integro-differential equation

The paper proposes a constructive method to solve a nonlinear boundary value problem for a Fredholm integro-differential equation. Using D.S. Dzhumabaev parametrization method, the problem under consideration is transformed into an equivalent boundary value problem for a system of nonlinear integro-differential equations with parameters on the subintervals. When applying the parametrization method to a nonlinear Fredholm integro-differential equation, the intermediate problem is a special Cauchy problem for a system of nonlinear integro-differential equations with parameters. By substitution the solution to the special Cauchy problem with parameters into the boundary condition and the continuity conditions of the solution to the original problem at the interior partition points, we construct a system of nonlinear algebraic equations in parameters. It is proved that the solvability of this system provides the existence of a solution to the original boundary value problem. The iterative methods are used to solve both the constructed system of algebraic equations in parameters and the special Cauchy problem. An algorithm for solving boundary value problem under consideration is provided.

Keywords: nonlinear Fredholm integro-differential equation, boundary value problem, special Cauchy problem, iterative process, isolated solution, algorithm, Dzhumabaev parametrization method.

Introduction

The research of initial and boundary value problems (BVPs) for integro-differential equations (IDEs) is devoted to the works of many authors [1–15]. Fredholm IDEs have a number of features that should be taken into account in setting problems for these equations and developing methods for solving them. By D.S. Dzhumabaev parametrization method [16] the new Δ_N general solution to linear Fredholm IDE is proposed in [17], the concept of the general solution is extended to Fredholm IDEs with nonlinear differential parts [18]. In [19–21], the criteria for solvability, unique solvability and conditions of well-posedness of linear boundary value problems for Fredholm IDEs are established.

On $[0,T]$ the boundary value problem for nonlinear Fredholm integro-differential equation (IDE) is ыconsidered:

$$\frac{dx}{dt} = A(t)x + \sum_{k=1}^{m} \varphi_k(t) \int_0^T \psi_k(\tau) f_k(\tau, x(\tau)) d\tau, \quad t \in [0,T], \quad x \in R^n, \tag{1}$$

$$g[x(0), x(T)] = 0, \tag{2}$$

where $n \times n$ matrices $A(t)$, $\varphi_k(t)$, $\psi_k(\tau)$ are continuous on $[0,T]$, $f_k : [0,T] \times R^n \to R^n$, $k = \overline{1,m}$ $\|x\| = \max\limits_{i=\overline{1,n}} |x_i|$.

Denote by $C\big([0,T], R^n\big)$ the space of continuous functions $x : [0,T] \to R^n$ with the norm $\|x\|_1 = \max\limits_{t\in[0,T]} \|x(t)\|$.

A solution to problem (1), (2) is a continuously differentiable on $[0,T]$ (at the points $t = 0$, $t = T$, equation (1) is satisfied by one-sided derivatives) function $x(t) \in C\big([0,T], R^n\big)$, which satisfies equation (1) and boundary condition (2).

The aim of the paper is to propose a constructive method for finding isolated solution to problem (1), (2).

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1 Scheme of the parametrization method

Let Δ_N be a partition of the interval $[0,T]$ into N parts by points $t_0 = 0 < t_1 < \ldots < t_N = T$.

The restriction of the function $x(t)$ on the rth interval $[t_{r-1}, t_r)$ denote by $x_r(t) : x_r(t) = x(t)$, $t \in [t_{r-1}, t_r)$, $r = \overline{1,N}$, and we reduce the problem (1), (2) to equivalent multi-point boundary value problem

$$\frac{dx_r}{dt} = A(t)x_r + \sum_{k=1}^{m} \varphi_k(t) \sum_{j=1}^{N} \int_{t_{r-1}}^{t_r} \psi_k(\tau) f_k(\tau, x_j(\tau)) d\tau, \quad t \in [t_{r-1}, t_r), \quad x_r \in R^n, \quad r = \overline{1,N}, \tag{3}$$

$$g[x_1(0), \lim_{t \to T-0} x_N(t)] = 0, \tag{4}$$

$$\lim_{t \to t_p - 0} x_p(t)] = x_{p+1}(t_p), \quad p = \overline{1, N-1}, \tag{5}$$

where equations (5) are the continuity conditions for solutions to problem (1), (2) at the interior points of partition Δ_N.

Denote by $C\big([0,T], \Delta_N, R^{nN}\big)$ the space consisting of all function systems $x[t] = \big(x_1(t), x_2(t), \ldots, x_N(t)\big)$, where functions $x_r : [t_{r-1}, t_r) \to R^n$, $r = \overline{1,N}$, are continuous and have finite left-sided limits $\lim\limits_{t \to t_r - 0} x_r(t)$, with the norm $\big\|x[\cdot]\big\|_2 = \max\limits_{r=\overline{1,N}} \sup\limits_{t \in [t_{r-1}, t_r)} \|x_r(t)\|$.

A solution to problem (3)–(5) is a function system $x^*[t] = (x_1^*(t), x_2^*(t), \ldots, x_N^*(t)) \in C([0,T], \Delta_N, R^{nN})$, where the function $x^*r(t)$ continuously differentiable on $[t_{r-1}, t_r)$, satisfies equation (3) for all $t \in [t_{r-1}, t_r)$, $r = \overline{1,N}$, and for $x_1^*(0)$, $\lim\limits_{t \to T-0} x_N^*(t)$, $\lim\limits_{t \to t_p - 0} x_p^*(t)$, $x_{p+1}^*(t_p)$, $p = \overline{1, N-1}$, there are equalities (4), (5).

We introduce additional parameters $\lambda_r = x_r(t_{r-1})$ and make substitutions $u_r(t) = x_r(t) - \lambda_r$, $r = \overline{1,N}$, then we obtain the multi-point boundary value problem with parameters

$$\frac{du_r}{dt} = A(t)[u_r + \lambda_r] + \sum_{k=1}^{m} \varphi_k(t) \sum_{j=1}^{N} \int_{t_{r-1}}^{t_r} \psi_k(\tau) f_k(\tau, u_j(\tau) + \lambda_j) d\tau, \quad t \in [t_{r-1}, t_r), \quad r = \overline{1,N}, \tag{6}$$

$$u_r(t_{r-1}) = 0, \quad r = \overline{1,N}, \tag{7}$$

$$g[\lambda_1, \lambda_N + \lim_{t \to T-0} u_N(t)] = 0, \tag{8}$$

$$\lambda_p + \lim_{t \to t_p - 0} u_p(t)] = \lambda_{p+1}, \quad p = \overline{1, N-1}. \tag{9}$$

A solution to problem (6)–(9) is a pair $(\lambda^*, u^*[t])$ with elements $\lambda^* = (\lambda_1^*, \lambda_2^*, \ldots, \lambda_N^*) \in R^{nN}$, $u^*[t] = (u_1^*(t), u_2^*(t), \ldots, u_N^*(t)) \in C([0,T], \Delta_N, R^{nN})$, where the function $u_r^*(t)$ continuously differentiable on $[t_{r-1}, t_r)$ satisfies differential equation (6) for all $t \in [t_{r-1}, t_r)$ (for $t = t_{r-1}$ equation (6) satisfies the right-hand derivative of the function $u_r(t)$,) condition (7), and for λ_1^*, $\lambda_N^* + \lim\limits_{t \to T-0} u_N^*(t)$, $\lambda_p^* + \lim\limits_{t \to t_p - 0} u_p^*(t) = \lambda_{p+1}^*$, $p = \overline{1, N-1}$, equalities (8), (9) hold.

If $(\lambda^*, u^*[t])$ is a solution to problem (6)–(9), then the function $x^*(t)$ defined by the equalities $x^*(t) = \lambda_r^* + u_r^*(t)$, $t \in [t_{r-1}, t_r)$, $r = \overline{1,N}$, $x^*(t) = \lambda_N^* + \lim\limits_{t \to T-0} u_N^*(t)$, is the solution to problem (1), (2). And vice versa, if $\widetilde{x}(t)$ is a solution to problem (1), (2), then the pair $(\widetilde{\lambda}, \widetilde{u}[t])$ with elements $\widetilde{\lambda} = (\widetilde{\lambda}_1, \widetilde{\lambda}_2, \ldots, \widetilde{\lambda}_N) \in R^{nN}$, $\widetilde{u}_1(t), \widetilde{u}_2(t), \ldots, \widetilde{u}_1(t)_N$, where $\widetilde{\lambda}_r = \widetilde{x}(t_{r-1})$, $\widetilde{u}_r(t) = \widetilde{x}(t) - \widetilde{\lambda}_r$, $t \in [t_{r-1}, t_r)$, $r = \overline{1,N}$, is the solution to problem (6)–(9).

Problem (6), (7) is the special Cauchy problem for the system of nonlinear Fredholm IDEs.

2 The solvability of problem (1), (2)

We will use the limit values of the solution to problem (6), (7) later on, when we turn to problem (1), (2). Therefore, it is reasonable to consider the special Cauchy problem on the closed subintervals:

$$\frac{dv_r}{dt} = A(t)[v_r + \lambda_r] + \sum_{k=1}^{m} \varphi_k(t) \sum_{j=1}^{N} \int_{t_{r-1}}^{t_r} \psi_k(\tau) f_k(\tau, v_j(\tau) + \lambda_j) d\tau, \quad t \in [t_{r-1}, t_r], \quad r = \overline{1,N}, \tag{10}$$

$$v_r(t_{r-1}) = 0, \quad r = \overline{1,N}. \tag{11}$$

Denote by $\widetilde{C}\big([0,T],\Delta_N,R^{nN}\big)$ the space consisting of all function systems $v[t] = \big(v_1(t), v_2(t), \ldots, v_N(t)\big)$, where functions $v_r : [t_{r-1}, t_r] \to R^n$, $r = \overline{1,N}$, are continuous, with the norm $\big\|v[\cdot]\big\|_3 = \max\limits_{r=\overline{1,N}} \max\limits_{t\in[t_{r-1},t_r]} \|v_r(t)\|$.

It is obvious that if for fixed value of parameter $\lambda = \widehat{\lambda}$ the function systems $u[t,\widehat{\lambda}]$ and $v[t,\widehat{\lambda}]$ are the solutions to problems (6), (7) and (10), (11) respectively, then the following equalities are valid:

$$u_r(t,\widehat{\lambda}) = v_r(t,\widehat{\lambda}), \quad t \in [t_{r-1}, t_r), \quad r = \overline{1,N}, \tag{12}$$

$$\lim_{t\to t_r-0} u_r(t,\widehat{\lambda}) = v_r(t_r,\widehat{\lambda}), \quad r = \overline{1,N}. \tag{13}$$

The sufficient conditions of the solvability problem (10), (11) are established in [22].

Employing (8), (9) and considering (12), (13), we get the system of nonlinear algebraic equations in parameters

$$g[\lambda_1, \lambda_N + v_N(t,\lambda_1,\lambda_2,\ldots,\lambda_N)] = 0, \tag{14}$$

$$\lambda_p + v_p(t,\lambda_1,\lambda_2,\ldots,\lambda_N)] = \lambda_{p+1}, \quad p = \overline{1,N-1}. \tag{15}$$

We rewrite system (14), (15) in the following form:

$$Q_*(\Delta_N, \lambda, v[t]) = 0. \tag{16}$$

Condition A. There exists $h > 0 : Nh = T$, $N \in \mathbb{N}$, such that the system of nonlinear equations $Q_*(\Delta_N,\lambda,0) = 0$ has the solution $\lambda^{(0)} = (\lambda_1^{(0)}, \lambda_2^{(0)}, \ldots, \lambda_N^{(0)}) \in R^{nN}$, and for $\lambda = \lambda^{(0)}$ the special Cauchy problem (10), (11) with the initial guess solution $v^{(0,0)}[t] = (0,0,\ldots,0)$, has the solution $v[t,\lambda^{(0)}] \in \widetilde{C}([0,T],\Delta_N,R^{nN})$.

Denote by $PC([0,T],\Delta_N,R^n)$ the space of piecewise continuous functions $x : [0,T] \to R^n$ with the possible discontinuity points t_s, $s = \overline{1,N-1}$, with the norm $\|x\|_4 = \sup\limits_{t\in[0,T]} \|x(t)\|$.

By the equalities $x^{(0)}(t) = \lambda_r^{(0)} + v_r^{(0)}(t)$, $t \in [t_{r-1}, t_r]$, $r = \overline{1,N}$, we define the piecewise continuous function $x^{(0)}(t)$ on $[0,T]$.

Choose $\rho_\lambda > 0$, $\rho_v > 0$, $\rho_x > 0$ we construct sets:

$$S(\lambda^{(0)},\rho_\lambda) = \{\lambda = (\lambda_1,\lambda_2,\ldots,\lambda_N) \in R^{nN} : \|\lambda - \lambda^{(0)}\| = \max_{r=\overline{1,N}} \|\lambda_r - \lambda_r^{(0)}\| < \rho_\lambda\},$$

$$S(v[t,\lambda^{(0)}],\rho_v) = \{v[t] \in C([0,T],\Delta_N,R^{nN}) : \|v[\cdot] - v[\cdot,\lambda^{(0)}]\|_2 < \rho_v\},$$

$$S(x^{(0)}(t),\rho_x) = \{x(t) \in PC([0,T],\Delta_N,R^n) : \|x - x^{(0)}\|_4 < \rho_x\}.$$

Theorem 1. Let $\lambda^* \in S(\lambda^{(0)},\rho_\lambda)$ be a solution to equation (16) and $v[t,\lambda^*] \in S(v[t,\lambda^{(0)}],\rho_v)$ be a solution to the special Cauchy problem (10), (11) for $\lambda = \lambda^*$. Then the function $x^*(t)$, defined by the equalities $x^*(t) = \lambda_r^* + v_r(t,\lambda^*)$, $t \in [t_{r-1}, t_r]$, $r = \overline{1,N}$, is a solution to problem (1), (2) and $x^*(t) \in S(x^{(0)}(t),\rho_x)$.

Using Theorem 2 [23; 45] to equation (16) we get the following assertion.

Theorem 2. Let the following conditions be fulfilled:

(i) the Jacobi matrix $\dfrac{\partial Q_*(\Delta_N;\lambda,v[t])}{\partial\lambda}$ is uniformly continuous in $S(\lambda^{(0)},\rho_\lambda)$;

(ii) $\dfrac{\partial Q_*(\Delta_N;\lambda,v[t])}{\partial\lambda}$ is invertible and $\Big\|\Big[\dfrac{\partial Q_*(\Delta_N;\lambda,v[t])}{\partial\lambda}\Big]^{-1}\Big\| \le \gamma^*$ for all $\lambda \in S(\lambda^{(0)},\rho_\lambda)$, γ^* is constant;

(iii) $\gamma^*\big\|Q_*(\Delta_N;\lambda^{(0)},v^{(0)}[t])\big\| < \rho_\lambda$.

Then there exists $\alpha_0 \ge 1$ such that for any $\alpha \ge \alpha_0$ the sequence $\lambda^{(k)}$, generated by the iterative process

$$\lambda^{(k+1)} = \lambda^{(k)} - \frac{1}{\alpha}\left[\frac{\partial Q_*(\Delta_N;\lambda^{(k)},v[t,\lambda^{(k)}])}{\partial\lambda}\right]^{-1} Q_*(\Delta_N;\lambda^{(k)},v[t,\lambda^{(k)}]), \quad k = 0,1,2,\ldots,$$

converges to λ^*, an isolated solution to equation (16) in $S(\lambda^{(0)},\rho_\lambda)$, and

$$\|\lambda^* - \lambda^{(0)}\| \le \big\|Q_*(\Delta_N;\lambda^{(0)},v[t,\lambda^{(0)}])\big\|.$$

Condition B. The functions $f_k(t,x)$, $k=\overline{1,m}$, $g(v,w)$ have uniformly continuous partial derivatives $\dfrac{\partial f_k(t,x)}{\partial x}$, $\dfrac{\partial g(v,w)}{\partial v}$, $\dfrac{\partial g(v,w)}{\partial w}$, in $G^{0,1}(\rho_x)=\{(t,x): t\in[0,T], \|x-x^{(0)}(t)\|<\rho_x\}$, $G^{0,2}(\rho_x)=\{(v,w)\in R^{2n}: \|v-x^{(0)}(0)\|<\rho_x, \|w-x^{(0)}(T)\|<\rho_x\}$ respectively, and

$$\Big\|\frac{\partial f_k(t,x)}{\partial x}\Big\|\le L_{k,0},\quad \Big\|\frac{\partial g(v,w)}{\partial v}\Big\|\le L_1,\quad \Big\|\frac{\partial g(v,w)}{\partial w}\Big\|\le L_2,$$

where $L_{k,0}$, $k=\overline{1,m}$, L_1, L_2 are const.

Let the function system $v[t,\lambda]$ be the solution to the special Cauchy problem (10), (11), i.e. the following equalities are valid:

$$\frac{dv_r(t,\lambda)}{dt}=A(t)[v_r(t,\lambda)+\lambda_r]+\sum_{k=1}^m\varphi_k(t)\sum_{j=1}^N\int_{t_{r-1}}^{t_r}\psi_k(\tau)f_k(\tau,v_j(\tau,\lambda)+\lambda_j)d\tau,\quad t\in[t_{r-1},t_r],\quad r=\overline{1,N}, \tag{17}$$

$$v_r(t_{r-1},\lambda)=0,\quad r=\overline{1,N}. \tag{18}$$

Condition B by Peano's theorem provides the existence of partial derivatives $\dfrac{\partial v_r(t,\lambda)}{\partial\lambda_i}$, $r,i=\overline{1,N}$, for all $\lambda\in S(\lambda^{(0)},\rho_\lambda)$. Differentiating (17), (18) with respect to λ_i, $i=\overline{1,N}$, yields

$$\frac{d}{dt}\Big(\frac{\partial v_r(t,\lambda)}{\partial\lambda_i}\Big)=A(t)\Big[\frac{\partial v_r(t,\lambda)}{\partial\lambda_i}+\sigma_{ri}\Big]+\sum_{k=1}^m\varphi_k(t)\sum_{j=1}^N\int_{t_{r-1}}^{t_r}\psi_k(\tau)\frac{\partial f_k(\tau,v_j(\tau,\lambda)+\lambda_j)}{\partial x}\frac{\partial v_j(\tau,\lambda)}{\partial\lambda_i}d\tau+$$
$$+\sum_{k=1}^m\varphi_k(t)\int_{t_{i-1}}^{t_i}\psi_k(\tau)\frac{\partial f_k(\tau,v_i(\tau,\lambda)+\lambda_i)}{\partial x}d\tau,\quad t\in[t_{r-1},t_r],\quad r=\overline{1,N},$$

$$\frac{\partial v_r(t_{r-1},\lambda)}{\partial\lambda_i}=0,\quad r,i=\overline{1,N},$$

where

$$\sigma_{ri}=\begin{cases}I, & r=i,\quad I \text{ is the identity matrix of dimension } n,\\ O, & r\ne i,\quad O \text{ is the } n\times n \text{ zero matrix.}\end{cases}$$

If we denote by $z_{ri}(t,\lambda)$ the partial derivative $\dfrac{\partial v_r(t,\lambda)}{\partial\lambda_i}=0$, $r,i=\overline{1,N}$, then for each $i=\overline{1,N}$ the function system $z_i[t,\lambda]=(z_1(t,\lambda),\ldots,(z_N(t,\lambda))$ is a solution to the linear special matrix Cauchy problem

$$\frac{dz_{ri}}{dt}=A(t)\big[z_{ri}+\sigma_{ri}\big]+\sum_{k=1}^m\varphi_k(t)\sum_{j=1}^N\int_{t_{r-1}}^{t_r}\psi_k(\tau)\frac{\partial f_k(\tau,v_j(\tau,\lambda)+\lambda_j)}{\partial x}z_{ji}(\tau,\lambda)d\tau+$$
$$+\sum_{k=1}^m\varphi_k(t)\int_{t_{i-1}}^{t_i}\psi_k(\tau)\frac{\partial f_k(\tau,v_i(\tau,\lambda)+\lambda_i)}{\partial x}d\tau,\quad t\in[t_{r-1},t_r],\quad r=\overline{1,N},$$

$$z_{ri}(t_{r-1},\lambda)=0,\quad r,i=\overline{1,N}.$$

Condition B and conditions of Theorem 2 [22] provide the existence of the Jacobi matrix

$$\frac{\partial Q_*(\Delta_N;\widehat{\lambda},\widehat{v}[t])}{\partial\lambda}=\begin{pmatrix}q_{1,1}(\widehat{\lambda}) & \ldots & q_{1,N-1}(\widehat{\lambda}) & q_{1,N}(\widehat{\lambda})\\ q_{2,1}(\widehat{\lambda}) & \ldots & q_{2,N-1}(\widehat{\lambda}) & q_{2,N}(\widehat{\lambda})\\ \ldots & \ldots & \ldots & \ldots\\ q_{N,1}(\widehat{\lambda}) & \ldots & q_{N,N-1}(\widehat{\lambda}) & q_{N,N}(\widehat{\lambda})\end{pmatrix} \tag{19}$$

for all $\widehat{\lambda}\in S(\lambda^{(0)},\rho_\lambda)$ and its uniform continuity in $S(\lambda^{(0)},\rho_\lambda)$. Here the components of $\dfrac{\partial Q_*(\Delta_N;\widehat{\lambda},v[t,\widehat{\lambda}])}{\partial\lambda}$ are the $n\times n$ matrices

$$q_{1,1}(\widehat{\lambda})=g_v'\big[\widehat{\lambda}_1,\widehat{\lambda}_N+v_N(T,\widehat{\lambda})\big]+g_w'\big[\widehat{\lambda}_1,\widehat{\lambda}_N+v_N(T,\widehat{\lambda})\big]z_{N,1}(T,\widehat{\lambda}),$$

$$q_{1,s}\big(\widehat{\lambda}\big) = g'_w\big[\widehat{\lambda}_1, \widehat{\lambda}_N + v_N(T,\widehat{\lambda})\big] z_{N,s}\big(T,\widehat{\lambda}\big), \quad s = \overline{2, N-1},$$

$$q_{1,N}\big(\widehat{\lambda}\big) = g'_w\big[\widehat{\lambda}_1, \widehat{\lambda}_N + v_N(T,\widehat{\lambda})\big]\big[I + z_{N,N}(T,\widehat{\lambda})\big],$$

$$q_{p,r}\big(\widehat{\lambda}\big) = z_{p-1,r}(t_{p-1},\widehat{\lambda}), \quad p \neq r, \quad p \neq r+1,$$

$$q_{p,p}\big(\widehat{\lambda}\big) = -I + z_{p-1,p}\big(t_{p-1},\widehat{\lambda}\big), \quad q_{p,p-1}\big(\widehat{\lambda}\big) = I + z_{p-1,p-1}\big(t_{p-1},\widehat{\lambda}\big), \quad p = \overline{2,N}, \; r = \overline{1,N},$$

where $z_i[t,\widehat{\lambda}] = \big(z_{1,i}(t,\widehat{\lambda}), \ldots, z_{N,i}(t,\widehat{\lambda})\big)$, $i = \overline{1,N}$, is the solution to the special Cauchy problem (10), (11) for $\lambda = \widehat{\lambda}$.

3 An algorithm for solving problem (6)–(9)

Assume that the conditions A, B hold and problem (10), (11) is well-posedness. For the initial approximation of the solution to problem (6)–(9), we take a pair $(\lambda^{(0)}, v[t,\lambda^{(0)}])$ and find the sequence $(\lambda^{(k)}, v[t,\lambda^{(k)}])$, $k \in \mathbb{N}$, according to the following algorithm:

Step 1. a) Employing the values of elements of the function system $v[t,\lambda^{(0)}]$, we compose the n vector and the $n \times n$ matrices:

$$Q_*\big(\Delta_N; \lambda^{(0)}, v[t,\lambda^{(0)}]\big) = \begin{pmatrix} g\big[\lambda_1^{(0)}, \lambda_N^{(0)} + v_N\big(T,\lambda^{(0)}\big)\big] \\ \lambda_1^{(0)} + v_1\big(t_1,\lambda^{(0)}\big) - \lambda_2^{(0)} \\ \cdots \\ \lambda_{N-1}^{(0)} + v_{N-1}\big(t_{N-1},\lambda^{(0)}\big) - \lambda_N^{(0)} \end{pmatrix},$$

$$P_{ri}^{(0)}(t) = A(t)\sigma_{ri} + \sum_{k=1}^{m} \varphi_k(t) \int\limits_{t_{i-1}}^{t_i} \psi_k(\tau) \frac{\partial f_k(\tau, v_i(\tau,\lambda^{(0)}) + \lambda_i^{(0)})}{\partial x} d\tau, \quad t \in [t_{r-1}, t_r], \quad r, i = \overline{1,N},$$

$$\Psi_{kj}^{(0)}(t) = \psi_k(t) \frac{\partial f_k(t, v_j(t,\lambda^{(0)}) + \lambda_j^{(0)})}{\partial x}, \quad t \in [t_{r-1}, t_r], \quad r, j = \overline{1,N}.$$

b) By solving N special matrix Cauchy problems for the system of linear IDEs

$$\frac{dz_{ri}}{dt} = A(t) z_{ri} + \sum_{k=1}^{m} \varphi_k(t) \sum_{j=1}^{N} \int_{t_{j-1}}^{t_j} \Psi_{kj}^{(0)}(t) z_{ji}(\tau) d\tau + P_{ri}^{(0)}(t), \quad t \in [t_{r-1}, t_r],$$

$$z_{ri}(t_{r-1}) = 0, \quad r, i = \overline{1,N},$$

we find the function systems

$$z_i\big[t,\lambda^{(0)}\big] = \big(z_{1i}(t,\lambda^{(0)}), \ldots, z_{Ni}(t,\lambda^{(0)})\big), \quad i = \overline{1,N}.$$

c) Construct the Jacobi matrix $\dfrac{\partial Q_*\big(\Delta_N; \lambda^{(0)}, v[t,\lambda^{(0)}]\big)}{\partial \lambda}$ by formula (19), where

$$q_{1,1}\big(\lambda^{(0)}\big) = g'_v\Big[\lambda_1^{(0)}, \lambda_N^{(0)} + v_N\big(T,\lambda^{(0)}\big)\Big] + g'_w\Big[\lambda_1^{(0)}, \lambda_N^{(0)} + v_N\big(T,\lambda^{(0)}\big)\Big] z_{N,1}\big(T,\lambda^{(0)}\big),$$

$$q_{1,s}\big(\lambda^{(0)}\big) = g'_w\Big[\lambda_1^{(0)}, \lambda_N^{(0)} + v_N\big(T,\lambda^{(0)}\big)\Big] z_{N,s}\big(T,\lambda^{(0)}\big), \quad s = \overline{2, N-1},$$

$$q_{1,N}\big(\lambda^{(0)}\big) = g'_w\Big[\lambda_1^{(0)}, \lambda_N^{(0)} + v_N\big(T,\lambda^{(0)}\big)\Big] \times \Big[I + z_{N,N}\big(T,\lambda^{(0)}\big)\Big],$$

$$q_{p,r}\big(\lambda^{(0)}\big) = z_{p-1,r}\big(t_{p-1},\lambda^{(0)}\big), \quad p \neq r, \quad p \neq r+1,$$

$$q_{p,p}\big(\lambda^{(0)}\big) = -I + z_{p-1,p}\big(t_{p-1},\lambda^{(0)}\big), \quad q_{p,p-1}\big(\lambda^{(0)}\big) = I + z_{p-1,p-1}\big(t_{p-1},\lambda^{(0)}\big), \quad p = \overline{2,N}, \quad r = \overline{1,N}.$$

Solve the system of linear algebraic equations

$$\frac{\partial Q_*\big(\Delta_N; \lambda^{(0)}, v[t,\lambda^{(0)}]\big)}{\partial \lambda} \Delta\lambda = -\frac{1}{\alpha} Q_*(\Delta_N; \lambda^{(0)}, v[t,\lambda^{(0)}]), \quad \Delta\lambda \in R^{nN},$$

for some $\alpha \geq 1$ and find $\Delta\lambda^{(0)}$. Determine $\lambda^{(1)}$ as follows:

$$\lambda^{(1)} = \lambda^{(0)} + \Delta\lambda^{(0)}.$$

d) Choose the function system $v\big[t, \lambda^{(0)}\big]$ as an initial guess solution to problem (10), (11) for $\lambda = \lambda^{(1)}$, and by iterative process [22; 53] find the function system $v\big[t, \lambda^{(1)}\big]$.

Step 2. a) Employing the values of elements of the function system $v[t, \lambda^{(1)}]$, we compose the n vector and the $n \times n$ matrices:

$$Q_*\big(\Delta_N; \lambda^{(1)}, v[t, \lambda^{(1)}]\big) = \begin{pmatrix} g\big[\lambda_1^{(1)}, \lambda_N^{(1)} + v_N\big(T, \lambda^{(1)}\big)\big] \\ \lambda_1^{(1)} + v_1\big(t_1, \lambda^{(1)}\big) - \lambda_2^{(1)} \\ \dots \\ \lambda_{N-1}^{(1)} + v_{N-1}\big(t_{N-1}, \lambda^{(1)}\big) - \lambda_N^{(1)} \end{pmatrix},$$

$$P_{ri}^{(1)}(t) = A(t)\sigma_{ri} + \sum_{k=1}^{m} \varphi_k(t) \int\limits_{t_{i-1}}^{t_i} \psi_k(\tau) \frac{\partial f_k(\tau, v_i(\tau, \lambda^{(1)}) + \lambda_i^{(1)})}{\partial x} d\tau, \quad t \in [t_{r-1}, t_r], \quad r, i = \overline{1, N},$$

$$\Psi_{kj}^{(1)}(t) = \psi_k(t) \frac{\partial f_k(t, v_j(t, \lambda^{(1)}) + \lambda_j^{(1)})}{\partial x}, \quad t \in [t_{r-1}, t_r], \quad r, j = \overline{1, N}.$$

b) By solving N special matrix Cauchy problems for the system of linear IDEs

$$\frac{dz_{ri}}{dt} = A(t)z_{ri} + \sum_{k=1}^{m} \varphi_k(t) \sum_{j=1}^{N} \int_{t_{j-1}}^{t_j} \Psi_{kj}^{(1)}(t) z_{ji}(\tau) d\tau + P_{ri}^{(1)}(t), \quad t \in [t_{r-1}, t_r],$$

$$z_{ri}(t_{r-1}) = 0, \quad r, i = \overline{1, N},$$

we find the function systems

$$z_i\big[t, \lambda^{(1)}\big] = \big(z_{1i}(t, \lambda^{(1)}), \dots, z_{Ni}(t, \lambda^{(1)})\big), \quad i = \overline{1, N}.$$

c) Construct the Jacobi matrix $\dfrac{\partial Q_*\big(\Delta_N; \lambda^{(1)}, v[t, \lambda^{(1)}]\big)}{\partial \lambda}$ by formula (19), where

$$q_{1,1}\big(\lambda^{(1)}\big) = g'_v\Big[\lambda_1^{(1)}, \lambda_N^{(1)} + v_N\big(T, \lambda^{(1)}\big)\Big] + g'_w\Big[\lambda_1^{(1)}, \lambda_N^{(1)} + v_N\big(T, \lambda^{(1)}\big)\Big] z_{N,1}\big(T, \lambda^{(1)}\big),$$

$$q_{1,s}\big(\lambda^{(1)}\big) = g'_w\Big[\lambda_1^{(1)}, \lambda_N^{(1)} + v_N\big(T, \lambda^{(1)}\big)\Big] z_{N,s}\big(T, \lambda^{(1)}\big), \quad s = \overline{2, N-1},$$

$$q_{1,N}\big(\lambda^{(1)}\big) = g'_w\Big[\lambda_1^{(1)}, \lambda_N^{(1)} + v_N\big(T, \lambda^{(1)}\big)\Big] \times \Big[I + z_{N,N}\big(T, \lambda^{(0)}\big)\Big],$$

$$q_{p,r}\big(\lambda^{(1)}\big) = z_{p-1,r}\big(t_{p-1}, \lambda^{(1)}\big), \quad p \neq r, \quad p \neq r+1, \quad q_{p,p}\big(\lambda^{(1)}\big) = -I + z_{p-1,p}\big(t_{p-1}, \lambda^{(1)}\big),$$

$$q_{p,p-1}\big(\lambda^{(1)}\big) = I + z_{p-1,p-1}\big(t_{p-1}, \lambda^{(1)}\big), \quad p = \overline{2, N}, \quad r = \overline{1, N}.$$

Solve the system of linear algebraic equations

$$\frac{\partial Q_*\big(\Delta_N; \lambda^{(1)}, v[t, \lambda^{(1)}]\big)}{\partial \lambda} \Delta\lambda = -\frac{1}{\alpha} Q_*(\Delta_N; \lambda^{(1)}, v[t, \lambda^{(1)}]), \quad \Delta\lambda \in R^{nN},$$

for some $\alpha \geq 1$ and find $\Delta\lambda^{(1)}$. Determine $\lambda^{(2)}$ as follows:

$$\lambda^{(2)} = \lambda^{(1)} + \Delta\lambda^{(1)}.$$

d) Choose the function system $v\big[t, \lambda^{(1)}\big]$ as an initial guess solution to problem (10), (11) for $\lambda = \lambda^{(2)}$, find the function system $v\big[t, \lambda^{(2)}\big]$.

Continuing this process, in the kth step of the algorithm we get a pair $\big(\lambda^{(k)}, v\big[t, \lambda^{(k)}\big]\big)$, $k = 1, 2, \dots$. The conditions of Theorem 2 [22; 53] and Theorem 2 ensure the convergence of this sequence to $\big(\lambda^*, v\big[t, \lambda^*\big]\big)$, the solution to problem (6)–(9), as $k \to \infty$.

Given $\varepsilon > 0$, the iterative process should be terminated if $\|\Delta\lambda^{(k)}\| < \varepsilon$. Theorem 2 yields an alternative termination criterion. If conditions of this theorem are fulfilled, then the inequality $\|\lambda^* - \lambda^{(k)}\| \leq \gamma^* \|Q_*(\Delta_N, \lambda^{(k)}, v[t, \lambda^{(k)}])\|$ is true. Therefore, the iterative process terminates if $\gamma^* \|Q_*(\Delta_N, \lambda^{(k)}, v[t, \lambda^{(k)}])\| < \varepsilon$.

Acknowledgments

This research is funded by the Science Committee of the Ministry of Education and Science of the Republic of Kazakhstan, Grant No. AP 09258829.

References

1 Быков Я.В. О некоторых задачах теории интегро-дифференциальных уравнений / Я.В. Быков. — Фрунзе: Киргиз. гос. ун-т, 1957. — 327 с.

2 Dehghan M. Rational pseudospectral approximation to the solution of a nonlinear integro-differential equation arising in modeling of the population growth / M. Dehghan, M. Shahini // Applied Mathematical Modelling. — 2015. — Vol. 39. — Iss. 3(18). — P. 5521–5530. DOI: https://doi.org/10.1016/j.apm.2015.01.001

3 Lakshmikantham V. Theory of Integro-Differential Equations / V. Lakshmikantham, M.R.M. Rao. — London: Gordon and Breach, 1995. — 375 p.

4 Pruss J. Evolutionary Integral equations and Applications / J. Pruss. — Basel etc.: Birkhäuser Verlag, 1993. — 366 p.

5 Boichuk A.A. Generalized inverse operators and Fredholm boundary value problems / A.A. Boichuk, A.M. Samoilenko. — Boston: SPV, Utrecht, 2004. — 333 p.

6 Dehghan M. The numerical solution of the non-linear integro-differential equations based on the meshless method / M. Dehghan, R. Salehi // J. Comput. Appl. Math. — 2012. — Vol. 236. — Iss.9. — P. 2367–2377. DOI: https://doi.org/10.1016/j.cam.2011.11.022

7 Kheybari S. A semi-analytical algorithm to solve systems of integro-differential equations under mixed boundary conditions / S. Kheybari, M.T. Darvishi, A.M. Wazwaz // J. Comput. Appl. Math. — 2017. — Vol. 317. — P. 72–89. DOI: https://doi.org/10.1016/j.cam.2016.11.029

8 Maleknejad K. Hybrid Legendre polynomials and Block-Pulse functions approach for nonlinear Volterra–Fredholm integro-differential equations / K. Maleknejad, B. Basirat, E. Hashemizadeh // Comput. Math. Appl. — 2013. — Vol. 61. — P. 2821–2828. DOI: https://doi.org/10.1016/j.camwa.2011.03.055

9 Pedas A. A discrete collocation method for Fredholm integro-differential equations with weakly singular kernels / A. Pedas, E. Tamme // Appl. Numer. Math. — 2011. — Vol. 61. — Iss. 6. — P. 738–751. DOI: https://doi.org/10.1016/j.apnum.2011.01.006

10 Shakeri F. A high order finite volume element method for solving elliptic partial integro-differential equations / F. Shakeri, M. Dehghan // Appl. Numer. Math. — 2013. — Vol. 65. — P. 105-118. DOI: https://doi.org/10.1016/j.apnum.2012.10.002

11 Susahab D.N. Efficient quadrature rules for solving nonlinear fractional integro-differential equations of the Hammerstein type / D.N. Susahab, S. Shahmorad, M. Jahanshahi // Appl. Numer. Math. — 2015. — Vol. 39. — Iss. 18. — P. 5452–5458. DOI: https://doi.org/10.1016/j.apm.2015.01.008

12 Turkyilmazoglu M. High-order nonlinear Volterra–Fredholm-Hammerstein integro-differential equations and their effective computation / M. Turkyilmazoglu // Appl. Math. Comput. — 2014. — Vol. 247. — P. 410–416. DOI: https://doi.org/10.1016/j.amc.2014.08.074

13 Yulan M. New algorithm for second order boundary value problems of integro-differential equation / W. Yulan, T. Chaolu, P. Ting // J. Comput. Appl. Math. — 2009. — Vol. 229. — P. 1–6. DOI: https://doi.org/10.1016/j.cam.2008.10.007

14 Zhao J. Compact finite difference methods for high order integro-differential equations / J. Zhao // Appl. Math. Comput. — 2013. — No. 221. — P. 66–78.

15 Wazwaz A.M. Linear and Nonlinear Integral Equations [Methods and Applications] / A.M. Wazwaz. — Higher Equacation Press, Beijing and Springer-Verlag Berlin Heidelberg, 2011. — 639 p.

16 Dzhumabaev D.S. Criteria for the unique solvability of a linear boundary-value problem for an ordinary differential equation / D.S. Dzhumabaev // — 1989. — No. 29. — P. 34–46.

17 Dzhumabaev D.S. New general solutions to linear Fredholm integro-differential equations and their applications on solving the boundary value problems / D.S. Dzhumabaev // J. Comput. Appl. Math. — 2018. — Vol. 327. — P. 79–108. DOI: https://doi.org/10.1016/j.cam.2017.06.010

18 Dzhumabaev D.S. New general solution to a nonlinear Fredholm integro-differential equation / D.S. Dzhumabaev, S.T. Mynbayeva // Eurasian Mathematical Journal. — 2019. — Vol. 10. — No. 4. — P. 23—32. DOI: https://doi.org/10.32523/2077-9879-2019-10-4-23-32

19 Dzhumabaev D.S. A Method for Solving the Linear Boundary Value Problem for an Integro Differential Equation / D.S. Dzhumabaev // Comput. Math. Math. Phys. — 2010 — Vol. 50. — No. 7. — P. 1150–1161. DOI: https://doi.org/10.1134/S0965542510070043

20 Dzhumabaev D.S. Necessary and Sufficient Conditions for the Solvability of Linear Boundary-Value Problems for the Fredholm Integro-differential Equations / D.S. Dzhumabaev // Ukrainian Mathematical Journal. — 2015. — Vol. 66. — No. 8. — P. 1200–1219. DOI: https://doi.org/10.1007/s11253-015-1003-6

21 Dzhumabaev D.S. On one approach to solve the linear boundary value problems for Fredholm integro-differential equations / D.S. Dzhumabaev // J. Comput. Appl. Math. — 2016. — Vol. 294. — P. 342–357. DOI: https://doi.org/10.1016/j.cam.2015.08.023

22 Dzhumabaev D.S. Iterative method for solving special Cauchy problem for the system of integro-differential equations with nonlinear integral part / D.S. Dzhumabaev, S.G. Karakenova // Kazakh Mathematical Journal. — 2019. — Vol. 19. — No. 2.— P. 49–58.

23 Dzhumabaev D.S. A parametrization method for solving nonlinear two-point boundary value problems / D.S. Dzhumabaev, S.M. Temesheva // Comput. Math. Math. Phys. — 2007. — No. 47. — P. 37–61. DOI: 10.1134/S096554250701006X

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Фредгольм сызықты емес интегралдық-дифференциалдық теңдеуі үшін шеттік есептің шешілімділігі туралы

Мақалада Фредгольм интегралдық-дифференциалдық теңдеуі үшін сызықты емес шеттік есепті шешудің конструктивті әдісі ұсынылған. Д.С. Джумабаевтың параметрлеу әдісін қолдана отырып, қарастырылып отырған есеп ішкі интервалдардағы параметрлі сызықты емес интегралдық-дифференциалдық теңдеулер жүйесі үшін эквивалентті шеттік есепке келтірілген. Фредгольм сызықты емес интегралдық-дифференциалдық теңдеуіне параметрлеу әдісін қолданған кезде аралық есеп параметрлі сызықты емес интегралдық-дифференциалдық теңдеулер жүйесі үшін арнайы Коши есебі болып табылады. Арнайы Коши есебінің шешімін шеттік шартқа және бастапқы есептің шешімін бөліктеудің ішкі нүктелеріндегі үзіліссіздік шарттарына қою арқылы параметрлерге қатысты сызықты емес алгебралық теңдеулер жүйесі құрылады. Бұл жүйенің шешілімділігі бастапқы шеттік есептің шешілімділігін қамтамасыз ететіндігіне негізделген. Параметрлерге қатысты алгебралық теңдеулер жүйесін және арнайы Коши есебін шешу үшін итерациялық әдістер қолданылады. Қарастырылып отырған шеттік есепті шешу алгоритмі ұсынылған.

Кілт сөздер: сызықты емес Фредгольм интегралдық-дифференциалдық теңдеуі, шеттік есеп, итерациялық процесс, оқшауланған шешім, алгоритм, Джумабаевтың параметрлеу әдісі.

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О разрешимости краевой задачи для нелинейного интегро-дифференциального уравнения Фредгольма

В статье предложен конструктивный метод решения нелинейной краевой задачи для интегро-дифференциального уравнения Фредгольма. С помощью метода параметризации Д.С. Джумабаева рассматриваемая задача преобразована в эквивалентную краевую задачу для системы нелинейных интегро-дифференциальных уравнений с параметрами на подынтервалах. При применении метода параметризации к нелинейному интегро-дифференциальному уравнению Фредгольма промежуточной задачей является специальная задача Коши для системы нелинейных интегро-дифференциальных уравнений с параметрами. Путем подстановки решения специальной задачи Коши с параметрами в граничное условие и условия непрерывности решения исходной задачи в точках внутреннего разбиения строится система нелинейных алгебраических уравнений по параметрам. Доказано, что разрешимость этой системы обеспечивает существование решения исходной краевой задачи. Итерационные методы использованы как для решения построенной системы алгебраических уравнений по параметрам, так и для решения специальной задачи Коши. Приведен алгоритм решения рассматриваемой краевой задачи.

Ключевые слова: нелинейное интегро-дифференциальное уравнение Фредгольма, краевая задача, специальная задача Коши, итерационный процесс, изолированное решение, алгоритм, метод параметризации Джумабаева.

References

1 Bykov, Ya.V. (1957). *O nekotorykh zadachakh teorii integro-differentsialnykh uravnenii [On some problems in the theory of integro-differential equations].* Frunze: Kirgizskii gosudarstvennyi unversitet [in Russian].

2 Dehghan, M., & Shahini, M. (2015). Rational pseudospectral approximation to the solution of a nonlinear integro-differential equation arising in modeling of the population growth. *Applied Mathematical Modelling, 3*(18), 5521–5530. DOI: https://doi.org/10.1016/j.apm.2015.01.001.

3 Lakshmikantham, V., & Rao, M.R.M. (1995). *Theory of Integro-Differential Equations.* London: Gordon and Breach.

4 Pruss, J. (1993). *Evolutionary Integral equations and Applications.* Basel etc.: Birkhäuser Verlag.

5 Boichuk, A.A., & Samoilenko, A.M. (2004). *Generalized inverse operators and Fredholm boundary value problems.* Boston: SPV, Utrecht.

6 Dehghan, M., & Salehi, R. (2012). The numerical solution of the non-linear integro-differential equations based on the meshless method. *J. Comput. Appl. Math., 236*(9), 2367–2377. DOI: https://doi.org/10.1016/j.cam.2011.11.022.

7 Kheybari, S., Darvishi, M.T., & Wazwaz, A.M. (2017). A semi-analytical algorithm to solve systems of integro-differential equations under mixed boundary conditions. *J. Comput. Appl. Math., 317,* 72–89. DOI: https://doi.org/10.1016/j.cam.2016.11.029.

8 Maleknejad, K., Basirat, B., & Hashemizadeh, E. (2011). Hybrid Legendre polynomials and Block-Pulse functions approach for nonlinear Volterra–Fredholm integro-differential equations. *Comput. Math. Appl., 61,* 2821–2828. DOI: https://doi.org/10.1016/j.camwa.2011.03.055.

9 Pedas A., & Tamme E. (2011). A discrete collocation method for Fredholm integro-differential equations with weakly singular kernels. *Appl. Numer. Math., 61*(6), 738–751. DOI: https://doi.org/10.1016/j.apnum.2011.01.006.

10 Shakeri F. (2013). A high order finite volume element method for solving elliptic partial integro-differential equations. *Appl. Numer. Math., 65,* 105–118. DOI: https://doi.org/10.1016/j.apnum.2012.10.002

11 Susahab D.N., Shahmorad S., & Jahanshahi M. (2015). Efficient quadrature rules for solving nonlinear fractional integro-differential equations of the Hammerstein type. *Applied Mathematical Modelling, 39*(18), 5452–5458. DOI: https://doi.org/10.1016/j.apm.2015.01.008.

12 Turkyilmazoglu, M. (2014). High-order nonlinear Volterra–Fredholm-Hammerstein integro-differential equations and their effective computation. *Appl. Math. Comput., 247,* 410–416. DOI: https://doi.org/10.1016/j.amc.2014.08.074.

13 Yulan, W., Chaolu, T., & Ting, P. (2009). New algorithm for second order boundary value problems of integro-differential equation. *J. Comput. Appl. Math., 229,* 1–6. DOI: https://doi.org/10.1016/j.cam.2008.10.007.

14 Zhao, J. (2013). Compact finite difference methods for high order integro-differential equations. *Appl. Math. Comput., 221,* 66–78. DOI: https://doi.org/10.1016/j.amc.2013.06.030

15 Wazwaz, A.M. (2011). *Linear and Nonlinear Integral Equations [Methods and Applications].* Higher Equacation Press, Beijing and Springer-Verlag Berlin Heidelberg.

16 Dzhumabaev, D.S. (1989). Criteria for the unique solvability of a linear boundary-value problem for an ordinary differential equation. *Comput. Math. Math. Phys., 29*, 34–46.

17 Dzhumabaev, D.S. (2018). New general solutions to linear Fredholm integro-differential equations and their applications on solving the boundary value problems. *J. Comput. Appl. Math., 327,* 79–108. DOI: https://doi.org/10.1016/j.cam.2017.06.010.

18 Dzhumabaev, D.S., & Mynbayeva, S.T. (2019). New general solution to a nonlinear Fredholm integro-differential equation. *Eurasian Mathematical Journal, 10*(4), 23–32. DOI: https://doi.org/10.32523/2077-9879-2019-10-4-23-32.

19 Dzhumabaev, D.S. (2010). A Method for Solving the Linear Boundary Value Problem for an Integro Differential Equation. *Comput. Math. Math. Phys., 50*(7), 1150–1161. DOI: 10.1134/S0965542510070043

20 Dzhumabaev, D.S. (2015). Necessary and Sufficient Conditions for the Solvability of Linear Boundary-Value Problems for the Fredholm Integro-differential Equations. *Ukrainian Mathematical Journal., 66*(8), 1200–1219. DOI: https://doi.org/10.1007/s11253-015-1003-6.

21 Dzhumabaev, D.S. (2016). On one approach to solve the linear boundary value problems for Fredholm integro-differential equations. *J. Comput. Appl. Math., 294,* 342–357. DOI: https://doi.org/10.1016/j.cam.2015.08.023.

22 Dzhumabaev, D.S., & Karakenova, S.G. (2019). Iterative method for solving special Cauchy problem for the system of integro-differential equations with nonlinear integral part. *Kazakh Mathematical Journal, 19*(2), 49–58.

23 Dzhumabaev, D.S., & Temesheva S.M. (2007). A parametrization method for solving nonlinear two-point boundary value problems. *Comput. Math. Math. Phys., 47,* 37-–61. DOI: 10.1134/S096554250701006X

DOI 10.31489/2022M1/35-42

UDC 517.988.68

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On the stability of the difference analogue of the boundary value problem for a mixed type equation

This paper considers a difference problem for a mixed-type equation, to which a problem of integral geometry for a family of curves satisfying certain regularity conditions is reduced. These problems are related to numerous applications, including interpretation problem of seismic data, problem of interpretation of X-ray images, problems of computed tomography and technical diagnostics. The study of difference analogues of integral geometry problems has specific difficulties associated with the fact that for finite-difference analogues of partial derivatives, basic relations are performed with a certain shift in the discrete variable. In this regard, many relations obtained in a continuous formulation, when transitioned to a discrete analogue, have a more complex and cumbersome form, which requires additional studies of the resulting terms with a shift. Another important feature of the integral geometry problem is the absence of a theorem for existence of a solution in general case. Consequently, the paper uses the concept of correctness according to A.N. Tikhonov, particularly, it is assumed that there is a solution to the problem of integral geometry and its differential-difference analogue. The stability estimate of the difference analogue of the boundary value problem for a mixed-type equation obtained in this work is vital for understanding the effectiveness of numerical methods for solving problems of geotomography, medical tomography, flaw detection, etc. It also has a great practical significance in solving multidimensional inverse problems of acoustics, seismic exploration.

Keywords: ill-posed problem, boundary value problem, mixed-type equation, stability estimate, difference problem, quadratic form.

Introduction

The research focuses on a difference-differential problem for a mixed-type equation, to which reduces the problem of integral geometry for a family of curves satisfying certain regularity conditions is reduced.

The problems of integral geometry consist of finding a function or a more complex quantity (differential form, tensor field, etc.) defined on a certain variety, through its integrals over a certain family of sub-variety of smaller dimension.

Some inverse problems for kinetic equations widely used in physics and astrophysics are closely related to problems of integral geometry. The problems of integral geometry refer to ill-posed problems of mathematical physics, the foundations of which were laid in the works [1–3]. These problems are associated with numerous applications (problem of computed tomography, inverse problems of acoustics, and seismic exploration).

The need to study differential-difference and finite-difference analogues of integral geometry problems was first expressed and formulated as a new promising direction by Academician M.M. Lavrentiev. Therefore, the study of differential-difference and finite-difference analogues of integral geometry problems is an urgent problem.

M.M. Lavrentiev and V.G. Romanov first showed in the work [4] that a number of inverse problems for hyperbolic equations are reduced to problems of integral geometry. Further, V.G. Romanov obtained uniqueness theorems and estimates of conditional stability to solve integral geometry problems for a fairly general family of curves on a plane invariant with respect to the rotation group [5], as well as for families of curves and hyper surfaces in n-dimensional space invariant with respect to parallel transfers of these objects along some plane [6].

A very general result on uniqueness and stability estimates for a special family of curves was obtained by R.G. Mukhometov. These stability estimates are based on reducing the integral geometry problem to an equivalent boundary value problem for a partial differential equation of mixed type [7].

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Methods and materials

Let D be a bounded simply connected domain in the plane with a smooth boundary Γ:

$$x = \xi(z), \qquad y = \eta(z), \qquad z \in [0, l], \qquad \xi(0) = \xi(l), \qquad \eta(0) = \eta(l),$$

where z is the length of the curve Γ. In the $\overline{D}$ there are smooth curves defined by the equations

$$x = \varphi(x_0, y_0, \theta, s), y = \psi(x_0, y_0, \theta, s), \tag{1}$$

where (x_0, y_0) is a point from which the curve exits at an angle θ, the variable parameter s is the curve length. The set of function definitions φ and ψ is the set, indeed

$$T = \{(x_0, y_0, \theta, s)/(x_0, y_0) \in \overline{D}, \theta \in [0, 2\pi], s \in [0, \tilde{l}(x_0, y_0, \theta)]\},$$

where $\tilde{l}(x_0, y_0, \theta)$ is the length of the part of the curve leaving the point (x_0, y_0) at an angle θ and lying between (x_0, y_0) and the point of intersection of the curve with the boundary.

Let the set of curves (1) be such that it can be regarded as a two-parameter family of curves $K(\gamma, z)$ satisfying the conditions as follows:

a) through any two different points from $\overline{D}$ single curve $K(\gamma, z)$ passes; each curve of the family $K(\gamma, z)$ intersects Γ at points $(\xi(z), \eta(z))$ and $(\xi(\gamma), \eta(\gamma))$, the other points do not lie on Γ; the lengths of all curves are uniformly bounded;

b)$\varphi \in C^3(T), \psi \in C^3(T)$, and all derivatives of these functions are uniformly bounded in T;

c) $\frac{1}{s}\frac{D(\varphi,\psi)}{D(\theta,s)} \geq c_1 > 0$, where c_1 is a constant;

d) $\varphi(x, y, 0, s) = \varphi(x, y, 2\pi, s), \psi(x, y, 0, s) = \psi(x, y, 2\pi, s)$, similar equalities are also valid for derivatives of these functions up to the third order inclusive.

Let $U(x,\, y) \in C^2(\overline{D})$ and

$$V(\gamma, z) = \int\limits_{K(\gamma,z)} U(x,\, y)\rho(x, y, z)ds; \quad \gamma \in [0, l], \quad z \in [0, l]. \tag{2}$$

The problem of integral geometry (2) is to find a function $U(x, y)$ in the domain $\overline{D}$ according to the given curves $K(\gamma, z)$ and functions $V(\gamma, z)$.

If the family $K(\gamma, z)$ satisfies the conditions a)-d), then problem (2) is equivalent to the following boundary value problem

$$\frac{\partial}{\partial z}\left(\frac{\partial W}{\partial x}\frac{\cos\theta}{\rho} + \frac{\partial W}{\partial y}\frac{\sin\theta}{\rho}\right) = 0, \quad (x, y, z) \in \Omega_1, \tag{3}$$

$$W(\xi(\gamma), \eta(\gamma), z) = V(\gamma, z), \qquad V(z, z) = 0, \qquad \gamma, z \in [0, l], \tag{4}$$

where $\rho(x, y, z)$ is a known function, $\Omega_1 = \Omega \backslash \{(\xi(z), \eta(z), z) : z \in [0, l]\}, \Omega = \overline{D} \times [0, l]$.

$K(x, y, z)$ is a part of the curve from the family $K(\gamma, z)$ connecting the points $(x, y) \in \overline{D}$ and $(\xi(z),\, \eta(z))$,

$$W(x,\, y,\, z) = \int\limits_{K(x,\, y,\, z)} U(x, y, z)\rho(x, y, z)ds.$$

$\theta(x, y, z)$ is an angle between the tangent to $K(x, y, z)$ at the point (x, y) and the x axis, the variable parameter s is the curve length.

The functions $W(x, y, z)$ and $\theta(x, y, z)$ have the following differential properties [7]:

Lemma 1. The function $W(x, y, z) \in C(\Omega)$ has continuous derivatives up to and including the second order on the set Ω_1.

Lemma 2. The derivative W_x, W_y, W_z are bounded in Ω_1, and W_{xz}, W_{yz}, W_{xy} in the neighborhood of any point of the form $(\xi(z),\, \eta(z),\, z)$ that can have a type singularity $\left[(x - \xi(z))^2 + (y - \eta(z))^2\right]^{-\frac{1}{2}}$.

Lemma 3. The function $\theta(x, y, z)$ is differentiable on the set Ω_1 and the derivative θ_z in the neighborhood of any point of the form $(\xi(z),\, \eta(z),\, z)$ has a type singularity $\left[(x - \xi(z))^2 + (y - \eta(z))^2\right]^{-\frac{1}{2}}$.

Assume that the requirements for the family of curves $K(\gamma, z)$ and the plane D necessary to bring problem (2) to problem (3), (4) are met. Let us also assume that any straight line that is parallel to the axis of the abscissa or ordinate can intersect the boundary of the domain D at no more than two points.

Let

$$a_1 = \inf_{(x,y)\in D} \{x\}, \; b_1 = \sup_{(x,y)\in D} \{x\},$$

$$a_2 = \inf_{(x,y)\in D} \{y\}, b_2 = \sup_{(x,y)\in D} \{y\},$$

$$h_j = (b_j - a_j)/N_j, \; j = 1, 2; h_3 = l/N_3,$$

where $N_j, j = 1, 2, 3$, are natural numbers.

Let ε satisfy the condition

$$0 < \varepsilon < \min \{(b_1 - a_1)/3, (b_2 - a_2)/3\},$$

$$D^\varepsilon = \left\{(x, y) \in D : \min_{(\alpha,\beta)\in} \rho((x, y), (\alpha, \beta)) > \varepsilon\right\},$$

$$R_h = \{(x_i, y_j), \; x_i = a_1 + ih_1, y_j = a_2 + jh_2, i = 0, 1, ..., N_1; j = 0, 1, ..., N_2\}.$$

The neighborhood $\aleph(ih_1, jh_2)$ of the point $(a_1 + ih_1, a_2 + jh_2)$ will be called a set consisting of the point itself $(a_1 + ih_1, a_2 + jh_2)$ and four points of the form $(a_1 + (i \pm 1)h_1, a_2 + (j \pm 1)h_2)$.

D^ε_h is a set of all points $(a_1 + ih_1, a_2 + jh_2)$ lying in $D^\varepsilon \cap R_h$ together with its neighborhood $\aleph(ih_1, jh_2)$.

Γ^ε_h - the set of all points $(a_1 + ih_1, a_2 + jh_2) \in D^\varepsilon_h$ such that the intersection (ih_1, jh_2) with the set $(D^\varepsilon \cap R_h)/D^\varepsilon_h$ is nonempty. Then,

$$\Delta^\varepsilon_h = \bigcup_{\substack{\varepsilon \\ h}} \aleph(ih_1, ih_2), \quad D_h = R_h \cap D.$$

Futher, we assume that the coefficients and a solution of problem (3)–(4) have the following properties:

$$W(x, y, z) \in C^3(\Omega^\varepsilon), \theta(x, y, z) \in C^2(\Omega^\varepsilon), \; \Omega^\varepsilon = \overline{D}^\varepsilon \times [0, l],$$

$$\rho(x, y, z) \in C^2(\Omega), \rho(x, y, z) > C^* > 0, \frac{\partial \theta}{\partial z} > \left|\frac{\partial \rho}{\partial z} \cdot \frac{1}{\rho}\right|.$$

We consider the following difference problem (depending on the parameter z): Find a function $\Phi_{i,j}(z)$, that satisfies the equation

$$\Phi_{\overset{0}{x}} \tfrac{A}{C} + \Phi_{\overset{0}{y}} \tfrac{B}{C} = U_{i,j}, \quad (a_1 + ih_1, a_2 + ih_2) \in D_h, \quad z \in [0, l] \tag{5}$$

and the boundary condition

$$\Phi_{i,j}(z) = F_{i,j}(z), \quad (a_1 + ih_1, a_2 + jh_2) \in \Delta^\varepsilon_h, \quad z \in [0, l], \tag{6}$$

here

$$\Phi_{i,j}(z) = (x_i, y_j, z) = (a_1 + ih_1, a_2 + jh_2, z),$$

$$U_{i,j} = U(x_i, y_j) = U(a_1 + ih_1, \quad a_2 + jh_2), \quad i = \overline{0, N_1}, \quad j = \overline{0, N_2},$$

$$\Phi_{\overset{0}{x}} = (F_{i+1,j} - F_{i-1,j})/2h_1, \quad \Phi_{\overset{0}{y}} = (F_{i,j+1} - F_{i,j-1})/2h_2,$$

$$A = \cos \theta_{i,j}(z), \quad B = \sin \theta_{i,j}(z), \quad \theta_{i,j}(z) = \theta(a_1 + ih_1, a_2 + jh_2, z), \quad C = \rho(a_1 + ih_1, a_2 + jh_2, z).$$

Note that in this formulation, information about the solution is given not only on the boundary Γ but also in some its ε neighborhood, which is due to the presence of type features $\left[(x - \xi(z))^2 + (y - \eta(z))^2\right]^{-\frac{1}{2}}$ in derivatives $\theta_z, W_{xz}, W_{yz}, W_{xy}$ in the neighborhood of any point of the form $(\xi(z), \eta(z), z)$ [7].

Results and Discussion

Theorem. Suppose that the solution to problem (5)-(6) exists. Let for all the $(x_i, y_j) \in D_h$ functions

$$\Phi_{i,j}(z) \in C^1[0,l], \quad \Phi_{i,j}(0) = \Phi_{i,j}(l),$$

$$F_{i,j}(z) \in C^1[0,l], \quad F_{i,j}(0) = F_{i,j}(l),$$

and the functions $C = \rho(a_1 + ih_1, a_2 + jh_2, z)$. $\theta_{i,j}(z)$ satisfy the conditions

$$\theta_{i,j}(0) = \theta_{i,j}(l), \frac{\partial \theta}{\partial z} > \left| \frac{\partial \rho}{\partial z} \cdot \frac{1}{\rho} \right|.$$

Then for all $N_j > 9,\ j = 1, 2$ there is an estimate

$$\sum_{D_h^\varepsilon} (\Phi_{\overset{0}{x}}^2 + \Phi_{\overset{0}{y}}^2) h_1 h_2 \le c_3 \int\limits_0^l \sum_{\Delta_h^\varepsilon} \left[F_{\overset{0}{x}}^2 h_1 + F_{\overset{0}{y}}^2 h_2 + \left(\frac{\partial F}{\partial z} \right)^2 (h_1 + h_2) \right] dz, \tag{7}$$

where c_3 is some positive constant that depends on the function $\rho(x, y, z)$ and the curves family $K(\gamma, z)$.

In estimation (7), it is assumed that with a decrease of h_1 and h_2, the parameter ε can also decrease, since c_3 does not depend on ε (the parameter ε was chosen solely to eliminate features that are present in the original continuous problem). Consequently, the smaller the grid is, the narrower the domain may be in which the feature is concentrated.

Proof. Using the methodology proposed in the papers [8], [9] both parts (5) are multiplied by $2C(-B\Phi_{\overset{0}{x}} + A\Phi_{\overset{0}{y}})\frac{\partial}{\partial z}$, the resulting equality is written in the form

$$J_1 + J_2 = 0. \tag{8}$$

Here

$$J_1 = J_2 = C(-B\Phi_{\overset{0}{x}} + A\Phi_{\overset{0}{y}}) \frac{\partial}{\partial z} (\Phi_{\overset{0}{x}} \frac{A}{C} + \Phi_{\overset{0}{y}} \frac{B}{C}).$$

Using the differentiation formula of the product of functions, we transform J_1:

$$\begin{gathered} J_1 = \tfrac{\partial}{\partial z} \left[\left(-B\Phi_{\overset{0}{x}} + A\Phi_{\overset{0}{y}} \right) \left(A\Phi_{\overset{0}{x}} + B\Phi_{\overset{0}{y}} \right) \right] + \\ + AB \tfrac{1}{C} \tfrac{\partial C}{\partial z} \Phi_{\overset{0}{x}}{}^2 - \tfrac{1}{C} \tfrac{\partial C}{\partial z} A^2 \Phi_{\overset{0}{x}} \Phi_{\overset{0}{y}} + \\ + \tfrac{1}{C} \tfrac{\partial C}{\partial z} B^2 \Phi_{\overset{0}{x}} \Phi_{\overset{0}{y}} - \tfrac{1}{C} \tfrac{\partial C}{\partial z} AB \Phi_{\overset{0}{y}}{}^2 + \tfrac{\partial \theta}{\partial z} A^2 \Phi_{\overset{0}{x}}{}^2 + \\ + AB\Phi_{\overset{0}{x}} \tfrac{\partial}{\partial z} \left(\Phi_{\overset{0}{x}} \right) + \tfrac{\partial \theta}{\partial z} AB \Phi_{\overset{0}{x}} \Phi_{\overset{0}{y}} - A^2 \Phi_{\overset{0}{x}} \tfrac{\partial}{\partial z} \left(\Phi_{\overset{0}{y}} \right) + \\ + \tfrac{\partial \theta}{\partial z} AB \Phi_{\overset{0}{x}} \Phi_{\overset{0}{y}} + B^2 \tfrac{\partial}{\partial z} \left(\Phi_{\overset{0}{x}} \right) \Phi_{\overset{0}{y}} + \\ + \tfrac{\partial \theta}{\partial z} B^2 \Phi_{\overset{0}{y}}{}^2 - AB \Phi_{\overset{0}{y}} \tfrac{\partial}{\partial z} \left(\Phi_{\overset{0}{y}} \right). \end{gathered} \tag{9}$$

Opening the brackets in J_2, we have

$$\begin{gathered} J_2 = -AB\Phi_{\overset{0}{x}} \tfrac{\partial}{\partial z} \left(\Phi_{\overset{0}{x}} \right) - \tfrac{\partial \theta}{\partial z} AB \Phi_{\overset{0}{x}} \Phi_{\overset{0}{y}} + \tfrac{1}{C} \tfrac{\partial C}{\partial z} B^2 \Phi_{\overset{0}{x}} \Phi_{\overset{0}{y}} - \\ - B^2 \Phi_{\overset{0}{x}} \tfrac{\partial}{\partial z} \left(\Phi_{\overset{0}{y}} \right) + \tfrac{\partial \theta}{\partial z} B^2 \Phi_{\overset{0}{x}} + \tfrac{1}{C} \tfrac{\partial C}{\partial z} AB \Phi_{\overset{0}{x}}{}^2 + \\ + A^2 \Phi_{\overset{0}{y}} \tfrac{\partial}{\partial z} \left(\Phi_{\overset{0}{x}} \right) - \tfrac{\partial \theta}{\partial z} AB \Phi_{\overset{0}{x}} \Phi_{\overset{0}{y}} - \tfrac{1}{C} \tfrac{\partial C}{\partial z} A^2 \Phi_{\overset{0}{x}} \Phi_{\overset{0}{y}} + \\ + AB \Phi_{\overset{0}{y}} \tfrac{\partial}{\partial z} \left(\Phi_{\overset{0}{y}} \right) + \tfrac{\partial \theta}{\partial z} A^2 \Phi_{\overset{0}{y}}{}^2 - \tfrac{1}{C} \tfrac{\partial C}{\partial z} AB \Phi_{\overset{0}{y}}{}^2. \end{gathered} \tag{10}$$

Substituting these expressions J_1, J_2 into (8) and denoting $D = \sin 2\theta = 2 \sin\theta \cos\theta = 2AB$, $E = \cos 2\theta = \cos^2\theta - \sin^2\theta = A^2 - B^2$, from (9), (10) we get

$$\begin{gathered} \left(\tfrac{\partial \theta}{\partial z} + \tfrac{1}{C} \tfrac{\partial C}{\partial z} D \right) \Phi_{\overset{0}{x}}{}^2 - 2\Phi_{\overset{0}{x}} \Phi_{\overset{0}{y}} \tfrac{1}{C} \tfrac{\partial C}{\partial z} E + \left(\tfrac{\partial \theta}{\partial z} - \tfrac{1}{C} \tfrac{\partial C}{\partial z} D \right) \Phi_{\overset{0}{y}}{}^2 + \\ + \Phi_{\overset{0}{y}} \tfrac{\partial}{\partial z} \left(\Phi_{\overset{0}{x}} \right) - \Phi_{\overset{0}{x}} \tfrac{\partial}{\partial z} \left(\Phi_{\overset{0}{y}} \right) + \tfrac{\partial}{\partial z} \left[\left(-B\Phi_{\overset{0}{x}} + A\Phi_{\overset{0}{y}} \right) \left(A\Phi_{\overset{0}{x}} + B\Phi_{\overset{0}{y}} \right) \right] = 0. \end{gathered} \tag{11}$$

It is not difficult to notice that

$$\frac{\partial}{\partial z}\left(\Phi_{\overset{0}{x}}\right) = \left(\frac{\partial \Phi}{\partial z}\right)_{\overset{0}{x}}, \qquad \frac{\partial}{\partial z}\left(\Phi_{\overset{0}{y}}\right) = \left(\frac{\partial \Phi}{\partial z}\right)_{\overset{0}{y}},$$

$$(uv)_{\overset{0}{x}} = u_{\overset{0}{x}}\, v + u v_{\overset{0}{x}} + \frac{h_1^2}{2}[u_x v_x]_{\bar{x}},$$

where

$$f_x = \tfrac{f_{i+1} - f_i}{h_1}, \quad f_{\bar{x}} = \tfrac{f_i - f_{i-1}}{h_1}.$$

Then,

$$\begin{gathered}\Phi_{\overset{0}{y}}\left(\tfrac{\partial \Phi}{\partial z}\right)_{\overset{0}{x}} - \Phi_{\overset{0}{x}}\left(\tfrac{\partial \Phi}{\partial z}\right)_{\overset{0}{y}} = \left[\Phi_{\overset{0}{y}}\left(\tfrac{\partial \Phi}{\partial z}\right)\right]_{\overset{0}{x}} - \left[\Phi_{\overset{0}{x}}\left(\tfrac{\partial \Phi}{\partial z}\right)\right]_{\overset{0}{y}} - \\ -\tfrac{h_1^2}{2}\left[\Phi_{\overset{0}{y}x}\left(\tfrac{\partial \Phi}{\partial z}\right)_x\right]_{\bar{x}} + \tfrac{h_1^2}{2}\left[\Phi_{\overset{0}{x}y}\left(\tfrac{\partial \Phi}{\partial z}\right)_y\right]_{\bar{y}},\end{gathered}$$

from (11) we obtain

$$\begin{gathered}J_3 + \tfrac{\partial}{\partial z}\left[\left(-B\Phi_{\overset{0}{x}} + A\Phi_{\overset{0}{y}}\right)\left(A\Phi_{\overset{0}{x}} + B\Phi_{\overset{0}{y}}\right)\right] + \left[\Phi_{\overset{0}{y}}\tfrac{\partial \Phi}{\partial z}\right]_{\overset{0}{x}} - \left[\Phi_{\overset{0}{x}}\tfrac{\partial \Phi}{\partial z}\right]_{\overset{0}{y}} - \\ -\tfrac{h_1^2}{2}\left[\Phi_{\overset{0}{y}x}\left(\tfrac{\partial \Phi}{\partial z}\right)_x\right]_{\bar{x}} + \tfrac{h_2^2}{2}\left[\Phi_{\overset{0}{x}y}\left(\tfrac{\partial \Phi}{\partial z}\right)_y\right]_{\bar{y}} = 0,\end{gathered} \tag{12}$$

where

$$J_3 = \left(\frac{\partial \theta}{\partial z} + \frac{1}{C}\frac{\partial C}{\partial z}D\right)\Phi_{\overset{0}{x}}^{\ 2} - 2\Phi_{\overset{0}{x}}\Phi_{\overset{0}{y}}\frac{1}{C}\frac{\partial C}{\partial z}E + \left(\frac{\partial \theta}{\partial z} - \frac{1}{C}\frac{\partial C}{\partial z}D\right)\Phi_{\overset{0}{y}}^{\ 2}.$$

Considering the expression J_3 as a quadratic form with respect to $\Phi_{\overset{0}{x}}$ and $\Phi_{\overset{0}{y}}$, it is not difficult to ensure that the determinant of this quadratic form is

$$\left(\frac{\partial \theta}{\partial z}\right)^2 - \left(\frac{1}{C}\frac{\partial C}{\partial z}\right)^2.$$

Then from the condition

$$\frac{\partial \theta}{\partial z} > \left|\frac{1}{C}\frac{\partial C}{\partial z}\right|$$

the positive definiteness of the quadratic form J_3 follows.

Using the inequality

$$ax^2 + 2bxy + cy^2 \geq \frac{2\left(ac - b^2\right)}{a + c + \sqrt{(a-c)^2 + 4b^2}}(x^2 + y^2),$$

which is true for a positive-definite quadratic form $ax^2 + 2bxy + cy^2$, we have

$$J_3 \geq \left(\frac{\partial \theta}{\partial z} - \left|\frac{1}{C}\frac{\partial C}{\partial z}\right|\right)\left(\Phi_{\overset{0}{x}}^{\ 2} + \Phi_{\overset{0}{y}}^{\ 2}\right). \tag{13}$$

Considering that

$$C = \rho(x, y, z), \qquad \rho(x, y, z) >^{*} > 0, \qquad \left(\frac{\partial \theta}{\partial z} - \left|\frac{1}{C}\frac{\partial C}{\partial z}\right|\right) > 0, \tag{14}$$

it is not difficult to make sure that there is such $c_2 > 0$ and there is an inequality

$$\int\limits_0^l \left(\frac{\partial \theta}{\partial z} - \left|\frac{1}{C}\frac{\partial C}{\partial z}\right|\right) C^2 dz \geq \frac{1}{c_2^2} > 0.$$

Further, summing over i, j using condition (6) and integrating over z, taking into account the formulas (13), (14) as well as the periodicity of functions $\Phi_{i,j}(z), \theta_{i,j}(z)$ over z and inequality $|ab| \leq (a^2 + b^2)/2$, from equality (12) after simple transformations, we obtain an estimate

$$\sum_{D_h^\varepsilon} (\Phi_{\overset{0}{x}}^2 + \Phi_{\overset{0}{y}}^2) h_1 h_2 \leq c_3 \int_0^l \sum_{\Delta_h^\varepsilon} \left[F_{\overset{0}{x}}^2 h_1 + F_{\overset{0}{y}}^2 h_2 + \left(\frac{\partial F}{\partial z} \right)^2 (h_1 + h_2) \right] dz,$$

in which c_3 depends on the functions $\rho(x, y, z)$ and the curves family $K(\gamma, z)$. So, the theorem is proved.

Conclusions

The stability estimate of the difference analogue of the boundary value problem for a mixed-type equation obtained in the work can be used to justify the convergence of numerical methods for solving problems of geotomography, medical tomography, flaw detection and is of great practical significance in solving multidimensional inverse problems of acoustics, seismic exploration.

References

1 Lavrentev M.M. Ill-posed problems of mathematical physics and analysis / M.M. Lavrent'ev, V.G. Romanov, S.P. Shishatskii // Translations of Mathematical Monographs. American Mathematical Society, Providence, R.I. — 1986. — Vol. 64.

2 Tikhonov A.N. Solutions of Ill-posed Problems / A.N. Tikhonov, V.Ya. Arsenin // V.H. Winston & Sons, a Division of Scripta Technica, Inc. — 1977.

3 Ivanov V.K. Theory of linear ill-posed problems and its applications / V.K. Ivanov, V.V. Vasin, V.P. Tanana // Berlin, Boston: De Gruyter. — 2013.

4 Лаврентьев М.М. О трех линеаризированных обратных задачах для гиперболических уравнений / М.М. Лаврентьев, В.Г. Романов // Докл. АН СССР. — 1966. — 171. — № 6. — С. 1279–1281.

5 Романов В.Г. О некоторых классах единственности решения задач интегральной геометрии / В.Г. Романов // Математические заметки. — 1974. — 16. — № 4. — С. 657–668.

6 Романов В.Г. О восстановлении функции через интегралы по семейству кривых / В.Г. Романов // Сиб. мат. журн. — 1967. — 8. — № 5. – С. 1206–1208.

7 Мухометов Р.Г. О задаче интегральной геометрии / Р.Г. Мухометов // Математические проблемы геофизики. — Новосибирск, 1975. — 6. — С. 212–252.

8 Кабанихин С.И. Об устойчивости конечно-разностного аналога двумерной задачи интегральной геометрии / С.И. Кабанихин, Г.Б. Баканов // Докл. АН СССР. — 1987. — 292. — № 1. — С. 25–29.

9 Bakanov G. On the stability of a differential-difference analogue of a two-dimensional problem of integral geometry / G. Bakanov // Filomat. — 2018. — 32(3). — P. 933–938.

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Аралас типті теңдеу үшін шекаралық есептің айырымдық аналогының орнықтылығы жайлы

Мақалада кейбір аралас типті теңдеу үшін айырымдық есепке келтірілетін регулярлық шарттарын қанағаттандыратын қисықтар үйірі үшін қойылған интегралдық геометрия есебі қарастырылды. Бұл есептер көптеген қосымшалармен байланысты, оның ішінде сейсмобарлау мәліметтерін интерпретациялау есептері, рентген суреттерін интерпретациялау, компьютерлік томография және техникалық диагностика есептері. Интегралдық геометрия есептерінің айырымдық аналогтарын зерттеудің өзіне тән күрделі тұстары бар, дербес туындылардың шектеулі-айырымдық аналогтары үшін негізгі қатынастар дискретті айнымалы бойынша белгілі бір ығысумен жүргізілуіне байланысты болады. Сондықтан, үзіліссіз қойылымда алынатын көптеген қатынастар дискретті аналогқа ауысқанда анағұрлым күрделі түрге ие болады және ығысу барысында туындайтын қосылғыштарға қатысты қосымша зерттеулерді талап етеді. Бұл есептердің тағы да бір ерекшелігі — жалпы жағдайда шешімнің бар болуы жайлы теорема жоқ. Осыған байланысты А.Н. Тихонов бойынша корректілік ұғымы қолданылады, яғни интегралдық геометрия есебі мен оның дифференциалдық-айырымдық аналогының шешімі бар болады деп жорамалданады. Сонымен қатар, аралас типті теңдеу үшін шекаралық есептің айырымдық аналогының алынған орнықтылық бағасы геотомография, медициналық томография, дефектоскопия және т.б. есептерді сандық әдістермен шешудің тиімділігін түсіну үшін өте маңызды. Сондай-ақ, акустика, сейсмобарлаудың көп өлшемді кері есептерін шешуде де үлкен практикалық мәні бар.

Кілт сөздер: корректілі емес есеп, шекаралық есеп, аралас типті теңдеу, орнықтылық бағасы, айырымдық есеп, квадратты форма.

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Об устойчивости разностного аналога граничной задачи для уравнения смешанного типа

В статье рассмотрена разностная задача для уравнения смешанного типа, к которой сводится задача интегральной геометрии для семейства кривых, удовлетворяющих некоторым условиям регулярности. Эти задачи связаны с многочисленными приложениями, в том числе с задачами интерпретации данных сейсморазведки, интерпретации рентгеновских снимков, компьютерной томографии и задачами технической диагностики. Исследование разностных аналогов задач интегральной геометрии имеет специфические трудности, связанные с тем обстоятельством, что для конечно-разностных аналогов частных производных основные соотношения выполняются с некоторым сдвигом по дискретной переменной. В связи с этим многие соотношения, получаемые в непрерывной постановке, при переходе к дискретному аналогу имеют более сложную и громоздкую форму, что требует дополнительных исследований возникающих слагаемых со сдвигом. Еще одной важной особенностью задачи интегральной геометрии является отсутствие теоремы существования решения в общем случае. В связи с этим в работе использовано понятие корректности по А.Н. Тихонову, а именно, предполагается, что решение задачи интегральной геометрии и ее дифференциально-разностного аналога существует. Полученная авторами оценка устойчивости разностного аналога граничной задачи для уравнения смешанного типа имеет важное значение для понимания эффективности численных методов решения задач геотомографии, медицинской томографии, дефектоскопии и т. д. Кроме того, имеет большое практическое значение при решении многомерных обратных задач акустики, сейсморазведки.

Ключевые слова: некорректная задача, краевая задача, уравнение смешанного типа, оценка устойчивости, разностная задача, квадратичная форма.

References

1 Lavrentev, M.M., Romanov, V.G., & Shishatskii, S.P. (1986). Ill-posed problems of mathematical physics and analysis. *Translations of Mathematical Monographs. American Mathematical Society, Providence,* 64.

2 Tikhonov, A.N., & Arsenin, V.Ya. (1977). *Solutions of Ill-posed Problems.* V.H. Winston & Sons, a Division of Scripta Technica, Inc.

3 Ivanov, V.K., Vasin, V.V., & Tanana, V.P. (2013). *Theory of linear ill-posed problems and its applications.* Berlin, Boston: De Gruyter.

4 Lavrentev, M.M., & Romanov, V.G. (1966). O trekh linearizirovannykh obratnykh zadachakh dlia giperbolicheskikh uravnenii [Three-linearized inverse problems for hyperbolic equations]. *Doklady Akademii nauk SSSR – Reports of the USSR Academy of Sciences, 171*(6), 1279–1281 [in Russian].

5 Romanov, V.G. (1974). O nekotorykh klassakh edinstvennosti resheniia zadach integralnoi geometrii [On some classes of uniqueness of solutions to problems of integral geometry]. *Matematicheskie zametki – Math notes, 16*(4), 657–668 [in Russian].

6 Romanov, V.G. (1967). O vosstanovlenii funktsii cherez integraly po semeistvu krivykh [Reconstruction of a function in terms of integrals over a family of curves]. *Sibirskii matematicheskii zhurnal – Siberian Mathematical Journal, 8*(5), 1206–1208 [in Russian].

7 Mukhometov, R.G. (1975). O zadache integralnoi geometrii [On a problem of inegral geometry]. *Matematicheskie problemy geofiziki – Mathematical Problems of Geophyzics. 6,* 212–252. Novosibirsk [in Russian].

8 Kabanikhin, S.I., & Bakanov, G.B. (1987). Ob ustoichivosti konechno-raznostnogo analoga dvumernoi zadachi integralnoi geometrii [On the stability of a finite-difference analogue of a two-dimensional problem of integral geometry]. *Doklady Akademii nauk SSSR – Reports of the USSR Academy of Sciences, 292*(1), 25–29 [in Russian].

9 Bakanov, G. (2018). On the stability of a differential-difference analogue of a two-dimensional problem of integral geometry. *Filomat, 32*(3), 933–938.

DOI 10.31489/2022M1/43-51

UDC 517.9

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A note on epidemiologic models: SIR modeling of the COVID-19 with variable coefficients

The coronavirus disease 2019 (COVID-19) has been responsible for over three million reported cases worldwide. The construction of an appropriate mathematical (epidemiological) model for this disease is a challenging task. In this paper, we first consider susceptible — infectious — recovered (SIR) model with constant parameters and obtain an approximate solution for the SIR model with varying coefficient as it is one of the simplest models and many models are derived from this framework. The numerical experiments confirm that the proposed formulation demonstrates similar characteristic behaviour with the well-known approximations.

Keywords: infectious diseases, COVID-19, mathematical modeling, SIR model, variable coefficients.

Introduction

The coronavirus disease (COVID-19) was pronounced as a major health hazard by World Health Organization (WHO) in late December 2019 [1]. At present, this disease is affecting over 200 countries and territories around the world, and the global number of COVID-19 cases is increasing rapidly. In early December of 2019, this infectious disease has first broken out in China. Although, the disease in China seems to be under control, there are still many infections around the world. The high rate of the infection spread and the number of fatalities makes the understanding of the current epidemiological models more important than ever before. A considerable amount of research works of different complexity levels from simple to complicated ones has been devoted to defeat the disease, which include a lot of problem parameters. The introduced models encountered in the literature are typically based on systems of ordinary differential equations (ODEs) or partial differential equations (PDEs) [2, 3]. Although the PDE models allow one to describe dynamics in time and space; they are not simple to formulate, analyze, and solve.

The most relevant mathematical models relating to the spread of the disease is the susceptible — infectious — recovered (SIR) model [4–8], susceptible — exposed — infectious — removed (SEIR) model [5, 9–11], the susceptible — infectious — susceptible (SIS) model [12, 13]. The SEIR, SIR and SIS models can also reflect the dynamics of different epidemics such as Human Immunodeficiency Virus (HIV), Severe Acute Respiratory Syndrome (SARS) and they have also been used to model the COVID-19 [8, 11]. There are also other strategies such as the logistic model [14, 15], the susceptible — asymptomatic — recovered — infected — isolated infected — quarantined susceptible ($SARII_qS_q$) model [16], the susceptible — unquarantined — quarantined — confirmed (SUQC) model [17], the susceptible — exposed — insusceptible — quarantined — recovered — death (SEIQRDP) model [18] to describe the trend of COVID-19 [19]. Although many studies use ordinary differential equations (ODEs) to predict the susceptible, infected, and recovered populations, it is worth mentioning the PDE models for the spread of an epidemic. The SIR model has been studied in [20] by constructing a hyperbolic Kolmogorov PDE for the discrete-stochastic model, in the large population limit. Moreover, the dynamics of SIR type reaction-diffusion epidemic model with specific nonlinear incidence rate has been investigated in [21]. The study of suitable PDE models for the COVID-19 case will be detailed in a forthcoming work. It should also be noted that complicated models need more effort as they include a lot of variables and require a detailed analysis for their validation which makes the procedure difficult in the absence of reliable data.

In this work, we consider well-known SIR model to simulate the process of COVID-19 which is proposed by Kermack and McKendrick [22]. There are different strategies to understand the predictions of this model and

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the behavior of its solutions. Kermack and McKendrick [22] reduced this problem with constant parameters to a single differential equation and derived an approximate solution in terms of a hyperbolic secant function. The classical SIR model contains two time-invariant coefficients: The transmission rate β and the removal (recovering) rate γ which neglects the time-varying property of β and γ. However, it is too simple to effectively predict the trend of the disease. Therefore, assuming the ratio of the transmission and removal rates to remain constant when both rates are functions of time variable t, we study a time-dependent SIR model and obtain approximate solution of such model which allows changing infection and removal rates for the latest COVID-19 data.

The rest of this paper is organized as follows. In Section 1, we briefly introduce the SIR model with constant coefficients and its approximate solution. We present approximate solution of SIR model with time-dependent coefficients in Section 2. In Section 3, we provide some numerical tests to illustrate the performance of proposed formulation for both constant and variable coefficient cases.

1 The SIR Model with Constant Coefficients

In 1927, a set of equations studied by Kermack and McKendrick [22] to investigate the dynamics of an infectious disease in three groups: Susceptible (S), infectious (I), and recovered (R) whose sizes are functions of time t, that is,

$$\begin{cases} \frac{dS(t)}{dt} = -\beta\, I(t)\, S(t), \qquad S(t_0) = S_0, \\ \frac{dI(t)}{dt} = \beta\, I(t)\, S(t) - \gamma I(t), \qquad I(t_0) = I_0, \\ \frac{dR(t)}{dt} = \gamma\, I(t), \qquad R(t_0) = R_0 \end{cases} \tag{1}$$

together with a fixed population size N,

$$S(t) + I(t) + R(t) = N. \tag{2}$$

Here, $S(t)$ represents the number of susceptible individuals not yet infected with disease at t, $I(t)$ stands for the number of infectious individuals who have been infected and are in danger of spreading the disease to the susceptibles, and $R(t)$ is the number of removed (and immune) or deceased individuals. The constant parameter β stands for the infection rate, and the average infectious period is $1/\gamma$ days. The initial conditions are given by: $S(t_0) = S_0, I(t_0) = I_0, R(t_0) = R_0 \geq 0$.

The mentioned simple system is appropriate for estimating the dynamics of the COVID-19 in different countries [23–25] by using the freely available statistics provided by the European Centre for Disease Prevention Control and World Health Organisation [26]. Therefore, this model has been taken as a background by many researchers for modeling COVID-19 in various countries of the world as it provides a simple procedure. The equations are generally solved numerically. Kermack and McKendrick [22] first reduced this problem to a single differential equation as in the following way: Using problem (1) and equality (2), we have

$$\frac{dR(t)}{dt} = \gamma(N - S(t) - R(t)) \tag{3}$$

and

$$\frac{dS}{dR} = -\frac{\beta}{\gamma}\, S$$

which is a separable differential equation and it can be solved for S and then substituted to the equation (3) to get:

$$\frac{dR(t)}{dt} = \gamma(N - S_0 e^{-\frac{\beta}{\gamma}(R(t)-R_0)} - R(t)).$$

Since it is not possible to find R as an explicit function, by assuming that $\frac{\beta}{\gamma}R$ is small compared with unity, the exponential term can be expanded in powers of $\frac{\beta}{\gamma}R$. Thus, we have

$$\frac{dR(t)}{dt} = \gamma\Big(N - S_0 - R_0 + \Big(\frac{\beta}{\gamma}S_0 - 1\Big)R(t) - \frac{\beta^2}{2\gamma^2}S_0 R(t)^2 + ...\Big). \tag{4}$$

Moreover, Kermack and McKendrick [22] neglegted some terms in (4) that is,

$$\frac{dR(t)}{dt} \approx \gamma\Big(N - S_0 - R_0 + \Big(\frac{\beta}{\gamma}S_0 - 1\Big)R(t) - \frac{\beta^2}{2\gamma^2}S_0 R(t)^2\Big),$$

and derived the following approximate solution of the SIR model for the removal rate, $\frac{dR}{dt}$, in terms of a hyperbolic function,

$$R(t) \approx \frac{\gamma^2}{\beta^2 S_0}\Big(\frac{\beta}{\gamma}S_0 - 1 + \sqrt{-\mu}\tanh\Big(\frac{\sqrt{-\mu}}{2}\gamma t - \phi\Big)\Big),$$

where

$$\begin{cases} \phi = \tanh^{-1}\frac{\frac{\beta}{\gamma}S_0 - 1}{\sqrt{-\mu}}, \\ \sqrt{-\mu} = \Big(\frac{\beta}{\gamma}S_0 - 1\Big)^2 + 2S_0 I_0 \frac{\beta^2}{\gamma^2}. \end{cases}$$

We note that the rate of infection β can be changed by vaccination or isolation of infected individuals and the rate of removal γ can be changed by the use of different medicines or treatment protocols. Moreover, changing infection rate β and removal rate γ for the latest coronavirus data may allow one to track the reproductivity of the COVID-19 through time and to assess the effectiveness of the control measures implemented by the public health authorities [27]. This can be achieved by using time-dependent $\beta(t)$ and $\gamma(t)$ functions, rather than constants β and γ which is the subject of the following section.

2 The SIR Model with Time-Dependent Coefficients

In this section, we consider a generalized version of the SIR model in which the infectious rate β and the removal rate γ may vary with respect to time when the ratio $\frac{\beta(t)}{\gamma(t)}$ remains constant. Replacing β and γ by $\beta(t)$ and $\gamma(t)$ in problem (1) yields,

$$\begin{cases} \frac{dS(t)}{dt} = -\beta(t)\, I(t)\, S(t), \qquad S(t_0) = S_0, \\ \frac{dI(t)}{dt} = \beta(t)\, I(t)\, S(t) - \gamma(t) I(t), \qquad I(t_0) = I_0, \\ \frac{dR(t)}{dt} = \gamma(t)\, I(t), \qquad R(t_0) = R_0. \end{cases} \tag{5}$$

By following the steps in Section 1, we wish to solve the following problem which is a Riccati differential equation that is quadratic in the unknown function R:

$$\frac{dR(t)}{dt} - \gamma(t)\Big(N - S_0 - R_0 + \Big(\frac{\beta(t)}{\gamma(t)}S_0 - 1\Big)R(t) - \frac{\beta(t)^2}{2\gamma(t)^2}S_0 R(t)^2\Big) \approx 0. \tag{6}$$

Then, it is easy to verify that the solution of problem (6) is given by,

$$R(t) = \frac{-1 + S_0 q - \sqrt{\psi}\tan\Big(\sqrt{\psi}\Big(C_1 + \int_0^t \frac{1}{2}\gamma(x)\, dx\Big)\Big)}{S_0 q^2}, \tag{7}$$

where

$$\begin{cases} q = \frac{\beta(t)}{\gamma(t)}, \\ \psi = -1 + S_0 q(q(-2N + 2R_0 + S_0) + 2), \\ C_1 = \frac{1}{\sqrt{\psi}}\arctan(\frac{S_0 q - 1}{\sqrt{\psi}}). \end{cases}$$

Once the unknown function $R(t)$ is calculated, $I(t)$ and $S(t)$ can be obtained by using problem (5):

$$I(t) = \frac{1}{\gamma(t)}\frac{dR(t)}{dt},$$

$$S(t) = S_0 e^{-qR(t)}.$$

We note that, due to the nature of the problem such a SIR model with time-dependent coefficients is much better to track the disease spread, control, and predict the future trend of the disease.

Remark 1. It is worth mentioning that the mean square analytical solution of a Riccati equation of random coefficients under some assumptions is studied in [28]. Moreover, the solution (7) can be seen as a solution of SIR model in a different form when the infectious rate β and the removal rate γ are constant. For more details, we refer the readers to [22].

3 Numerical Results

3.1 Experiments with Constant Coefficients

In this case, we report some numerical experiments and compare the performance of two numerical solutions: The solution by Kermack and McKendrick and the present formulation for the SIR model with constant coefficients. We remark that the numerical results are encouraging and the proposed formulation has similar features with the solution obtained by Kermack and McKendrick.

Experiment 1a:

We first investigate the following test problem in [6],

$$\begin{cases} \frac{dS(t)}{dt} = -\beta\, I(t)\, S(t), \qquad S(0) = S_0, \\ \frac{dI(t)}{dt} = \beta\, I(t)\, S(t) - \gamma\, I(t), \qquad I(0) = I_0, \\ \frac{dR(t)}{dt} = \gamma\, I(t), \qquad R(0) = R_0. \end{cases}$$

In this study, the authors study the spread of ongoing COVID-19 when $t \in [0, 200]$; $\beta = 2/14$; $\gamma = 1/14$; $S_0 = 0.999$; $I_0 = 0.001$; $R_0 = 0$; $N = 1$ (normalized version). In Figure 1, we represent elevation plots of the solutions obtained with the present formulation and the Kermack and McKendrick formulation. The results of the numerical experiments have similar features with the strategies which shows that it can be used to estimate COVID-19 epidemic trend.

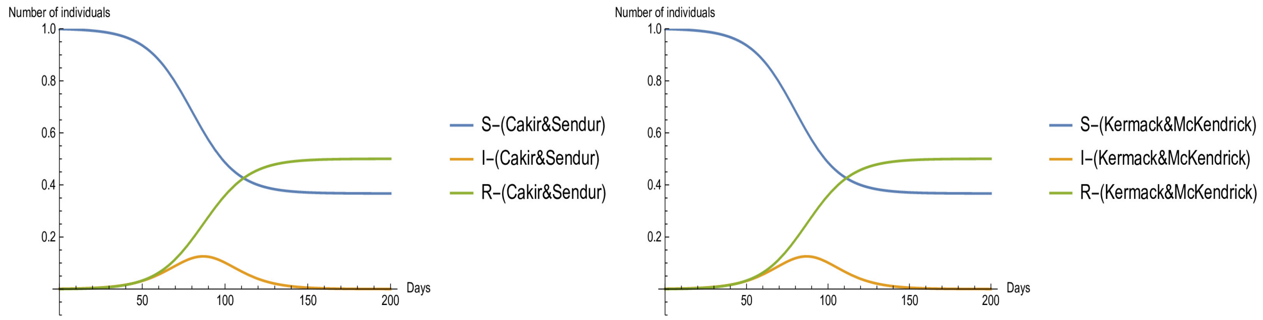

Figure 1. SIR model solutions. Left: Solution by Cakir and Sendur, Right: Solution by Kermack and McKendrick

Experiment 1b:

We consider the test problem in [25]. In this study, the data for the COVID-19 disease outbreak is adjusted the Kermack and McKendrick approximation of the SIR model. We set the problem parameters to find the solution of the SIR model for different countries in Table 1. We present only the elevation plots for China in Figure 2 as the data for other countries produces similar features in capturing the behavior of the solution. The results show that the SIR model is a good choice to get a better understanding of COVID-19.

Table 1

The parameters to solve SIR model for different countries

Country	γ	β	S_0	I_0
China	0.08	4.564 e-06	85631	5
Spain	0.08	8.416 e-07	265551	1
Italy	0.08	8.597 e-07	258511	1
France	0.08	1.096 e-06	179659	3
Germany	0.08	1.053 e-06	206003	1
Argentina	0.019	5.784 e-07	155575	200
Mexico	0.0908	3.174 e-07	479575	13

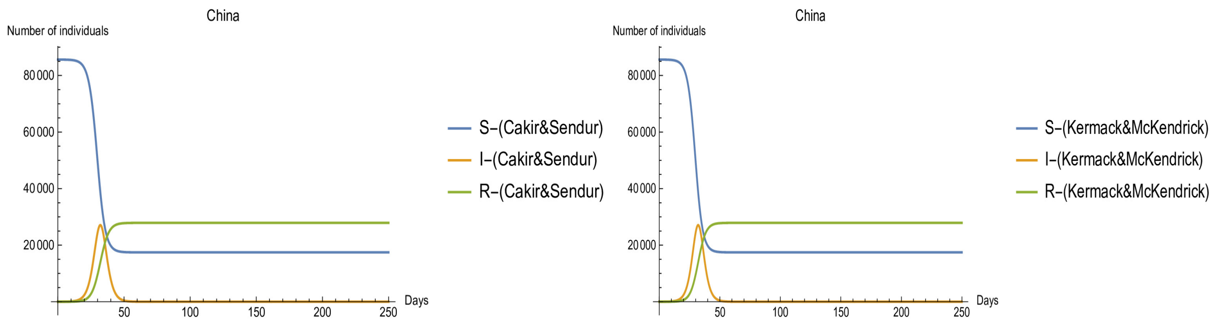

Figure 2. SIR model solutions for China. Left: Solution by Cakir and Sendur, Right: Solution by Kermack and McKendrick

3.2 Experiments with Time-Dependent Coefficients

In this case, we report some numerical experiments to display the performance of the present formulation for the SIR model with time dependent coefficients when $\beta(t) = q\,p\,t^r$; $\gamma(t) = p\,t^r$ for several values of p, q, r. With the above choice, the solution can be rewritten in the following form:

$$R(t) = \frac{-1 + qS_0 - \sqrt{\varphi}\tan\left(\frac{p\sqrt{\varphi}}{2(r+1)}t^{r+1} + \tan^{-1}\left(\frac{qS_0-1}{\sqrt{\varphi}}\right)\right)}{q^2 S_0},$$

where

$$\varphi = -1 + qS_0(q(-2N + 2R_0 + S_0) + 2).$$

Once the unknown function $R(t)$ is calculated, $I(t)$ and $S(t)$ can be obtained by following the steps in Section 2.

Experiment 2a:

We investigate the following test problem with time-dependent coefficients:

$$\begin{cases} \frac{dS(t)}{dt} = -\beta(t)\; I(t)\; S(t), \qquad S(t_0) = S_0, \\ \frac{dI(t)}{dt} = \beta(t)\; I(t)\; S(t) - \gamma(t)I(t), \qquad I(t_0) = I_0, \\ \frac{dR(t)}{dt} = \gamma(t)\; I(t), \qquad R(t_0) = R_0, \end{cases}$$

when $t \in [0, 200]$; $t_0 = 0$; $S_0 = 0.999$; $I_0 = 0.001$; $R_0 = 0$.

In Figure 3, we first set $q = 2$, $p = 0.02$ and illustrate the behavior of the solution obtained with the present formulation for increasing values of $r : 0 < r \leq 1$. We observe high number of infectious individuals at later stages when r is smaller and high number of infected individuals at early stage when r is increasing.

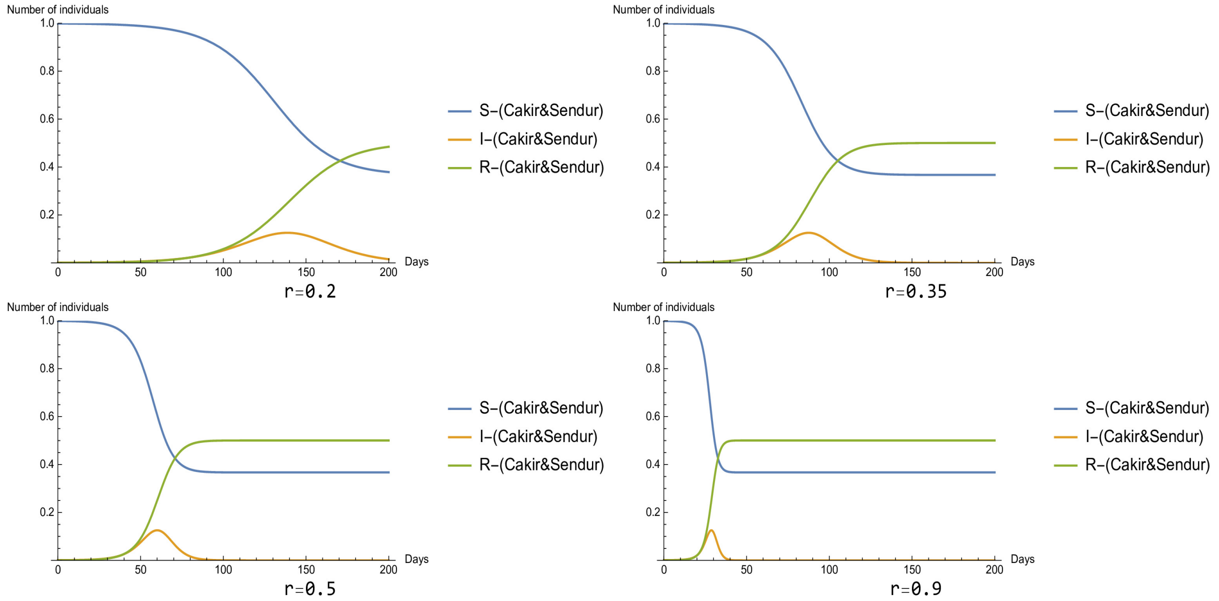

Figure 3. SIR model solutions when $\beta(t) = 2\gamma(t)$; $\gamma(t) = t^r/50$

Experiment 2b:

Next, we investigate the behavior of the solution with respect to the ratio $q = \frac{\beta}{\gamma}$. In Figure 4, we set $p = 1/100$, $r = 0.5$ and demonstrate the behavior of the solution obtained with the present formulation for increasing values of q. We note that relatively more susceptible individuals can complete the disease process without being infected when $1.4 < q < 1.6$. This situation can be explained with the existence of high-quality health care services, individuals' protection awareness and high rates of COVID-19 vaccinations. Moreover, we observe that more individuals have been infected and are in danger of spreading the disease to the susceptible for increasing values of q. The numerical results are encouraging and the approximate solution captures the characteristic behavior of the problem.

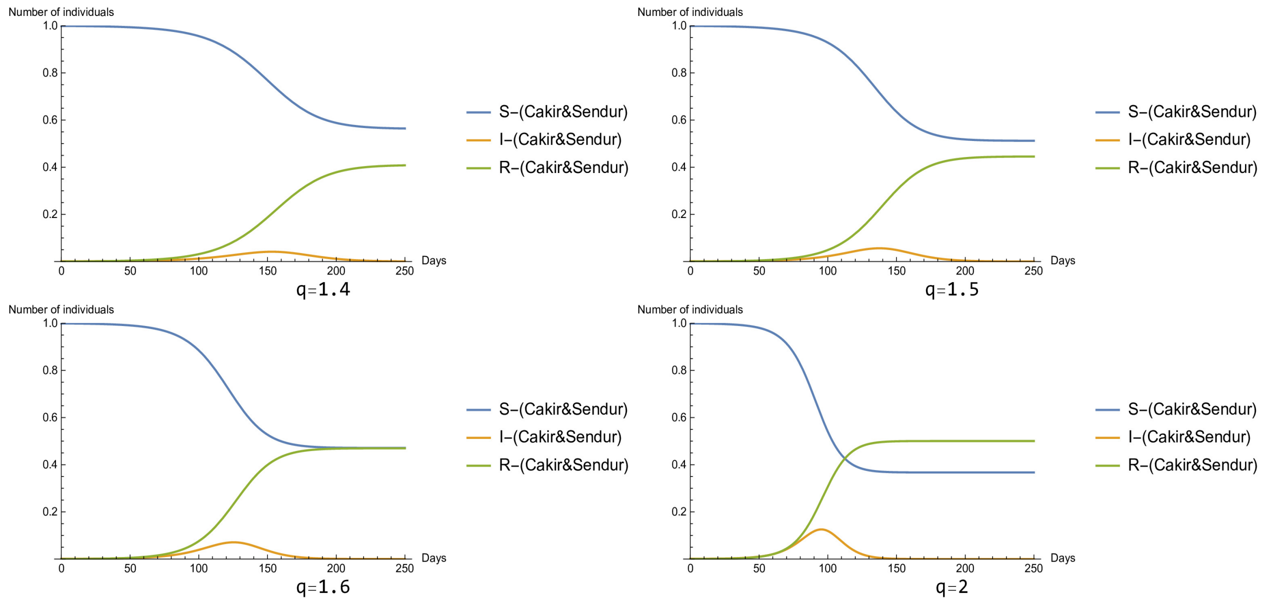

Figure 4. SIR model solutions when $\beta(t) = q\,\gamma(t)$; $\gamma(t) = t^{0.5}/100$

Conclusions

The main advantage of SIR models comes from its ability to establish a balance between simplicity and usefulness. Therefore, we investigate the approximate solutions of the SIR epidemiological model which has been widely used for over 100 years. Numerical experiments confirm the good performance of the proposed formulation for a wide range of problem configurations. Consequently, the SIR model captures some features of the COVID-19 behavior and thus, it could provide guidance for the evolution of the pandemic with only two parameters. Moreover the coefficients for the transition to the infectious or recovered (removed) compartment, namely β and γ, do not remain fixed during the spread of the disease. Indeed, the transmission coefficients could be different for various cases, such as active tourism season, Christmas, the start and end periods of education, festivals, periods in which measures are applied tightly or loosened. For this reason, the SIR model with time-dependent coefficients seems much better to analyze the trend of the disease.

We note that, recently, many remarkable complicated models including a lot of parameters have been used to understand the COVID-19 cases. However, it is not easy to determine which mathematical model describes the COVID-19 outbreak best. Furthermore, a simpler model is not better or worse than a more complicated model and using complicated models may not be more reliable compared to using a simpler model. The investigation of various suitable models for the COVID-19 case, a comparison of such models ranging from simple to more complicated ones for specific countries and the highlight of their strengths and weaknesses in different situations can be considered as a future work. We also note that many studies use ordinary differential equations (ODEs) to predict the susceptible, infected, and recovered populations for the COVID-19 case. It is also remarkable to consider the spatial effects in the spread of epidemics for the mobility of people within a country and the regional levels of risk (effects of transboundary spread, face mask requirement, quarantine, lockdown, etc., among county clusters). This situation can be modelled by partial differential equations (PDEs) and it is a subject of a new research.

References

1 WHO. (2020). Coronavirus disease (COVID-19) Weekly Epidemiological Update and Weekly Operational Update. *who.int.* Retrieved from https://www.who.int/emergencies/diseases/novel-coronavirus-2019/situation-reports

2 Murray, J.D. (2003). *Mathematical biology II: spatial models and biomedical application.* Springer, Berlin.

3 Viguerie, A., Veneziani, A., Lorenzo, G., Baroli, D., Aretz-Nellesen, N., Patton, A., & Auricchio, F. et al. (2020). Diffusion-reaction compartmental models formulated in a continuum mechanics framework: application to COVID-19, mathematical analysis, and numerical study. *Computational Mechanics, 66*(5), 1131–1152.

4 Anderson, R.M., Anderson, B., & May, R.M. (1992). *Infectious Diseases of Humans: Dynamics and Control.* Oxford Univ. Press.

5 Bjornstad, O.N. (2018). *Epidemics: Models and Data Using R.* Springer.

6 Bjornstad, O.N., Shea, K., Krzywinski, M., & Altman, N. (2020). Modeling infectious epidemics. *Nature methods, 17*, 455–456.

7 Cai, Y., Kang, Y., & Wang, W. (2017). A stochastic sirs epidemic model with nonlinear incidence rate. *Appl. Math. Comput., 305*, 221–240.

8 Chen, Y., Lu, P., Chang, C.S., & Liu, T.H. (2020). A time-dependent SIR model for COVID-19 with undetectable infected persons. *IEEE Transactions on Network Science and Engineering, 7*(4), 3279–3294.

9 Almeida, R. (2018). Analysis of a fractional seir model with treatment. *Appl. Math. Lett., 84*, 56–62.

10 Al-Rahman El-Nor Osman, M., Adu, I.K., & Yang, C. (2017). A simple seir mathematical model of malaria transmission. *Asian Res. J. Math., 7*, 1–22.

11 Tang, B., Wang, X., & Li, Q. (2020). Estimation of the transmission risk of the 2019-ncov and its implication for public health interventions. *J. Clin. Med., 9*(2), 462.

12 Liu, J., Pare, P.E., Du, E., & Sun, Z. (2019). A networked SIS disease dynamics model with a waterborne pathogen. *American Control Conference (ACC)*, 2735–2740.

13 Van den Driessche, P., & Watmough, J. (2000). A simple SIS epidemic model with a backward bifurcation. *J. Math. Biol., 40*, 525–540.

14 Aviv-Sharon, E., & Aharoni, A. (2020). Generalized logistic growth modeling of the COVID-19 pandemic in Asia. *Infectious Disease Modelling, 5*, 502–509.

15 Cakir, Z., & Savas, H.B. (2020). A Mathematical Modelling Approach in the Spread of the Novel 2019 Coronavirus SARS-CoV-2 (COVID-19) Pandemic. *Electron. J. Gen. Med., 17*(4), em205.

16 Sarkar, K., Khajanchi, S., & Nieto, J.J. (2020). Modeling and forecasting the COVID-19 pandemic in India. *Chaos, Solitons & Fractals, 139*, 110049.

17 Zhao S., & Chen H. (2020). Modeling the epidemic dynamics and control of COVID-19 outbreak in China. *Quant. Biol., 8*, 11–19.

18 Peng, L., Yang, W., Zhang, D., Zhuge, C., & Hong, L. (2020). Epidemic analysis of COVID-19 in China by dynamical modeling. *arXiv preprint*, arXiv: 2002.06563.

19 Roda, W.C., Varughese, M.B., Han, D., & Li, M.Y. (2020). Why is it difficult to accurately predict the COVID-19 epidemic? *Infect. Dis. Model, 5*, 271–281.

20 Chalub, F.A., & Souza, M.O. (2011). The SIR epidemic model from a PDE point of view. *Mathematical and Computer Modelling, 53*(7-8), 1568–1524.

21 Lotfi, E.M., Maziane, M., Hattaf, K., & Yousfi, N. (2014). Partial differential equations of an epidemic model with spatial diffusion. *International Journal of Partial Differential Equations, 2014*, 1–7.

22 Kermack, W.O., & McKendrick, A.G. (1927). Contribution to the mathematical theory of epidemics. *Proc. Roy. Soc. Lond A., 115*, 700–721.

23 Cooper, I., Mondal, A., & Antonopoulos, C.G. (2020). A SIR model assumption for the spread of COVID-19 in different communities. *Chaos, Solitons & Fractals, 139*, 110057.

24 Postnikov, E.B. (2020). Estimation of COVID-19 dynamics "on a back-of-envelope": Does the simplest SIR model provide quantitative parameters and predictions? *Chaos, Solitons & Fractals, 135*, 109841.

25 Rojas, S. (2020). Comment on Estimation of COVID-19 dynamics "on a back-of-envelope": Does the simplest SIR model provide quantitative parameters and predictions? *Chaos, Solitons & Fractals: X*, 100047.

26 ECDC. (2020). *ecdc.europa.eu.* Retrieved from https://www.ecdc.europa.eu/en/publications-data/downloadtodays-data-geographic-distribution-covid-19-cases-worldwide

27 Sun, H., Qiu, Y., Yan, H., Huang, Y., Zhu, Y., Gu, J., & Chen, S.X. (2020). Tracking reproductivity of COVID-19 epidemic in China with varying coefficient SIR model. *Journal of Data Science, 18*(3), 455–472.

28 Licea, J.A., Villafuerte, L., & Chen-Charpentier, B.M. (2013). Analytic and numerical solutions of a Riccati differential equation with random coefficients. *Journal of computational and applied mathematics, 239*, 208–219.

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Эпидемиологиялық модельдер туралы мақала: өзгермелі коэффициенттері бар SIR COVID–19 модельдеу

2019 жылғы коронавирус ауруы (COVID-19) бүкіл әлемде тіркелген үш миллионнан астам жағдайға себеп болды. Бұл аурудың жеткілікті математикалық (эпидемиологиялық) моделін құру қиын міндет. Мақалада алдымен тұрақты параметрлері бар "сезімтал — жұқпалы — қалпына келтірілген" (SIR) моделі қарастырылған және өзгермелі коэффициенті бар SIR моделіне жуық шешім алынған, өйткені бұл қарапайым модельдердің бірі және көптеген модельдер осы құрылымның туындысы болып табылады. Сандық эксперименттер ұсынылған тұжырымның белгілі жуықтаулармен ұқсас сипаттағы тәртібін көрсететінін растайды.

Кілт сөздер: жұқпалы аурулар, COVID-19, математикалық модельдеу, SIR моделі, айнымалы коэффициенттер.

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Заметка об эпидемиологических моделях: моделирование SIR COVID–19 с переменными коэффициентами

Коронавирусная болезнь 2019 года (COVID–19) стала причиной более трех миллионов зарегистрированных случаев заболевания во всем мире. Построение адекватной математической (эпидемиологической) модели этого заболевания является сложной задачей. В статье рассмотрена модель «восприимчивый — заразный — выздоровевший» (SIR) с постоянными параметрами и получено приближенное решение для модели SIR с переменным коэффициентом, поскольку это одна из самых простых моделей, и многие модели являются производными от этой структуры. Численные эксперименты подтверждают, что предложенная формулировка показывает сходное характерное поведение с известными приближениями.

Ключевые слова: инфекционные болезни, COVID–19, математическое моделирование, SIR–модель, переменные коэффициенты.

DOI 10.31489/2022M1/52-65

UDC 512.815.1

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On simple modules with singular highest weights for $\mathfrak{so}_{2l+1}(\mathbb{K})$

In this paper, we study formal characters of simple modules with singular highest weights over classical Lie algebras of type B over an algebraically closed field of characteristic $p \geq h$, where h is the Coxeter number. Assume that the highest weights of these simple modules are restricted. We have given a description of their formal characters. In particular, we have obtained some new examples of simple Weyl modules. In the restricted region, the representation theory of algebraic groups and its Lie algebras are equivalent. Therefore, we can use the tools of the representation theory of semisimple and simply-connected algebraic groups in positive characteristic. To describe the formal characters of simple modules, we construct Jantzen filtrations of Weyl modules of the corresponding highest weights.

Keywords: Lie algebra, simple module, algebraic group, Weyl module, Jantzen filtration.

Introduction

The study of the structure of Weyl modules is one of the central questions of the representation theory of simply-connected and semisimple algebraic groups in positive characteristics. If the structure of the Weyl modules is known, then it is easy to describe the formal characteristics of simple modules associated with them. There is a remarkable Lusztig's conjecture, which facilitates to describe the characters of simple modules. Its validity is proved for sufficiently large characteristics of the ground field. In [1], Fiebig gave the upper bound of the exceptional characteristics for Lusztig's character formula. It depends on the root system and is far from the Coxeter number of the root system. Furthermore, the description of the Kazhdan-Lusztig polynomials for the antidominant elements of the affine Weyl group that appear in the Lusztig's character formula is due to the complicated calculations. In the restricted region, they are known only for small groups, such as $SL_2(\mathbb{K})$, $SL_3(\mathbb{K})$, $Sp_4(\mathbb{K})$, G_2, $SL_4(\mathbb{K})$, $Sp_6(\mathbb{K})$ и $SO_7(\mathbb{K})$. For nonrestricted elements, they are computed in some special cases. Thus, the problem of a description of the formal characters of simple modules using the structure of Weyl modules still remains of current interest.

Let G be a semisimple, simply-connected algebraic group of type B_l over an algebraically closed field of characteristic $p \geq h$, where h is the Coxeter number, and $\mathfrak{g}$ be a Lie algebra of G. In this paper, for all $l > 2$, we give the structure of the Weyl modules with singular highest weights for G with highest weights defined by the dominant elements of the following subsets of the affine Weyl group W_p of G :

$$Y_1 = \{y_{-1} = 1,\ y_i = s_0 s_1 s_2 \cdots s_i \,|\, i = 0, 1, \cdots, l\};$$

$$Y_2 = \{y_{l+j} = s_0 s_1 s_2 \cdots s_l s_{l-1} \cdots s_{l-j} \,|\, j = 1, 2, \cdots, l-2\};$$

$$Z_1 = \{z_{-1} = s_0,\ z_0 = 1,\ z_i = y_i s_0 \,|\, i = 1, 2, \cdots, l\};$$

$$Z_2 = \{z_{l+j} = y_{l+j} s_0 \,|\, j = 1, 2, \cdots, l-2\}.$$

Here $s_0, s_1, \cdots, s_l$ are the generators of W_p. The affine Weyl group W_p is a Coxeter group of type $\widetilde{B}_l$ with the following defending relations:

$$(s_i s_j)^{m_{ij}} = 1,\ s_0^2 = 1,\ (s_0 s_i)^2 = 1\ (i \neq 1),\ (s_0 s_1)^4 = 1, \qquad (1)$$

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where $i,\, j \in \{1,\, 2,\, \cdots,\, l\}$ and

$$m_{ij} = \begin{cases} 1, & \text{if } i = j; \\ 2, & \text{if } |i-j| > 1; \\ 3, & \text{if } |i-j| = 1 \text{ and } (i,j) \notin \{(l-1,l),\, (l,l-1)\}; \\ 4, & \text{if } (i,j) \in \{(l-1,l),\, (l,l-1)\}. \end{cases}$$

Let R be an irreducible root system of type B_l, $\alpha_1, \alpha_2, \cdots, \alpha_l$ be the simple roots, and $\omega_1, \omega_2, \cdots, \omega_l$ be the fundamental weights. Denote by h the Coxeter number of R. Let us shortly discuss the well-known results on the structure of the Weyl modules for simply-connected and semisimple algebraic groups of type B_l. In [2] and [3], Braden and Jantzen described the structure of the Weyl modules with highest restricted weights of the algebraic group of type B_2. The simplicity of the Weyl module $V(s_0 \cdot \nu)$ with the dominant highest weight $s_0 \cdot \nu$, where $\nu \in \overline{C}_1 \setminus C_1$, was proved by Rudakov in [4, Theorem 2] for semisimple Lie algebras over a field of characteristic $p \geq h$. In [5], O'Halloran obtained a set of Weyl modules with a simple radical and described their structure. In small characteristics, the Weyl modules with restricted highest weights were described for $B_4\,(p = 2)$ by Dowd and Sin [6], for $B_3\,(p = 2, 3)$ by Ye and Zhou [7], [8]. In [9], Arslan and Sin studied the nonrestricted case $V(2\omega_l)$ and the Weyl modules with the fundamental weights in characteristic $p = 2$ for B_l whith $l \geq 2$. The structure of the Weyl modules with highest weights in $\{r\omega_1 \,|\, 0 \leq r \leq p-1\}$ was calculated by Cardinali and Pasini [10]. For the group of type B_4 over a field of characteristic $p > 0$, in [11], Wiggins calculated the structure of the Weyl modules with the highest weights in $\{r\omega_4 \,|\, 0 < r \leq p-1\}$. A similar result was obtained by Cavallin for the groups of type B_l over a field of characteristic $p > 2$ and for the highest weights in $\{2\omega_1,\, \omega_1 + 2\omega_l,\, \omega_1 + \omega_j \,|\, 2 \leq j \leq l\}$ [12].

1 Notation and formulation of the main results

Before starting to formulate our results, we introduce some notation and useful facts. Basically, we will use standard notation. Let R be an irreducible root system of type B_l and let G be a simply-connected and semisimple algebraic group with root system R over an algebraically closed field $\mathbb{K}$ of characteristic $p \geq h$, where h is Coxeter number of R. We assume that $R \subset \mathbb{R}^l$, where $\mathbb{R}$ is the field of real numbers. On $\mathbb{R}^l$ there is the usual euclidean inner product $(\cdot,\cdot)$. This leads to the natural pairing $\langle \cdot,\cdot \rangle : \mathbb{R}^l \times \mathbb{R}^l \to \mathbb{R}$ given by $\langle \lambda, \mu \rangle = (\lambda, \mu^\vee)$, where $\mu^\vee = \frac{2}{(\mu,\mu)}\mu$. If $\Delta = \{\alpha_1, \alpha_2, \cdots, \alpha_l\}$ is the set of simple roots and $\{\varepsilon_i \,|\, i = 1,\, 2,\, \cdots,\, l\}$ is the orthonormal basis of $\mathbb{R}^l$ then the positive roots of R can be seen as a set [13]:

$$R^+ = \begin{array}{l} \{\alpha_i + \alpha_{i+1} + \cdots + \alpha_j = \varepsilon_i - \varepsilon_{j+1} \mid 1 \leq i \leq j \leq l-1\} \cup \\ \{\alpha_i + \cdots + \alpha_l = \varepsilon_i \,|\, i = 1,\, 2,\, \cdots,\, l\} \cup \\ \{\alpha_i + \cdots + \alpha_j + 2\alpha_{j+1} + \cdots + 2\alpha_l = \varepsilon_i + \varepsilon_{j+1} \,|\, 1 \leq i \leq j < l\}. \end{array} \tag{2}$$

Let $T \subseteq G$ be a maximal torus, and B be the Borel subgroup corresponding to the negative roots. We denote by U the unipotent radical of B. The set $X(T)$ of additive characters for T can be seen as a subset of $\mathbb{R}^l$ with basis $\omega_1,\, \omega_2,\, \cdots, \omega_l$ satisfying $\langle \omega_i, \alpha_j \rangle = \delta_{ij}$. The set $X(T)$ also has the following property:

$$X(T) = \{\lambda \in \mathbb{R}^l \,|\, \langle \lambda, \alpha \rangle \in \mathbb{Z} \text{ for all } \alpha \in R\}.$$

Any rational G-module can be considered as the direct sum of T-modules:

$$V = \bigoplus_{\lambda \in X(T)} V_\lambda,$$

where $V_\lambda = \{v \in V \,|\, tv = \lambda(t)v, \text{ for all } t \in T\}$. If $V_\lambda \neq 0$ we say λ is a weight of V, and V_λ is a weight subspace of V. In this case, the vectors of V_λ will be called weight vectors.

Let

$$X(T)^+ = \{\lambda \in X(T) \,|\, \langle \lambda, \alpha \rangle \geq 0 \text{ for all } \alpha \in R^+\}$$

be the set of dominant weights. If $\rho \in X(T)^+$ is the half-sum of the positive roots, then it is easy to prove that

$$\rho = \omega_1 + \omega_2 + \cdots + \omega_l. \tag{3}$$

We define the formal characters of V by

$$[V] = \sum_{\lambda \in X(T)} \dim_k V_\lambda e^\lambda \in \mathbb{Z}(X(T)) = \bigoplus_{\lambda \in X(T)} \mathbb{Z}e^\lambda.$$

Let $\mathbb{Z}$ be the set of integers, and $m_1, m_2, \cdots, m_l \in \mathbb{Z}$. If $\lambda = \sum_{i=1}^{l} m_i\omega_i \in X(T)$ then, by (2) and (3), we get

$$\langle \lambda + \rho, \alpha \rangle = \begin{cases} m_i + 1, \text{ if } \alpha = \alpha_i, \ i = 1, 2, \cdots, l; \\ \\ m_i + \cdots + m_j + (j - i + 1), \\ \text{if } \alpha = \alpha_i + \cdots + \alpha_j, \ 1 \leq i < j = 1, 2, \cdots, l-1; \\ \\ 2m_i + \cdots + 2m_{l-1} + m_l + 2l - 2i + 1, \\ \text{if } \alpha = \alpha_i + \cdots + \alpha_l, \ i = 1, 2, \cdots, l-1; \\ \\ m_i + \cdots + m_j + 2m_{j+1} + \cdots + 2m_{l-1} + m_l + 2l - i - j, \\ \text{if } \alpha = \alpha_i + \cdots + \alpha_j + 2\alpha_{j+1} + \cdots + 2\alpha_l, \ 1 \leq i \leq j < l. \end{cases} \quad (4)$$

Let $\lambda \in X(T)^+$ and $H^0(\lambda)$ be the vector space over $\mathbb{K}$ of all regular functions $f : G \to \mathbb{K}$ satisfying:

$$f(bg) = \lambda(b^{-1})f(g), \text{ for all } b \in B, \ g \in G.$$

We define on $H^0(\lambda)$ a G-module structure given by

$$gf(h) = f(hg), \text{ where } f \in H^0(\lambda), \ g, h \in G.$$

Also, it is well-known that $H^0(\lambda) = Ind_B^G \mathbb{K}_\lambda$, where $\mathbb{K}_\lambda$ is a one dimensional B-module defined by $\lambda \in X(T)^+$ via the isomorphism $B/U \cong T$. Let $L(\lambda)$ be a maximal semisimple submodule (socle) of $H^0(\lambda)$. Each $L(\lambda)$ is a simple G-module and every simple G-module is isomorphic to $L(\lambda)$ for some $\lambda \in X(T)^+$. The Weyl module $V(\lambda)$ with the highest weight $\lambda \in X(T)^+$ is isomorphic to $H^0(-w_0(\lambda))^*$, where w_0 is the maximal element of the Weyl group W for R. There is the following Weyl character formula:

$$\chi(\lambda) := [V(\lambda)] = [H^0(\lambda)] = \frac{\sum_{w \in W} (-1)^{l(w)} e^{w(\lambda+\rho)}}{\sum_{w \in W} (-1)^{l(w)} e^{w(\rho)}}.$$

Let V be a G-module. We define *a composition coefficient* $[V : L(\lambda)]$ for $\lambda \in X(T)^+$ such that

$$[V] = \sum_{\lambda \in X(T)^+} [V : L(\lambda)][L(\lambda)].$$

If $[V : L(\lambda)] \neq 0$ then we say that $L(\lambda)$ is a *composition factor of* V.

For $\alpha \in R^+$ and $n \in \mathbb{Z}$ let us define the affine reflections $s_{\alpha,n}$ on $X(T)$ by [14, II.6.1]

$$s_{\alpha,n} \cdot \lambda = \lambda - \langle \lambda + \rho, \alpha \rangle + np\alpha \text{ for all } \lambda \in X(T).$$

By W_p denote the *affine Weyl group* generated by all $s_{\alpha,n}$ with $\alpha \in R^+$ and $n \in \mathbb{Z}$. The usual finite Weyl group W of R appears as the subgroup of W_p generated by the reflections $s_{\alpha,0}$ with $\alpha \in R^+$.

Let $\alpha_0 = \omega_1 = \varepsilon_1$ be the unique highest short root of R. We will use the following notation: $s_{\alpha_i,0} := s_i$ for all $i \in \{1, 2, \cdots, l\}$ and $s_0 := s_{\alpha_0,1}$. Then the set of generators of W is $S = \{s_i \,|\, i = 1, 2, \cdots, l\}$ and the set of generators of W_p is $S_p = S \cup \{s_0\}$.

We will also use the affine hyperplanes and the affine alcoves. For $\alpha \in R^+$ and $n \in \mathbb{Z}$ we define

$$H_{\alpha,n} = \{v \in \mathbb{R}^l \,|\, \langle v + \rho, \alpha \rangle = np\}.$$

The set of affine alcoves A is defined as the set of connected components of

$$\mathbb{R}^l \setminus (\bigcup_{\alpha \in R_+, \, n \in \mathbb{Z}} H_{\alpha,n}).$$

The fundamental alcove $C_1 \in A$ is defined by

$$C_1 = \{v \in \mathbb{R}^l \,|\, 0 < \langle v+\rho, \alpha\rangle < p \text{ for all } \alpha \in R^+\}.$$

We denote by $\overline{C}_1$ a closure of C_1.

Let $W_p^+ \subset W_p$ be the set of *dominant elements* defined by

$$W_p^+ = \{w \in W_p \,|\, w \cdot \nu \in X(T)^+ \text{ for any } \nu \in C_1\}.$$

The stabilizer $st(\lambda)$ of $\lambda \in X(T)$ is the set

$$st(\lambda) = \{w \in W_p \,|\, w \cdot \lambda = \lambda\}.$$

If $st(\lambda) \cap S_p = \emptyset$ we say λ is a *regular weight*, otherwise it is called a *singular weight.* Let $\lambda = w \cdot \nu$, where $w \in W_p$ and $\nu \in \overline{C}_1$. It is known that λ is a regular weight if and only if $\nu \in C_1$.

Next we introduce some notation for singular weights. Let $H_0 := H_{\alpha_0,1}$ and $H_i := H_{\alpha_i,0}$ for all $i \in \{1,\, 2,\, \cdots,\, l\}$. By $\nu_{i_1,\, i_2,\, \cdots,\, i_m}$ denote any element of $\overline{C}_1 \setminus C_1$ satisfying the following conditions:

1) $\nu_{i_1,\, i_2,\, \cdots,\, i_m} \in H_{i_1} \cap H_{i_2} \cap \cdots \cap H_{i_m}$
2) $i_1,\, i_2,\, \cdots,\, i_m \in \{0,\, 1,\, \cdots,\, l\}$;
3) $i_1 < i_2 < \cdots < i_m$;
4) $m \in \{1,\, 2,\, \cdots,\, l+1\}$.

Let $\nu \in \overline{C}_1 \setminus C_1$. Denote by $\overline{w}_\nu$ the (left) coset of the stabilizer $st(\nu)$ containing the element $w \in W_p^+$. Then $W_p^+ = \cup_{w \in W_p^+} \overline{w}_\nu$. An action of $\overline{w}_\nu$ on ν defined by $\overline{w}_\nu \cdot \nu = u \cdot \nu$ for any $u \in \overline{w}_\nu$. If $\overline{w}_\nu \cdot \nu \in X(T)^+$ we say $\overline{w}_\nu$ is *dominant for* ν. Then, up to isomorphism, $\overline{w}_\nu$ defines a simple G-module (respectively, a Weyl module) with highest weight $\overline{w}_\nu \cdot \nu$. We will use notation $\overline{w}$ for the coset $\overline{w}_\nu$ when ν is fixed.

Let $W' \subset W_p^+$ and $\nu \in \overline{C}_1 \setminus C_1$. By definition, put

$$W'^+_\nu := \{\overline{w}_\nu \,|\, \overline{w}_\nu \cdot \nu \in X(T)^+ \text{ and } w \in W'\}.$$

We say that W'^+_ν is *the set of dominant elements of* W' *for* ν.

Now we formulate the main results of this paper.

Theorem 1. Let G be a simply-connected and semisimple algebraic group of type $B_l\,(l > 2)$ over an algebraically closed field $\mathbb{K}$ of characteristic $p \geq h$, where $h = 2l$ is the Coxeter number. Suppose that $\nu \in \overline{C}_1 \setminus C_1$ and $(Y_1 \cup Y_2 \cup Z_1 \cup Z_2)^+_\nu \neq \emptyset$. If $\overline{w}_\nu \in (Y_1 \cup Y_2 \cup Z_1 \cup Z_2)^+_\nu$ then $\chi(\overline{w}_\nu \cdot \nu) = [L(\overline{w}_\nu \cdot \nu)]$ except in the following cases:

(a) $\chi(\overline{y_i} \cdot \nu_0) = [L(\overline{y_i} \cdot \nu_0)] + [L(\overline{y_{i-1}} \cdot \nu_0)]$, where $i \in \{2,\, 3,\, \cdots,\, 2l-2\}$;
(b) $\chi(\overline{z_2} \cdot \nu_1) = [L(\overline{z_2} \cdot \nu_1)] + [L(\overline{y_0} \cdot \nu_1)]$;
(c) $\chi(\overline{z_i} \cdot \nu_1) = [L(\overline{z_i} \cdot \nu_1)] + [L(\overline{z_{i-1}} \cdot \nu_1)]$, where $i \in \{3,\, 4,\, \cdots,\, 2l-3\}$;
(d) $\chi(\overline{z_{2l-2}} \cdot \nu_1) = [L(\overline{z_{2l-2}} \cdot \nu_1)] + [L(\overline{z_{2l-3}} \cdot \nu_1)] + [L(\overline{y_{2l-2}} \cdot \nu_1)]$;
(e) $\chi(\overline{z_{i-1}} \cdot \nu_i) = [L(\overline{z_{i-1}} \cdot \nu_i)] + [L(\overline{y_{i-1}} \cdot \nu_i)]$, where $i \in \{2,\, 3,\, \cdots,\, l\}$;
(f) $\chi(\overline{z_{2l-i-1}} \cdot \nu_i) = [L(\overline{z_{2l-i-1}} \cdot \nu_i)] + [L(\overline{y_{2l-i-1}} \cdot \nu_i)]$, where $i \in \{2,\, 3,\, \cdots,\, l-1\}$.

A result similar to Theorem 1 was also obtained for simply-connected and semisimple algebraic groups of type D [15] and C [16].

Corollary 1. Let $\mathfrak{g}$ be a simple classical Lie algebra of type $B_l\,(l > 2)$ over an algebraically closed field $\mathbb{K}$ of characteristic $p \geq h$, where h is the Coxeter number. Suppose that $\nu \in \overline{C}_1 \setminus C_1$ and $(Y_1 \cup Y_2 \cup Z_1 \cup Z_2)^+_\nu \neq \emptyset$. If $\overline{w}_\nu \in (Y_1 \cup Y_2 \cup Z_1 \cup Z_2)^+_\nu$ then $[L(\overline{w}_\nu \cdot \nu)] = \chi(\overline{w}_\nu \cdot \nu)$ except in the following cases:

(a) for all $i \in \{2,\, 3,\, \cdots,\, 2l-2\}$,

$$[L(\overline{y_i} \cdot \nu_0)] = \sum_{j=0}^{i} (-1)^{i-j} \chi(\overline{y_j} \cdot \nu_0);$$

(b) $[L(\overline{z_2} \cdot \nu_1)] = -\chi(\overline{y_0} \cdot \nu_1) + \chi(\overline{z_2} \cdot \nu_1))$;
(c) for all $i \in \{3,\, 4,\, \cdots,\, 2l-3\}$,

$$[L(\overline{z_i} \cdot \nu_1)] = (-1)^{i-1} \chi(\overline{y_0}\nu_1) + \sum_{j=2}^{i} (-1)^{i-j} \chi(\overline{z_j} \cdot \nu_1);$$

(d)$[L(\overline{z_{2l-2}}\cdot\nu_1)] = (-1)^{2l-3}\chi(\overline{y_0}\cdot\nu_1) + \sum_{j=2}^{i}(-1)^{2l-2-j}\chi(\overline{z_j}\cdot\nu_1) - \chi(\overline{y_{2l-2}}\cdot\nu_1)$;
(e) $[L(\overline{z_{i-1}}\cdot\nu_i)] = -\chi(\overline{y_{i-1}}\cdot\nu_i) + \chi(\overline{z_{i-1}}\cdot\nu_i)$, where $i\in\{2, 3, \cdots, l\}$;
(f) $[L(\overline{z_{2l-i-1}}\cdot\nu_i)] = -\chi(\overline{y_{2l-i-1}}\cdot\nu_i) + \chi(\overline{z_{2l-i-1}}\cdot\nu_i)$, where $i\in\{2, 3, \cdots, l-1\}$.

2 Preliminary results

Let $V(\lambda)$ be a Weyl module with highest weight $\lambda\in X(T)^+$. Then there is a filtration of submodules

$$V(\lambda) = V(\lambda)^0 \supset V(\lambda)^1 \supset V(\lambda)^2 \supset\cdots \tag{5}$$

such that $V(\lambda)/V(\lambda)^1\cong L(\lambda)$ and

$$\sum_{j>0}[V(\lambda)^j] = \sum_{\alpha\in R^+}\sum_{0<np<\langle\lambda+\rho,\alpha\rangle}\nu_p(np)\chi(s_{\alpha,n}\cdot\lambda), \tag{6}$$

where $\nu_p(m) = max\{i\in\mathbb{N}\,|\,p^i|m\}$ [14, II.8.19]. The filtration (5) is called the *Jantzen filtration* and (6) is called *Jantzen's sum formula.*

Let $\nu = a_1\omega_1 + a_2\omega_2 + \cdots + a_l\omega_l\in X(T)$, where $a_j\in\mathbb{Z}$ for all $j\in\{1, 2, \cdots, l\}$.

Lemma 1. Let $y_i\in Y_1\cup Y_2$ and $\nu = \sum_{i=1}^{l}a_i\omega_i$, where $a_1, \cdots, a_l\in\mathbb{Z}$. Then
(a) $y_0\cdot\nu = (p - a_1 - 2\sum_{i=2}^{l-1}a_i - a_l - 2l + 1)\omega_1 + \sum_{i=2}^{l}a_i\omega_i$;
(b) for all $i\in\{1, \cdots, l-2\}$,

$$y_i\cdot\nu = (p - \textstyle\sum_{j=1}^{i}a_j - 2\sum_{j=i+1}^{l-1}a_j - a_l - 2l + i)\omega_1 + \\ \textstyle\sum_{j=2}^{i}a_{j-1}\omega_j + (a_i + a_{i+1} + 1)\omega_{i+1} + \sum_{j=i+2}^{l}a_j\omega_j;$$

(c) $y_{l-1}\cdot\nu = (p - \sum_{j=1}^{l}a_j - l - 1)\omega_1 + \sum_{j=2}^{l-1}a_{j-1}\omega_j + (2a_{l-1} + a_l + 2)\omega_l$;
(d) $y_l\cdot\nu = (p - \sum_{j=1}^{l-1}a_j - l)\omega_1 + \sum_{j=2}^{l-1}a_{j-1}\omega_j + (2a_{l-1} + a_l + 2)\omega_l$;
(e) for all $i\in\{1, \cdots, l-2\}$,

$$y_{l+i}\cdot\nu = (p - \textstyle\sum_{j=1}^{l-i-1}a_j - l + i)\omega_1 + \\ \textstyle\sum_{j=2}^{l-i-1}a_{j-1}\omega_j + (a_{l-i-1} + a_{l-i} + 1)\omega_{l-i} + \sum_{j=l-i+1}^{l}a_j\omega_j.$$

Proof. (a) By (4), we have

$$y_0\cdot\nu = \nu - (\langle\nu+\rho,\alpha_0\rangle - p)\alpha_0 = \nu + (p - 2\textstyle\sum_{i=1}^{l-1}a_i - a_l - 2l + 1)\omega_1 = \\ (p - a_1 - 2\textstyle\sum_{i=2}^{l-1}a_i - a_l - 2l + 1)\omega_1 + \sum_{i=2}^{l}a_i\omega_i.$$

(b) We use induction on i. By (4),

$$s_1\cdot\nu = (-a_1 - 2)\omega_1 + (a_1 + a_2 + 1)\omega_2 + \sum_{i=3}^{l}a_i\omega_i.$$

Then

$$y_1\cdot\nu = s_1\cdot\nu - (\langle s_1\cdot\nu + \rho,\alpha_0\rangle - p)\alpha_0 = \\ s_1\cdot\nu + (p - 2\textstyle\sum_{i=2}^{l-1}a_i - a_l - 2l + 3)\omega_1 = \\ (p - a_1 - 2\textstyle\sum_{i=2}^{l-1}a_i - a_l - 2l + 1)\omega_1 + (a_1 + a_2 + 1)\omega_2 + \sum_{i=3}^{l}a_i\omega_i.$$

Therefore, the statement is true for $i = 1$.

Suppose that the statement is true for all $i < t$, where $t\le l-2$. By (4),

$$s_t\cdot\nu = \sum_{j=1}^{t-2}a_j\omega_j + (a_{t-1} + a_t + 1)\lambda_{t-1} + (-a_t - 2)\omega_t + (a_t + a_{t+1} + 1)\omega_{t+1} + \sum_{j=t+2}^{l}a_j\omega_j.$$

By the induction hypothesis,

$$y_{t-1}\cdot\nu = (p - \textstyle\sum_{j=1}^{t-1}a_j - 2\sum_{j=t}^{l-1}a_j - 2l + t - 1)\omega_1 + \\ \textstyle\sum_{j=2}^{t-1}a_{j-1}\omega_j + (a_{t-1} + a_t + 1)\omega_t + \sum_{j=t+1}^{l}a_j\omega_j.$$

Then

$$y_t\cdot\nu=y_{t-1}\cdot(s_t\cdot\nu)=(p-\sum_{j=1}^{t-2}a_j-(a_{t-1}+a_t+1)-2(-a_t-2)-$$
$$2(a_t+a_{t+1}+1)-2\sum_{j=t+2}^{l-1}a_j-2l+t-1)\omega_1+\sum_{j=2}^{t-1}a_{j-1}\omega_j+$$
$$(a_{t-1}+a_t+1-a_t-2+1)\omega_t+(a_t+a_{t+1}+1)\omega_{t+1}+\sum_{j=t+2}^{l}a_j\omega_j=$$
$$(p-\sum_{j=1}^{t}a_j-2\sum_{j=t+1}^{l-1}-2l+t)\omega_1+\sum_{j=2}^{t}a_{j-1}\omega_j+$$
$$(a_{t-1}+a_t+1)\omega_{t+1}+\sum_{j=t+2}^{l}a_j\omega_j.$$

(c) By (4) we have

$$s_{l-1}\cdot\nu=\sum_{j=1}^{l-3}a_j\omega_j+(a_{l-2}+a_{l-1}+1)\lambda_{l-2}+(-m_{l-1}-2)\omega_{l-1}+(2a_{l-1}+a_l+2)\omega_l.$$

Then, the statement (b) for $i=l-2$ gives us

$$y_{l-1}\cdot\nu=y_{l-2}\cdot(s_{l-1}\cdot\nu)=(p-\sum_{j=1}^{l-3}a_j-(a_{l-2}+a_{l-1}+1)-$$
$$2(-a_{l-1}-2)-(2a_{l-1}+a_l+2)-l-2)\omega_1+$$
$$\sum_{j=2}^{l-2}a_{j-1}\omega_j+(a_{l-2}+a_{l-1}+1-a_{l-1}-2+1)\omega_{l-1}+(2a_{l-1}+a_l+2)\omega_l+$$
$$\sum_{j=t+2}^{l}a_j\omega_j=(p-\sum_{j=1}^{l}a_j-l-1)\omega_1+\sum_{j=2}^{l-1}a_{j-1}\omega_j+(2a_{l-1}+a_l+2)\omega_l.$$

(d) By (4),

$$s_l\cdot\nu=\sum_{j=1}^{l-2}a_j\omega_j+(a_{l-1}+a_l+1)\omega_{l-1}+(-a_l-2)\omega_l.$$

Then using the statement (c), we have

$$y_l\cdot\nu=y_{l-1}\cdot(s_l\cdot\nu)=(p-\sum_{j=1}^{l-2}a_j-(a_{l-1}+a_l+1)-$$
$$(-a_l-2)-l-1)\omega_1+\sum_{j=2}^{l-1}a_{j-1}\omega_j+(2(a_{l-1}+a_l+1)+(-a_l-2)+2)\omega_l=$$
$$(2a_{l-1}+a_l+2)\omega_l+\sum_{j=t+2}^{l}a_j\omega_j=$$
$$(p-\sum_{j=1}^{l-1}a_j-l)\omega_1+\sum_{j=2}^{l-1}a_{j-1}\omega_j+(2a_{l-1}+a_l+2)\omega_l.$$

(e) By (4),

$$s_{l-1}\cdot\nu=\sum_{j=1}^{l-2}a_j\omega_j+(a_{l-2}+a_{l-1}+1)\omega_{l-2}+(-a_{l-1}-2)\omega_{l-1}+(2a_{l-1}+a_l+2)\omega_l.$$

Then using the statement (d), we have

$$y_{l+1}\cdot\nu=y_l\cdot(s_{l-1}\cdot\nu)=(p-\sum_{j=1}^{l-2}a_j-l+1)\omega_1+\sum_{j=2}^{l-2}a_{j-1}\omega_j+(a_{l-2}+a_{l-1}+1)\omega_{l-1}+a_l\omega_l.$$

Therefore, the statement is true for $i=1$.

Suppose that the statement is true for all $i<t$, where $t\le l-2$. By (4)

$$s_{l-t}\cdot\nu=\sum_{j=1}^{l-t-2}a_j\omega_j+(a_{l-t-1}+a_{l-t}+1)\omega_{l-t-1}+$$
$$(-a_{l-t}-2)\omega_{l-t}+(a_{l-t}+a_{l-t+1}+1)\omega_{l-t+1}+\sum_{j=l-t+2}^{l}a_j\omega_j.$$

By the induction hypothesis,

$$y_{l+t-1}\cdot\nu=(p-\sum_{j=1}^{l-t}a_j-l+t-1)\omega_1+$$
$$\sum_{j=2}^{l-t}a_{j-1}\omega_j+(a_{l-t}+a_{l-t+1}+1)\omega_{l-t+1}+\sum_{j=l-t+2}^{l}a_j\omega_j.$$

Then

$$y_{l+t}\cdot\nu=y_{l+t-1}\cdot(s_{l-t}\cdot\nu)=(p-\sum_{j=1}^{l-t-2}a_j-(a_{l-t-1}+a_{l-t}+1)-$$
$$(-a_{l-t}-2)-l+t-1)\omega_1\sum_{j=2}^{l-t-1}a_{j-1}\omega_j+(a_{l-t-1}+a_{l-t})\omega_{l-t}+$$
$$(-a_{l-t}-2+a_{l-t}+a_{l-t+1}+1+1)\omega_{l-t+1}+\sum_{j=l-t+2}^{l}a_j\omega_j=$$
$$(p-\sum_{j=1}^{l-t-1}a_j-l+t)\omega_1+\sum_{j=2}^{l-t-1}a_{j-1}\omega_j+$$
$$(a_{l-t-1}+a_{l-t})\omega_{l-t}+\sum_{j=l-t+1}^{l}a_j\omega_j.$$

□

Lemma 2. Let $z_i \in Z_1 \cup Z_2$ and $\nu = \sum_{i=1}^{l} a_i\omega_i$, where $a_1, a_2, \cdots, a_l \in \mathbb{Z}$. Then
(a) $z_1 \cdot \nu = a_1\omega_1 + (p - a_1 - a_2 - 2\sum_{i=3}^{l-1} a_i - a_l - 2l + 2)\omega_2 + \sum_{i=3}^{l} a_i\omega_i$;
(b) for all $i \in \{2, 3, \cdots, l-2\}$,

$$z_i \cdot \nu = (\sum_{j=1}^{i} a_j + i - 1)\omega_1 + (p - a_1 - 2\sum_{j=2}^{l-1} a_j - a_l - 2l + 1)\omega_2 + \sum_{j=3}^{i} a_{j-1}\omega_j + (a_i + a_{i+1} + 1)\omega_{i+1} + \sum_{j=i+2}^{l} a_j\omega_j;$$

(c) $z_{l-1} \cdot \nu = (\sum_{j=1}^{l-1} a_j + l - 2)\omega_1 + (p - a_1 - 2\sum_{j=2}^{l-1} a_j - a_l - 2l + 1)\omega_2 + \sum_{j=3}^{l-1} a_{j-1}\omega_j + (2a_{l-1} + a_l + 2)\omega_l$;
(d) $z_l \cdot \nu = (\sum_{j=1}^{l} a_j + l - 1)\omega_1 + (p - a_1 - 2\sum_{j=2}^{l-1} a_j - a_l - 2l + 1)\omega_2 + \sum_{j=3}^{l-1} a_{j-1}\omega_j + (2a_{l-1} + a_l + 2)\omega_l$;
(e) for all $i \in \{1, \cdots, l-2\}$,

$$z_{l+i} \cdot \nu = (\sum_{j=1}^{l-i-1} a_j + 2\sum_{j=l-i}^{l-1} a_j + a_l + l + i - 1)\omega_1 + (p - a_1 - 2\sum_{j=2}^{l-1} a_j - a_l - 2l + 1)\omega_2 + \sum_{j=3}^{l-i-1} a_{j-1}\omega_j + (a_{l-i-1} + a_{l-i} + 1)\omega_{l-i} + \sum_{j=l-i+1}^{l} a_j\omega_j.$$

Proof. By definition, $z_i = y_i s_0$ for all $i \in \{1, 2, \cdots, 2l-2\}$. Then $z_i \cdot \nu = y_i \cdot (s_0 \cdot \nu)$ for all $i \in \{1, 2, \cdots, 2l-2\}$. Since $s_0 = y_0$, using the statement (a), we have

$$s_0 \cdot \nu = (p - a_1 - 2\sum_{i=2}^{l-1} a_i - a_l - 2l + 1)\omega_1 + \sum_{i=2}^{l} a_i\omega_i.$$

Therefore,

$$z_i \cdot \nu = y_i \cdot ((p - a_1 - 2\sum_{i=2}^{l-1} a_i - a_l - 2l + 1)\omega_1 + \sum_{i=2}^{l} a_i\omega_i). \quad (7)$$

Thus, for all $i \in \{1, 2, \cdots, 2l-2\}$, the statement of the lemma for z_i follows from the corresponding statement for y_i of the Lemma 1 and from (7). □

Now we find a system of generators of a stabilizer of the elements $\nu_{i_1,i_2,\cdots,i_m} \in \overline{C}_1 \setminus C_1$.

Lemma 3. Let S_ν be a system of generators of a stabilizer of $\nu \in \overline{C}_1 \setminus C_1$. If $\nu = \nu_{i_1,i_2,\cdots,i_m}$ then $S_\nu = \{s_{i_1}, s_{i_2}, \cdots, s_{i_m}\}$.

In particular, if $m = 1$ then $S_{\nu_i} = \{s_i\}$ for all $i \in \{0, 1, 2, \cdots, l\}$.

Proof. The generators $s_0, s_1, \cdots, s_l$ of W_p act on ν as follows:

$$s \cdot \nu = \begin{cases} \nu - (\langle \nu + \rho, \alpha_0 \rangle - p)\alpha_0 & \text{if } s = s_0 \\ \nu - \langle \nu + \rho, \alpha_i \rangle \alpha_i & \text{if } i \in \{1, 2, \cdots, l\} \end{cases} \quad (8)$$

If $i_1 = 0$ then by definition $\nu_{0,i_2,\cdots,i_m} \in H_0 \cap H_{i_2} \cap \cdots \cap H_{i_m}$. Then

$$\langle \nu_{0,i_2,\cdots,i_m} + \rho, \alpha_0 \rangle = p$$

and

$$\langle \nu_{0,i_2,\cdots,i_m} + \rho, \alpha_i \rangle = 0$$

for all $i \in \{i_2, \cdots, i_m\}$. Therefore, by (8), the condition

$$s \in S_{\nu_{0,i_2,\cdots,i_m}} = \{s \in S_p \,|\, s \cdot \nu_{0,i_2,\cdots,i_m} = \nu_{0,i_2,\cdots,i_m}\}$$

yields $s \in \{s_0, s_{i_2}, \cdots, s_{i_m}\} \subset S_p$.

If $i_1 \neq 0$ then by definition $\nu_{i_1,i_2,\cdots,i_m} \in H_{i_1} \cap H_{i_2} \cap \cdots \cap H_{i_m}$. Then

$$\langle \nu_{i_1,i_2,\cdots,i_m} + \rho, \alpha_i \rangle = 0$$

for all $i \in \{i_1, i_2, \cdots, i_m\}$. Therefore, by (8), the condition

$$s \in S_{\nu_{i_1,i_2,\cdots,i_m}} = \{s \in S_p \,|\, s \cdot \nu_{i_1,i_2,\cdots,i_m} = \nu_{i_1,i_2,\cdots,i_m}\}$$

yields $s \in \{s_{i_1}, s_{i_2}, \cdots, s_{i_m}\} \subset S_p$. □

Lemma 4. Let $w \in Y_1 \cup Y_2$, $\nu \in \overline{C}_1 \setminus C_1$ and $w \cdot \nu \in X(T)^+$.
(a) If $w = 1$ then $\nu \in \{\nu_0\}$.
(b) If $w = y_0$ then $\nu \in \{\nu_1, \nu_0\}$.
(c) If $w = y_1$ then $\nu \in \{\nu_1, \nu_2, \nu_0, \nu_{0,2}\}$.
(d) If $w = y_i$, where $i \in \{2, 3, \cdots, l-2\}$, then $\nu \in \{\nu_i, \nu_{i+1}, \nu_0, \nu_{0,i}, \nu_{0,i+1}\}$.
(e) If $w = y_{l-1}, y_l$ then $\nu \in \{\nu_{l-1}, \nu_l, \nu_0, \nu_{0,l-1}\}$.
(f) If $w = y_{l+i}$, where $i \in \{1, 2, \cdots, l-2\}$, then

$$\nu \in \{\nu_{l-i-1}, \nu_{l-i}, \nu_0, \ \nu_{0,l-i-1}, \ \nu_{0,l-i}\}.$$

Proof. (a) We prove that $\nu = \nu_{i_1,i_2,\cdots,i_m} \in X(T)^+$ if and only if $m = 1$ and $i_1 = 0$. Indeed, if $m = 1$ and $i_1 = 0$ then ν belongs to the upper closure of C_1, since $\langle \nu_0 + \rho, \alpha_i \rangle \neq 0$ for all $i \in \{1, 2, \cdots, l\}$ and $\langle \nu_0 + \rho, \alpha_0 \rangle = p$. Therefore $0 < \langle \nu_0 + \rho, \alpha \rangle \leq p$ for all $\alpha \in \Delta$.

Conversely, if $\nu = \nu_{i_1,i_2,\cdots,i_m} \in X(T)^+$ then $\langle \nu_{i_1,i_2,\cdots,i_m} + \rho, \alpha \rangle \neq 0$ for all $\alpha \in R^+$. In particular, $\langle \nu_{i_1,i_2,\cdots,i_m} + +\rho, \alpha \rangle \neq 0$ for all $\alpha \in \{\alpha_{i_1}, \alpha_{i_2}, \cdots, \alpha_{i_m}\}$. Then

$$i_1 = i_2 = \cdots = i_m = 0,$$

since

$$\nu_{i_1,i_2,\cdots,i_m} \in \overline{C}_1 \setminus C_1.$$

Therefore, by the conditions 2) and 3) of the definition of $\nu_{i_1,i_2,\cdots,i_m}$, we get $m = 1$ and $i_1 = 0$. This implies $\nu = \nu_0$.

(b) Let $\nu = \nu_{i_1,i_2,\cdots,i_m} = \sum_{i=1}^{l} a_i \omega_i$ and $\nu \notin H_0$. Then $\langle \nu_{i_1,i_2,\cdots,i_m} + \rho, \alpha \rangle \neq 0$ for all $\alpha \in \{\alpha_{i_1}, \alpha_{i_2}, \cdots, \alpha_{i_m}\}$. This condition yields $a_{i_1} = a_{i_2} = \cdots = a_{i_m} = -1$. By the statement (b) of Lemma 1, $y_0 \cdot \nu \in X(T)^+$ if and only if $i_1 = i_2 = \cdots = i_m = 1$. Then by the conditions 2) and 3) of the definition of $\nu_{i_1,i_2,\cdots,i_m}$, we get $m = 1$ and $i_1 = 1$. This implies that $\nu = \nu_1$.

If $\nu \in H_0$ then $i_1 = 0$ and $\langle \nu_{0,i_2,\cdots,i_m} + \rho, \alpha_0 \rangle = p$. Using (5) we get $2\sum_{j=1}^{l-1} a_j + a_l + 2l - 1 = p$. Then by the statement (b) of Lemma 1, $y_0 \cdot \nu \in X(T)^+$ if and only if $i_2 = i_3 = \cdots = i_m = 0$. Then by the conditions 2) and 3) of the definition of $\nu_{0,i_2,\cdots,i_m}$, we get $m = 1$ and $i_1 = 0$. Therefore, $\nu = \nu_0$.

Other statements are easily proved similarly as the previous statement. □

For the elements of $Z_1 \cup Z_2$, using Lemma 2, we have the following

Lemma 5. Let $w \in Z_1 \cup Z_2$, $\nu \in \overline{C}_1 \setminus C_1$ and $w \cdot \nu \in X(T)^+$.
(a) If $w = z_1$ then $\nu \in \{\nu_2, \nu_0, \nu_{0,2}\}$.
(b) If $w = z_2$ then $\nu \in \{\nu_1, \nu_2, \nu_3, \nu_0, \nu_{1,3}, \ \nu_{0,2}, \nu_{0,3}\}$.
(c) If $w = z_i$, where $i \in \{3, 4, \cdots, l-2\}$, then

$$\nu \in \{\nu_1, \nu_i, \nu_{i+1}, \nu_{1,i}, \nu_{1,i+1}, \nu_0, \nu_{0,i}, \nu_{0,i+1}\}.$$

(d) If $w = z_{l-1}, z_l$ then $\nu \in \{\nu_1, \nu_{l-1}, \nu_l, \nu_{1,l-1}, \nu_{1,l}, \nu_0, \nu_{0,l-1}\}$.
(e) If $w = z_{l+i}$, where $i \in \{1, 2, \cdots, l-3\}$, then

$$\nu \in \{\nu_1, \nu_{l-i-1}, \nu_{l-i}, \nu_{1,l-i-1}, \nu_{1,l-i}, \nu_0, \nu_{0,l-i-1}, \nu_{0,l-i}\}.$$

(f) If $w = z_{2l-2}$ then $\nu \in \{\nu_1, \nu_2, \nu_{1,2}, \nu_0, \nu_{0,1}, \nu_{0,2}\}$.

By Lemmas 4 and 5, if $(Y_1 \cup Y_2 \cup Z_1 \cup Z_2)^+_\nu \neq \emptyset$ then

$$\nu \in \{\nu_0, \nu_l, \nu_i, \nu_{1,i+1}, \nu_{0,i} \,|\, i = 1, 2, \cdots, l-1\}.$$

We calculate the stabilizers of these elements ν.

Lemma 6. The following statements hold:
(a) for all $i \in \{0, 1, \cdots, l\}$, $st(\nu_i) = \{1, s_i\}$;
(b) for all $i \in \{3, 4, \cdots, l\}$, $st(\nu_{1,i}) = \{1, s_1, s_i, s_1 s_i\}$;
(c)for all $i \in \{2, 3, \cdots, l-1\}$, $st(\nu_{0,i}) = \{1, s_0, s_i, s_0 s_i\}$;
(d) $st(\nu_{1,2}) = \{1, s_1, s_2, s_1 s_2, s_2 s_1, s_1 s_2 s_1\}$;
(e) $st(\nu_{0,1}) = \{1, s_0, s_1, s_1 s_0, s_0 s_1, s_1 s_0 s_1, s_0 s_1 s_0, s_1 s_0 s_1 s_0\}$.

Proof. It follows from the defining relations (1) of the affine Weyl group W_p and Lemma 3. □

Using Lemma 6, we can easily describe $(Y_1 \cup Y_2 \cup Z_1 \cup Z_2)^+_\nu$ for all ν listed above. Below, we often omit the index ν of the element $\overline{x}_\nu$ when ν is a fixed element of $\overline{C}_1 \setminus C_1$.

Lemma 7. Consider the elements in $\{\nu_0, \nu_l, \nu_i, \nu_{1,i+1}, \nu_{0,i} \,|\, i = 1, 2, \cdots, l-1\}$. The following hold:

(a) $(Y_1 \cup Y_2 \cup Z_1 \cup Z_2)^+_{\nu_0} = \{\overline{y_i} \,|\, i = 0,\, 1,\, \cdots\, 2l-2\}$, where $\overline{y_0} = \{1,\, y_0\}$ and $\overline{y_i} = \{y_i,\, z_i\}$;

(b) $(Y_1 \cup Y_2 \cup Z_1 \cup Z_2)^+_{\nu_1} = \{\overline{y_0},\, \overline{y_{2l-2}},\, \overline{z_i} \,|\, i = 2,\, 3,\, \cdots\, 2l-2\}$, where $\overline{y_0} = \{y_0,\, y_1\}$, $\overline{y_{2l-2}} = \{y_{2l-2},\, y_{2l-2}s_1\}$ and $\overline{z_i} = \{z_i,\, z_i s_1\}$;

(c) for all $i \in \{2,\, 3,\, \cdots,\, l-1\}$,

$$(Y_1 \cup Y_2 \cup Z_1 \cup Z_2)^+_{\nu_i} = \{\overline{y_{i-1}},\, \overline{z_{i-1}},\, \overline{y_{2l-i-1}},\, \overline{z_{2l-i-1}}\},$$

where $\overline{y_{i-1}} = \{y_{i-1},\, y_i\}$, $\overline{z_{i-1}} = \{z_{i-1},\, z_i\}$, $\overline{y_{2l-i-1}} = \{y_{2l-i-1},\, y_{2l-i}\}$ and $\overline{z_{2l-i-1}} = \{z_{2l-i-1},\, z_{2l-i}\}$;

(d) $(Y_1 \cup Y_2 \cup Z_1 \cup Z_2)^+_{\nu_l} = \{\overline{y_{l-1}},\, \overline{z_{l-1}}\}$, where $\overline{y_{l-1}} = \{y_{l-1},\, y_l\}$ и $\overline{z_{l-1}} = \{z_{l-1},\, z_l\}$;

(e) $(Y_1 \cup Y_2 \cup Z_1 \cup Z_2)^+_{\nu_{1,2}} = \{\overline{z_{2l-3}}\}$, where

$$\overline{z_{2l-3}} = \{z_{2l-3},\, z_{2l-2},\, z_{2l-3}s_1,\, z_{2l-2}s_1,\, z_{2l-3}s_1s_2,\, z_{2l-2}s_1s_2\};$$

(f) for all $i \in \{3,\, 4,\, \cdots,\, l-1\}$, $(Y_1 \cup Y_2 \cup Z_1 \cup Z_2)^+_{\nu_{1,i}} = \{\overline{z_{i-1}},\, \overline{z_{2l-i-1}}\}$, where

$$\overline{z_{i-1}} = \{z_{i-1},\, z_i,\, z_{i-1}s_1,\, z_i s_1\},\quad \overline{z_{2l-i-1}} = \{z_{2l-i-1},\, z_{2l-i},\, z_{2l-i-1}s_1,\, z_{2l-i}s_1\};$$

(g) $(Y_1 \cup Y_2 \cup Z_1 \cup Z_2)^+_{\nu_{1,l}} = \{\overline{z_{l-1}}\}$, where $\overline{z_{l-1}} = \{z_{l-1},\, z_l,\, z_{l-1}s_1,\, z_l s_1\}$.

(h) $(Y_1 \cup Y_2 \cup Z_1 \cup Z_2)^+_{\nu_{0,1}} = \{\overline{y_{2l-2}}\}$, where

$$\overline{y_{2l-2}} = \{y_{2l-2},\, z_{2l-2},\, y_{2l-2}s_1,\, z_{2l-2}s_1,\, y_{2l-2}s_1s_0,\, z_{2l-2}s_1s_0,\, y_{2l-2}s_1s_0s,\, z_{2l-2}s_1s_0s_1\};$$

(l) for all $i \in \{2,\, 3,\, \cdots,\, l-1\}$,

$$(Y_1 \cup Y_2 \cup Z_1 \cup Z_2)^+_{\nu_{0,i}} = \{\overline{y_{i-1}},\, \overline{y_{2l-i-1}}\},$$

where $\overline{y_{i-1}} = \{y_{i-1},\, y_i,\, z_{i-1},\, z_i\}$ and $\overline{y_{2l-i-1}} = \{y_{2l-i-1},\, y_{2l-i},\, z_{2l-i-1},\, z_{2l-i}\}$.

Proof. Let $w \in Y_1 \cup Y_2 \cup Z_1 \cup Z_2$. By definition,

$$\overline{w}_\nu = \{wx \,|\, x \in st(\nu)\}. \tag{9}$$

Then using (9) and Lemmas 4 – 6, we obtain the required statements. □

3 Proof of the Theorem 1

Using Lemma 7 and the sum formula (6), we can easily prove Theorem 1.

By sum formula (6), for all cases listed in the statements (d)–(l) of Lemma 7,

$$\sum_{j>0} [V(\overline{w} \cdot \nu)^j] = 0.$$

In the following cases $\sum_{j>0}[V(\overline{w} \cdot \nu)^j]$ is also trivial:

1) $\overline{w} \in \{\overline{y_0},\, \overline{y_1}\} \subset (Y_1 \cup Y_2 \cup Z_1 \cup Z_2)^+_{\nu_0}$;

2) $\overline{w} \in \{\overline{y_0},\, \overline{z_{2l-2}}\} \subset (Y_1 \cup Y_2 \cup Z_1 \cup Z_2)^+_{\nu_1}$;

3) $\overline{w} \in \{\overline{y_{i-1}},\, \overline{y_{2l-i-1}}\} \subset (Y_1 \cup Y_2 \cup Z_1 \cup Z_2)^+_{\nu_i}$, where $i \in \{2,\, 3,\, \cdots,\, l-1\}$;

4) $\overline{w} = \overline{y_{l-1}} \in (Y_1 \cup Y_2 \cup Z_1 \cup Z_2)^+_{\nu_l}$.

Therefore, in all these cases $\chi(\overline{w} \cdot \nu) = [L(\overline{w} \cdot \nu)]$.

Thus, it remains to prove only the statements (a)–(f).

(a) By the statement (a) of Lemma 7, $\overline{y_0}_{\nu_0} = \{1,\, y_0\}$ and $\overline{y_i}_{\nu_0} = \{y_i,\, z_i\}$ for all $i \in \{1,\, 2,\, \cdots,\, 2l-2\}$. Therefore, $\chi(\nu_0) = \chi(y_0 \cdot \nu_0)$ and $\chi(y_i \cdot \nu_0) = \chi(z_i)$ for all $i \in \{1,\, 2,\, \cdots,\, 2l-2\}$. Then using the sum formula (6), we have

$$\sum_{j>0} [V(\overline{y_i} \cdot \nu_0)^j] = \sum_{k=2}^{i} (-1)^{i-k} \chi(\overline{y_{k-1}} \cdot \nu_0) \tag{10}$$

for all $i \in \{2, 3, \cdots, 2l-2\}$. If $i = 2$ then by (10),

$$\sum_{j>0}[V(\overline{y}_2 \cdot \nu_0)^j] = \chi(\overline{y_1} \cdot \nu_0) = [L(\overline{y_1} \cdot \nu_0)].$$

This implies that $\chi(\overline{y_2} \cdot \nu_0) = [L(\overline{y_2} \cdot \nu_0)] + [L(\overline{y_1} \cdot \nu_0)]$.

Now suppose that the statement (a) is true for all $i < t$, where $t \leq 2l-2$. Then by (10),

$$\sum_{j>0}[V(\overline{y_t} \cdot \nu_0)^j] = \sum_{k=2}^{i}(-1)^{t-k}([L(\overline{y_{k-1}} \cdot \nu_0)] + [L(\overline{y_{k-2}} \cdot \nu_0)]) = [L(\overline{y_{t-1}} \cdot \nu_0)].$$

This yields $\chi(\overline{y_t} \cdot \nu_0) = [L(\overline{y_t} \cdot \nu_0)] + [L(\overline{y_{t-1}} \cdot \nu_0)]$. So, the statement (a) is true for all $i \in \{2, 3, \cdots 2l-2\}$.

(b). By the statement (b) of Lemma 7, $\chi(\nu_1) = \chi(z_1 \cdot \nu_1) = 0$. Then by (6),

$$\begin{gathered}\sum_{j>0}[V(\overline{z_i} \cdot \nu_1)^j] = (-1)^i\chi(\overline{y_0} \cdot \nu_1) + \\ \sum_{k=2}^{i-1}(-1)^{i-k-1}\chi(\overline{z_k} \cdot \nu_1) + \delta(i = 2l-2)\chi(\overline{y_{2l-2}} \cdot \nu_1)\end{gathered} \tag{11}$$

for all $i \in \{2, 3, \cdots, 2l-2\}$. If $i = 2$ then using (11), we get

$$\sum_{j>0}[V(\overline{z_2} \cdot \nu_1)^j] = \chi(\overline{y_0} \cdot \nu_1) = [L(\overline{y_0} \cdot \nu_1)].$$

This yields the statement (b).

(c). We use (11) and the induction on i. If $i = 3$ then by (11) and by the statement (b) of this Theorem 1, we have

$$\sum_{j>0}[V(\overline{z_3} \cdot \nu_1)^j] = -\chi(\overline{y_0} \cdot \nu_1) + \chi(\overline{z_2} \cdot \nu_1) = [L(\overline{z_2} \cdot \nu_1)].$$

This yields $\chi(\overline{z_3} \cdot \nu_1) = [L(\overline{z_3} \cdot \nu_1)] + [L(\overline{z_2} \cdot \nu_1)]$.

Now suppose that the statement (c) is true for all $i < t$, where $t \leq 2l-3$. Then by (11),

$$\begin{gathered}\sum_{j>0}[V(\overline{z_t} \cdot \nu_0)^j] = (-1)^t\chi(\overline{y_0} \cdot \nu_1) + \sum_{k=2}^{t-1}(-1)^{t-k-1}\chi(\overline{z_k} \cdot \nu_1) = \\ (-1)^t[L(\overline{y_0} \cdot \nu_1)] + (-1)^{t-3}([L(\overline{z_2} \cdot \nu_1)] + [L(\overline{y_0} \cdot \nu_1)]) + \\ \sum_{k=3}^{t-1}(-1)^{t-k-1}([L(\overline{z_k} \cdot \nu_1)] + [L(\overline{z_{k-1}} \cdot \nu_1)]) = [L(\overline{z_{t-1}} \cdot \nu_1)].\end{gathered}$$

It follows that $\chi(\overline{z_t} \cdot \nu_0) = [L(\overline{z_t} \cdot \nu_0)] + [L(\overline{z_{t-1}} \cdot \nu_0)]$. Therefore, the statement (c) is true for all $i \in \{3, 4, \cdots 2l-3\}$.

(d). By (11),

$$\sum_{j>0}[V(\overline{z_{2l-2}} \cdot \nu_0)^j] = \chi(\overline{y_0} \cdot \nu_1) + \sum_{k=2}^{2l-3}(-1)^{2l-k-3}\chi(\overline{z_k} \cdot \nu_1) + \chi(\overline{y_{2l-2}}).$$

Using the previous statements (b) and (c) of this Theorem 1, we obtain

$$\begin{gathered}\sum_{j>0}[V(\overline{z_{2l-2}} \cdot \nu_0)^j] = [L(\overline{y_0} \cdot \nu_1)] + (-1)^{2l-5}([L(\overline{z_2} \cdot \nu_1)] + [L(\overline{y_0} \cdot \nu_1)]) + \\ \sum_{k=3}^{2l-3}(-1)^{2l-k-3}([L(\overline{z_k} \cdot \nu_1)] + [L(\overline{z_{k-1}} \cdot \nu_1)]) + [L(\overline{y_{2l-2}} \cdot \nu_1)] = \\ [L(\overline{z_{2l-3}} \cdot \nu_1)] + [L(\overline{y_{2l-2}} \cdot \nu_1)].\end{gathered}$$

It follows that $\chi(\overline{z_{2l-2}} \cdot \nu_1) = [L(\overline{z_{2l-2}} \cdot \nu_1)] + [L(\overline{z_{2l-3}} \cdot \nu_1)] + [L(\overline{y_{2l-2}} \cdot \nu_1)]$.

(e). Let $i \in \{2, 3, \cdots, l\}$. By the statement (c) of Lemma 7,

$$\chi(\nu_i) = \chi(y_j \cdot \nu_i) = \chi(z_j \cdot \nu_i) = 0$$

for all $j \in \{0, 1, \cdots, i-2\}$. Then by (6),

$$\sum_{j>0}[V(\overline{z_{i-1}} \cdot \nu_i)^j] = \chi(\overline{y_{i-1}} \cdot \nu_i)$$

for all $i \in \{2, 3, \cdots, l\}$. So, for all $i \in \{2, 3, \cdots, l\}$,

$$\chi(\overline{z_{i-1}} \cdot \nu_i) = [L(\overline{z_{i-1}} \cdot \nu_i)] + [L(\overline{y_{i-1}} \cdot \nu_i)].$$

(f). Let $i \in \{2, 3, \cdots l-1\}$. By the statement (c) of Lemma 7,

$$\chi(\nu_i) = \chi(y_j \cdot \nu_i) = \chi(z_j \cdot \nu_i) = 0$$

for all $j \in \{0, 1, \cdots, i-2\} \cup \{i+1, i+2, \cdots, 2l-i-1\}$ and $\chi(\overline{z_{i-1}}) = \chi(\overline{z_i})$. Then by (6),

$$\sum_{j>0} [V(\overline{z_{2l-i-1}} \cdot \nu_i)^j] = \chi(\overline{y_{2l-i-1}} \cdot \nu_i)$$

for all $i \in \{2, 3, \cdots, l-1\}$. So, for all $i \in \{2, 3, \cdots l-1\}$,

$$\chi(\overline{z_{2l-i-1}} \cdot \nu_i) = [L(\overline{z_{2l-i-1}} \cdot \nu_i)] + [L(\overline{y_{2l-i-1}} \cdot \nu_i)].$$

The proof of Theorem 1 is complete. □

Remark 1. Let $(Y_1 \cup Y_2 \cup Z_1 \cup Z_2)^+_\nu \neq \emptyset$. If ν lies in the intersection of two hyperplanes then, by Theorem 1, all Weyl modules with highest weights $\overline{w} \cdot \nu$ with $\overline{w} \in (Y_1 \cup Y_2 \cup Z_1 \cup Z_2)^+_\nu$ are simple.

From the proof of Theorem 1 we immediately obtain the following

Corollary 2. Let G be a simply-connected and semisimple algebraic group of type $B_l \, (l > 2)$ over an algebraically closed field $\mathbb{K}$ of characteristic $p \geq h$, where h is the Coxeter number. Then the Weyl modules with the following highest weights are simple:

(a) ν_0, $\overline{y_1} \cdot \nu_0$;
(b) $\overline{y_0} \cdot \nu_1$, $\overline{z_2} \cdot \nu_1$, $\overline{z_{2l-2}} \cdot \nu_1$;
(c) $\overline{y_{i-1}} \cdot \nu_i$, where $i \in \{2, 3, \cdots, l\}$;
(d) $\overline{y_{2l-i-1}} \cdot \nu_i$, where $i \in \{2, 3, \cdots, l-1\}$;
(e) $\overline{z_{2l-3}} \cdot \nu_{1,2}$,
(f) $\overline{z_{i-1}} \cdot \nu_{1,i}$, $\overline{z_{2l-i-1}} \cdot \nu_{1,i}$, where $i \in \{3, 4, \cdots, l-1\}$;
(g) $\overline{z_{l-1}} \cdot \nu_{1,l}$;
(h) $\overline{y_{2l-2}} \cdot \nu_{0,1}$;
(l) $\overline{y_{i-1}} \cdot \nu_{0,i}$, $\overline{y_{2l-i-1}} \cdot \nu_{0,i}$, where $i \in \{2, 3, \cdots, l-1\}$.

Remark 2. It is known that, in the restricted region, the differential of each simple G-module is a simple $\mathfrak{g}$-module, where $\mathfrak{g}$ is the Lie algebra of G. Therefore, Corollary 2 generalizes the Rudakov simplicity criterion [4, Theorem 2] for semisimple Lie algebras of type B_l. The Weyl module $V(\overline{y_0} \cdot \nu_1)$ satisfies the Rudakov simplicity criterion. In all other cases the highest weights obtained in Corollary 2 do not satisfy the Rudakov simplicity criterion.

Using Lemmas 1 and 2, one can easily describe highest weights of the simple Weyl modules listed in Corollary 2. For example, by definition, $\overline{y_0} \cdot \nu_1 = y_0 \cdot \nu_1$, and ν_1 satisfies the condition $\langle \nu_1 + \rho, \alpha_1 \rangle = 0$, since $\nu_1 \in H_1$. If we write $\nu_1 = \sum_{j=1}^{l} a_j \omega_j$ then the above condition yields $a_1 = -1$. Then by the statement (a) of Lemma 1,

$$\overline{y_0} \cdot \nu_1 = y_0 \cdot \nu_1 = (p - 2\sum_{j=2}^{l-1} a_j - a_l - 2l + 2)\omega_1 + \sum_{j=2}^{l} a_j \omega_j,$$

where $2\sum_{j=2}^{l-1} a_j + a_l + 2l - 3 < p$.

Remark 3. The simplicity of the following Weyl modules for the algebraic group of type B_4 was proved in [11, Theorem 1, (d)]:

- $V((p-4)\omega_4) = V(\overline{z_2} \cdot \nu_{1,3})$, where $\nu_{1,3} = -\omega_1 - \omega_3 + (p-4)\omega_4 \in H_1 \cap H_3$;
- $V((p-5)\omega_4) = V(\overline{y_1} \cdot \nu_{0,2})$, where $\nu_{0,2} = -\omega_2 + (p-5)\omega_4 \in H_0 \cap H_2$;
- $V((p-6)\omega_4) = V(\overline{y_0} \cdot \nu_1)$, where $\nu_1 = -\omega_1 + (p-6)\omega_4 \in H_1$;
- $V((p-7)\omega_4) = V(\nu_0)$, where $\nu_0 = (p-7)\omega_4 \in H_0$.

Thus, Corollary 2 gives several new examples of simple Weyl modules for the algebraic groups of type B_l.

Acknowledgments

This research was funded by Science Committee grant of the Ministry of Education and Science of the Republic of Kazakhstan (Grant No. AP08855935).

References

1 Fiebig P. An upper bound on the exceptional characteristics for Lusztig's character formula / P. Fiebig // J. Reine Angew. Math. — 2012. — Iss. 673. — P. 1–31.

2 Braden B. Restricted representations of classical Lie algebras of type A_2 and B_2 / B. Braden // Bull. Amer. Math. Soc. — 1967. — 73. — No. 3. — P. 482–486.

3 Jantzen J.C. Darstellungen halbeinfascher Gruppen und kontravariante Formen / J.C.Jantzen // J. Reine Angew. Math. — 1977. — 290. — P. 117–141.

4 Рудаков А.Н. Размерности некоторых неприводимых представлений полупростых алгебр Ли классического типа над полем конечной характеристики / А.Н.Рудаков // Тр. Семинара И.Г. Петровского. — 1978. — № 3. — С. 147–160.

5 O'Halloran J. Weyl modules and the cohomology of Chevalley groups/ J.O'Halloran // Amer. J. Math. — 1981. — 103. — No. 2. — P. 399–410.

6 Dowd M.F. On representations of algebraic groups in characteristic two/M.F. Dowd, P. Sin // Comm. Algebra. — 1996. — 24. — No. 8. — P. 2597–2686.

7 Ye J. Irreducible Characters of Algebraic Groups in Characteristic Two (II) / J. Ye, Z. Zhou // Algebra Colloquium. — 1999. — 6. — No 4. — P. 401–411.

8 Ye J. Irreducible Characters of Algebraic Groups in Characteristic Three / J. Ye, Z. Zhou // Commun. Algebra. — 2001. — 29. — No. 1. — P. 221–223.

9 Arslan O. Some simple modules for classical groups and p-ranks of orthogonal and Hermitian geometries / O. Arslan, P. Sin // J. Algebra. — 2011. — 327. — No. 1. — P. 141–169.

10 Cardinali I. On certain submodules of Weyl modules for $SO(2n+1;\mathbb{F})$ with $char\,(\mathbb{F}) = 2$/I. Cardinali, A. Pasini // J. Group Theory. — 2014. — 17. — No. 4. — P. 559–588.

11 Wiggins E. Weyl modules for simple algebraic groups of type B_4 and D_4 / E. Wiggins // Comm. Algebra. — 2016. — 44. — No. 9. — P. 4001–4025.

12 Cavallin M. Structure of certain Weyl modules for the spin groups / M. Cavallin // https://arxiv.org/pdf/1704.01803 — 2017.

13 Bourbaki N. Elements of Mathematics. Lie groups and Lie algebras: Chapters 4–6 / N. Bourbaki. — Berlin Heidelberg: Springer — 2011. — 300 p.

14 Jantzen J.C. Representations of algebraic groups, second ed., Mathematical Surveys and Monographs, Vol. 107., Amer. Math. Soc., 2003.

15 Ibraev Sh.Sh. Submodule structure of some Weyl modules and the cohomology for $SO_{2l}(\mathbb{K})$ / Sh.Sh. Ibraev // Comm. Algebra. — 2019. — 47. — No 9. — P. 3735–3764.

16 Ibraev Sh.Sh. On Weyl modules with singular highest weights for algebraic groups of tye C_l/ Sh.Sh. Ibraev // Kazakh Math. Journal. — 2019. — 19. — No. 4. — P. 79–97.

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$\mathfrak{so}_{2l+1}(\mathbb{K})$ үшін сингуляр үлкен салмақты жәй модульдер туралы

Мақалада сипаттамасы $p \geq h$, мұндағы h – Кокстер саны болатын алгебралық тұйық өрістегі B түріндегі классикалық Ли алгебрасының сингуляр үлкен салмақты жәй модульдерінің формалды характерлері зерттелді. Бұл жәй модульдердің үлкен салмақтары шектелген деп есептелінеді. Олардың формалды характерлерінің сипаттамасы берілді. Дербес жағдайда, жәй Вейль модульдерінің жаңа мысалдары алынды. Шектелген салмақтар жағдайында алгебралық группалар мен олардың Ли алгебраларының көріністер теориясы эквивалентті. Сондықтан, оң сипаттамалы өрістердегі жартылай жәй бірбайланысқан алгебралық группалардың көріністер теориясының әдістері қолданылған. Жәй модульдердің формалды характерлерін сипаттау үшін үлкен салмағы сәйкес келетін Вейль модульдерінің Янцен фильтрациясын құру пайдаланылады.

Кілт сөздер: Ли алгебрасы, жәй модуль, алгебралық группа, Вейль модулі, Янцен фильтрациясы.

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О простых модулях со старшим сингулярным весом для $\mathfrak{so}_{2l+1}(\mathbb{K})$

В статье изучены формальные характеры простых модулей со старшим сингулярным весом классической алгебры Ли типа B над алгебраически замкнутым полем характеристики $p \geq h$, где h – число Кокстера. Предположено, что старшие веса этих простых модулей ограничены. Авторами описаны их формальные характеры. В частности, получены новые примеры простых модулей Вейля. В области ограниченных весов теории представлений алгебраических групп и их алгебр Ли эквивалентны. По этой причине можно применять инструменты теории представлений полупростых односвязных алгебраических групп в положительной характеристике. Для описания формальных характеров простых модулей использована конструкция фильтрации Янцена модулей Вейля соответствующих старших весов.

Ключевые слова: алгебра Ли, простой модуль, алгебраическая группа, модуль Вейля, фильтрация Янцена.

References

1 Fiebig, P. (2012). An upper bond on the exceptional characteristics for Lusztig's character formula *J. Reine Angew. Math., 673*, 1–31.

2 Braden, B. (1967). Restricted representations of classical Lie algebras of type A_2 and B_2 *Bull. Amer. Math. Soc., 73*(3), 482–486.

3 Jantzen, J.C. (1977). Darstellungen halbeinfascher Gruppen und kontravariante Formen *J. Reine Angew. Math., 290*, 117–141.

4 Rudakov, A.N. (1978). Razmernosti nekotorykh neprivodimykh predstavlenii poluprostykh algebr Li klassicheskogo tipa nad polem konechnoi kharakteristiki [Dimensions of some irreducible representations of semisimple Lie algebras of classical type over fields of finite characteristic]. *Trudy Seminara imeni I.G. Petrovskogo — Proceedings of the seminar of I.G. Petrovsky, 3*, 147–160 [in Russian].

5 O'Halloran, J. (1981). Weyl modules and cohomology of Chevalley groups *Amer. J. Math., 103*(2), 399–410.

6 Dowd, M.F., & Sin, P. (1996). On representations of algebraic groups in characteristic two *Comm. Algebra., 24*(8), 2597–2686.

7 Ye, J., & Zhou, Z. (1999). Irreducible Characters of Algebraic Groups in Characteristic Two (II) *Algebra Colloquium, 6*(4), 401–411.

8 Ye, J., & Zhou, Z. (2001). Irreducible Characters of Algebraic Groups in Characteristic Three *Commun. Algebra, 29*(1), 221–223.

9 Arslan, O., & Sin, P. (2011). Some simple modules for classical groups and p-ranks of orthogonal and Hermitian geometries *J. Algebra, 327*(1), 141–169.

10 Cardinali, I., & Pasini, A. (2014). On certain submodules of Weyl modules for $SO(2n+1;\mathbb{F})$ with $char\,(\mathbb{F}) = 2$ *J. Group Theory, 17*(4), 559–588.

11 Wiggins, E. (2016). Weyl modules for simple algebraic groups of type B_4 and D_4 *Comm. Algebra, 44*(9), 4001–4025.

12 Cavallin, M. (2017). Structure of certain Weyl modules for the Spin groups. *https://arxiv.org/pdf/1704.01803.*

13 Bourbaki, N. (2011). *Elements of Mathematics. Lie groups and Lie algebras: Chapters 4–6.* Berlin Heidelberg: Springer.

14 Jantzen, J.C. (2003). Representations of algebraic groups, second ed. *Mathematical Surveys and Monographs, 107*, Amer. Math. Soc.

15 Ibraev, Sh.Sh. (2019) Submodule structure of some Weyl modules and the cohomology for $SO_{2l}(\mathbb{K})$ *Comm. Algebra, 47*(9), 3735–3764.

16 Ibraev, Sh.Sh. (2019). On Weyl modules with singular highest weights for algebraic groups of tye C_l *Kazakh Math. Journal, 19*(4), 79–97.

DOI 10.31489/2022M1/66-73

UDC 517.51

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Interpolation of nonlinear integral Urysohn operators in net spaces

In this paper, we study the interpolation properties of the net spaces $N_{p,q}(M)$, in the case when M is a sufficiently general arbitrary system of measurable subsets from $\mathbb{R}^n$. The integral Urysohn operator is considered. This operator generalizes all linear, integral operators, and non-linear integral operators. The Urysohn operator is not a quasilinear or subadditive operator. Therefore, the classical interpolation theorems for these operators do not hold. A certain analogue of the Marcinkiewicz-type interpolation theorem for this class of operators is obtained. This theorem allows to obtain, in a sense, a strong estimate for Urysohn operators in net spaces from weak estimates for these operators in net spaces with local nets. For example, in order for the Urysohn integral operator in a net space, where the net is the set of all balls in $\mathbb{R}^n$, it is sufficient for it to be of weak type for net spaces, where the net is concentric balls.

Keywords: interpolation spaces, net spaces, Urysohn integral operators.

Introduction

Let (V,ν), (U,μ) are measurable spaces and $Z(U)$, $M(V)$ are normed spaces of ν-measurable and μ-measurable functions, respectively. Let $K:\mathbb{R}\times U\times V\to\mathbb{R}$, and the operator $T: Z(U)\to M(V)$ is defined by the following equality: For any $f\in Z(U)$

$$T(f,y)=\int_U K(f(x),x,y)d\mu \quad , \quad y\in V \tag{1}$$

and assume that this integral exists and is finite for almost all $y\in V$. This operator is called the Urysohn integral operator.

In the paper [1], new interpolation theorems were proved for these operators in Morrey-type spaces. Analogs of the interpolation theorems of Marcinkiewicz-Calderon, Stein-Weiss, Petre were obtained.

In this paper, we study the interpolation properties of the net spaces $N_{p,q}(M)$. Also, we prove a certain analogue of the Marcinkiewicz-type interpolation theorem for the Urysohn operator (1). We use the ideas developed in [1–3], where an interpolation theorem of Marcinkiewicz type for Morrey spaces was obtained.

Let in $\mathbb{R}^n$ is given n-dimensional Lebesgue measure μ, M is an arbitrary system of measurable subsets from $\mathbb{R}^n$. For a function $f(x)$, defined and integrable on each e from M, we define the function

$$\bar{f}(t,M)=\sup_{\substack{e\in M\\ |e|\ge t}}\frac{1}{|e|}\left|\int_e f(x)dx\right|, \quad t>0,$$

where the supremum is taken over all $e\in M$, whose measure is $|e|\overset{\text{def}}{=}\mu e\ge t$. In the case, when $\sup\{|e|: e\in M\}=\alpha<\infty$ and $t>\alpha$ assuming that $\bar{f}(t,M)=0$.

Let p,q parameters satisfy the conditions $0<p\le\infty$, $0<q\le\infty$. We define the net spaces $N_{p,q}(M)$, as the set of all functions f, such that for $q<\infty$

$$\|f\|_{N_{p,q}(M)}=\left(\int_0^\infty\left(t^{\frac{1}{p}}\bar{f}(t,M)\right)^q\frac{dt}{t}\right)^{\frac{1}{q}}<\infty,$$

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and for $q = \infty$

$$\|f\|_{N_{p,\infty}(M)} = \sup_{t>0} t^{\frac{1}{p}} \bar{f}(t, M) < \infty.$$

These spaces were introduced in the work [4]. Net spaces have found important applications in various problems of harmonic analysis, operator theory and the theory of stochastic processes [5–13].

Marcinkiewicz-type interpolation theorem for Urysohn operators

A family of measurable sets $G = \{G_t\}_{t>0}$ is called a local net if it satisfies the following conditions $G_t \subset G_s$ for $t \le s$ and $|G_t| = t$. An example of a local net is the set $\{B_t(x)\}_{t>0}$ of all balls centered at the point x.

Let $G = \{G_t\}_{t>0}$ be a local net. We define the net $F_{G,\Omega} = \bigcup_{x\in\Omega} G + x$, where $G + x = \{G_t + x\}_{t>o}$. The net $F_{G,\Omega}$ will be called the generated by local net G and the set Ω.

Example. Let $\Omega = \mathbb{R}^n$, $G = \{Q_t\}_{t>0}$ be a set of cubes centered at 0, then $F_{G,\Omega} = \{Q_t + x\}_{t>0,\, x\in\mathbb{R}^n}$ is the set of all cubes in $\mathbb{R}^n$.

Lemma 1. Let T be the Urysohn operator of the form (1), then for an arbitrary function $f \in Z(U)$ from the domain and for any μ measurable set $w \subset U$ the following condition holds:

$$T(f, y) = T(f\chi_w, y) + T(f\chi_{U\setminus w}, y) - T(0, y).$$

Proof. Due to the additivity of the integral with respect to measure

$$T(f, y) - T(f\chi_w, y) = \int_U K(f(x), xy)d\mu - \int_U K(f(x)\chi_w(x), x, y)d\mu$$

$$= \int_{U\setminus w} K(f(x)\chi_{U\setminus w}(x), x, y)d\mu + \int_w K(f(x)\chi_w(x), x, y)d\mu$$

$$- \int_{U\setminus w} K(0, x, y)d\mu - \int_w K(f(x)\chi_w(x), x, y)d\mu$$

$$= \int_U K(f(x)\chi_{U\setminus w}(x), x, y)d\mu - \int_w K(0, x, y)d\mu - \int_{U\setminus w} K(0, x, y)d\mu$$

$$= \int_U K(f(x)\chi_{U\setminus w}(x), x, y)d\mu - \int_U K(0, x, y)d\mu = T(f\chi_{U\setminus w}, y) - T(0, y).$$

Lemma 2. (Hardy's inequalities) Let $\mu > 0, -\infty < \nu < \infty$ and $0 < \sigma \le \tau \le \infty$, then the following inequalities hold

$$\left(\int_0^\infty \left(y^{-\mu}\left(\int_0^y \left(r^{-\nu}|g(r)|\right)^\sigma \frac{dr}{r}\right)^{\frac{1}{\sigma}}\right)^\tau \frac{dy}{y}\right)^{\frac{1}{\tau}} \le (\mu\sigma)^{-\frac{1}{\sigma}} \left(\int_0^\infty \left(y^{-\mu-\nu}|g(y)|\right)^\tau \frac{dy}{y}\right)^{\frac{1}{\tau}}$$

and

$$\left(\int_0^\infty \left(y^{\mu}\left(\int_0^y \left(r^{-\nu}|g(r)|\right)^\sigma \frac{dr}{r}\right)^{\frac{1}{\sigma}}\right)^\tau \frac{dy}{y}\right)^{\frac{1}{\tau}} \le (\mu\sigma)^{-\frac{1}{\sigma}} \left(\int_0^\infty \left(y^{\mu-\nu}|g(y)|\right)^\tau \frac{dy}{y}\right)^{\frac{1}{\tau}}.$$

Theorem 1. Let $\Omega \subset \mathbb{R}^n$, $G = \{G_t\}_{t>0}$ is the local net, $F = \bigcup_{x\in\Omega} G + x$. Let $0 < p_0 < p_1 < \infty$ and $0 < q_0, q_1 \le \infty$, $q_0 \ne q_1$, $0 < \theta < 1$, $1 \le \tau \le \infty$,

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \qquad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}.$$

If for the Urysohn operator T and some $M_0, M_1 > 0$ the following inequalities hold

$$\|T(f) - T(0)\|_{N_{q_i,\infty}(G+x)} \le M_i \|f\|_{N_{p_i,1}(G+x)}, \; i = 0, 1, \; x \in \Omega, \tag{2}$$

then

$$\|T(f) - T(0)\|_{N_{q,\tau}(F)} \le c M_0^{1-\theta} M_1^\theta \|f\|_{N_{p,\tau}(F)}, \tag{3}$$

for all functions $f \in N_{p,\tau}(F)$, where $c > 0$ depends only on the parameters $p_0, p_1, q_0, q_1, p, q, \tau, \theta$.

Proof.

Let $1 \le \tau < \infty$, $f \in N_{p,\tau}(F)$, for arbitrary $x \in \Omega$, $s > 0$, we define the functions

$$f_{0,s} = f\chi_{G_s+x}, \quad f_{1,s} = f - f_{0,s},$$

where χ_{G_s+x} denotes the characteristic function of the set $G_s + x$. It is easily seen that $f_{0,s} \in N_{p_0,1}$ and $f_{1,s} \in N_{p_1,1}$. Then $f = f_{0,s} + f_{1,s}$ and

$$\sup_{\xi\ge t}\frac{1}{|G_\xi|}\left|\int_{G_\xi+x}(T(f,y)-T(0,y))dy\right|$$

$$=\sup_{\xi\ge t}\frac{1}{|G_\xi|}\left|\int_{G_\xi+x}(T(f\chi_{G_s+x},y)+T(f\chi_{\mathbb{R}^n\setminus G_s+x},y)-2T(0,y))dy\right|$$

$$\le \sup_{\xi\ge t}\frac{1}{|G_\xi|}\left|\int_{G_\xi+x}(T(f_{0,s},y)-T(0,y))dy\right|$$

$$+\sup_{\xi\ge t}\frac{1}{|G_\xi|}\left|\int_{G_\xi+x}(T(f_{1,s},y)-T(0,y))dy\right| = I_1 + I_2.$$

First, we estimate I_1, according to the inequality (2) we have

$$I_1 = \sup_{\xi\ge t}\frac{1}{|G_s|}\left|\int_{G_\xi+x}(T(f_{0,s},y)-T(0,y))dy\right|$$

$$\le t^{-\frac{1}{q_0}}\sup_{r>0} r^{\frac{1}{q_0}}\sup_{\xi\ge r}\frac{1}{|G_\xi|}\left|\int_{G_\xi+x}(T(f_{0,s},y)-T(0,y))dy\right|$$

$$= t^{-\frac{1}{q_0}}\|(T(f_{0,s},y)-T(0,y))\|_{N_{q_0,\infty}(G+x)} \le M_0 t^{-\frac{1}{q_0}}\|f_{0,s}\|_{N_{p_0,1}(G+x)}$$

$$= M_0 t^{-\frac{1}{q_0}}\left(\int_0^s r^{\frac{1}{p_0}}\sup_{\xi\ge r}\frac{1}{|G_\xi|}\left|\int_{G_\xi+x}f_{0,s}(y)dy\right|\frac{dr}{r} + \int_s^\infty r^{\frac{1}{p_0}}\sup_{\xi\ge r}\frac{1}{|G_\xi|}\left|\int_{G_\xi+x}f_{0,s}(y)dy\right|\frac{dr}{r}\right).$$

Let $0 < r \le s$, if $\xi \le s$, $y \in G_\xi + x$, we have $f_{0,s}(y) = f(y)\chi_{G_s+x} = f(y)$, if $\xi > s$, then

$$\left|\int_{G_\xi+x}f_{0,s}(y)dy\right| = \left|\int_{G_s+x}f(y)dy\right|.$$

By the first integral, we have the following,

$$\int_0^s r^{\frac{1}{p_0}}\sup_{\xi\ge r}\frac{1}{|G_\xi}\left|\int_{G_\xi+x}f_{0,s}(y)dy\right|\frac{dr}{r} = \int_0^s r^{\frac{1}{p_0}}\sup_{s\ge\xi\ge r}\frac{1}{|G_\xi}\left|\int_{G_\xi+x}f(y)dy\right|\frac{dr}{r}$$

$$\le \int_0^s r^{\frac{1}{p_0}}\sup_{\xi\ge r}\frac{1}{|G_\xi}\left|\int_{G_\xi+x}f(y)dy\right|\frac{dr}{r} \le \int_0^s r^{\frac{1}{p_0}}\bar{f}(r,F)\frac{dr}{r}.$$

By the second integral, we have

$$\int_s^\infty r^{\frac{1}{p_0}}\sup_{\xi\ge r}\frac{1}{|G_\xi}\left|\int_{G_\xi+x}f_{0,s}(y)dy\right|\frac{dr}{r} = \int_s^\infty r^{\frac{1}{p_0}}\sup_{\xi\ge r}\frac{1}{|G_\xi}\left|\int_{G_s+x}f(y)dy\right|\frac{dr}{r}$$

$$= \left|\int_{G_s+x}f(y)dy\right|\int_s^\infty r^{\frac{1}{p_0}}\sup_{\xi\ge r}\frac{1}{|G_\xi|}\frac{dr}{r} = \left|\int_{G_s+x}f(y)dy\right|\int_s^\infty r^{\frac{1}{p_0}-1}\frac{dr}{r}$$

$$= p_0' s^{\frac{1}{p_0}}\frac{1}{|G_s|}\left|\int_{G_s+x}f(y)dy\right| \le p_0' s^{\frac{1}{p_0}}\bar{f}(s,F).$$

Thus, we get

$$I_1 \lesssim M_0 t^{-\frac{1}{q_0}}\left(\int_0^s r^{\frac{1}{p_0}}\bar{f}(r,F)\frac{dr}{r} + s^{\frac{1}{p_0}}\bar{f}(s,F)\right).$$

We estimate I_2 in a similar way applying the inequality (2), we obtain

$$I_2 = \sup_{s\geq t} \frac{1}{|G_s|}\Big| \int_{G_s+x} (T(f_{1,s},y) - T(0,y))dy\Big|$$
$$\leq t^{-\frac{1}{q_1}} \sup_{r>0} r^{\frac{1}{q_1}} \sup_{s\geq r} \frac{1}{|G_s|}\Big| \int_{G_s+x} (T(f_{1,s},y) - T(0,y))dy\Big|$$
$$= t^{-\frac{1}{q_1}} \|(T(f_{1,s},y) - T(0,y))\|_{N_{q_1,\infty}(G+x)} \leq M_1 t^{-\frac{1}{q_1}} \|f_{1,s}\|_{N_{p_1,1}(G+x)}$$
$$= M_1 t^{-\frac{1}{q_1}} \bigg(\int_0^\infty r^{\frac{1}{p_1}} \sup_{s\geq r} \frac{1}{|G_s|}\Big| \int_{G_s+x} f_{1,s}(y)dy\Big| \frac{dr}{r}\bigg)$$
$$= M_1 t^{-\frac{1}{q_1}} \bigg(\int_0^s r^{\frac{1}{p_1}} \sup_{\xi\geq r} \frac{1}{|G_\xi|}\Big| \int_{G_\xi+x} f_{1,s}(y)dy\Big| \frac{dr}{r} + \int_s^\infty r^{\frac{1}{p_1}} \sup_{\xi\geq r} \frac{1}{|G_\xi|}\Big| \int_{G_\xi+x} f_{1,s}(y)dy\Big| \frac{dr}{r}\bigg)$$
$$= M_1 t^{-\frac{1}{q_1}} \Big(J_1 + J_2 \Big).$$

To estimate J_1, J_2 note that

$$\Big| \int_{G_\xi+x} f_{1,s}(y)dy\Big| = \begin{cases} 0,\ \xi\leq s, \\ \Big| \int_{G_\xi+x\setminus G_s+x} f(y)dy\Big|,\ \xi > s \end{cases}$$

$$= \begin{cases} 0,\ \xi\leq s, \\ \Big| \int_{G_\xi+x} f(y)dy - \int_{G_s+x} f(y)dy\Big| \leq \Big| \int_{G_\xi+x} f(y)dy\Big| + \Big| \int_{G_s+x} f(y)dy\Big|,\ \xi > s. \end{cases}$$

Further,

$$J_1 \leq \int_0^s r^{\frac{1}{p_1}} \sup_{\xi\geq r} \frac{1}{|G_\xi|}\bigg(\Big| \int_{G_\xi+x} f(y)dy\Big| + \Big| \int_{G_s+x} f(y)dy\Big|\bigg)\frac{dr}{r}$$
$$\leq \int_0^s r^{\frac{1}{p_1}} \bigg(\bar{f}(s,F) + \Big| \int_{G_s+x} f(y)dy\Big| \sup_{\xi\geq r}\frac{1}{|G_\xi|}\bigg)\frac{dr}{r}$$
$$\leq 2\bar{f}(s,F) \int_0^s r^{\frac{1}{p_1}} \frac{dr}{r} = 2p_1 s^{\frac{1}{p_1}} \bar{f}(s,F),$$

and

$$J_2 \leq \int_s^\infty r^{\frac{1}{p_1}} \sup_{\xi\geq r} \frac{1}{|G_\xi|}\bigg(\Big| \int_{G_\xi+x} f(y)dy\Big| + \Big| \int_{G_s+x} f(y)dy\Big|\bigg)\frac{dr}{r}$$
$$\leq \int_s^\infty r^{\frac{1}{p_1}} \bigg(\bar{f}(s,F) + \Big| \int_{G_s+x} f(y)dy\Big| \sup_{\xi\geq r}\frac{1}{|G_\xi|}\bigg)\frac{dr}{r} \leq \int_s^\infty r^{\frac{1}{p_1}} \bar{f}(r,F)\frac{dr}{r}$$
$$+\Big| \int_{G_s+x} f(y)dy\Big| \int_s^\infty r^{\frac{1}{p_1}} \sup_{\xi\geq r}\frac{1}{|G_\xi|}\frac{dr}{r} = \int_s^\infty r^{\frac{1}{p_1}} \bar{f}(r,F)\frac{dr}{r}$$
$$+\Big| \int_{G_s+x} f(y)dy\Big| \frac{s^{\frac{1}{p_1}-1}}{p_1-1} p_1 \lesssim \int_s^\infty r^{\frac{1}{p_1}} \bar{f}(r,F)\frac{dr}{r} + s^{\frac{1}{p_1}} \bar{f}(s,F).$$

Combining the estimates, we obtain the following estimate

$$I_2 \lesssim M_1 t^{\frac{1}{q_1}} \bigg(\int_s^\infty r^{\frac{1}{p_1}} \bar{f}(r,F)\frac{dr}{r} + s^{\frac{1}{p_1}} \bar{f}(s,F)\bigg).$$

So, we got the following estimate

$$\sup_{s\geq t} \frac{1}{|G_s|}\Big| \int_{G_s+x} (T(f,y) - T(0,y))dy\Big| \lesssim M_0 t^{-\frac{1}{q_0}} \bigg(\int_0^s r^{\frac{1}{p_0}} \bar{f}(r,F)\frac{dr}{r} + s^{\frac{1}{p_0}} \bar{f}(s,F)\bigg)$$

$$+M_1 t^{-\frac{1}{q_1}}\left(\int_s^\infty r^{\frac{1}{p_1}}\bar{f}(r,F)\frac{dr}{r}+s^{\frac{1}{p_1}}\bar{f}(s,F)\right).$$

Assuming that $s=ct^\gamma$, where $\gamma=\left(\frac{1}{q_1}-\frac{1}{q_0}\right)\Big/\left(\frac{1}{p_1}-\frac{1}{p_0}\right)$, then, taking into account the above estimates, we obtain

$$\|T(f)-T(0)\|_{N_{q,\tau}(F)}=\left(\int_0^\infty\left(t^{\frac{1}{q}}\sup_{\substack{s\ge t\\ x\in\mathbb{R}^n}}\frac{1}{|G_s|}\left|\int_{G_s+x}f(x)dx\right|\right)^\tau\frac{dt}{t}\right)^{\frac{1}{\tau}}$$

$$\lesssim M_0A_1+M_0A_2+M_1A_3+M_1A_4,$$

where

$$A_1=\left(\int_0^\infty\left(t^{\frac{1}{q}-\frac{1}{q_0}}\int_0^{ct^\gamma}r^{\frac{1}{p_0}}\bar{f}(r,F)\frac{dr}{r}\right)^\tau\frac{dt}{t}\right)^{\frac{1}{\tau}},$$

$$A_2=\left(\int_0^\infty\left(t^{\frac{1}{q}-\frac{1}{q_0}}(ct^\gamma)^{\frac{1}{p_0}}\bar{f}(ct^\gamma,F)\right)^\tau\frac{dt}{t}\right)^{\frac{1}{\tau}},$$

$$A_3=\left(\int_0^\infty\left(t^{\frac{1}{q}-\frac{1}{q_1}}\int_{ct^\gamma}^\infty r^{\frac{1}{p_1}}\bar{f}(r,F)\frac{dr}{r}\right)^\tau\frac{dt}{t}\right)^{\frac{1}{\tau}},$$

$$A_4=\left(\int_0^\infty\left(t^{\frac{1}{q}-\frac{1}{q_1}}(ct^\gamma)^{\frac{1}{p_1}}\bar{f}(ct^\gamma,F)\right)^\tau\frac{dt}{t}\right)^{\frac{1}{\tau}}.$$

Using the change of variable $ct^\gamma=y$, we get

$$A_1=\gamma^{-\frac{1}{\tau}}c^{-\theta\left(\frac{1}{p_1}-\frac{1}{p_0}\right)}B_1,\quad A_2=\gamma^{-\frac{1}{\tau}}c^{-\theta\left(\frac{1}{p_1}-\frac{1}{p_0}\right)}B_2,$$

$$A_3=\gamma^{-\frac{1}{\tau}}c^{(1-\theta)\left(\frac{1}{p_1}-\frac{1}{p_0}\right)}B_2,\quad A_4=\gamma^{-\frac{1}{\tau}}c^{(1-\theta)\left(\frac{1}{p_1}-\frac{1}{p_0}\right)}B_3,$$

where

$$B_1=\left(\int_0^\infty\left(y^{\theta\left(\frac{1}{p_1}-\frac{1}{p_0}\right)}\int_0^y r^{\frac{1}{p_0}}\bar{f}(r,F)\frac{dr}{r}\right)^\tau\frac{dy}{y}\right)^{\frac{1}{\tau}},$$

$$B_2=\left(\int_0^\infty\left(y^{\frac{1}{p}}\bar{f}(y,F)\right)^\tau\frac{dy}{y}\right)^{\frac{1}{\tau}}=\|f\|_{N_{p,\tau}(F)},$$

$$B_3=\left(\int_0^\infty\left(y^{-(1-\theta)\left(\frac{1}{p_1}-\frac{1}{p_0}\right)}\int_y^\infty r^{\frac{1}{p_1}}\bar{f}(r,F)\frac{dr}{r}\right)^\tau\frac{dy}{y}\right)^{\frac{1}{\tau}},$$

$$B_4=\left(\int_0^\infty\left(y^{\frac{1}{p}}\bar{f}(y,F)\right)^\tau\frac{dy}{y}\right)^{\frac{1}{\tau}}=\|f\|_{N_{p,\tau}(F)}.$$

To estimate B_1,B_3 we apply Hardy's inequalities from the lemma 2 and we obtain

$$B_1\lesssim\left(\int_0^\infty\left(y^{\theta\left(\frac{1}{p_1}-\frac{1}{p_0}\right)+\frac{1}{p_0}}\bar{f}(r,F)\right)^\tau\frac{dy}{y}\right)^{\frac{1}{\tau}}\lesssim\|f\|_{N_{p,\tau}(F)},$$

$$B_3\lesssim\left(\int_0^\infty\left(y^{(\theta-1)\left(\frac{1}{p_1}-\frac{1}{p_0}\right)+\frac{1}{p_1}}\bar{f}(r,F)\right)^\tau\frac{dy}{y}\right)^{\frac{1}{\tau}}\lesssim\|f\|_{N_{p,\tau}(F)}.$$

Thus, from the above estimates, the following estimate was obtained

$$\|T(f)-T(0)\|_{N_{q,\tau}(F)}\lesssim\left(M_0c^{-\theta\left(\frac{1}{p_1}-\frac{1}{p_0}\right)}+M_1c^{(1-\theta)\left(\frac{1}{p_1}-\frac{1}{p_0}\right)}\right)\|f\|_{N_{p,\tau}(F)},$$

where the corresponding constants depend only on $p_0, p_1, q_0, q_1, p, q, \tau$ and θ.

Let $c = \left(\frac{M_1}{M_0}\right)^{\frac{p_0 p_1}{p_1 - p_0}}$, then

$$\|T(f) - T(0)\|_{N_{q,\tau}(F)} \lesssim M_0^{1-\theta} M_1^{\theta} \|f\|_{N_{p,\tau}(F)}.$$

Consequently, we obtain the required estimate (3). The theorem is proved.

Corollary. Let F be the set of all balls in $\mathbb{R}^n$, F_x – the set of all balls centered at the point $x \in \mathbb{R}^n$. Let $0 < p_0 < p_1 < \infty$ and $0 < q_0, q_1 \leq \infty$, $q_0 \neq q_1$, $0 < \theta < 1$, $1 \leq \tau \leq \infty$,

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \qquad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}.$$

If the following inequalities hold for the Urysohn operator T and some $M_0, M_1 > 0$

$$\|T(f) - T(0)\|_{N_{q_i,\infty}(G+x)} \leq M_i \|f\|_{N_{p_i,1}(G+x)}, \ i = 0, 1, \ x \in \mathbb{R}^n,$$

then for all functions $f \in N_{p,\tau}(F)$, holds

$$\|T(f) - T(0)\|_{N_{q,\tau}(F)} \leq c M_0^{1-\theta} M_1^{\theta} \|f\|_{N_{p,\tau}(F)},$$

where $c > 0$ depends only on parameters $p_0, p_1, q_0, q_1, p, q, \tau, \theta$.

Acknowledgments

This research was supported by the Ministry of Education and Science of the Republic of Kazakhstan (Project No. AP08856479).

References

1 Буренков В.И. Интерполяционные теоремы для нелинейных операторов в общих пространствах типа Морри и их приложения / В.И. Буренков, Е.Д. Нурсултанов // Тр. МИАН. — 2021. — 312. — С. 124–149.

2 Burenkov V.I. Marcinkiewicz-type interpolation theorem and estimates for convolutions for Morrey-type spaces / V.I. Burenkov, D.K. Chigambayeva, E.D. Nursultanov // Eurasian Math. J. — 2018. — 9. — No. 2. — P. 82–88.

3 Burenkov V.I. Interpolation theorems for nonlinear urysohn integral operators in general Morrey-type spaces / V.I. Burenkov, E.D. Nursultanov // Eurasian Math. J. — 2020. — 11. No. 4. — P. 87–94.

4 Нурсултанов Е.Д. Сетевые пространства и неравенства типа Харди–Литтлвуда / Е.Д. Нурсултанов // Мат. сб. — 1998. — 189. — № 3. — С. 399–419.

5 Akylzhanov R. $L_p - L_q$ multipliers on locally compact groups / R. Akylzhanov, M. Ruzhansky // J. Fun. Anal. — 2020. — 278. — No. 3. — P. 108324.

6 Akylzhanov R. Net spaces on lattices, Hardy-Littlewood type inequalities, and their converses/ R. Akylzhanov, M. Ruzhansky // Eurasian Math. J. — 2017. — 8. — No. 3. — P. 10–27.

7 Akylzhanov R. Hardy-Littlewood, Hausdorff-Young-Paley inequalities, and $L_p - L_q$ Fourier multipliers on compact homogeneous manifolds / R. Akylzhanov, M. Ruzhansky, E.D. Nursultanov // J. Math. Anal. Appl. — 2019. — 479. — No. 2. — P. 1519–1548.

8 Костюченко А.Г. Теория управления «катастрофами» / А.Г. Костюченко, Е.Д. Нурсултанов // УМН. — 1998. — 53. — № 3. — С. 628, 629.

9 Нурсултанов Е.Д. Нижние и верхние оценки нормы мультипликаторов кратных тригонометрических рядов Фурье в пространствах Лебега / Е.Д. Нурсултанов, Н.Т. Тлеуханова // Функц. анализ и его прил. — 2000. — 34. — № 2. — С. 151–153.

10 Нурсултанов Е.Д. О коэффициентах кратных рядов Фурье из L_p-пространств / Е.Д. Нурсултанов // Изв. РАН. Сер. мат. — 2000. — 64. — № 1. — С. 95–122.

11 Нурсултанов Е.Д. О применении интерполяционных методов к исследованию свойств функций многих переменных / Е.Д. Нурсултанов // Мат. заметки. — 2004. — 75. — № 3. — С. 372–383.

12 Nursultanov E.D. Net spaces and boundedness of integral operators / E.D. Nursultanov, S. Tikhonov // Geom. Anal. — 2011. — 21. — P. 950–981.

13 Nursultanov E.D. Interpolation methods for stochastic processes spaces / E.D. Nursultanov, T.U. Aubakirov // Abstr. Appl. Anal. — 2013. — 2013. — P. 1–12.

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Сызықтық емес интегралды Урысон операторларының торлы кеңістіктеріндегі интерполяциясы

Мақалада $N_{p,q}(M)$ торлы кеңістіктерінің интерполяциялық қасиеттері зерттелген, мұндағы M – $\mathbb{R}^n$ жиынының өлшенетін ішкі жиындардың жеткілікті жалпы ерікті жүйесі. Интегралды Урысон операторы қарастырылған. Бұл оператор барлық сызықтық, интегралды операторларды, сондай-ақ сызықты емес интегралды операторларды жалпылайды. Урысон операторы, әдетте, квазисызықты немесе субаддитивті оператор емес, сондықтан бұл операторлар үшін классикалық интерполяциялық теоремалар орындалмайды. Осы операторлар класы үшін Марцинкевич типіндегі интерполяциялық теоремасының белгілі бір аналогы алынды. Бұл теорема белгілі бір мағынада локальды торы бар торлы кеңістіктердегі Урысон операторлары үшін әлсіз бағалаулардан торлы кеңістіктердегі осы операторлар үшін күшті бағалау алуға мүмкіндік береді. Мысалы, тор $\mathbb{R}^n$-дегі барлық шарлар жиынтығы болатын торлы кеңістігінде Урысон интегралды операторы болу үшін, оның тор концентрлі шарлар болатын торлы кеңістіктер үшін әлсіз типті болуы жеткілікті.

Кілт сөздер: интерполяция кеңістіктері, торлы кеңістіктер, Урысон операторлары.

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Интерполяция нелинейных интегральных операторов Урысона в сетевых пространствах

В статье изучены интерполяционные свойства сетевых пространств $N_{p,q}(M)$, в случае когда M есть достаточно общая произвольная система измеримых подмножеств из $\mathbb{R}^n$. Рассмотрен интегральный оператор Урысона. Данный оператор обобщает все линейные, интегральные, а также нелинейные интегральные операторы. Оператор Урысона, вообще говоря, не является квазилинейным либо субаддитивным, поэтому классические интерполяционные теоремы для этих операторов не имеют места. Получен некий аналог интерполяционной теоремы типа Марцинкевича для этого класса операторов. Настоящая теорема позволяет получать в некотором смысле сильную оценку для операторов Урысона в сетевых пространствах из слабых оценок для них. Так, например, для того, чтобы был интегральный оператор Урысона в сетевом пространстве, где сеть есть множество всех шаров в $\mathbb{R}^n$, достаточно, чтобы он был слабого типа для сетевых пространств, где сеть есть концентрические шары.

Ключевые слова: интерполяционные пространства, сетевое пространство, операторы Урысона, интегральный оператор.

References

1 Burenkov, V.I., & Nursultanov, E.D. (2021). Interpoliatsionnye teoremy dlia nelineinykh operatorov v obshchikh prostranstvakh tipa Morri i ikh prilozheniia [Interpolation Theorems for Nonlinear Operators in General Morrey-Type Spaces and Their Applications]. *Trudy MIAN — Proc. Steklov Inst. Math., 312*, 124–149 [in Russian].

2 Burenkov, V.I., Chigambayeva, D.K., & Nursultanov, E.D. (2018). Marcinkiewicz-type interpolation theorem and estimates for convolutions for Morrey-type spaces. *Eurasian Math. J., 9* (2), 82–88.

3 Burenkov, V. I., & Nursultanov, E.D. (2020). Interpolation theorems for nonlinear urysohn integral operators in general Morrey-type spaces. *Eurasian Math. J., 11* (4), 87–94 .

4 Nursultanov, E.D. (1998). Setevye prostranstva i neravenstva tipa Khardi–Littlvuda [Net spaces and inequalities of Hardy-Littlewood type]. *Matematicheskii sbornik, 189* (3), 399–419 [in Russian].

5 Akylzhanov, R., & Ruzhansky, M. (2020). $L_p - L_q$ multipliers on locally compact groups. *J. Fun. Anal., 278* (3), 108324.

6 Akylzhanov, R., & Ruzhansky, M. (2017). Net spaces on lattices, Hardy-Littlewood type inequalities, and their converses. *Eurasian Math. J., 8* (3), 10–27.

7 Akylzhanov, R., Ruzhansky, M., & Nursultanov, E.D. (2019). Hardy-Littlewood, Hausdorff-Young-Paley inequalities, and $L_p - L_q$ Fourier multipliers on compact homogeneous manifolds. *J. Math. Anal. Appl., 479* (2), 1519–1548.

8 Nursultanov, E.D., & Kostyuchenko, A.G. (1998). Teoriia upravleniia «katastrofami» [Theory of control of "catastrophes"]. *Russ. Math. Surv, 53* (3), 628–629 [in Russian].

9 Nursultanov, E.D., & Tleukhanova, N.T. (2000). Nizhnie i verkhnie otsenki normy multiplikatorov kratnykh trigonometricheskikh riadov Fure v prostranstvakh Lebega [Lower and upper bounds for the norm of multipliers of multiple trigonometric Fourier series in Lebesgue spaces]. *Func. Anal. Appl., 34* (2), 151–153 [in Russian].

10 Nursultanov, E.D. (2000). O koeffitsientakh kratnykh riadov Fure iz L_p-prostranstv [On the coefficients of multiple Fourier series in L_p - spaces]. *Izvestiia RAN. Seriia Matematika — Izvestiia Mathematics, 64* (1), 93–120 [in Russian].

11 Nursultanov, E.D. (2004). O primenenii interpoliatsionnykh metodov k issledovaniiu svoistv funktsii mnogikh peremennykh [Application of Interpolational Methods to the Study of Properties of Functions of Several Variables]. *Mathematicheskii zametki — Mathematical Notes, 75*, 341–351 [in Russian].

12 Nursultanov, E.D., & Tikhonov, S. (2011). Net spaces and boundedness of integral operators. *J. Geom. Anal., 21*, 950–981.

13 Nursultanov, E.D., & Aubakirov, T.U. (2013). Interpolation methods for stochastic processes spaces. *Abstr. Appl. Anal., 2013*, 1–12.

DOI 10.31489/2022M1/74-82

UDC 517.95

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Boundary value problem for the heat equation with a load as the Riemann-Liouville fractional derivative

A boundary value problem for a fractionally loaded heat equation is considered in the first quadrant. The loaded term has the form of the Riemann-Liouville's fractional derivative with respect to the time variable, and the order of the derivative in the loaded term is less than the order of the differential part. The study is based on reducing the boundary value problem to a Volterra integral equation. The kernel of the obtained integral equation contains a special function, namely, the Wright function. The kernel is estimated, and the conditions for the unique solvability of the integral equation are obtained.

Keywords: loaded equation, fractional derivative, Volterra integral equation, Wright function, unique solvability.

Introduction

The study of fractional differential equations has been the subject of intense research attention [1–7]. This is due both to the development of the fractional integration and differentiation theory, and to the use of the apparatus of fractional integration and differentiation in various fields of science. Considerable interest in the study of fractional differential equations, among other things, is also fueled by various applications in physics, mechanics, and simulation [8–14]. Of particular note are some recent applications of the fractional diffusion equation to economics and financial modeling (see e.g., [15]). Monographs [16–18] contain vast bibliographies concerning the issue. Also, an important section in the theory of differential equations is the class of loaded equations. The study of loaded partial differential equations has a long history and occupies an important place in the modern theory of differential equations. In [19], on numerous examples A.M. Nakhushev showed the practical and theoretical importance of studies on loaded equations. In [20–23], the theory of loaded equations was further developed. In [22, 23] loaded differential equations are considered as weak or strong perturbations of differential equations depending on the derivative order of the loaded summand.

In the works [24–27], BVPs with a loaded heat equation are investigated, when the loaded term is represented in the form of a fractional derivative. In [24, 25], the load moves with a constant velocity. The loaded term is the trace of the fractional order derivative on the line $x = t$. It is represented as a Riemann-Liouville fractional derivative. The obtained Volterra singular integral equation has a nonempty spectrum for certain values of the fractional derivative order. Volterra integral equations of the second kind with singularities in the kernel arising from the study were considered in [26, 27] In the papers [28, 29], the loaded term is represented in the form of the Caputo fractional derivative with respect to the time variable and the spatial variable, and the derivative order of the loaded term is less than the order of the differential part.

In this paper, a BVP is considered in the open right upper quadrant. The problem is reduced to an integral equation that, in some cases, belongs to the pseudo-Volterra type, and its solvability depends on the order of differentiation in the loaded term and the behavior of the load line in a neighborhood of the origin. The BVP is reduced to a Volterra integral equation of the second kind with a kernel containing a special function. The solvability of the integral equation in the class of continuous functions is established depending on the nature of the load for small values of time.

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The article has 4 sections. Section 1 contains notations some previously known concepts and several auxiliary assertions. In Section 2, we formulate the problem we are going to solve. In Section 3, the problem is reduced to an integral equation. In Section 4, we study the resulting integral equation by evaluating its kernel and formulate the corresponding results on the solvability of the problem.

1 Preliminaries

Let us first recall some previously known concepts and results. The first one is the definition of the Riemann-Liouville fractional derivative.

Definition 1 ([1]). Let $f(t) \in L_1[a,b]$. Then, the Riemann-Liouville derivative of the order β is defined by the following formula

$$ {}_rD^{\beta}_{a,t} f(t) = \frac{1}{\Gamma(n-\beta)} \frac{d^n}{dt^n} \int_a^t \frac{f(\tau)}{(t-\tau)^{\beta-n+1}} d\tau, \; \beta, a \in R, \; n-1 < \beta < n. \tag{1} $$

From formula (1) it follows that

$$ {}_rD^{0}_{a,t} f(t) = f(t), \quad {}_rD^{n}_{a,t} f(t) = f^{(n)}(t), \quad n \in N. \tag{2} $$

In [30], when considering the limiting cases of the order of the fractional derivative in the loaded term of the equation, formula (2) is used to investigate the continuity in the order of the fractional derivative.

We study boundary value problems for the loaded heat equation when the loaded term is represented in the form of a fractional derivative. The considered problem is reduced to an integral equation by inverting the differential part.

In the domain $Q = \{(x,t) \mid x > 0, \quad t > 0\}$ the solution to the boundary value problem ([31]; 57) of heat conduction

$$ u_t = a^2 u_{xx} + F(x,t), $$
$$ u\,|_{t=0} = f(x), \quad u\,|_{x=0} = g(x), $$

is described by the formula

$$ u(x,t) = \int_0^{\infty} G(x,\xi,t)\, f(\xi)\, d\xi + \int_0^t H(x,t-\tau)\, g(\tau)\, d\tau + $$
$$ + \int_0^t \int_0^{\infty} G(x,\xi,t-\tau)\, F(\xi,\tau)\, d\xi d\tau, \tag{3} $$

where

$$ G(x,\,\xi,\,t) = \frac{1}{2\sqrt{\pi\, a\, t}} \left\{ \exp\left(-\frac{(x-\xi)^2}{4\,a\,t} \right) - \exp\left(-\frac{(x+\xi)^2}{4\,a\,t} \right) \right\}, $$
$$ H(x,\,t) = \frac{1}{2\sqrt{\pi\, a}\, t^{3/2}} \exp\left(-\frac{x^2}{4\,a\,t} \right). $$

The Green's function $G(x,\xi,t-\tau)$ satisfies the relation

$$ \int_0^{\infty} G(x,\xi,t)\, d\xi = \operatorname{erf}\left(\frac{x}{2\sqrt{t}} \right), \tag{4} $$

where

$$ \operatorname{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-\xi^2} d\xi. \tag{5} $$

Fractional calculus can be considered as a "laboratory" for special functions.

We get a reduced integral equation with a kernel containing the Wright function. Accordingly, we determine the conditions for the solvability of this equation using the kernel estimate from the works [32, 33].

ϕ is the Wright function:

$$ \phi(a,b;\, z) = \sum_{k=0}^{\infty} \frac{z^k}{k!\,\Gamma(ak+b)} \qquad (a > -1). \tag{6} $$

The differentiation formula is valid:

$$\left(\frac{d}{dz}\right)^n \phi(\alpha,\beta;z) = \phi(\alpha,\alpha+n\beta;z), n \in N. \tag{7}$$

For all $\alpha \in]0,1[, \beta \in R, x>0, y>0$ the following inequality holds

$$\left|y^{\beta-1}\phi\left(-\alpha,\beta;-xy^{-\alpha}\right)\right| \leq Cx^{-\theta}y^{\beta+\alpha\theta-1}, \tag{8}$$

where

$$\theta \geq \begin{cases} 0, (-\beta) \notin N\bigcup\{0\}, \\ -1, (-\beta) \in N\bigcup\{0\}. \end{cases}$$

2 Statement of the fractionally loaded BVP of heat conduction

In the domain $Q = \{(x,t) : x>0, t>0\}$ we consider a BVP

$$u_t - u_{xx} + \lambda\left\{{}_rD^{\beta}_{0,t}u(x,t)\right\}|_{x=\gamma(t)} = f(x,t), \tag{9}$$

$$u(x,0) = 0, \quad u(0,t) = 0, \tag{10}$$

where λ is a complex parameter, ${}_rD^{\beta}_{0,\,t}\,u(x,\,t)$ is the Riemann-Liouville derivative (1) of an order β, $0<\beta<1$, $\gamma(t)$ is a continuous increasing function, $\gamma(0)=0$.

The problem is studied in the class of continuous functions.

Let us introduce the notation

$$D^{\nu}_{at}g(t) = \frac{1}{\Gamma(-\nu)}\int_a^t \frac{g(\xi)\,d\xi}{(t-\xi)^{\nu+1}}, \quad \nu<0.$$

When $\nu=0$ $D^0_{at}g(t) = g(t)$ then

$$D^{\nu}_{at}g(t) = \frac{d^n}{dt^n}D^{\nu-n}_{at}g(t), \quad n-1<\nu\leq n, \quad n\in N.$$

$a=0$, $n=1$, $\nu=\beta \Rightarrow$

$${}_rD^{\beta}_{0t}u(x,t) = \frac{d}{dt}D^{\beta-1}_{0t}u(x,t) \tag{11}$$

or

$${}_rD^{\beta}_{0t}u(x,t) = \frac{d}{dt}\left(\frac{1}{\Gamma(1-\beta)}\int_0^t \frac{u(x,\tau)\,d\tau}{(t-\tau)^{\beta}}\right). \tag{12}$$

The derivative in the loaded term of equation (9) is determined by the formula (12).

3 Reducing the boundary value problem to an integral equation

According to the formula (3) a solution to BVP (9)–(10) can be represented as

$$u(x,t) = -\lambda\int_0^t\int_0^{\infty} G(x,\xi,t-\tau)\,\mu(\tau)\,d\xi d\tau + f_1(x,t), \tag{13}$$

where

$$\mu(t) = \left\{{}_rD^{\beta}_{0t}u(x,t)\right\}|_{x=\gamma(t)} \tag{14}$$

$$f_1(x,t) = \int_0^t\int_0^{+\infty} G(x,\xi,t-\tau)\,f(\xi,\tau)\,d\xi d\tau. \tag{15}$$

According to the formula (4) and

$$e^{-\xi^2} = \sqrt{\pi}\phi\left(-\frac{1}{2},\frac{1}{2},-2\xi\right), \tag{16}$$

where

$$\phi\left(a,b,z\right)=\sum_{\kappa=0}^{\infty}\frac{z^{\kappa}}{\kappa!\Gamma\left(a\kappa+b\right)},\quad a>-1,\quad b\in C \tag{17}$$

is the Wright function (6), we have [34]

$$\operatorname{erf}\left(z\right)=2\int_{0}^{z}\phi\left(-\frac{1}{2},\frac{1}{2},-2\xi\right)d\xi=1-\phi\left(-\frac{1}{2},1,-2z\right). \tag{18}$$

Indeed, since

$$\Gamma\left(1-z\right)\cdot\Gamma\left(z\right)=\frac{\pi}{\sin\pi z}\Rightarrow$$

$$\Gamma\left(-\frac{\kappa}{2}+\frac{1}{2}\right)=\frac{\pi}{\Gamma\left(\frac{\kappa}{2}+\frac{1}{2}\right)\sin\left(\frac{\pi\kappa}{2}+\frac{\pi}{2}\right)}=\frac{\pi}{\Gamma\left(\frac{\kappa}{2}+\frac{1}{2}\right)\cos\frac{\pi\kappa}{2}};$$

$$\cos\frac{\pi\kappa}{2}=\begin{cases}0, & \text{if}\quad \kappa=2n+1,\\ \left(-1\right)^{n}, & \text{if}\quad \kappa=2n.\end{cases}$$

$$\Rightarrow\frac{1}{\Gamma\left(-\frac{\kappa}{2}+\frac{1}{2}\right)}=\begin{cases}0, & \text{if}\quad \kappa=2n+1,\\ \dfrac{(-1)^{n}\Gamma\left(\frac{\kappa}{2}+\frac{1}{2}\right)}{\pi}, & \text{if}\quad \kappa=2n,\end{cases}$$

where $n=0;1;2;...$, then from (17) we have:

$$\phi\left(-\frac{1}{2},\frac{1}{2},-2\xi\right)=\sum_{\kappa=0}^{\infty}\frac{(-2\xi)^{\kappa}}{\kappa!\Gamma\left(-\frac{\kappa}{2}+\frac{1}{2}\right)}=\sum_{n=0}^{\infty}\frac{\left(-1\right)^{n}\Gamma\left(n+\frac{1}{2}\right)}{(2n)!\cdot\pi}\cdot\left(-2\xi\right)^{2n}=$$

$$\left\|\Gamma\left(n+\frac{1}{2}\right)=\frac{(2n)!}{4^{n}\cdot n!}\sqrt{\pi}\right\|$$

$$=\frac{1}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{\left(-1\right)^{n}}{4^{n}\cdot n!}\cdot 4^{n}\left(\xi^{2}\right)^{n}=\frac{1}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{\left(-\xi^{2}\right)^{n}}{n!}=\frac{1}{\sqrt{\pi}}e^{-\xi^{2}}.$$

We obtain formula (16).

From (5) we have:

$$\operatorname{erf}\left(z\right)=\frac{2}{\sqrt{\pi}}\int_{0}^{z}e^{-\xi^{2}}d\xi=2\int_{0}^{z}\phi\left(-\frac{1}{2},\frac{1}{2};-2\xi\right)d\xi=-\int_{0}^{-2z}\phi\left(-\frac{1}{2},\frac{1}{2};\zeta\right)d\zeta=$$

$$=-\sum_{\kappa=0}^{\infty}\int_{0}^{2z}\frac{\zeta^{\kappa}}{\kappa!\cdot\Gamma\left(-\frac{\kappa}{2}+\frac{1}{2}\right)}d\zeta=-\sum_{\kappa=0}^{\infty}\frac{\zeta^{\kappa+1}}{(\kappa+1)!\cdot\Gamma\left(-\frac{\kappa+1}{2}+1\right)}\Big|_{\zeta=0}^{\zeta=-2z}=$$

$$=-\sum_{\kappa=0}^{\infty}\frac{(-2z)^{\kappa+1}}{(\kappa+1)!\cdot\Gamma\left(-\frac{\kappa+1}{2}+1\right)}=-\sum_{n=0}^{\infty}\frac{(-2z)^{n}}{n!\cdot\Gamma\left(-\frac{n}{2}+1\right)}+1=1-\phi\left(-\frac{1}{2},1,-2z\right).$$

We get formula (18).

Then, taking into account formulas (16) and (18), representation (13) can be rewritten as:

$$u\left(x,t\right)=-\lambda\int_{0}^{t}K\left(\frac{x}{2\sqrt{t-\tau}}\right)\mu\left(\tau\right)d\tau+f_{1}\left(x,t\right), \tag{19}$$

where

$$K\left(\frac{x}{2\sqrt{t-\tau}}\right)=1-\phi\left(-\frac{1}{2},1,-\frac{x}{\sqrt{t-\tau}}\right) \tag{20}$$

and $\mu(t)$ and $f_1(t)$ are defined by formulas (14) and (15), respectively.

For formula (19) we implement the fractional differentiation formula of order β $(0 < \beta < 1)$ in the sense of Riemann-Liouville.

Denote $K\left(\frac{x}{2\sqrt{t}}\right) = g(x,t)$.

As:

$$\int_0^t K\left(\frac{x}{2\sqrt{t-\tau}}\right)\mu(\tau)\,d\tau = \int_0^t g(x,t-\tau)\,\mu(\tau)\,d\tau = (g(x,t) * \mu(t))(t),$$

and

$$\frac{d}{dt}(g*\mu)(t) = \left(\frac{dg}{dt}*\mu\right)(t) + g\,|_{t=0}\cdot\mu(t),$$

then by formulas (11), (7) and (20) we have

$$D_{0t}^{\beta}\left(\int_0^t K\left(\frac{x}{2\sqrt{t-\tau}}\right)\mu(\tau)\,d\tau\right) = D_{0t}^{\beta}\left(K\left(\frac{x}{2\sqrt{t}}\right)*\mu(t)\right) =$$

$$= D_{0t}^{\beta}\left(1-\phi\left(-\frac{1}{2},1,-\frac{x}{\sqrt{t}}\right)\right)*\mu(t) + \mathrm{K}\left(\frac{x}{2\sqrt{t}}\right)\Bigg|_{t=0}\cdot\mu(t). \quad (21)$$

As

$$D_{0t}^{\beta}(1) = \frac{1}{\Gamma(1-\beta)}\cdot t^{-\beta},$$

then when $0<\beta<1$

$$D_{0t}^{\beta}t^{\mu} = \frac{\Gamma(\mu+1)}{\Gamma(\mu-\beta+1)}t^{\mu-\beta}.$$

From here

$$D_{0t}^{\beta}\phi\left(-\frac{1}{2},1;-x\cdot t^{-\frac{1}{2}}\right) = \sum_{n=0}^{\infty}\frac{(-x)^n}{n!\cdot\Gamma\left(-\frac{n}{2}+1\right)}D_{0t}^{\beta}\left(t^{-\frac{n}{2}}\right) =$$

$$= \sum_{n=0}^{\infty}\frac{(-x)^n}{n!\cdot\Gamma\left(-\frac{n}{2}+1\right)}\frac{\Gamma\left(-\frac{n}{2}+1\right)}{\Gamma\left(-\frac{n}{2}+1-\beta\right)}\cdot t^{-\frac{n}{2}-\beta} = t^{-\beta}\cdot\sum_{n=0}^{\infty}\frac{\left(-\frac{x}{\sqrt{t}}\right)^n}{n!\cdot\Gamma\left(-\frac{n}{2}+1-\beta\right)} =$$

$$= t^{-\beta}\cdot\phi\left(-\frac{1}{2},1-\beta;-\frac{x}{\sqrt{t}}\right). \quad (22)$$

$$g(x,t)\,|_{t=0} = \mathrm{K}\left(\frac{x}{2\sqrt{t}}\right)\Bigg|_{t=0} = \left(1-\phi\left(-\frac{1}{2},1;-\frac{x}{\sqrt{t}}\right)\right)\Bigg|_{t=0}. \quad (23)$$

Since in the given problem (9), (10) the line along which the load is moving has the form $x=\gamma(t)$, and $\gamma(t)$ increases and $\gamma(0)=0$ then there are different cases of behavior for $\frac{x}{\sqrt{t}}\big|_{x=\gamma(t)}$ when $t\to 0$.

Let $0<x=\gamma(t)\sim t^{\omega}$ when $t\to 0$. Then $\frac{x}{\sqrt{t}}\to+\infty$ when $t\to 0$, if $\omega<\frac{1}{2}$.

Cases $\omega>\frac{1}{2}$ and $\omega=\frac{1}{2}$ we consider later.

From [12, p. 6] we have an asymptotic expansion for $z\to+\infty$:

$$\phi\left(-\frac{1}{2},1;-z\right) = e^{-\frac{z^2}{4}}\left[\sum_{j=0}^{m}A_j\cdot 2^{2j+1}\cdot z^{-2j-1} + O\left(2^{2m}\cdot z^{-2m-1}\right)\right].$$

Then if $\omega<\frac{1}{2}$ for formula (23) we get when $t\to 0$

$$g(x,t) = \left(1-\phi\left(-\frac{1}{2},1;-\frac{x}{\sqrt{t}}\right)\right)\to 1. \quad (24)$$

So, applying to (19) the fractional differentiation of the order β by formula (11) taking into account the formula (21)–(24), when $x=\gamma(t)$, where $\gamma(t))\sim t^{\omega}$ when $t\to 0$, $\omega<\frac{1}{2}$, we get when $\lambda\neq -1$

$$\mu(t) + \frac{\lambda}{\lambda+1}\int_0^t K(t,\tau)\,\mu(\tau)\,d\tau = f_3(t), \quad (25)$$

where

$$f_3(t) = \frac{\lambda}{\lambda+1} D_{0t}^{\beta}\left(f_1(x,t)\right)\Bigg|_{x=\gamma(t)}, \tag{26}$$

$$K(t,\tau) = \frac{1}{\Gamma(1-\beta)(t-\tau)^{\beta}} - \frac{1}{(t-\tau)^{\beta}} \cdot \phi\left(-\frac{1}{2}, 1-\beta; -\frac{\gamma(t)}{\sqrt{t-\tau}}\right). \tag{27}$$

4 Integral equation research. Main result

Let us estimate kernel (27) of integral equation (25). The Wright function for $\forall\alpha \in (0;1)$, $b \in R$, $x > 0$, $y > 0$ satisfies the inequality (8) [16]

$$\left|y^{b-1} \cdot \phi\left(-\alpha, b; -xy^{-\alpha}\right)\right| \leq C \cdot x^{-\theta} \cdot y^{b+\alpha\theta-1},$$

where $\theta \geq 0$, when $-b \notin N \bigcup \{0\}$.

Then

$$\left|\frac{1}{(t-\tau)^{\beta}} \cdot \phi\left(-\frac{1}{2}, 1-\beta; -\frac{\gamma(t)}{\sqrt{t-\tau}}\right)\right| \leq C\left(\gamma(t)\right)^{-\theta}(t-\tau)^{-\beta+\frac{\theta}{2}}, \theta \geq 0.$$

At $\theta = 0$ we obtain:

$$|K(t,\tau)| \leq \left(\frac{1}{\Gamma(1-\beta)} + 1\right) \cdot (t-\tau)^{-\beta},$$

when $0 < \beta < 1$.

From here we get that the kernel of the integral equation has an integrable singularity if $\gamma(t)) \sim t^{\omega}$ when $t \to 0$, $\omega < \dfrac{1}{2}$.

Thus, the following theorem has been proved.

Theorem. Integral equation (25) with kernel (27) for $0 < \beta < 1$ and with $\gamma(t) \sim t^{\omega}$ in the neighborhood of $t = 0$ is uniquely solvable in the class of continuous functions for any continuous right-hand side $f_3(t)$ defined by formula (26), if $0 \leq \omega < \dfrac{1}{2}$.

This result coincided with the result obtained in [30].

Conclusions

According to the theorem, the integral equation (25) has a kernel with a weak singularity. Therefore, to find a unique solution to the equation (25) in the class of continuous functions, we can apply the method of successive approximations. After finding the solution $\mu(\tau)$ to equation (25), the solution to the original boundary value problem is found uniquely by formula (13). For the boundary value problem, the loaded term is a weak perturbation.

In other cases of values of the parameters β and ω, the method of successive approximations is not applicable for solving the integral equation (25). It is possible that the corresponding homogeneous equation will have nontrivial solutions for some values of the parameter λ, i.e. the spectrum of the problem will appear. Then the load can be interpreted as a strong perturbation. The existence and uniqueness of solutions to the integral equation depends on the fractional derivative order of the loaded summand. For $\lambda = -1$, BVP (9), (10) is reduced to The Volterra integral equation of the first kind.

Acknowledgments

This research was funded by the Science Committee of the Ministry of Education and Science of the Republic of Kazakhstan (Grant No. AP09259780, 2021–2023.)

References

1 Samko, S.G., Kilbas, A.A., & Marichev, O.I. (1993). *Fractional Integrals and Derivatives. Theory and Application.* Gordon and Breach: New York.

2 Le Mehaute, A., Tenreiro Machado, J.A., Trigeassou, J.C., & Sabatier, J. (2005). *Fractional Differentiation and its Applications.* Bordeaux Univ: Bordeaux.

3 Podlubny, I. (2002). Geometric and physical interpretation of fractional integration and fractional differentiation. *Fract. Calculus Appl. Anal., 5,* 367–386. an: 1042.26003.

4 Heymans, N., & Podlubny, I. (2006). Physical interpretation of initial conditions for fractional differential equations with Riemann-Liouville fractional derivatives. *Rheologica Acta, 45*(5), 765–772. DOI: 10.1007/S00397-005-0043-5.

5 Feng, M., Zhang, X., & Ge, W. (2011). New existence results for higher-order nonlinear fractional differential equations with integral boundary conditions. *Bound. Value Probl. Art., 720702,* 20. DOI: 10.1155/2011/720702.

6 Cao Labora, D., Rodriguez-Lopez, R., & Belmekki, M. (2020). Existence of solutions to nonlocal boundary value problems for fractional differential equations with impulses. *Electronic Journal of Differential Equations, 15,* 16.

7 Yusuf, A., Qureshi, S., Inc, M., Aliyu, A.I., Baleanu, D., & Shaikh, A.A. (2018). Two-strain epidemic model involving fractional derivative with Mittag-Leffler kernel. *Chaos: An Interdisciplinary Journal of Nonlinear Science, 28*(12), 123121. DOI: 10.1063/1.5074084.

8 Nakhushev, A.M. (2003). *Fractional Calculus and Its Applications.* Fizmatlit: Moscow.

9 Uchaikin, V.V. (2008). *Method of Fractional Derivatives.* Artishok: Ulyanovsk.

10 Tarasov, V.E. (2010). *Fractional Dynamics: Application of Fractional Calculus to Dynamics of Particles, Fields and Media.* Higher Education Press: Beijing, China; Springer: Berlin/Heidelberg.

11 Mainardi, F. (2010). *Fractional Calculus and Waves in Linear Viscoelasticity: An Introduction to Mathematical Models.* World Scientific: Singapore.

12 Atanacković, T.M., Pilipović, S., Stanković, B., & Zorica, D. (2014). *Fractional Calculus with Applications in Mechanics.* ISTE: London, UK; Wiley: Hoboken.

13 Tarasov, V. (Ed.) (2019). *Handbook of Fractional Calculus with Applications.* Vol. 4: Applications in Physics, Part A; De Gruyter: Berlin.

14 Tarasov, V. (Ed.) (2019). *Handbook of Fractional Calculus with Applications.* Vol. 5: Applications in Physics, Part B; De Gruyter: Berlin.

15 Korbel, J., & Luchko, Y. (2016). Modeling of financial processes with a space-time fractional diffusion equation of varying order. *Fract. Calc. Appl. Anal., 19*, 1414–1433.

16 Pskhu, A.V. (2005). *Partial Differential Equations of Fractional Order.* Science: Moscow, Russia.

17 Kilbas, A.A., Srivastava, H.M., & Trujillo, J.J. (2006). *Theory and Applications of Fractional Differential Equations.* Elsevier: Amsterdam.

18 Kochubei, A., & Luchko, Y. (Eds.) (2019) *Handbook of Fractional Calculus with Applications.* Vol. 2: Fractional Differential Equations; De Gruyter: Berlin.

19 Nakhushev, A.M. (1983). Loaded equations and their applications. *Diff. equations, 19*(1), 86–94.

20 Amangaliyeva, M.M., Akhmanova, D.M., Dzhenaliev, M.T., & Ramazanov, M.I. (2011). Boundary value problems for a spectrally loaded heat operator with load line approaching the time axis at zero or infinity. *Differential Equations, 47*(2), 231–243. DOI: 10.1134/S0012266111020091

21 Dzhenaliev, M.T., & Ramazanov, M.I. (2006). On the boundary value problem for the spectrally loaded heat conduction operator. *Siberian Mathematical Journal, 47*(3), 433–451. DOI: 10.1007/s11202-006-0056-z

22 Dzhenaliev, M.T., & Ramazanov, M.I. (2007). On a boundary value problem for a spectrally loaded heat operator: I *Differential Equations, 43*(4), 513–524. DOI: 10.1134/S0012266107040106

23 Dzhenaliev, M.T., & Ramazanov, M.I. (2007). On a boundary value problem for a spectrally loaded heat operator: II *Differential Equations, 43*(6), 806–812. DOI: 10.1134/S0012266107060079

24 Attayev, A.Kh., Iskakov, S.A., Karshigina, G.Zh., & Ramazanov, M.I. (2014). The first boundary problem for heat conduction equation with a load of fractional order. I. *Bulletin of the Karaganda University-mathematics, 76*(4), 11–16.

25 Iskakov, S.A., Ramazanov, M.I., & Ivanov, I.A. (2015). The first boundary problem for heat conduction equation with a load of fractional order. II. *Bulletin of the Karaganda University-mathematics, 78*(2), 25–30.

26 Amangaliyeva, M.M., Jenaliyev, M.T., Kosmakova, M.T., & Ramazanov, M.I. (2015). Uniqueness and non-uniqueness of solutions of the boundary value problems of the heat equation. *AIP Conference Proceedings, 1676,* 020028. DOI: 10.1063/1.4930454.

27 Amangaliyeva, M.M., Jenaliyev, M.T., Kosmakova, M.T., & Ramazanov, M.I. (2014). On the spectrum of Volterra integral equation with the incompressible kernel. *AIP Conference Proceedings, 1611,* 127–132. DOI: 10.1063/1.4893816.

28 Kosmakova, M.T., & Kasymova, L.Zh. (2019). To solving the heat equation with fractional load. *Journal of Mathematics, Mechanics and Computer Science, 104*(4), 50-62. DOI: 10.26577/JMMCS-2019-4-m6.

29 Ramazanov, M.I., Kosmakova, M.T., & Kasymova, L.Zh. (2020). On a Problem of Heat Equation with Fractional Load. *Lobachevskii Journal of Mathematics, 41*(9), 1873–1885. DOI: 10.1134/S199508022009022X

30 Kosmakova, M.T., Iskakov, S.A., & Kasymova, L.Zh. (2021). To solving the fractionally loaded heat equation. *Bulletin of the Karaganda University-mathematics, 1*(101), 65–77. DOI 10.31489/ 2021M1/65-77.

31 Polyanin, A.D. (2002). *Handbook of Linear Partial Differential Equations for Engineers and Scientists.* Chapman and Hall/CRC: New York-London.

32 Wright, E.M. (1933). On the coefficients of power series having exponential singularities. *J. London Math. Soc., 8*, 71–79.

33 Pskhu, A.V. (2003). Solution of a Boundary Value Problem for a Fractional Partial Differential Equation. *Differential Equations, 39*(8), 1150–1158. DOI: 10.1023/B:DIEQ.0000011289.79263.02

34 Gorenflo, R., Luchko, Yu., & Mainardi, F. (1999). Analytical properties and applications of the Wright function. *Fractional Calculus and Applied Analysis, 2, 4,* 383–414.

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Риман-Лиувилль бөлшек туындысы түріндегі жүктememен берілген жылуөткізгіштік теңдеуі үшін шекаралық есеп

Бірінші квадрантта үздіксіз функциялар класында бөлшекті-жүктелген жылуөткізгіштік теңдеуі үшін шекаралық есеп қарастырылған. Жүктелген қосылғыш уақытша айнымалы бойынша Риман-Лиувиллдің бөлшек туындысы түрінде болады, ал жүктелген қосылғыштағы туынды реті дифференциалдық бөліктің ретінен аз болады. Зерттеу шеттік есепті Вольтерр интегралдық теңдеуіне келтіруге негізделген. Алынған интегралдық теңдеудің ядросында арнайы функция бар, атап айтқанда Райт функциясы. Ядро бағаланып, интегралдық теңдеудің біркелкі шешілу шарттары алынды.

Кілт сөздер: жүктелген теңдеу, бөлшек туынды, Вольтеррдің интегралдық теңдеуі, Райт функциясы, бірмәнді шешімділік.

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Граничная задача для уравнения теплопроводности с нагрузкой в виде дробной производной Римана–Лиувилля

В первом квадранте рассмотрена краевая задача для дробно-нагруженного уравнения теплопроводности в классе непрерывных функций. Нагруженное слагаемое имеет форму дробной производной Римана–Лиувилля по временной переменной, и порядок производной в нагруженном слагаемом меньше порядка дифференциальной части. Исследование основано на сведении краевой задачи к интегральному уравнению Вольтерра. Ядро полученного интегрального уравнения содержит специальную функцию, а именно, функцию Райта. Произведена оценка ядра, и получены условия однозначной разрешимости интегрального уравнения.

Ключевые слова: нагруженное уравнение, дробная производная, интегральное уравнение Вольтерра, функция Райта, однозначная разрешимость.

DOI 10.31489/2022M1/83-95

UDC 517.95

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Boundary value problem for fractional diffusion equation in a curvilinear angle domain

We consider a boundary value problem for the fractional diffusion equation in an angle domain with a curvilinear boundary. Existence and uniqueness theorems for solutions are proved. It is shown that Holder continuity of the curvilinear boundary ensures the existence of solutions. The uniqueness is proved in the class of functions that vanish at infinity with a power weight. The solution to the problem is constructed explicitly in terms of the solution of the Volterra integral equation.

Keywords: noncylindrical domain, curvilinear angle domain, boundary value problem, fractional diffusion equation.

Introduction and problem statement

Consider the equation

$$\left(\frac{\partial^\alpha}{\partial y^\alpha} - \frac{\partial^2}{\partial x^2}\right) u(x,y) = f(x,y), \qquad (0 < \alpha \le 1) \tag{1}$$

where $\frac{\partial^\alpha}{\partial y^\alpha}$ denotes a fractional derivative with respect to y of order α with the origin at the point $x = 0$.

The fractional diffusion equations and their generalizations have attracted great attention in recent years. Research of (1) began in works [1–4].

To give an idea of the variety of problems considered for this equation and the multiplicity of approaches to studying them, we mention [5–33]. An overview is provided in [31]. A more detailed survey can be found in the article [34]. We also point out the monographs [35–37], which reflect many of these approaches and contain vast bibliographies concerning the issue.

Interest in the study of fractional differential equations is also fueled by applications in physics and simulation (see e.g. [38–41]).

Fractional differentiation is given in the Riemann-Liouville sense [38], i.e.

$$\frac{\partial^\alpha}{\partial y^\alpha} u(x,y) = D^\alpha_{0y} u(x,y) = \frac{\partial}{\partial y} D^{\alpha-1}_{0y} u(x,y)$$

and

$$D^{\alpha-1}_{0y} u(x,y) = \frac{1}{\Gamma(1-\alpha)} \int_0^y u(t)(y-t)^{-\alpha}\, dt.$$

We will consider the equation (1) in a curivilinear angle domain Ω that is defined by

$$\Omega = \{(x,y) :\ y > 0,\ x > z(y)\},$$

where $z(y)$ is a non-decreasing continuous function such that $z(0) = 0$.

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A function $u(x, y)$ is called a regular solution of equation (1) in the domain Ω if $y^{1-\alpha}u(x, y) \in C\left(\overline{\Omega}\right)$, and, moreover, $u(x, y)$ has continuous derivatives in Ω with respect to x up to the second order, and the function $D_{0y}^{\alpha-1}u(x, y)$ is continuously differentiable as function of y for a fixed x at interior points of Ω, and $u(x, y)$ satisfies equation (1) at all points of Ω.

Our purpose is to solve the following problem: *find a regular solution of the equation* (1) *in the domain* Ω *satisfying initial and boundary value conditions*

$$\lim_{y\to 0} D_{0y}^{\alpha-1}u(x, y) = \tau(x) \qquad (x > 0) \tag{2}$$

and

$$u(z(y), y) = \varphi(y) \qquad (y > 0) \tag{3}$$

where $\tau(x)$ and $\varphi(y)$ are given continuous functions.

The problems in domains with curvilinear boundary were considered in [42], [26]. For the problems for parabolic equations in angles domains, we refer to [43–46].

1 Notations and preliminaries

In what follows, we use the denotations

$$w_\mu(x, y) = x^{\mu-1}\phi\left(-\beta, \mu; -|x|\, y^{-\beta}\right), \tag{4}$$

$$w(x, y) = w_0(x, y), \qquad \text{and} \qquad \beta = \frac{\alpha}{2}.$$

In (4), ϕ denotes the Wright function [47], [48],

$$\phi\,(a, b;\, z) = \sum_{k=0}^{\infty} \frac{z^k}{k!\,\Gamma(ak + b)} \qquad (a > -1).$$

The asymptotics of the Wright function can be given in the form [48]

$$\phi(-\beta, \mu; -t) = \exp\left(-\rho\, t^{\sigma}\right)\left[A\, t^{\delta} + O\left(t^{\delta-\sigma}\right)\right] \qquad (t \to \infty), \tag{5}$$

where $\beta \in (0, 1)$, $\sigma = \frac{1}{1-\beta}$, $\rho = (1-\beta)\beta^{\frac{\beta}{1-\beta}}$, and $\delta = \frac{1-2\mu}{2(1-\beta)}$.

In particular, formula (5) implies

$$|w_\mu(x, y)| \le C\,|x|^{-\theta} y^{\beta\theta+\mu-1}, \tag{6}$$

where

$$\theta \ge \left\{\begin{array}{ll} 0, & (-\mu) \notin \mathbb{N}\cup\{0\}, \\ -1, & (-\mu) \in \mathbb{N}\cup\{0\}, \end{array}\right. \qquad \text{and} \qquad C = C(\beta, \mu, \theta).$$

Here and subsequently, by C we denote positive constants, which may be different in different cases, indicating in parentheses the parameters on which they depend, if necessary, $C = C(\alpha, \beta, ...)$.

The differentiation formulas for the Wright function [48]

$$\frac{d}{dx}\,\phi(-\beta,\, \mu;\, x) = \phi(-\beta,\, \mu-\beta;\, x)$$

and [49]

$$D_{0y}^{\nu}\left[y^{\mu-1}\phi\left(-\beta,\, \mu;\, -\frac{c}{y^{\beta}}\right)\right] = y^{\mu-\nu-1}\phi\left(-\beta,\, \mu-\nu;\, -\frac{c}{y^{\beta}}\right) \quad (c > 0)$$

give

$$\frac{\partial}{\partial x} w_\mu(x,y) = -sign(x) w_{\mu-\beta}(x,y), \qquad D^\nu_{0y} w_\mu(x,y) = w_{\mu-\nu}(x,y), \tag{7}$$

and

$$\left(D^\alpha_{0y} - \frac{\partial^2}{\partial x^2}\right) w_\mu(x,y) = 0 \qquad (\mu \in \mathbb{R}, \quad |x| > 0, \quad y > 0).$$

We need later the following two statements.

Lemma 1. Let $\psi(t) \in L(0,y)$, let the function $\psi(t)$ be left continuous at the point $t = y$, and let

$$0 \le z(y) - z(t) \le C(y-y)^\delta \qquad (\delta > \beta, \ 0 < t < y).$$

Then

$$\lim_{\substack{x \to z(y) \\ x > z(y)}} \int_0^y \psi(t) w(x - z(t), y - t)\, dt = \psi(y) + \int_0^y \psi(t) w(z(y) - z(t), y - t)\, dt.$$

Lemma 2. Let $t^{1-\alpha} q(t) \in C[0,y]$, let $q(t)$ be Hölder continuous in a left neighborhood of y with an exponent $\delta > \alpha$,

$$|q(t) - q(y)| < C(y-t)^\delta \qquad (y - \varepsilon < t < y, \quad \varepsilon > 0),$$

and let

$$t^{1-\alpha} q(t) \le y^{1-\alpha} q(y) \qquad \text{for all} \quad t \in (0,y).$$

Then

$$(D^\alpha_{0t} q)_{t=y} \ge 0,$$

and $(D^\alpha_{0t} q)_{t=y} = 0$ if and only if $q(t) = Ct^{\alpha-1}$.

The proof of Lemma 1 is given in [26]. Lemma 2 is proved in [42] (see also [50]).

2 Existence theorem

Theorem 1. Let

$$0 \le z(y_2) - z(y_1) \le C\,(y_2 - y_1)^\delta \qquad (0 \le y_1 < y_2, \ \delta > \beta), \tag{8}$$

$$\tau(x) \in C[0,\infty), \qquad y^{1-\alpha}\varphi(y) \in C[0,\infty), \qquad \tau(0) = \Gamma(\alpha)\left[y^{1-\alpha}\varphi(y)\right]_{y=0}, \tag{9}$$

$$y^{1-\alpha} f(x,y) \in C(\overline{\Omega}),$$

$$\lim_{x\to\infty} \tau(x) \exp\left(-\omega x^{\frac{2}{2-\alpha}}\right) = 0, \tag{10}$$

$$\lim_{x\to\infty} y^{1-\alpha} f(x,y) \exp\left(-\omega x^{\frac{2}{2-\alpha}}\right) = 0, \tag{11}$$

for every $\omega > 0$, and let the function $f(x,y)$ be representable in the form

$$f(x,y) = D^{-\delta}_{0y} g(x,y) \tag{12}$$

for some $\delta > \beta$ and $y^{1-\alpha+\delta} g(x,y) \in C(\overline{\Omega})$.

Then there exists a solution to problem (1), (2), and (3); it can be represented in the form:

$$u(x,y) = \int_0^y \psi(t) w(x - z(t), y - t)\, dt + T(x,y) + F(x,y), \tag{13}$$

where

$$T(x,y) = \frac{1}{2}\int_0^\infty \tau(s)w_\beta(x-s,y)\,ds, \qquad F(x,y) = \frac{1}{2}\int_0^y \int_{z(t)}^\infty f(s,t)w_\beta(x-s,y-t)\,ds\,dt,$$

and the function $\psi(x)$ is a solution of the integral equation

$$\psi(y) + \int_0^y \psi(t)w(z(y)-z(t),y-t)\,dt = \varphi(y) - T(z(y),y) - F(z(y),y). \tag{14}$$

Proof. Consider separately each of three summands on the right side of (13), namely $F(x,y)$, $T(x,y)$, and $\Psi(x,y)$, where

$$\Psi(x,y) = \int_0^y \psi(t)w(x-z(t),y-t)\,dt.$$

Let us start with $F(x,y)$. By (5), (6), and (7), with (11) and (12), we get

$$|F(x,y)| \le Cy^{3\beta-1},$$

and

$$\frac{\partial^2}{\partial x^2}F(x,y) = \frac{1}{2}\frac{\partial}{\partial x}\int_0^y \int_{z(t)}^\infty \mathrm{sign}(x-s)\,g(s,t)w_\delta(x-s,y-t)\,ds\,dt =$$

$$= -\int_0^y \frac{(y-t)^{\delta-1}}{\Gamma(\delta)}g(x,t)ds\,dt + \frac{1}{2}\int_0^y \int_{z(t)}^\infty g(s,t)w_{\delta-\beta}(x-s,y-t)\,ds\,dt =$$

$$= -f(x,y) + \frac{1}{2}D_{0y}^\alpha \int_0^y \int_{z(t)}^\infty f(s,t)w_\beta(x-s,y-t)\,ds\,dt.$$

This means that

$$y^{1-\alpha}F(x,y) \in C\left(\overline{\Omega}\right), \qquad \lim_{y\to 0} D_{0y}^{\alpha-1}F(x,y) = 0, \tag{15}$$

and

$$\left(D_{0y}^\alpha - \frac{\partial^2}{\partial x^2}\right)F(x,y) = f(x,y).$$

Consider now $T(x,y)$. By (5), (7), and (10), one can check that

$$\left(D_{0y}^\alpha - \frac{\partial^2}{\partial x^2}\right)T(x,y) = 0.$$

Rewrite $T(x,y)$ in the form

$$T(x,y) = \frac{1}{2}\int_0^\infty \left[\tau(s)-\tau(x)\right]w_\beta(x-s,y)\,ds + \frac{\tau(x)}{2}\int_0^\infty w_\beta(x-s,y)\,ds.$$

Using (5), (6), and (7), leads to

$$\int_0^\infty w_\beta(x-s,y)\,ds = \left(\int_0^x + \int_x^\infty\right)w_\beta(x-s,y)\,ds = \frac{2y^{\alpha-1}}{\Gamma(\alpha)} - w_\alpha(x,y)$$

and

$$\left|\int_0^\infty \left[\tau(s)-\tau(x)\right]w_\beta(x-s,y)\,ds\right| \le \left(\int_0^{x-\varepsilon} + \int_{x+\varepsilon}^\infty\right)|\tau(s)-\tau(x)|\,w_\beta(x-s,y)\,ds +$$

$$+C\sup_{t\in(x-\varepsilon,x+\varepsilon)}|\tau(t)-\tau(y)|\int_{x-\varepsilon}^{x+\varepsilon}w_\beta(x-s,y)\,ds\le Cy^{\alpha-1}\left[y^\theta+\omega(\varepsilon)\right],$$

where $\theta>0$ and $\omega(\varepsilon)\to 0$ as $\varepsilon\to 0$. This gives that

$$y^{1-\alpha}T(x,y)=C\left(\overline{\Omega}\setminus\{0,0\}\right)\qquad\text{and}\qquad D_{0y}^{\alpha-1}T(x,y)=\tau(x).$$

Let us examine the behavior of $T(x,y)$ in a neighborhood of $(0,0)$. Assume that $\sigma=\sigma(y)$ is a continuous function defined in a right neighborhood of the point $y=0$ such that $(\sigma(y),y)\in\Omega$ (i.e. $\sigma(y)>z(y)$) and $\sigma(y)\to 0$ as $y\to 0$. It is easy to see that

$$T(\sigma(y),y)=\frac{y^{\alpha-1}}{2}\int_0^{\sigma/y^\beta}\tau\left(sy^\beta\right)\phi\left(-\beta,\beta;\,s-\frac{\sigma}{y^\beta}\right)ds+$$
$$+\frac{y^{\alpha-1}}{2}\int_0^{\infty}\tau\left(sy^\beta+\sigma\right)\phi\left(-\beta,\beta;\,-s\right)ds$$

and

$$\lim_{y\to 0}y^{1-\alpha}T(\sigma(y),y)=\frac{\tau(0)}{\Gamma(\alpha)}-\frac{\tau(0)}{2}\lim_{y\to 0}\phi\left(-\beta,\alpha;\,-\frac{\sigma(y)}{y^\beta}\right).\tag{16}$$

Finally, consider $\Psi(x,y)$. By (6) and (8), we have

$$|w(z(y)-z(t),y-t)|\le C|z(y)-z(t)|(y-t)^{-\beta-1}\le C(y-t)^{\delta-\beta-1}.$$

This means that integral equation (14) has a unique solution $\psi(y)$. A simple computation gives

$$\left(D_{0y}^{\alpha}-\frac{\partial^2}{\partial x^2}\right)\Psi(x,y)=0,$$

and

$$\lim_{y\to 0}D_{0y}^{\alpha-1}\Psi(x,y)=0\qquad\text{and}\qquad y^{1-\alpha}\Psi(x,y)=C\left(\overline{\Omega}\setminus\{0,0\}\right).$$

Now we investigate the behavior of $\Psi(x,y)$ in a neighborhood of $(0,0)$. It follows from formulas (9), (14), (15), (16) that

$$y^{1-\alpha}\psi(y)\in C[0,\infty)\qquad\text{and}\qquad\lim_{y\to 0}y^{1-\alpha}\psi(y)=\frac{\tau(0)}{2\Gamma(\alpha)}.\tag{17}$$

Set $\psi_0(y)=y^{1-\alpha}\psi(y)$, $(\sigma(y),y)\in\Omega$ $(\sigma(y)>z(y))$, and $\sigma(y)\to 0$ as $y\to 0$. It is easy to see that

$$\Psi(\sigma(y),y)=\int_0^y\psi(t)\left[w(\sigma(y)-z(t),y-t)-w(\sigma(y)-z(y),y-t)\right]dt+$$
$$+\int_0^y t^{1-\alpha}\left[\psi_0(t)-\psi_0(y)\right]w(\sigma(y)-z(y),y-t)\,dt+$$
$$+\psi_0(y)\int_0^y t^{1-\alpha}w(\sigma(y)-z(y),y-t)\,dt.$$

This yields that

$$\lim_{y\to 0}y^{1-\alpha}\Psi(\sigma(y),y)=\psi_0(0)\lim_{y\to 0}\phi\left(-\beta,\alpha;\,-\frac{\sigma(y)-z(y)}{y^\beta}\right).$$

Taking into account (16), (17), and

$$\lim_{y\to 0}\phi\left(-\beta,\alpha;\,-\frac{\sigma(y)-z(y)}{y^\beta}\right)=\lim_{y\to 0}\phi\left(-\beta,\alpha;\,-\frac{\sigma(y)}{y^\beta}\right),$$

we get

$$y^{1-\alpha}\left[\Psi(x,y)+T(x,y)\right]\in C(\overline{\Omega})$$

and, consequently,

$$y^{1-\alpha}u(x,y)\in C(\overline{\Omega}).$$

To complete the proof, it remains to show that the function (13) satisfies (3). Indeed, by Lemma 1 we have

$$u(z(y),y)=\lim_{x\to z(y)}\Psi(x,y)+T\left(z(y),y\right)+F\left(z(y),y\right)=$$

$$=\psi(y)+\Psi(z(y),y)+T\left(z(y),y\right)+F\left(z(y),y\right).$$

Due to (14) we get

$$\psi(y)+\Psi(z(y),y)=\varphi(y)-T\left(z(y),y\right)-T\left(z(y),y\right).$$

Combining the last two equations leads to (3).

3 Solution uniqueness

Theorem 2. Let $z(y)\in C[0,\infty)$. There exists at most one regular solution to the problem (1), (2), and (3), satisfying

$$\lim_{x\to\infty}\sup_{0<y<T}\left|y^{1-\alpha}u(x,y)\right|=0 \tag{18}$$

for every $T>0$.

Proof. Let $u(x,y)$ be a solution of the homogeneous problem

$$\left(\frac{\partial^\alpha}{\partial y^\alpha}-\frac{\partial^2}{\partial x^2}\right)u(x,y)=0,\qquad \left(D_{0y}^{\alpha-1}u\right)_{y=0}=0,\qquad u(z(y),y)=0. \tag{19}$$

Set

$$\Omega_T=\Omega\cap\{y\le T\}\quad\text{for some}\quad T>0,\quad\text{and}\quad v(x,y)=y^{1-\alpha}u(x,y).$$

By (18), (19), and the equality $\left(D_{0y}^{\alpha-1}u\right)_{y=0}=\Gamma(\alpha)v(x,0)$, we can conclude that there is a point $(\xi,\eta)\in\Omega_T$ such that

$$v(\xi,\eta)\ge v(x,y)\qquad\text{for all}\quad (x,y)\in\Omega_T. \tag{20}$$

Lemma 2, with (19) and (20), yields that

$$\left[D_{0y}^{\alpha}u(\xi,y)\right]_{y=\eta}\ge 0\quad\text{and}\quad u_{xx}(\xi,\eta)\le 0.$$

This means that $\left[D_{0y}^{\alpha}u(\xi,y)\right]_{y=\eta}=0$, and consequently $u(\xi,y)=Cy^{\alpha-1}$. Taking into account that $v(x,0)=0$, this implies that $C=0$. Thus, we get

$$u(x,y)\le 0\qquad\text{for all}\quad (x,y)\in\Omega_T.$$

Similarly, considering the function $v(x,y)=-y^{1-\alpha}u(x,y)$ gives

$$u(x,y)\ge 0\qquad\text{for all}\quad (x,y)\in\Omega_T.$$

Thus, we can conclude that $u(x,y)=0$ for all $(x,y)\in\Omega_T$. The arbitrary choice of the number T implies that $u(x,y)=0$ in Ω.

Acknowledgments

The study was supported by the Ministry of Science and Education of the Republic of Kazakhstan (Grant Projects AP09259780 (2021–2023) and AP09258892 (2021–2023)).

References

1 Wyss W. The fractional diffusion equation / W. Wyss // J. Math. Phys. — 1986. — № 27. — P. 2782–2785.

2 Schneider W.R. Fractional diffusion and wave equations / W.R. Schneider, W. Wyss // J. Math. Phys. — 1989. — № 30. — P. 134–144.

3 Kochubei A.N. Diffusion of fractional order / A.N. Kochubei // Differ. Equ. — 1990. — № 26. — P. 485–492.

4 Fujita Ya. Integrodifferential equation which interpolates the heat equation and the wave equation I, II / Ya. Fujita // Osaka J. Math. — 1990. — № 27. — P. 309–321, 797–804.

5 Mainardi F. The fundamental solutions for the fractional diffusion-wave equation / F. Mainardi // Appl. Math. Lett. — 1996. — № 9. — P. 23–28.

6 Engler H. Similiraty solutions for a class of hyperbolic integrodifferential equations / H. Engler // Differential Integral Equations. — 1997. — № 10. — P. 815–840.

7 Шогенов В.Х. Обобщенное уравнение переноса и дробные производные / В.Х. Шогенов, С.К. Кумыкова, М.Х. Шхануков-Лафишев // Докл. Нац. акад. наук Украины. — 1997. — №‘12. — С. 47–54.

8 Buckwar E. Invariance of a partial differential equation of fractional order under the Lie group of scaling transformations / E. Buckwar, Yu. Luchko // J. Math. Anal. Appl. — 1998. — № 227. — P. 81–97.

9 Gorenflo R. Wright functions as scale-invariant solutions of the diffusion-wave equation / R. Gorenflo, Yu. Luchko, F. Mainardi // J. Comput. Appl. Math. — 2000. — № 118. — P. 175–191.

10 Gorenflo R. Mapping between solutions of fractional diffusion-wave equations / R. Gorenflo, A. Iskenderov, Yu. Luchko // Fract. Calcul. and Appl. Anal. — 2000. — № 3. — P. 75–86.

11 Agrawal O.P. Solution for a fractional diffusion-wave equation defined in a bounded domain / O.P. Agrawal // Nonlinear Dynam. — 2002. — № 29. — P. 145–155.

12 Pskhu A.V. Solution of Boundary Value Problems for the Fractional Diffusion Equation by the Green Function Method / A.V. Pskhu // Differ. Equ. — 2003. — № 39. — P. 1509–1513.

13 Pskhu A.V. Solution of the First Boundary Value Problem for a Fractional-Order Diffusion Equation / A.V. Pskhu // Differ. Equ. — 2003. — № 39. — P. 1359–1363.

14 Eidelman S.D. Cauchy problem for fractional diffusion equations / S.D. Eidelman, A.N. Kochubei // Journal of Differential Equations. — 2004. — № 199. — P. 211–255.

15 Voroshilov A.A. The Cauchy problem for the diffusion-wave equation with the Caputo partial derivative / A.A. Voroshilov, A.A. Kilbas // Differ. Equ. — 2006. — № 42. — P. 638–649.

16 Pskhu A.V. The fundamental solution of a diffusion-wave equation of fractional order / A.V. Pskhu // Izv. Math. – 2009. – № 73. – P. 351–392

17 Liu J.J. A backward problem for the time-fractional diffusion equation / J.J. Liu, M. Yamamoto // Applicable Analysis. — 2010. — № 89(11). — P. 1769–1788.

18 Sakamoto K. Initial value boundary value problems for fractional diffusion-wave equations and applications to some inverse problems / K. Sakamoto, M. Yamamoto // J. Math. Anal. Appl. — 2011. — № 382. — P. 426–447.

19 Kemppainen J. Properties of the single layer potential for the time fractional diffusion equation / J. Kemppainen // Journal of Integral Equations and Applications. — 2011. — № 23. — P. 541–563.

20 Kochubei A.N. Asymptotic properties of solutions of the fractional diffusion-wave equation / A.N. Kochubei // Fract. Calcul. and Appl. Anal. — 2014. — № 17. — P. 881–896.

21 Al-Refai M. Asymptotic properties of solutions of the fractional diffusion-wave equation / M. Al-Refai, Yu. Luchko // Fract. Calcul. and Appl. Anal. — 2014. — № 17. — P. 483–498.

22 Мамчуев М.О. Необходимые нелокальные условия для диффузионно-волнового уравнения / М.О. Мамчуев // Вестн. Самар. ун-та. Естественнонауч. сер. — 2014. — № 7. — С. 45–59.

23 Tuan N.H. A regularization method for time-fractional linear inverse diffusion problems / N.H. Tuan, M. Kirane, V.C.H. Luu, Bin-B. Mohsin // Electronic Journal of Differential Equations. — 2016. — № 290. — P. 1–18.

24 Псху А.В. Уравнение дробной диффузии с оператором дискретно распределенного дифференцирования / А.В. Псху // Сибир. мат. электрон. изв. — 2016. — № 13. — С. 1078–1098.

25 Mamchuev M.O. Solutions of the main boundary value problems for a loaded second-order parabolic equation with constant coefficients / M.O. Mamchuev // Differ. Equ. — 2016. — № 52. — P. 789–797.

26 Псху А.В. Первая краевая задача для дробного диффузионно-волнового уравнения в нецилиндрической области / А.В. Псху // Изв. РАН. Сер. мат. — 2017. — № 81. — С. 1212–1233.

27 Mamchuev M.O. Solutions of the main boundary value problems for the time-fractional telegraph equation by the green function method / M.O. Mamchuev // Fract. Calc. Appl. Anal. — 2017. — № 20. — P. 190–211.

28 Масаева О.Х. Задача Дирихле для нелокального волнового уравнения с производной Римана-Лиувилля / О.Х. Масаева // Вестн. КРАУНЦ. Физ.-мат. науки. — 2019. — № 27. — С. 6–11.

29 Псху А.В. Функция Грина первой краевой задачи для дробного диффузионно-волнового уравнения в многомерной прямоугольной области / А.В. Псху // Итоги науки и техники. Сер. Современная математика и ее приложения. Тематические обзоры. — 2019. — № 167. — С. 52–61.

30 Alimov S. Inverse problem of determining an order of the Caputo time-fractional derivative for a subdiffusion equation / S. Alimov, R. Ashurov // Journal of Inverse and Ill-posed Problems. — 2020. — № 28. — P. 651–658.

31 Pskhu A.V. Green Functions of the First Boundary-Value Problem for a Fractional Diffusion–Wave Equation in Multidimensional Domains / A.V. Pskhu // Mathematics. — 2020. — № 8(4). — P. 464.

32 Pskhu A.V. Stabilization of solutions to the Cauchy problem for fractional diffusion-wave equation / A.V. Pskhu // Journal of Mathematical Sciences. — 2020. — № 250. — P. 800–810.

33 Pskhu A.V. Fractional Diffusion-Wave Equation with Application in Electrodynamics / A.V. Pskhu, S. Rekhviashvili // Journal of Mathematical Sciences. — 2020. — № 8. — P. 2086.

34 Kilbas A.A. Partial fractional differential equations and some of their applications / A.A. Kilbas // Analysis. — 2010. — № 30. — P. 35–66.

35 Pskhu A.V. Partial differential equations of fractional order / A.V. Pskhu. — Nauka: Moscow, 2005.

36 Kilbas A.A. Theory and applications of fractional differential equations / A.A. Kilbas, H.M. Srivastava, J.J. Trujillo. — North-Holland Math. Stud. Elsevier: Amsterdam, Vol. 204, 2006.

37 Kochubei A. Handbook of Fractional Calculus with Applications. Volume 2: Fractional Differential Equations / A. Kochubei, Y. Luchko (Eds.). — De Gruyter: Berlin, Germany, 2019.

38 Nakhushev A.M. Fractional calculus and its applications / A.M. Nakhushev. — Fizmatlit: Moscow, 2003.

39 Uchaikin V.V. Method of Fractional Derivatives / V.V. Uchaikin. — Artishok: Ulyanovsk, 2008.

40 Atanacković T.M. Fractional Calculus with Applications in Mechanics / T.M. Atanacković, S. Pilipović, B. Stanković, D. Zorica. — ISTE, Wiley: London, Hoboken, 2014.

41 Tarasov V. (Ed.) Handbook of Fractional Calculus with Applications. Volumes 4, 5: Applications in Physics, Parts A, B / V. Tarasov (Ed.). — De Gruyter: Berlin, Germany, 2019.

42 Pskhu A.V. On a boundary value problem for a fractional partial differential equation in a domain with curvilinear boundary / A.V. Pskhu // Differential Equations. — 2015. — № 51(8). — P. 1072–1078.

43 Рамазанов М.И. О сингулярном интегральном уравнении Вольтерра краевой задачи теплопроводности в вырождающейся области / М.И. Рамазанов, Н.К. Гульманов // Вестн. Удмурт. ун-та. Математика. Механика. Компьютерные науки. — 2021. — № 31(2). — С. 241–252.

44 Ramazanov M.I. Solution of a two-dimensional boundary value problem of heat conduction in a degenerating domain / M.I. Ramazanov, N.K. Gulmanov // Journal of Mathematics, Mechanics and Computer Science. — 2021. — № 111(3). — P. 65–78.

45 Amangalieva M.M. On one homogeneous problem for the heat equation in an infinite angular domain / M.M. Amangalieva, M.T. Dzhenaliev, M.T. Kosmakova, M.I. Ramazanov // Siberian Mathematical Journal. — 2015. — № 56(6). — P. 982–995.

46 Jenaliyev M.T. On a Homogeneous Parabolic Problem in an Infinite Corner Domain / M.T. Jenaliyev, M.I. Ramazanov // Filomat. — 2018. — № 32(3). — P. 965–974.

47 Wright E.M. On the coefficients of power series having exponential singularities / E.M. Wright // J. London Math. Soc. — 1933. — № 8. — P. 71–79.

48 Wright E.M. The generalized Bessel function of order greater than one / E.M. Wright // Quart. J. Math., Oxford Ser. — 1940. — № 11. — P. 36–48.

49 Pskhu A.V. Solution of a Boundary Value Problem for a Fractional Partial Differential Equation / A.V. Pskhu // Differential Equations. — 2003. — № 39. — P. 1092–1099.

50 Pskhu A.V. Nakhushev extremum principle for a class of integro-differential operators / A.V. Pskhu // Fract. Calc. Appl. Anal. — 2020. — № 23. — P. 1712–1722.

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Қисық сызықты бұрыштық облыстағы бөлшек ретті диффузиялық теңдеу үшін шеттік есеп

Мақалада қисық сызықты бұрыштық облыстағы бөлшек ретті диффузиялық теңдеудің шеттік есебі зерттелген. Қарастырылып отырған облыстағы есептің шешімінің бар болуы мен жалғыздығы туралы теоремалар дәлелденді. Гёльдер бойынша қисық сызықты шекараның үздіксіздігі шешімдердің бар болуын қамтамасыз ететіндігін көрсеткен. Шешімнің жалғыздығы шексіздікте дәрежелік салмақпен нөлге айналатын функциялар класында дәлелденді. Есептің шешуі Вольтерраның интегралдық теңдеуінің шешуі арқылы айқын түрде құрылады.

Кілт сөздер: цилиндрлік емес облыс, қисық сызықты бұрыштық облыс, шеттік есеп, бөлшек ретті диффузиялық теңдеу.

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Краевая задача для дробного диффузионного уравнения в криволинейной угловой области

В статье рассмотрена и доказана теорема существования и единственности рассматриваемой краевой задачи. Показано, что непрерывность криволинейной границы по Гёльдеру обеспечивает существование решений. Единственность решения задачи доказана в классе функций, обращающихся в нуль на бесконечности со степенным весом. Вычисление ответа задачи построено в явном виде через решение интегрального уравнения Вольтерра.

Ключевые слова: нецилиндрическая область, криволинейная угловая область, краевая задача, дробное диффузионное уравнение.

References

1 Wyss, W. (1986). The fractional diffusion equation. *J. Math. Phys., 27*, 2782–2785.

2 Schneider, W.R., & Wyss, W. (1989). *Fractional diffusion and wave equations. J. Math. Phys., 30*, 134–144.

3 Kochubei, A.N. (1990). Diffusion of fractional order. *Differ. Equ., 26*, 485–492.

4 Fujita, Ya. (1990). Integrodifferential equation which interpolates the heat equation and the wave equation I, II. *Osaka J. Math., 27*, 309–321, 797–804.

5 Mainardi, F. (1996). The fundamental solutions for the fractional diffusion-wave equation. *Appl. Math. Lett., 9*, 23–28.

6 Engler, H. (1997). Similiraty solutions for a class of hyperbolic integrodifferential equations. *Differential Integral Equations, 10*, 815–840.

7 Shogenov, V.Kh., Kumykova, S.K., & Shkhanukov-Lafishev, M.Kh. (1997). Obobshchennoe uravnenie perenosa i drobnye proizvodnye [The generalized transport equation and fractional derivatives]. *Doklady Natsionalnoi akademii nauk Ukrainy — Reports of the National Academy of Sciences of Ukraine, 12*, 47–54 [in Russian].

8 Buckwar, E., & Luchko, Yu. (1998). Invariance of a partial differential equation of fractional order under the Lie group of scaling transformations. *J. Math. Anal. Appl., 227*, 81–97.

9 Gorenflo, R., Luchko, Yu., & Mainardi, F. (2000). Wright functions as scale-invariant solutions of the diffusion-wave equation. *J. Comput. Appl. Math., 118*, 175–191.

10 Gorenflo, R., Iskenderov, A. & Luchko, Yu. (2000). Mapping between solutions of fractional diffusion-wave equations. *Fract. Calcul. and Appl. Anal., 3*, 75–86.

11 Agrawal, O.P. (2002). Solution for a fractional diffusion-wave equation defined in a bounded domain. *Nonlinear Dynam., 29*, 145–155.

12 Pskhu, A.V. (2003). Solution of Boundary Value Problems for the Fractional Diffusion Equation by the Green Function Method. *Differ. Equ., 39*, 1509–1513.

13 Pskhu, A.V. (2003). Solution of the First Boundary Value Problem for a Fractional-Order Diffusion Equation. *Differ. Equ., 39*, 1359–1363.

14 Eidelman, S.D., & Kochubei, A.N. (2004). Cauchy problem for fractional diffusion equations. *Journal of Differential Equations, 199*, 211–255.

15 Voroshilov, A.A., & Kilbas, A.A. (2006). The Cauchy problem for the diffusion-wave equation with the Caputo partial derivative. *Differ. Equ., 42*, 638–649.

16 Pskhu, A.V. (2009). The fundamental solution of a diffusion-wave equation of fractional order. *Izv. Math., 73*, 351–392.

17 Liu, J.J., & Yamamoto, M. (2010). A backward problem for the time-fractional diffusion equation. *Applicable Analysis., 89*(11), 1769–1788.

18 Sakamoto, K., & Yamamoto, M. (2011). Initial value boundary value problems for fractional diffusion-wave equations and applications to some inverse problems. *J. Math. Anal. Appl., 382*, 426–447.

19 Kemppainen, J. (2011). Properties of the single layer potential for the time fractional diffusion equation. *Journal of Integral Equations and Applications, 23*, 541–563.

20 Kochubei, A.N. (2014). Asymptotic properties of solutions of the fractional diffusion-wave equation. *Fract. Calcul. and Appl. Anal., 17*, 881–896.

21 Al-Refai, M., & Luchko, Yu. (2014). Asymptotic properties of solutions of the fractional diffusion-wave equation. *Fract. Calcul. and Appl. Anal., 17*, 483–498.

22 Mamchuev, M.O. (2014). Neobkhodimye nelokalnye usloviia dlia diffuzionno-volnovogo uravneniia [Necessary non-local conditions for a diffusion-wave equation]. *Vestnik Samarskogo universiteta. Estestvennonauchnaia seriia — Journal of Samara University. Natural science Series, 7*, 45–59 [in Russian].

23 Tuan, N.H., Kirane, M., Luu, V.C.H., & Mohsin, Bin-B. (2016). A regularization method for time-fractional linear inverse diffusion problems. *Electronic Journal of Differential Equations, 290*, 1–18.

24 Pskhu, A.V. (2016). Uravnenie drobnoi diffuzii s operatorom diskretno raspredelennogo differentsirovaniia [Fractional diffusion equation with discretely distributed differentiation operator]. *Sibirskie matematicheskie elektronnye izvestiia — Siberian Electronic Mathematical Reports, 13*, 1078–1098 [in Russian].

25 Mamchuev, M.O. (2016). Solutions of the main boundary value problems for a loaded second-order parabolic equation with constant coefficients. *Differ. Equ., 52*, 789–797.

26 Pskhu, A.V.(2017). Pervaia kraevaia zadacha dlia drobnogo diffuzionno-volnovogo uravneniia v netsilindricheskoi oblasti [The first boundary-value problem for a fractional diffusion-wave equation in a non-cylindrical domain]. *Izvestiia RAN. Seriia matematicheskaia — Izvestiya: Mathematics, 81*, 1212–1233 [in Russian].

27 Mamchuev, M.O. (2017). Solutions of the main boundary value problems for the time-fractional telegraph equation by the Green function method. *Fract. Calc. Appl. Anal., 20*, 190–211.

28 Masaeva, O.Kh. (2019). Zadacha Dirikhle dlia nelokalnogo volnovogo uravneniia s proizvodnoi Rimana–Liuvillia [Dirichlet problem for a nonlocal wave equation with Riemann–Liouville derivative]. *Vestnik KRAUNC. Fiziko-matematicheskie nauki — Bulletin KRASEC. Physical and Mathematical Sciences, 27*, 6–11 [in Russian].

29 Pskhu, A.V. (2019). Funktsiia Grina pervoi kraevoi zadachi dlia drobnogo diffuzionno-volnovogo uravneniia v mnogomernoi priamougolnoi oblasti [Green function of the first boundary-value

problem for the fractional diffusion wave equation in a multidimensional rectangular domain]. *Itogi nauki i tekhniki. Seriia Sovremennaia matematika i ee prilozheniia. Tematicheskie obzory — Results of science and technology. Series "Modern mathematics and its applications. Thematic reviews", 167*, 52–61 [in Russian].

30 Alimov, S., & Ashurov, R. (2020). Inverse problem of determining an order of the Caputo time-fractional derivative for a subdiffusion equation. *Journal of Inverse and Ill-posed Problems, 28*, 651–658.

31 Pskhu, A.V. (2020). Green Functions of the First Boundary-Value Problem for a Fractional Diffusion–Wave Equation in Multidimensional Domains. *Mathematics, 8*(4), 464.

32 Pskhu, A.V. (2020). Stabilization of solutions to the Cauchy problem for fractional diffusion-wave equation. *Journal of Mathematical Sciences, 250*, 800–810.

33 Pskhu, A.V., & Rekhviashvili, S. (2020). Fractional Diffusion-Wave Equation with Application in Electrodynamics. *Journal of Mathematical Sciences, 8*, 2086.

34 Kilbas, A.A. (2010). Partial fractional differential equations and some of their applications. *Analysis, 30*, 35–66.

35 Pskhu, A.V. (2005). *Partial differential equations of fractional order.* Nauka: Moscow.

36 Kilbas, A.A., Srivastava, H.M., & Trujillo, J.J. (2006). *Theory and applications of fractional differential equations, 204.* North-Holland Math. Stud. Elsevier: Amsterdam.

37 Kochubei, A., & Luchko, Y. (Eds.). (2019). *Handbook of Fractional Calculus with Applications. Volume 2: Fractional Differential Equations.* De Gruyter: Berlin, Germany.

38 Nakhushev, A.M. (2003). *Fractional calculus and its applications.* Fizmatlit: Moscow.

39 Uchaikin, V.V. (2008). *Method of Fractional Derivatives.* Artishok: Ulyanovsk.

40 Atanacković, T.M., Pilipović, S., Stanković, B., & Zorica, D. *Fractional Calculus with Applications in Mechanics.* ISTE, Wiley: London, Hoboken.

41 Tarasov, V. (Ed.). (2019). *Handbook of Fractional Calculus with Applications. Volumes 4, 5: Applications in Physics, Parts A, B.* De Gruyter: Berlin, Germany.

42 Pskhu, A.V. (2015). On a boundary value problem for a fractional partial differential equation in a domain with curvilinear boundary. *Differential Equations, 51*(8), 1072–1078.

43 Ramazanov, M.I., & Gulmanov, N.K. (2021). O singuliarnom integralnom uravnenii Volterra kraevoi zadachi teploprovodnosti v vyrozhdaiushcheisia oblasti [On the singular Volterra integral equation of the boundary value problem for heat conduction in a degenerating domain]. *Vestnik Udmurtskogo universiteta. Matematika. Mekhanika. Kompiuternye nauki — The Bulletin of Udmurt University. Mathematics. Mechanics. Computer Science, 31*(2), 241–252 [in Russian].

44 Ramazanov, M.I., & Gulmanov, N.K. (2021). Solution of a two-dimensional boundary value problem of heat conduction in a degenerating domain. *Journal of Mathematics, Mechanics and Computer Science, 111*(3), 65–78.

45 Amangalieva, M.M., Dzhenaliev, M.T., Kosmakova, M.T., & Ramazanov, M.I. On one homogeneous problem for the heat equation in an infinite angular domain. *Siberian Mathematical Journal, 56*(6), 982–995.

46 Jenaliyev, M.T., & Ramazanov, M.I. (2018). On a Homogeneous Parabolic Problem in an Infinite Corner Domain *Filomat, 32*(3), 965–974.

47 Wright, E.M. (1933). On the coefficients of power series having exponential singularities. *J. London Math. Soc., 8*, 71–79.

48 Wright, E.M. (1940). The generalized Bessel function of order greater than one. *Quart. J. Math., Oxford Ser., 11*, 36–48.

49 Pskhu, A.V. (2003). Solution of a Boundary Value Problem for a Fractional Partial Differential Equation. *Differential Equations, 39*, 1092–1099.

50 Pskhu, A.V. (2020). Nakhushev extremum principle for a class of integro-differential operators. *Fract. Calc. Appl. Anal., 23*, 1712–1722.

DOI 10.31489/2022M1/96-101
UDC 517.58

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Semi-integration of certain algebraic expressions

The theory of fractional calculus developed rapidly as the applications of this branch are extensive nowadays. There is no discipline of modern engineering and science that remains untouched by the techniques of fractional calculus. In fact, one could argue that real world processes are fractional order systems in general. In this article, we obtain the semi-integrals of certain algebraic functions in terms of difference of two complete elliptic integrals of different kinds by using series manipulation technique.

Keywords: hypergeometric functions, complete elliptic integrals, Pochhammer symbol, semi-integration.

1 Introduction, definitions and preliminaries

Fractional Calculus is the integration and differentiation of non-integer (fractional) order. The concept of fractional operators has been introduced almost simultaneously with the development of the classical ones. The idea of differentiation (and integration) to a non-integer order has appeared surprisingly early in the history of the Calculus. It is mentioned in a letter dated September 30, 1695, from G.W. Leibniz to G.A. L'Hôpital, and in another letter dated May 28, 1697, from Leibniz to J. Wallis. This consequently attracted the interest of many well-known mathematicians, including Euler, Liouville, Laplace, Riemann, Grünwald, Letnikov, and many others [1; 284].

In 1731, L. Euler extended the derivative formula in general form ([2; 80, Eq.(2.37)], [1; 285, Eq.(5)]):

$$D_x^{\mu}\left\{x^{\lambda}\right\}=\frac{d^{\mu}}{dx^{\mu}}x^{\lambda}=\frac{\Gamma(\lambda+1)}{\Gamma(\lambda+1-\mu)}x^{\lambda-\mu},$$

where μ is not restricted to integer values and μ may be an arbitrary complex number and $\Gamma(1+\lambda),\ \Gamma(1+\lambda-\mu)$ are well-defined. When μ is positive real number, then above formula stands for fractional differentiation and when μ is negative real number, then above formula represents fractional integration.

In this paper, we shall use the following standard notations:

$$\mathbb{N}:=\ \{1,2,3,\cdots\}\,;\quad \mathbb{N}_0:=\mathbb{N}\bigcup\{0\}\,;\quad \mathbb{Z}_0^-:=\mathbb{Z}^-\bigcup\{0\}=\{0,-1,-2,-3,\cdots\}\,.$$

The symbols $\mathbb{C},\ \mathbb{R},\ \mathbb{N},\ \mathbb{Z},\ \mathbb{R}^+$ and $\mathbb{R}^-$ denote the sets of complex numbers, real numbers, natural numbers, integers, positive and negative real numbers, respectively.

The classical Pochhammer symbol $(\alpha)_p$ $(\alpha,p\in\mathbb{C})$ is defined by ([3; 22, Eq.(1), p.32, Q.N.(8) and Q.N.(9)], see also [1; 23, Eq.(22) and Eq.(23)]).

A natural generalization of the Gaussian hypergeometric series ${}_2F_1[\alpha,\beta;\gamma;z]$ is accomplished by introducing any arbitrary number of numerator and denominator parameters [1; 42, Eq.(1)].

Each of the following results will be needed in our present study.
Some complete elliptic integrals [4; 321, Eq.(25)]:

$$\mathbf{B}(x)=\int_0^{\frac{\pi}{2}}\frac{\cos^2\theta}{\sqrt{(1-x^2\ \sin^2\theta)}}\,d\theta=\frac{\pi}{4}\,{}_2F_1\left[\begin{array}{c}\frac{1}{2},\ \frac{1}{2};\\ \\ 2;\end{array}\ x^2\right];\quad |x|<1,$$

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$$\mathbf{C}(x) = \int_0^{\frac{\pi}{2}} \frac{\sin^2\theta\cos^2\theta}{\left(\sqrt{(1-x^2\ \sin^2\theta)}\right)^3}\,d\theta = \frac{\pi}{16}\,{}_2F_1\left[\begin{array}{c}\frac{3}{2},\ \frac{3}{2};\\ \\ 3;\end{array} x^2\right];\quad |x|<1,$$

$$\mathbf{D}(x) = \int_0^{\frac{\pi}{2}} \frac{\sin^2\theta}{\sqrt{(1-x^2\ \sin^2\theta)}}\,d\theta = \frac{\pi}{4}\,{}_2F_1\left[\begin{array}{c}\frac{1}{2},\ \frac{3}{2};\\ \\ 2;\end{array} x^2\right];\quad |x|<1.$$

Complete elliptic integral of second kind [4; 317, Eq.(2)]:

$$\mathbf{E}(x) = \int_0^{\frac{\pi}{2}} \sqrt{(1-x^2\ \sin^2\theta)}\,d\theta = \frac{\pi}{2}\,{}_2F_1\left[\begin{array}{c}\frac{1}{2},\ -\frac{1}{2};\\ \\ 1;\end{array} x^2\right];\quad |x|<1.$$

Complete elliptic integral of first kind [4; 317, Eq.(1)]:

$$\mathbf{K}(x) = \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{(1-x^2\ \sin^2\theta)}} = \frac{\pi}{2}\,{}_2F_1\left[\begin{array}{c}\frac{1}{2},\ \frac{1}{2};\\ \\ 1;\end{array} x^2\right];\quad |x|<1.$$

See [3; 70, Q.N.(10)]:

$${}_2F_1\left[\begin{array}{c}\alpha,\ \alpha-\frac{1}{2};\\ \\ 2\alpha;\end{array} z\right] = \left(\frac{2}{1+\sqrt{(1-z)}}\right)^{2\alpha-1}, \tag{1}$$

where $|z|<1$ and $2\alpha \in \mathbb{C}\backslash\mathbb{Z}_0^-$.

$${}_2F_1\left[\begin{array}{c}\alpha,\ \alpha+\frac{1}{2};\\ \\ 2\alpha;\end{array} z\right] = \frac{1}{\sqrt{(1-z)}}\left(\frac{2}{1+\sqrt{(1-z)}}\right)^{2\alpha-1}, \tag{2}$$

where $|z|<1$ and $2\alpha \in \mathbb{C}\backslash\mathbb{Z}_0^-$.

Special value of the hypergeometric function [5; 474, Entry(100)]:

$${}_2F_1\left[\begin{array}{c}2,\ \frac{1}{2};\\ \\ 3;\end{array} x\right] = \frac{4}{3x^2}\left\{2-(2+x)\sqrt{1-x}\right\}.$$

Considering the work of Abramowitz *et al.* [6], Andrews [7, 1], Gradshteyn *et al.* [8], Magnus *et al.* [9], Srivastava *et al.* [10] and Qureshi *et al.* [11], we aim at obtaining the semi-integrals of certain algebraic functions. In Section 2, semi-integrals of some algebraic functions are mentioned in terms of difference of two complete elliptic integrals of different kinds. In Section 3, their proofs are given by using series manipulation technique.

2 Some results involving semi-integration

In this section, we obtain the semi-integration of some algebraic expressions in terms of difference of two complete elliptic integrals of different kinds.

$$\frac{d^{-\frac{1}{2}}}{dx^{-\frac{1}{2}}}\left\{\frac{-\sqrt{x}}{\sqrt{(1-x)}\ \left(1+\sqrt{(1-x)}\right)^2}\right\} = \frac{2}{\sqrt{\pi}}\ \left\{\mathbf{B}(\sqrt{x})-\mathbf{D}(\sqrt{x})\right\}. \tag{3}$$

$$\frac{d^{-\frac{1}{2}}}{dx^{-\frac{1}{2}}}\left\{\frac{-1}{x^{\frac{5}{2}}}\left(\frac{20-17x+x^2}{\sqrt{(1-x)}}-20+7x\right)\right\} = \frac{2}{\sqrt{\pi}}\ \left\{\mathbf{B}(\sqrt{x})-4\mathbf{C}(\sqrt{x})\right\}. \tag{4}$$

$$\frac{d^{-\frac{1}{2}}}{dx^{-\frac{1}{2}}}\left\{\frac{2-(2+x)\sqrt{(1-x)}}{x^{\frac{3}{2}}}\right\}=\frac{4}{\sqrt{\pi}}\left\{\mathbf{B}(\sqrt{x})-\frac{1}{2}\mathbf{E}(\sqrt{x})\right\}. \tag{5}$$

$$\frac{d^{-\frac{1}{2}}}{dx^{-\frac{1}{2}}}\left\{\frac{-\sqrt{x}}{\sqrt{(1-x)}\left(1+\sqrt{(1-x)}\right)^2}\right\}=\frac{4}{\sqrt{\pi}}\left\{\mathbf{B}(\sqrt{x})-\frac{1}{2}\mathbf{K}(\sqrt{x})\right\}. \tag{6}$$

$$\frac{d^{-\frac{1}{2}}}{dx^{-\frac{1}{2}}}\left\{\frac{3\sqrt{x}}{\sqrt{(1-x)}\left(1+\sqrt{(1-x)}\right)^3}\right\}=\frac{8}{\sqrt{\pi}}\left\{\mathbf{C}(\sqrt{x})-\frac{1}{4}\mathbf{D}(\sqrt{x})\right\}. \tag{7}$$

$$\frac{d^{-\frac{1}{2}}}{dx^{-\frac{1}{2}}}\left\{\sqrt{x}\;{}_4F_3\left[\begin{array}{c}\frac{1}{2},\ 2,\ \frac{73+\sqrt{145}}{32},\ \frac{73-\sqrt{145}}{32};\\ 4,\ \frac{41+\sqrt{145}}{32},\ \frac{41-\sqrt{145}}{32};\end{array}\ x\right]\right\}=\frac{8}{\sqrt{\pi}}\left\{\mathbf{C}(\sqrt{x})-\frac{1}{8}\mathbf{E}(\sqrt{x})\right\}. \tag{8}$$

$$\frac{d^{-\frac{1}{2}}}{dx^{-\frac{1}{2}}}\left\{\sqrt{\left(\frac{x}{1-x}\right)}\ \frac{(8\sqrt{(1-x)}-x+8)}{\left(1+\sqrt{(1-x)}\right)^4}\right\}=\frac{16}{\sqrt{\pi}}\left\{\mathbf{C}(\sqrt{x})-\frac{1}{8}\mathbf{K}(\sqrt{x})\right\}. \tag{9}$$

$$\frac{d^{-\frac{1}{2}}}{dx^{-\frac{1}{2}}}\left\{\frac{x^2-x+2\left(1-\sqrt{(1-x)}\right)}{x^{\frac{3}{2}}\sqrt{(1-x)}}\right\}=\frac{4}{\sqrt{\pi}}\left\{\mathbf{D}(\sqrt{x})-\frac{1}{2}\mathbf{E}(\sqrt{x})\right\}. \tag{10}$$

$$\frac{d^{-\frac{1}{2}}}{dx^{-\frac{1}{2}}}\left\{\frac{\sqrt{x}}{\sqrt{(1-x)}\left(1+\sqrt{(1-x)}\right)^2}\right\}=\frac{4}{\sqrt{\pi}}\left\{\mathbf{D}(\sqrt{x})-\frac{1}{2}\mathbf{K}(\sqrt{x})\right\}. \tag{11}$$

$$\frac{d^{-\frac{1}{2}}}{dx^{-\frac{1}{2}}}\left\{\sqrt{\left(\frac{x}{1-x}\right)}\right\}=\frac{2}{\sqrt{\pi}}\left\{\mathbf{K}(\sqrt{x})-\mathbf{E}(\sqrt{x})\right\}. \tag{12}$$

3 Demonstration of the semi-integrals

Proof of the result (3):

$$\frac{d^{-\frac{1}{2}}}{dx^{-\frac{1}{2}}}\left\{\frac{-\sqrt{x}}{\sqrt{1-x}\,(1+\sqrt{1-x})^2}\right\}=\frac{-1}{4}\frac{d^{-\frac{1}{2}}}{dx^{-\frac{1}{2}}}\left\{\sqrt{x}\,\frac{1}{\sqrt{1-x}}\left(\frac{2}{1+\sqrt{1-x}}\right)^2\right\}. \tag{13}$$

Using equation (2) in equation (13), we get

$$\frac{d^{-\frac{1}{2}}}{dx^{-\frac{1}{2}}}\left\{\frac{-\sqrt{x}}{\sqrt{1-x}\,(1+\sqrt{1-x})^2}\right\}=\frac{-1}{4}\frac{d^{-\frac{1}{2}}}{dx^{-\frac{1}{2}}}\left\{\sqrt{x}\;{}_2F_1\left[\begin{array}{c}\frac{3}{2},\ 2;\\ 3;\end{array}\ x\right]\right\}=$$

$$=\frac{-1}{4}\sum_{n=0}^{\infty}\frac{\left(\frac{3}{2}\right)_n(2)_n}{(3)_n\ n!}\ \frac{d^{-\frac{1}{2}}}{dx^{-\frac{1}{2}}}\ x^{n+\frac{1}{2}}=$$

$$=\frac{-1}{4}\sum_{n=0}^{\infty}\frac{\left(\frac{3}{2}\right)_n(2)_n}{(3)_n}\ \frac{\Gamma\left(n+\frac{3}{2}\right)}{\Gamma\left(n+2\right)}\ \frac{x^{n+1}}{n!}=$$

$$=\frac{-x\sqrt{\pi}}{8}\sum_{n=0}^{\infty}\frac{\left(\frac{3}{2}\right)_n\left(\frac{3}{2}\right)_n}{(3)_n}\ \frac{x^n}{n!}=$$

$$= \frac{-x\sqrt{\pi}}{8} \; {}_2F_1\left[\begin{matrix} \frac{3}{2}, \; \frac{3}{2}; \\ \\ 3; \end{matrix} \; x \right]. \tag{14}$$

From right-hand side of equation (3), we have

$$\frac{2}{\sqrt{\pi}} \; \{\mathbf{B}(\sqrt{x}) - \mathbf{D}(\sqrt{x})\} = \frac{\sqrt{\pi}}{2} \left\{ {}_2F_1\left[\begin{matrix} \frac{1}{2}, \; \frac{1}{2}; \\ \\ 2; \end{matrix} \; x \right] - {}_2F_1\left[\begin{matrix} \frac{1}{2}, \; \frac{3}{2}; \\ \\ 2; \end{matrix} \; x \right]\right\} =$$

$$= \frac{\sqrt{\pi}}{2} \left\{ \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{2}\right)_n \; x^n}{(2)_n \, n!} - \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n \left(\frac{3}{2}\right)_n \; x^n}{(2)_n \, n!} \right\} =$$

$$= \frac{\sqrt{\pi}}{2} \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n \; x^n}{(2)_n \, n!} \; \left\{ \left(\frac{1}{2}\right)_n - \left(\frac{3}{2}\right)_n \right\} =$$

$$= -\sqrt{\pi} \; \sum_{n=1}^{\infty} \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{2}\right)_n \; x^n}{(2)_n \, (n-1)!}.$$

Replacing n by $n+1$, we get

$$\frac{2}{\sqrt{\pi}} \; \{\mathbf{B}(\sqrt{x}) - \mathbf{D}(\sqrt{x})\} = \frac{-x\sqrt{\pi}}{8} \; \sum_{n=0}^{\infty} \frac{\left(\frac{3}{2}\right)_n \left(\frac{3}{2}\right)_n}{(3)_n} \; \frac{x^n}{n!} =$$

$$= \frac{-x\sqrt{\pi}}{8} \; {}_2F_1\left[\begin{matrix} \frac{3}{2}, \; \frac{3}{2}; \\ \\ 3; \end{matrix} \; x \right]. \tag{15}$$

From equations (14) and (15), we arrive at the result (3).

Proof of the results (4) to (8):

The proof of the results (4) to (8) follow the same steps as in the proof of the result (3). So we omit the details.

Proof of the result (9):

$$\frac{d^{-\frac{1}{2}}}{dx^{-\frac{1}{2}}} \; \left\{ \sqrt{\frac{x}{1-x}} \; \frac{(8\,\sqrt{1-x} - x + 8)}{\left(1+\sqrt{1-x}\right)^4} \right\} = \frac{d^{-\frac{1}{2}}}{dx^{-\frac{1}{2}}} \; \left\{ \sqrt{x} \; \left(\frac{8\sqrt{1-x} - (x-8)}{\left(1+\sqrt{1-x}\right)^4 \sqrt{1-x}} \right) \right\} =$$

$$= \frac{d^{-\frac{1}{2}}}{dx^{-\frac{1}{2}}} \; \left\{ \sqrt{x} \; \left(\frac{8\sqrt{1-x}\left(1+\sqrt{1-x}\right) + 7x}{\left(1+\sqrt{1-x}\right)^4 \sqrt{1-x}} \right) \right\} =$$

$$= \frac{d^{-\frac{1}{2}}}{dx^{-\frac{1}{2}}} \; \left\{ \sqrt{x} \; \left(\frac{8}{\left(1+\sqrt{1-x}\right)^3} + \frac{7x}{\left(1+\sqrt{1-x}\right)^4 \sqrt{1-x}} \right) \right\} =$$

$$= \frac{d^{-\frac{1}{2}}}{dx^{-\frac{1}{2}}} \; \left\{ \sqrt{x} \; \left(\frac{2}{1+\sqrt{1-x}} \right)^3 + \frac{7x}{16\sqrt{1-x}} \left(\frac{2}{1+\sqrt{1-x}} \right)^4 \right\}. \tag{16}$$

Using equations (1) and (2) in equation (16), we get

$$\frac{d^{-\frac{1}{2}}}{dx^{-\frac{1}{2}}} \; \left\{ \sqrt{\frac{x}{1-x}} \; \frac{(8\,\sqrt{1-x} - x + 8)}{\left(1+\sqrt{1-x}\right)^4} \right\} = \frac{d^{-\frac{1}{2}}}{dx^{-\frac{1}{2}}} \; \left\{ \sqrt{x} \; {}_2F_1\left[\begin{matrix} \frac{3}{2}, \; 2; \\ \\ 4; \end{matrix} \; x \right] + \frac{7x}{16} \; {}_2F_1\left[\begin{matrix} \frac{5}{2}, \; 3; \\ \\ 5; \end{matrix} \; x \right] \right\} =$$

$$= \sum_{n=0}^{\infty} \frac{\left(\frac{3}{2}\right)_n (2)_n}{(4)_n \;\; n!} \; \frac{d^{-\frac{1}{2}}}{dx^{-\frac{1}{2}}} \; x^{n+\frac{1}{2}} + \frac{7}{16} \; \sum_{n=0}^{\infty} \frac{\left(\frac{5}{2}\right)_n (3)_n}{(5)_n \;\; n!} \; \frac{d^{-\frac{1}{2}}}{dx^{-\frac{1}{2}}} \; x^{n+\frac{3}{2}} =$$

$$= \sum_{n=0}^{\infty} \frac{\left(\frac{3}{2}\right)_n (2)_n}{(4)_n} \frac{\Gamma\left(n+\frac{3}{2}\right)}{\Gamma(n+2)} \frac{x^{n+1}}{n!} + \frac{7}{16} \sum_{n=0}^{\infty} \frac{\left(\frac{5}{2}\right)_n (3)_n}{(5)_n} \frac{\Gamma\left(n+\frac{5}{2}\right)}{\Gamma(n+3)} \frac{x^{n+2}}{n!} =$$

$$= \frac{x\sqrt{\pi}}{2} \sum_{n=0}^{\infty} \frac{\left(\frac{3}{2}\right)_n \left(\frac{3}{2}\right)_n}{(4)_n} \frac{x^n}{n!} + \frac{21x^2\sqrt{\pi}}{128} \sum_{n=0}^{\infty} \frac{\left(\frac{5}{2}\right)_n \left(\frac{5}{2}\right)_n}{(5)_n} \frac{x^n}{n!} =$$

$$= \frac{x\sqrt{\pi}}{2} \; {}_2F_1\left[\begin{array}{c} \frac{3}{2}, \frac{3}{2}; \\ 4; \end{array} x\right] + \frac{21x^2\sqrt{\pi}}{128} \; {}_2F_1\left[\begin{array}{c} \frac{5}{2}, \frac{5}{2}; \\ 5; \end{array} x\right]. \tag{17}$$

From right-hand side of equation (9), we have

$$\frac{16}{\sqrt{\pi}} \left\{\mathbf{C}(\sqrt{x}) - \frac{1}{8}\mathbf{K}(\sqrt{x})\right\} = \sqrt{\pi}\left\{ {}_2F_1\left[\begin{array}{c} \frac{3}{2}, \frac{3}{2}; \\ 3; \end{array} x\right] - {}_2F_1\left[\begin{array}{c} \frac{1}{2}, \frac{1}{2}; \\ 1; \end{array} x\right]\right\} =$$

$$= \sqrt{\pi}\left\{ \sum_{n=0}^{\infty} \frac{\left(\frac{3}{2}\right)_n \left(\frac{3}{2}\right)_n x^n}{(3)_n \, n!} - \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{2}\right)_n x^n}{(1)_n \, n!} \right\} =$$

$$= \sqrt{\pi} \sum_{n=0}^{\infty} \frac{x^n}{n!} \left\{ \frac{\left(\frac{3}{2}\right)_n \left(\frac{3}{2}\right)_n}{(3)_n} - \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{2}\right)_n}{(1)_n} \right\} =$$

$$= \frac{5\sqrt{\pi}}{2} \sum_{n=1}^{\infty} \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{2}\right)_n \left(\frac{12}{7}\right)_n x^n}{\left(\frac{5}{7}\right)_n (3)_n (n-1)!}.$$

Replacing n by $n+1$, we get

$$\frac{16}{\sqrt{\pi}} \left\{\mathbf{C}(\sqrt{x}) - \frac{1}{8}\mathbf{K}(\sqrt{x})\right\} = \frac{x\sqrt{\pi}}{2} \sum_{n=0}^{\infty} \frac{\left(\frac{3}{2}\right)_n \left(\frac{3}{2}\right)_n}{(4)_n} \frac{x^n}{n!} \left(1 + \frac{7n}{12}\right) =$$

$$= \frac{x\sqrt{\pi}}{2} \left\{ \sum_{n=0}^{\infty} \frac{\left(\frac{3}{2}\right)_n \left(\frac{3}{2}\right)_n}{(4)_n} \frac{x^n}{n!} + \frac{7}{12} \sum_{n=1}^{\infty} \frac{\left(\frac{3}{2}\right)_n \left(\frac{3}{2}\right)_n}{(4)_n} \frac{x^n}{(n-1)!} \right\} =$$

$$= \frac{x\sqrt{\pi}}{2} \sum_{n=0}^{\infty} \frac{\left(\frac{3}{2}\right)_n \left(\frac{3}{2}\right)_n}{(4)_n} \frac{x^n}{n!} + \frac{21x^2\sqrt{\pi}}{128} \sum_{n=0}^{\infty} \frac{\left(\frac{5}{2}\right)_n \left(\frac{5}{2}\right)_n}{(5)_n} \frac{x^n}{n!} =$$

$$= \frac{x\sqrt{\pi}}{2} \; {}_2F_1\left[\begin{array}{c} \frac{3}{2}, \frac{3}{2}; \\ 4; \end{array} x\right] + \frac{21x^2\sqrt{\pi}}{128} \; {}_2F_1\left[\begin{array}{c} \frac{5}{2}, \frac{5}{2}; \\ 5; \end{array} x\right]. \tag{18}$$

From equations (17) and (18), we arrive at the result (9).

Proof of the results (10), (11) and (12):

The results (10), (11) and (12) are obtained in a similar manner by following the same steps as in the proof of the result (3) and making use of the equation (2). So we omit the details here.

4 Concluding remarks and observations

In this paper, we have obtained the semi-integrals of certain algebraic functions in terms of difference of two complete elliptic integrals of different kinds by using series manipulation technique. We conclude this paper with the remark that the semi-integrals of various other functions can be derived in an analogous manner. Moreover, the results deduced above are expected to lead some potential applications in several fields of Applied Mathematics, Statistics and Engineering Sciences.

Conflicts of interest: The authors declare that there are no conflicts of interest.

References

1 Srivastava, H.M., & Manocha, H.L. (1984). *A Treatise on Generating Functions.* Chichester, Brisbane and Toronto, New York: Halsted Press (Ellis Horwood Limited, Chichester), John Wiley and Sons.

2 Andrews, L.C. (1998). *Special Functions of Mathematics for Engineers.* Reprint of the 1992 Second Edition, SPIE Optical Engineering Press. Bellingham, W.A. Oxford: Oxford University Press.

3 Rainville, E.D. (1971). *Special Functions.* Bronx, New York: The Macmillan Co. Inc.; Reprinted by Chelsea publ. Co.

4 Erdélyi, A., Magnus, W., Oberhettinger, F., & Tricomi, F.G. (1955). *Higher Transcendental Functions.* Vol. II. Toronto and London, New York: McGraw-Hill Book Company.

5 Prudnikov, A.P., Brychknov, Yu. A., & Marichev, O.I. (1990). *Integrals and Series. Vol. III: More special functions.* (G.G.Gould, Trans.). Philadelphia London, Paris, Montreux, Tokyo, Melbourne, New York: Gordon and Breach Science Publishers (Original work published 1986).

6 Abramowitz, M., & Stegun, I.A. (1992). *Handbook of Mathematical Functions with Formulas, Graphs and Mathematical Tables.* Reprint of the 1972 Edition. New York: Dover Publications, Inc.

7 Andrews, L.C. (1985). *Special Functions for Engineers and Applied Mathematicians.* New York: Macmillan Publishing Company.

8 Gradshteyn, I.S., & Ryzhik, I.M. (2007). *Table of Integrals, Series and Products:* Seventh Edition. New York: Academic Press.

9 Magnus, W., Oberhettinger, F., & Soni, R.P. (1966). *Some Formulas and Theorems for the Special Functions of Mathematical Physics:* Third Enlarged Edition. New York: Springer-Verlag.

10 Srivastava, H.M., & Choi, J. (2011). *Zeta and q-Zeta Functions and Associated Series and Integrals.* Elsevier.

11 Qureshi, M.I, Majid, J., & Bhat, A.H. (2021). Solutions of some Nielsen-type integrals through hypergeometric technique with applications. *Punjab University J. Math., 52*(7), 45–59.

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Кейбір алгебралық өрнектерді жартылай интегралдау

Бөлшекті есептеу теориясы тез дамып келеді, себебі қазіргі уақытта осы облыстың қосымшалары өте кең. Қазіргі заманғы техника мен ғылымның бірде-бір пәні бөлшекті есептеу әдістерінен тыс қалған жоқ. Шын мәнінде, нақты әлем процестері бөлшекті ретті жүйелер деп айтуға болады. Мақалада алгебралық функциялардың жартылай интегралдарын қатарлармен манипуляциялау техникасын қолдана отырып, әр түрлі екі толық эллиптикалық интегралдардың айырымы тұрғысынан алынған.

Кілт сөздер: гипергеометриялық функциялар, толық эллиптикалық интегралдар, Похгаммер символы, жартылайинтегралдау.

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Полуинтегрирование некоторых алгебраических выражений

Теория дробного исчисления быстро развивается, так как приложения этой области в настоящее время очень широки. Ни одна дисциплина современной техники и науки, в целом, не остается нетронутой методами дробного исчисления. На самом деле можно утверждать, что процессы реального мира являются системами дробного порядка. В статье получены полуинтегралы некоторых алгебраических функций в терминах разности двух полных эллиптических интегралов разных видов с помощью техник и манипулирования рядами.

Ключевые слова: гипергеометрические функции, полные эллиптические интегралы, символ Похгаммера, полуинтегрирование.

DOI 10.31489/2022M1/102-108

UDC 517.9

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The Schrödinger equations generated by q-Bessel operator in quantum calculus

In this paper, we obtain exact solutions of a new modification of the Schrödinger equation related to the Bessel q-operator. The theorem is proved on the existence of this solution in the Sobolev-type space $W_q^2(\mathbb{R}_q^+)$ in the q-calculus. The results on correctness in the corresponding spaces of the Sobolev-type are obtained. For simplicity, we give results involving fractional q-difference equations of real order $a > 0$ and given real numbers in q-calculus. Numerical treatment of fractional q-difference equations is also investigated. The obtained results can be used in this field and be supplement for studies in this field.

Keywords: q-integral, q-Jackson integral, q-difference operator q-derivative, the q-Bessel Fourier transform, the Sobolev type space,the Schrödinger equation, q-Bessel operator.

Introduction

The origin of the q-difference calculus plays an important role due to their numerous applications and its importance in mathematics and other scientific fields. This calculus can be traced back to the works in [1, 2] by F. Jackson and R.D. Carmichael [3] from the beginning of the twentieth century, while basic definitions and properties can be found e.g. in the monographs [4, 5]. Recently, the q-difference calculus has been proposed by W. Al-Salam [6] and R.P. Agarwal [7]. Today, maybe due to the explosion in research within the fractional differential calculus setting, new developments in this theory of fractional q-difference calculus have been addressed extensively by several researchers.

The Schrödinger equation is the fundamental equation of the science of submicroscopic phenomena known as quantum mechanics. This equation studied by the Austrian physicist Erwin Schrödinger in 1926 [8]. Moreover, it is widely used in modern science in such areas as quantum information and econophysics [9, 10].

Nowadays, the several methods and techniques have been developed to study exact and approximate analytical solutions to the modern models of the Schrödinger equation for a better understanding of its dynamical behavior [11, 12]. The Exact solutions of the Schrödinger equation play an important role not only from a pure mathematical point of view but also for the conceptual understanding of the physical phenomena.

The paper is organized as follows: The main results are presented and proved in Section 2. To not disturb these presentations we include some necessary Preliminaries in Section 1.

1 Preliminaries

Throughout this paper, we assume that $0 < q < 1$. We start by recalling some basic notation in the q-calculus, see e.g. the books [4] and [13].

Let $\alpha \in \mathbb{R}$. Then a q-real number $[\alpha]_q$ is defined by

$$[\alpha]_q := \frac{1 - q^\alpha}{1 - q},$$

where $\lim\limits_{q \to 1} \frac{1-q^\alpha}{1-q} = \alpha$.

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We introduce for $k \in \mathbb{N}$:

$$(a;q)_0 = 1,\ (a;q)_n = \prod_{k=0}^{n-1}\left(1-q^k a\right),\ (a;q)_\infty = \lim_{n\to\infty}(a,q)_n,\ (a;q)_\alpha = \frac{(a;q)_\infty}{(q^\alpha a;q)_\infty}.$$

The q-analogue of the binomial coefficients $[n]_q!$ are defined by

$$[n]_q! = \begin{cases} 1, & \text{if } n=0, \\ [1]_q \times [2]_q \times \cdots \times [n]_q, & \text{if } n \in \mathrm{N}. \end{cases}$$

The q^2-differential operator is defined by (see [14] and [15])

$$\partial_q f(x) = \frac{f(q^{-1}x) + f(-q^{-1}x) - f(qx) + f(-qx) - 2f(-x)}{2x(1-q)},$$

where $x \neq 0$.

Note that if f is differentiable at x, then $\lim\limits_{q\to 1}\partial_q f(x) = f'(x)$.

A repeated application of the q^2-analogue differential operator n times is denoted by:

$$\partial_q^0 f = f,\ \ \partial_q^{n+1} f = \partial_q\left(\partial_q^n f\right).$$

The definite q-integral or the q-Jackson integral of a function f is defined by the formula (see [1] and [2])

$$\int_0^x f(t)d_qt := (1-q)x\sum_{k=0}^{\infty} q^k f(q^k x), \quad x \in (0,b),$$

and the improper q-integral of a function $f(x) : [0,\infty) \to \mathbb{R}$, is defined by the formula

$$\int_0^\infty f(t)d_qt := (1-q)\sum_{k=-\infty}^{\infty} q^k f(q^k).$$

We denote $\mathbb{R}_q^+ = \{q^k, k \in \mathbb{Z}\}$ and define

$$L^p_{\alpha,q}\left(\mathbb{R}_q^+\right) := \{f : \|f\|_{p,\alpha,q} = \left(\int_0^\infty |f(x)|^p x^{2\alpha+1} d_qx\right)^{\frac{1}{p}} < \infty\}.$$

Definition 1. (see [16] and [14], [17]) The q-Bessel Fourier transform is defined for $f \in L^1_{\alpha,q}(\mathbb{R}_q^+)$, by

$$\mathcal{F}_{q,\alpha}(f)(\lambda) = c_{q,\alpha}\int_0^\infty f(x) j_\alpha(\lambda x;q^2) x^{2\alpha+1} d_qx, \quad 0 < x < \infty \tag{1}$$

and its inverse $\mathcal{F}^{-1}_{q,\alpha}g(x)$ is given by

$$\mathcal{F}^{-1}_{q,\alpha}g(x) = c_{q,\alpha}\int_0^\infty F_{q,\alpha}(g)(\lambda) j_\alpha(\lambda x;q^2)\lambda^{2\alpha+1} d_q\lambda, \quad 0 < x < \infty, \tag{2}$$

for $g \in L^1_{\alpha,q}(\mathbb{R}_q^+)$, where $c_{q,\alpha} = \frac{(1+q)^{-\alpha}}{\Gamma_{q^2}(\alpha+1)}$.

Definition 2. (See [18]) For $1 \leq p < \infty$, we define the Sobolev type space associated with the q-Bessel Fourier transform $W^p_q(\mathbb{R}_q^+)$ equipped with the norms

$$\|u\|^2_{W^p_q(\mathbb{R}_q^+)} := \left(\int_0^\infty \left(1+|\lambda|^2\right)^{\frac{p}{2}} |\mathcal{F}_{q,\alpha}(u)(\lambda)|^2 d_q\lambda\right)^2.$$

Let $0 < T < \infty$. We also introduce the spaces $C^k\left([0,T];W_q^p(\mathbb{R}_q^+)\right)$ and $C^k\left([0,T];L_q^p(\mathbb{R}_q^+)\right)$ defined by the finiteness of the norms

$$\|u\|_{C^k\left([0,T];W_q^s(\mathbb{R}_q^+)\right)} : \sum_{n=0}^{k} \max_{0\le t\le T} \|\partial_t^n u(t,\cdot)\|_{W_q^p(\mathbb{R}_q^+)}$$

and

$$\|u\|_{C^k\left([0,T];L_q^p(\mathbb{R}_q^+)\right)} := \sum_{n=0}^{k} \max_{0\le t\le T} \|\partial_t^n u(t,\cdot)\|_{L_q^p(\mathbb{R}_q^+)},$$

respectively.

For $\lambda \in \mathbb{C}$, the function $j_\alpha(\lambda x; q^2)$ is the unique even solution of the problem

$$\begin{cases} \Delta_{q,\alpha} f(x) = -\lambda^2 f(x), \\ f(0) = 1, \end{cases}$$

where

$$\Delta_{q,\alpha} f(x) = \frac{1}{|x|^{2\alpha+1}} \partial_q [|x|^{2\alpha+1} \partial_q f(x)].$$

Moreover, if f and $\Delta_{q,\alpha} f$ are in $L^1_{\alpha,q}(\mathbb{R}_{q,+})$, then (see e.g. [14] and [17]):

$$\mathcal{F}_{q,\alpha}(\Delta_{q,\alpha} f)(\lambda) = -\lambda^2 \mathcal{F}_{q,\alpha}(f)(\lambda). \tag{3}$$

Theorem 1. 1) (Plancherel formula [17]) For all $f \in \partial_{q,*}(\mathbb{R})$, we have

$$\|\mathcal{F}_{q,\alpha}(f)\|_{2,\alpha,q} = \|f\|_{2,\alpha,q}. \tag{4}$$

2) (Plancherel theorem) The q-Bessel transform can be uniquely extended to an isometric isomorphism on $L^2_{q,\alpha}(\mathbb{R}_q^+)$ with $\mathcal{F}^{-1}_{q,\alpha} = \mathcal{F}_{q,\alpha}$.

2 Main problem

We consider the Schrödinger equation generated by the q-Bessel operator $\Delta_{q,\alpha} f$ in the following form:

$$\partial_t u(t,x) - i\Delta_{q,\alpha,x} u(t,x) = f(t,x), \quad (t,x) \in [0,T] \times \mathbb{R}_q^+, \tag{5}$$

$$u(0,x) = \varphi(x), \quad x \in \mathbb{R}_q^+. \tag{6}$$

where the function φ is given functions listed above.

Theorem 2. Let $0 < \alpha < 1$. Suppose that $f \in C^1\left([0,T], L^2(\mathbb{R}_q^+)\right)$ and $\varphi \in W_q^2(\mathbb{R}_q^+)$. Then the problem (5)-(6) has a unique solution $u \in C^1\left([0,T], L^2(\mathbb{R}_q^+)\right) \bigcap C\left([0,T], W_q^2(\mathbb{R}_q^+)\right)$ and can be represented by formula

$$\begin{aligned} u(t,x) &= c_{q,\alpha}^2 \int_0^\infty \int_0^\infty \exp(-i\lambda^2 t)\varphi(x) j_\alpha(\lambda x; q^2) x^{2\alpha+1} j_\alpha(\lambda x; q^2) \lambda^{2\alpha+1} d_q x d_q \lambda \\ &+ c_{q,\alpha}^2 \int_0^t \int_0^\infty \int_0^\infty \exp(-i\lambda^2 (t-s)) f(s,x) j_\alpha(\lambda x; q^2) x^{2\alpha+1} j_\alpha(\lambda x; q^2) \lambda^{2\alpha+1} d_q x d_q \lambda ds. \end{aligned}$$

Proof. We assume that

$$\mathcal{F}_{q,\alpha}\left(u(t,\cdot)\right)(\lambda) = U(t), \quad \mathcal{F}_{q,\alpha}\left(\varphi\right) = \widehat{\varphi}(\lambda), \quad \mathcal{F}_{q,\alpha}\left(f(t,\cdot)\right)(\lambda) = F(t)$$

for fixed λ. Let us prove the existence first. Taking the q-Bessel Fourier transform $\mathcal{F}_{q,\alpha}$ (see (1)) on both sides of (5)–(6) we have a simple initial value problem (IVP) of linear ODE:

$$U'(t) + i\lambda^2 U(t) = F(t) \tag{7}$$

$$U(0) = \widehat{\varphi}(\lambda) \tag{8}$$

and $0 < t < T$. The solution of the problem (7)–(8) is given by

$$U(t) = \widehat{\varphi}(\lambda)\exp(-i\lambda^2 t) + \int_0^t \exp(-i\lambda^2(t-s))F(s)ds. \tag{9}$$

Now by using the inverse q-Bessel Fourier transform $\mathcal{F}^{-1}_{q,\alpha}$ in (2) to (9), we obtain the formula for the solution of the problem (5)–(6), given by

$$\begin{aligned} u(t,x) &= c^2_{q,\alpha}\int_0^\infty\int_0^\infty \exp(-i\lambda^2 t)\varphi(x)j_\alpha(\lambda x;q^2)x^{2\alpha+1}j_\alpha(\lambda x;q^2)\lambda^{2\alpha+1}d_qxd_q\lambda \\ &+ c^2_{q,\alpha}\int_0^t\int_0^\infty\int_0^\infty \exp(-i\lambda^2(t-s))f(s,x)j_\alpha(\lambda x;q^2)x^{2\alpha+1}j_\alpha(\lambda x;q^2)\lambda^{2\alpha+1}d_qxd_q\lambda ds. \end{aligned}$$

Let $\varphi \in W^2_q(R^+_q)$ and $f \in C\left([0,T];W^2_q(R^+_q)\right)$. Using $|\exp(-z)| < 1$ for $z \in \mathbb{C}$, Parseval's identity(see (4)) and the relation (9) in the following form:

$$\begin{aligned} |\mathcal{F}_{q,\alpha}(u(t,\cdot))|^2 &\le \left|\widehat{\varphi}(\lambda)\exp(-i\lambda^2 t)\right|^2 + \left|\int_0^t \exp(-i\lambda^2(t-s))F(s)ds\right|^2 \\ &\le |\widehat{\varphi}(\lambda)|^2 + t\int_0^t |\mathcal{F}_{q,\alpha}(f(t,\cdot))(\lambda)|^2 ds. \end{aligned}$$

Hence, using the Parseval's identity (4) and (3)

$$\begin{aligned} \|u(t,\cdot)\|^2_{2,\alpha,q} &= \|\mathcal{F}_{q,\alpha}(u(t,\cdot))\|^2_{2,\alpha,q} \\ &= \int_0^\infty |\mathcal{F}_{q,\alpha}(u(t,\cdot))(\lambda)|^2 \lambda^{2\alpha+1}d_q\lambda \\ &\le \|\mathcal{F}_{q,\alpha}(\varphi)\|^2_{2,\alpha,q} + t\int_0^t \|\mathcal{F}_{q,\alpha}(f(s,\cdot))\|^2_{2,\alpha,q}ds \\ &\le \|\varphi\|^2_{2,\alpha,q} + t\int_0^t \max_{0\le s\le T}\|f(s,\cdot)\|^2_{2,\alpha,q}ds \\ &\le \|\varphi\|^2_{2,\alpha,q} + T^2\|f\|^2_{C\left([0,T];L^2_q(\mathbb{R}^+_q)\right)} < \infty. \end{aligned} \tag{10}$$

Then,

$$\begin{aligned}
\|u(t,\cdot)\|_{W_q^2(\mathbb{R}_q^+)} &= \int\limits_0^\infty (1+\lambda^2)\left|\mathcal{F}_{q,\alpha}\left(u(t,\cdot)\right)(\lambda)\right|^2 d_q\lambda \\
&\leq \int\limits_0^\infty (1+\lambda^2)\left|\widehat{\varphi}(\lambda)\right|^2 d_q\lambda + t\int\limits_0^t\int\limits_0^\infty (1+\lambda^2)\left|\mathcal{F}_{q,\alpha}\left(f(s,\cdot)\right)(\lambda)\right|^2 d_q\lambda ds \\
&\leq \|\varphi\|_{W_q^2(\mathbb{R}_q^+)} + \int\limits_0^t \max_{0\leq s\leq T}\|\widehat{f}(s,\cdot)\|_{W_q^2(\mathbb{R}_q^+)}ds \\
&\leq \|\varphi\|_{W_q^2(\mathbb{R}_q^+)} + \|\widehat{f}\|_{C\left([0,T],W_q^2(R_q^+)\right)} < \infty. \qquad (11)
\end{aligned}$$

From this, $\|u\|_{C\left([0,T],W_q^2(R_q^+)\right)} < \infty$.

Finally, using the relation (9) and the Parseval's identity (4) we have

$$\begin{aligned}
\|\partial_t u(t,\cdot)\|_{W_q^2(\mathbb{R}_q^+)} &\leq \int\limits_0^\infty (1+\lambda^2)\left|\mathcal{F}_{q,\alpha}\left(\varphi\right)(\lambda)\right|^2 d_q\lambda + \int\limits_0^\infty \left|\mathcal{F}_{q,\alpha}\left(f(t,\cdot)\right)(\lambda)\right|^2 d_q\lambda \\
&\leq \|\varphi\|_{W_q^2(\mathbb{R}_q^+)} + \|\widehat{f}(t,\cdot)\|_{L_q^2(\mathbb{R}_q^+)} \\
&\leq \|\varphi\|_{W_q^2(\mathbb{R}_q^+)} + \|\widehat{f}\|_{C\left([0,T],L_q^2(R_q^+)\right)} < \infty. \qquad (12)
\end{aligned}$$

From (10), (11) and (12) we conclude that the solluation $u \in C\left([0,T],W_q^2(R_q^+)\right) \cup C\left([0,T],L_q^2(R_q^+)\right)$ is unique. Assume that there are two different solutions $u(t,x)$ and $v(t,x)$ of Problem (5) and (6) such that

$$\begin{cases} \partial_t u(t,x) - i\Delta_{q,\alpha,x}u(t,x) = f(t,x), & (t,x)\in[0,T]\times\mathbb{R}_q^+, \\ u(0,x) = \varphi(x), & x\in\mathbb{R}_q^+, \end{cases}$$

and

$$\begin{cases} \partial_t \nu(t,x) - i\Delta_{q,\alpha,x}\nu(t,x) = f(t,x), & (t,x)\in[0,T]\times\mathbb{R}_q^+, \\ v(0,x) = \varphi(x), & x\in\mathbb{R}_q^+. \end{cases}$$

Denote $W(t,x)\equiv u(t,x)-v(t,x)$. Then the function $W(t,x)$ is the solution of the following problem.

$$\begin{cases} \partial_t W(t,x) - i\Delta_{q,\alpha,x}W(t,x) = 0, & (t,x)\in[0,T]\times\mathbb{R}_q^+, \\ W(0,x) = 0, & x\in\mathbb{R}_q^+. \end{cases}$$

From (10) it follows that $W(t,x)\equiv 0$. Hence, $u(t,x)\equiv v(t,x)$. This contradiction shows that our assumption is wrong so the solution is unique. The proof is complete.

Acknowledgments

The authors are grateful to the referee for general advices, which improved the final version of the paper. The study was supported by Ministry of Education and Science of the Republic of Kazakhstan, Grant AP08052208.

References

1 Jackson, F.H. (1908). On q-functions and a certain difference operator. *Trans., Roy. Soc. Edin., 46*, 253–281.

2 Jackson, F.H. (1910). On a q-definite integrals. *Quart. J. Pure Appl. Math., 41*, 193–203.

3 Carmichael, R.D. (1912). The general theory of linear q-difference equations. *Amer. J. Math., 34*, 147–168.

4 Cheung, P., & Kac, V. (2000). *Quantum calculus, Edwards Brothers.* Inc., Ann Arbor, MI, USA.

5 Ernst, T. (2002). *A new method of q-calculus.* Doctoral thesis, Uppsala University.

6 Al-Salam, W. (1966/1967). Some fractional q-integrals and q-derivatives. *Proc. Edinb. Math. Soc., 15*, 135–140.

7 Agarwal, R.P. (1969). Certain fractional q-integrals and q-derivatives. *Proc. Camb. Philos. Soc., 66*, 365–370.

8 Schrödinger, E. (1926). An Undulatory Theory of the Mechanics of Atoms and Molecules. *Phys. Rev., 28*, 6, 1049-1070.

9 Schrödinger, E. (1926). Quantisierung als Eigenwertproblem *Ann. Phys., 385*, 437–490.

10 Schrödinger, E. (1926). Quantisierung als Eigenwertproblem *Ann. Phys., 384*, 489–527.

11 Robinet, R.W. (2006). *Quantum Mechanics.* Oxford Univ. Press, Oxford.

12 Jia, C.-S., Lin, P.-Y., & Sun, L.-T. (2002). A new η-pseudo-Hermitian complex potential with PT symmetry. *Physics Letters A, 298*(2–3), 78–82.

13 Ernst, T. (2012). *A comprehensive treatment of q-calculus.* Birkhäuser/Springer, Basel AG, Basel.

14 Rubin, R.L. (2007). Duhamel solutions of non-homogeneous q^2-analogue wave equations. *Proc. Amer. Math. Soc., 135*, 777–785.

15 Rubin, R.L. (1997). A q^2-analogue operator for q^2-analogue Fourier analysis. *J. Math. Anal. Appl., 212*, 571–582.

16 Fitouhi, A., & Bettaieb, R.H. (2008). Wavelet transforms in the q^2-analogue Fourier analysis. *Math. Sci. Res. J., 12*, 202–214.

17 Bettaibi, N., & Bettaieb, R.H. (2006). q-Analogue of the Dunkl transform on the real line. *arXiv: 0801.0069v1 math.QA 29 Dec.*

18 Saoudi, A., & Fitouhi, A. (2015). On q^2-analogue Sobolev type spaces. *Le Matematiche, 70*, 63–77.

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Кванттық есептеуде q-Бессель операторымен алынған Шредингер теңдеулері

Мақалада q-Бессель операторымен байланысты Шредингер теңдеуінің жаңа модификациясының нақты шешімдері алынған. Бұл шешімнің Соболев типтес $W_q^2(\mathbb{R}_q^+)$ кеңістігіндегі q-есептеуде бар екендігі туралы теорема дәлелденген. Соболев типтес тиісті кеңістіктердегі дұрыстығы туралы нәтижелер алынды. Қарапайымдылық үшін $a > 0$ нақты ретті бөлшек q-айырымдық теңдеулерін және q-есептеудегі берілген нақты сандарды қамтитын нәтижелер берілген. Бөлшек q-айырымдық теңдеулерін сандық өңдеу де зерттелді. Алынған нәтижелер жаңа және әдебиеттердегі белгілі нәтижелерді жақсартады және толықтырады.

Кілт сөздер: q-интеграл, q-Джексон интегралы, q-айырмашылық операторы, q-туынды, q-Фурье Бессель түрлендіруі, Соболев типтес кеңістік, Шредингер теңдеуі, q-Бессель операторы.

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Уравнения Шредингера, порожденные q-оператором Бесселя в квантовом исчислении

В статье даны получены точные решения новой модификации уравнения Шредингера, связанные с q-оператором Бесселя. Доказана теорема о существовании этого решения в пространстве соболевского типа $W_q^2(\mathbb{R}_q^+)$ в q-исчислении. Получены результаты о корректности в соответствующих пространствах соболевского типа. Для простоты авторами приведены результаты, связанные с дробными q-разностными уравнениями действительного порядка $a > 0$ и заданными вещественными числами в q-исчислении. Исследована численная обработка дробных q-разностных уравнений. Полученные результаты являются новыми и дополняют известные ранее в литературе.

Ключевые слова: q-интеграл, q-интеграл Джексона, q-разностный оператор, q-производная, q-преобразование Бесселя–Фурье, пространство соболевского типа, уравнение Шредингера, q-оператор Бесселя.

DOI 10.31489/2022M1/109-117

UDC 517.9

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Separability of the third-order differential operator given on the whole plane

In this paper, in the space $L_2(R^2)$, we study a third-order differential operator with continuous coefficients in $R(-\infty,+\infty)$. Here, these coefficients can be unlimited functions at infinity. In addition under some restrictions on the coefficients, the bounded invertibility of the given operator is proved and a coercive estimate is obtained, i.e. separability is proved.

Keywords: resolvent, third-order differential operator, separability.

1 Introduction

Third-order partial differential equations are the basis of mathematical models of many phenomena and processes. Significant literature is devoted to the solvability of boundary value problems for third-order differential equations [1–6] and cited papers there.

Consider the differential operator

$$Lu + \lambda u = \frac{\partial u}{\partial y} + R_2(y)\frac{\partial^3 u}{\partial x^3} + R_1(y)\frac{\partial u}{\partial x} + R_0(y)u + \lambda u \tag{1}$$

initially defined on $C^\infty_{0,\pi}(R^2)$, $\lambda \geq 0$.

$C^\infty_{0,\pi}$ is a set consisting of infinitely differentiable finite functions in R^2.

We assume that the coefficients of operator (1) $R_0(y), R_1(y), R_2(y)$ satisfy the conditions:

i) $R_0(y) \geq \delta_0) > 0, R_1(y) \geq \delta_1 > 0, -R_2(y) \geq \delta_2 > 0$ are continuous functions in $R(-\infty,+\infty)$;

ii) $\mu_0 = \sup\limits_{|y-t|\leq 1} \frac{R_0(y)}{R_0(t)} < \infty, \mu_1 = \sup\limits_{|y-t|\leq 1} \frac{R_1(y)}{R_1(t)} < \infty, \mu_2 = \sup\limits_{|y-t|\leq 1} \frac{R_2(y)}{R_2(t)} < \infty.$

It is easy to verify that the operator $L + \lambda I$ admits closure in $L_2(R^2)$, which we also denote by $L + \lambda I$.

It should be noted that the issue of the existence of a bounded operator $(L + \lambda I)^2$ of a closed operator $L+\lambda I$ in $L_2(R^2)$ is equivalent to the following problem: Find a unique solution of $(L+\lambda I)u = f(x, y) \in L_2(R^2)$ belonging to $L_2(R^2)$, i.e. $u \in L_2(R^2)$. In this case, the closed operator $L + \lambda I$ generates a problem without initial conditions ([7], Chapter III, Section 4).

Recently, there has been an increased interest in differential operators with unbounded coefficients [8–14].

In [15], the linearized Korteweg-de Vries operator was studied, which generates the so-called periodic problem without initial conditions on the strip.

In contrast to [15], we study the separability of the third-order differential operator defined on the whole plane.

Theorem 1. Let the condition i) be fulfilled. Then the operator $L + \lambda I$ is continuously invertible in $L_2(R^2)$ for $\lambda \geq 0$.

Following the papers [8, 9], we introduce the following definition.

Definition 1. We called the operator L is separable in $L_2(R^2)$ if the estimate

$$\left\|\frac{\partial u}{\partial y}\right\|_2 + \left\|R_2(y)\frac{\partial^3 u}{\partial x^3}\right\|_2 + \left\|R_1(y)\frac{\partial u}{\partial x}\right\|_2 + \|R_0(y)u\|_2 \leq C(\|Lu\|_2 + \|u\|_2),$$

holds for $u \in D(L)$, where C is independent of $u(x, y)$, $\|\cdot\|_2$ is the norm of $L_2(R^2)$.

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Theorem 2. Let conditions i) - ii) be fulfilled. Then the operator L is separable.

Example. Let $R_0(y) = |y|^2 + 1$, $R_1(y) = e^{100|y|}$, $R_2(y) = -e^{1000|y|}$, $-\infty < y < \infty$. It is easy to verify that all the conditions of Theorem 2 are satisfied. Consequently, the operator L is separable, i.e.

$$\left\|\frac{\partial u}{\partial y}\right\|_{L_2(R^2)} + \left\|-e^{1000|y|}\frac{\partial^3 u}{\partial x^3}\right\|_{L_2(R^2)} + \left\|e^{100|y|}\frac{\partial u}{\partial x}\right\|_{L_2(R^2)} + \left\|(|y|^2+1)u\right\|_{L_2(R^2)} \le C(\|Lu\|_{L_2(R^2)} + \|u\|_{L_2(R^2)}),$$

where C is a constant.

2 Auxiliary lemmas and inequalities

Lemma 2.1. Let the condition i) be fulfilled and $\lambda \ge 0$. Then the inequality

$$\|(L+\lambda I)u\|_{L_2(R^2)} \ge (\delta_0 + \lambda)\|u\|_{L_2(R^2)}, \tag{2}$$

holds for all $u \in D(L)$, where $\delta_0 > 0$.

The proof follows from the functional $< (L+\lambda I)u, u >$, where $< \cdot, \cdot >$ is the scalar product in $L_2(R^2)$, $u \in D(L)$.

Consider the operator

$$(l_{t,j} + \lambda I)z = z'(y) + (-it^3 R_{2,j}(y) + itR_{1,j}(y) + R_{0,j}(y))z(y), \ (-\infty < t < \infty)$$

where $R_{2,j}(y)$, $R_{1,j}(y)$, $R_{0,j}(y)$ are bounded periodic functions of the same period $\Delta_j = (j-1, j+1)$, $j = 0, \pm 1, \pm 2$, $z(y) \in C_0^\infty(R)$, $-\infty < t < \infty$, $z(y) = u(y) + i\vartheta(y)$.

It is easy to verify that the operator $l_{t,j}$ admits closure in $L_2(R)$, which we also denote by $l_{t,j}$.

Lemma 2.2. Let the condition i) be fulfilled. Then the estimate

$$\|(l_{t,j} + \lambda I)z\|_2 \ge (\delta_0 + \lambda)\|z\|_2 \tag{3}$$

holds for all $z(y) \in D(l_{t,j} + \lambda I)$, $\|\cdot\|_2$ is the norm of $L_2(R)$.

Proof. Lemma 2.2 is proved in the same way as estimate (2) of Lemma 2.1.

Lemma 2.3. Let the condition i) be fulfilled. Then the operator $(l_{t,j} + \lambda I)$ has a continuous inverse operator $(l_{t,j} + \lambda I)^{-1}$ defined on the whole $L_2(R)$.

Proof. By the estimate (3) it suffices to show that the range is dense in $L_2(R)$.

Let us prove it by contradiction. Let us assume that the range is not dense in $L_2(R)$. Then there exists an element $\vartheta \in L_2(R)$ such that $< (l_{t,j} + \lambda I)u, \vartheta >= 0$ for all $u \in D(l_{t,j})$. This follows that

$$(l_{t,j} + \lambda I)^*\vartheta = -\vartheta' + (it^3 R_{2,j}(y) - itR_{1,j}(y) + R_{0,j}(y))\vartheta = 0. \tag{4}$$

in the sense of the theory of generalized functions. Now, using the periodicity of the functions $R_0(y)$, $R_1(y)$, $R_2(y)$, we have that $(it^3 R_{2,j}(y) - itR_{1,j}(y) + R_{0,j}(y))\vartheta \in L_2(R)$. Given this and from (4) it follows that $\vartheta \in W_2^1(R)$, where $W_2^1(R)$ is the Sobolev space. The general theory of the embedding theorems implies that

$$\lim_{|y|\to\infty} \vartheta(y) = 0. \tag{5}$$

Taking into account equality (5) and repeating the arguments used in the proof of the estimate (3), we obtain

$$\|(l_{t,j} + \lambda I)^*\vartheta\|_2 \ge \delta_0 \|\vartheta\|_2. \tag{6}$$

From estimates (4), (6) it follows that $\vartheta = 0$. Lemma 2.4 is proved.

Let $\{\varphi_j\}_{j=-\infty}^{\infty} \in C_0^\infty(R)$ is a set of functions such that $\varphi_j(y) \ge 0$, $\sup p\varphi_j \subseteq \Delta_j (j \in Z)$, $\sum\limits_{j=-\infty}^{\infty} \varphi_j^2(y) = 1$.

Here we note that any point $y \in R$ can belong to no more than three segments from the system of segments $\{supp\varphi_j\}$ [9, 10].

Assume that

$$K_\lambda f = \sum_{j=-\infty}^{\infty} \varphi_j(y)(l_{t,j} + \lambda I)^{-1}\varphi_j f,$$

$$B_\lambda f = \sum_{j=-\infty}^{\infty} \varphi_j'(y)(l_{t,j}+\lambda I)^{-1}\varphi_j f, \quad f \in C_0^\infty(R), \quad \lambda \geq 0.$$

It is easy to verify that

$$(l_t+\lambda I)K_\lambda f = f + \sum_j \varphi_j'(y)(l_{t,j}+\lambda I)^{-1}\varphi_j f, \tag{7}$$

where

$$(l_t+\lambda I)z = -z'(y) + (-it^3R_2(y) + itR_1(y) + R_0(y))z, \ z \in D(l_t).$$

Lemma 2.4. Let the condition i) be fulfilled. Then there exists a number $\lambda_0 > 0$ such that $\|B_\lambda\|_{2\to 2} < 1$ for all $\lambda \geq \lambda_0$.

Proof. Only functions φ_{j-1}, φ_j, φ_{j+1} are nonzero in the interval $\overline{\Delta_j}(j \in Z)$, consequently

$$\|B_\lambda f\|^2_{L_2(R)} = \int_{-\infty}^{\infty} |\sum_{j=-\infty}^{\infty} \varphi_j'(y)(l_{t,j}+\lambda I)^{-1}\varphi_j f|^2 dy \leq \sum_{j=-\infty}^{\infty} \int_{\Delta_j} |\sum_{k=j-1}^{j+1} [\varphi_k'(y)(l_{t,k}+\lambda I)^{-1}\varphi_k f]|^2 dy.$$

From the last inequality and by using the obvious inequality $(a+b+c)^2 \leq 3(a^2+b^2+c^2)$ and estimate (3), we have

$$\|B_\lambda f\|^2_{L_2(R)} \leq \sum_{j=-\infty}^{\infty} \int_{\Delta_j} |\sum_{k=j-1}^{j+1} [\varphi_k'(l_{t,k}+\lambda I)^{-1}\varphi_k f]|^2 dy \leq 9 \sum_{j=-\infty}^{\infty} \left\|\varphi_j'(l_{t,j}+\lambda I)^{-1}\varphi_j f\right\|^2_{L_2(\Delta_j)} \leq$$

$$\leq 9 \sum_{j=-\infty}^{\infty} \left\|\varphi_j'(l_{t,j}+\lambda I)^{-1}\varphi_j f\right\|^2_{L_2(R)} \leq 9 \cdot c \sum_{j=-\infty}^{\infty} \left\|(l_{t,j}+\lambda I)^{-1}\right\|^2_{L_2(R)\to L_2(R)} \cdot \|\varphi_j f\|^2_{L_2(R)} \leq$$

$$\leq \frac{9 \cdot c}{(\delta_0+\lambda)^2} \cdot \int_{-\infty}^{\infty} (\sum_j \varphi_j^2)|f|^2 dy = \frac{9 \cdot c}{(\delta_0+\lambda)^2} \cdot \|f\|^2_{L_2(R)}.$$

Hence

$$\|B_\lambda\|_{L_2(R)\to L_2(R)} \leq \frac{9 \cdot c}{(\delta_0+\lambda)^2}. \tag{8}$$

From (8) it follows that there exists a number $\lambda_0 > 0$, such that $\lambda \geq \lambda_0$, $\|B_\lambda\|_{L_2(R)\to L_2(R)} < 1$. Lemma 2.4 is proved.

Now consider the initial operator

$$(l_t+\lambda I)z = z'(y) + (-it^3R_2(y) + itR_1(y) + R_0(y))z(y),$$

where $z(y) = u(y) + i\vartheta(y)$, $z(y) \in C_0^\infty(R)$, $-\infty < t < \infty$, $(R = (-\infty, \infty))$.

Lemma 2.5. Let the condition i) be fulfilled. Then the estimate

$$\|(l_t+\lambda I)z\|_2 \geq (\delta_0+\lambda)\|z\|_2 \tag{9}$$

holds for all $z \in D(l_t)$.

Proof. The proof follows from the functional $< (l_t+\lambda I)z, z >$, $z \in D(l_t)$.

Lemma 2.6. Let the condition i) be fulfilled. Then there is a number λ_0 such that operator $l_t + \lambda I$ is boundedly invertible for $\lambda \geq \lambda_0$ and the equality

$$(l_t+\lambda I)^{-1} = K_\lambda(I - B_\lambda)^{-1} \tag{10}$$

holds for the inverse operator $(l_t+\lambda I)^{-1}$.

Proof. Using estimates (7), (9) and Lemma 2.4, we obtain the proof of Lemma 2.6.

On the existence of the resolvent. Proof of Theorem 1.

In this subsection, we prove Theorem 1. Firstly, we define the following definition:

Definition 2. The function $u \in L_2(R^2)$ is called a solution of the equation $(L+\lambda I)u = f$ in $L_2(R^2)$ if there exists a sequence $\{u_n\}_{n=1}^{\infty} \subset C_0^{\infty}(R^2)$ such that

$$\|u_n - u\|_2 \to 0, \;\; \|(L+\lambda I)u - f\|_2 \to 0, \;\; as \;\; n \to \infty.$$

Consider the equation

$$(L+\lambda I)u = \frac{\partial u}{\partial y} + R_2(y)\frac{\partial^3 u}{\partial x^3} + R_1(y)\frac{\partial u}{\partial x} + R_0(y)u + \lambda u = f \in C_0^{\infty}(R^2) \tag{11}$$

Applying the Fourier transform to the equation (11) with respect to the variable x, we obtain

$$(l_t+\lambda I)u = -\widetilde{u}'(t,y) + (-it^3R_2(y) + itR_1(y) + R_0(y))\widetilde{u} = \widetilde{f}(t,y), \tag{12}$$

where $\widetilde{u}(t,y)$, $\widetilde{f}(t,y)$ are the Fourier transform of functions $u(x,y)$ and $f(x,y)$ with respect to the variable x. Further, we denote the Fourier transform by $F_{x\to t}$ and the Fourier inverse formula by $F_{t\to x}^{-1}$.

Hence, the problem of solving of the equation (11) turns into the problem of solving of the equation (12). Therefore, according to Lemma 2.6, we have

$$\widetilde{u} = (l_t+\lambda I)\widetilde{f} = K_\lambda(I-B\lambda)^{-1}\widetilde{f}.$$

By using the inverse operator $F_{t\to x}^{-1}$, we find

$$u(x,y) = F_{t\to x}^{-1}\widetilde{u} = F_{t\to x}^{-1}(l_\tau+\lambda I)^{-1}\widetilde{f}. \tag{13}$$

The set $C_0^{\infty}(R^2)$ is dense in $L_2(R^2)$. From here and passing to the limit, through the boundedness and continuity of the Fourier transform, we obtain a proof for any $f(x,y) \in L_2(R^2)$. The uniqueness follows from Lemma 2.1. Theorem 1 is proved.

On the separability of the operator. Proof of theorem 2

To prove separability, first, we give the following lemmas.

Lemma 2.7. Let $z(y) \in D(l_{t,j}+\lambda I)$ and $z(y) = u(y) + i\vartheta(y)$, then $it^3R_2(y)z(y) \in L_2(R)$ if and only if $t^3R_2(y)u(y) \in L_2(R)$ and $t^3R_2(y)\vartheta(y) \in L_2(R)$.

Proof. The proof of Lemma 2.7 is obvious.

Remark. This Lemma is also true for $it^3R_{1,j}(y)z(y)$.

Using this lemma, we consider the operator

$$(l_{t,j}+\lambda I)u = u'(y) + (-it^3R_{2,j}(y) + itR_{1,j}(y) + R_0(y) + \lambda)u(y)$$

on the set of infinitely differentiable, finite and real-valued functions.

Lemma 2.8. Let the condition i) be fulfilled. Then the estimates:

$$\|(l_{t,j}+\lambda I)u(y)\|_2 \geq R_0(y_j)\,\|u\|_2\,,\; n = 0, \pm1, \pm2..., \; where \; R_0(y_j) = \min_{y\in\overline{\Delta_j}} R_{0,j}(y); \tag{14}$$

$$\|(l_{t,j}+\lambda I)u(y)\|_2 \geq |t|R_1(\overline{y_j})\,\|u\|_2\,,\; n = 0, \pm1, \pm2..., \; where \; R_1(\overline{y_j}) = \min_{y\in\overline{\Delta_j}} R_{1,j}(y); \tag{15}$$

$$\|(l_{t,j}+\lambda I)u(y)\|_2 \geq |t|^3R_2(\overline{\overline{y_j}})\,\|u\|_2 \;\; n = 0, \pm1, \pm2..., \; where \; R_2(\overline{\overline{y_j}}) = \min_{y\in\overline{\Delta_j}} |R_{2,j}(y)| \tag{16}$$

hold for any $u \in D(l_{t,j}+\lambda I)$.

Proof. Let $u(y) \in C_0^{\infty}(R)$. It is easy to verify that $\int_{-\infty}^{\infty} u'(y)u(u)dy = 0$ and reproducing the computations used in the proof of Lemma 2.1, we have

$$|<(l_{t,j}+\lambda I)u, u>| = |\int_{-\infty}^{\infty}(-it^3R_{2,j}(y) + itR_{1,j}(y) + R_{0,j}(y) + \lambda)u^2dy|. \tag{17}$$

From (17) we obtain

$$| < (l_{t,j} + \lambda I)u, u > | \geq | \int_{-\infty}^{\infty} (R_{0,j}(y) + \lambda)|u|^2 dy| \geq \min_{y \in \overline{\Delta_j}} R_0(y) \, \|u\|_2^2 . \tag{18}$$

Using the Cauchy-Bunyakovsky inequality, from (18) we obtain

$$\|l_{t,j} + \lambda I)u\|_2 \geq R_0(y_j) \, \|u\|_2 , \tag{19}$$

where $R_0(y_j) = \min_{y \in \overline{\Delta_j}} R_0(y)$.

From (19) we obtain the proof of inequality (14) of Lemma 2.8. Inequalities (15) and (16) can be proved in the same way as inequality (14). Lemma 2.8 is proved.

Lemma 2.9. Let the condition i) be fulfilled and $\lambda \geq \lambda_0$, $\alpha = 0, 1, 2, 3$, $p(y)$ is a continuous function defined on R. Then the estimate

$$\left\|p(y)|t|^{\alpha}(l_t + \lambda I)^{-1}\right\|^2_{L_2(R) \to L_2(R)} \leq c(\lambda) \sup_{j \in Z} \left\|p(y)|t|^{\alpha} \varphi_j (l_{t,j} + \lambda I)^{-1}\right\|^2_{L_2(\Delta_j) \to L_2(\Delta_j)} \tag{20}$$

holds, where $-\infty < t < \infty$

Proof. Let $f \in C_0^{\infty}(R)$. From the representation (10), considering the properties of the functions φ_j $(j \in Z)$ we have

$$\left\|p(y)|t|^{\alpha}(l_t + \lambda I)^{-1} f\right\|^2_{L_2(R)} = \left\|p(y)|t|^{\alpha} K_{\lambda}(I - B_{\lambda})^{-1} f\right\|^2_{L_2(R)} =$$

$$= \int_{-\infty}^{\infty} |p(y)|t|^{\alpha} \sum_{\{j\}} \varphi_j (l_{t,j} + \lambda I)^{-1} \varphi_j (I - B_{\lambda})^{-1} f|^2 dy.$$

From the construction it follows that on the interval$\Delta_j (j \in Z)$, only functions φ_{j-1}, φ_j, φ_{j+1} are nonzero, therefore

$$\left\|p(y)|t|^{\alpha}(l_t + \lambda I)^{-1} f\right\|^2_{L_2(R)} \leq \sum_{j=-\infty}^{\infty} \int_{\Delta_j} |p(y)|t|^{\alpha} \sum_{j-1}^{j+1} \varphi_j (l_{t,j} + \lambda I)^{-1} \varphi_j (I - B_{\lambda})^{-1} f|^2 dy \leq$$

$$\leq 9 \sup_{j \in Z} \left\|p(y)|t|^{\alpha} \varphi_j (l_{t,j} + \lambda I)^{-1}\right\|^2_{L_2(\Delta_j)} \cdot \int_{-\infty}^{\infty} (\sum_j \varphi_j^2) |(I - B_{\lambda})^{-1} f|^2 dy. \tag{21}$$

As is known $(\sum_j \varphi_j^2) = 1$, then from (21) we obtain

$$\left\|p(y)|t|^{\alpha}(l_t + \lambda I)^{-1} f\right\|^2_{L_2(R)} \leq 9 \sup_{j \in Z} \left\|p(y)|t|^{\alpha} \varphi_j (l_{t,j} + \lambda I)^{-1}\right\|^2_{L_2(\Delta_j)} \cdot \int_{-\infty}^{\infty} |(I - B_{\lambda})^{-1} f|^2 dy \leq$$

$$\leq 9 \sup_{j \in Z} \left\|p(y)|t|^{\alpha} \varphi_j (l_{t,j} + \lambda I)^{-1}\right\|^2_{L_2(\Delta_j)} \cdot \left\|(I - B_{\lambda})^{-1}\right\|^2_{2 \to 2} \cdot \|f\|_2^2 . \tag{22}$$

From Lemma 2.4 it follows that $\|I - B_{\lambda}\|^2_{2 \to 2} < c(\lambda)$. From this and (22), we obtain

$$\left\|p(y)|t|^{\alpha}](l_{t,j} + \lambda I)^{-1}\right\|_{L_2(R) \to L_2(R)} \leq 9 \cdot c(\lambda) \sup_{j \in Z} \left\|p(y)|t|^{\alpha} \varphi_j (l_{t,j} + \lambda I)^{-1}\right\|^2_{L_2(\Delta_j)} .$$

Lemma 2.9 is proved.

Lemma 2.10. Let the conditions i)-ii) be fulfilled. Then the estimates

$$\left\|R_0(y)(l_t + \lambda I)^{-1}\right\|_{L_2(R) \to L_2(R)} \leq C_0 < \infty; \tag{23}$$

$$\left\|R_1(y)|t|(l_t + \lambda I)^{-1}\right\|_{L_2(R) \to L_2(R)} \leq C_1 < \infty; \tag{24}$$

$$\left\| R_2(y)|t|^3 (l_t+\lambda I)^{-1} \right\|_{L_2(R)\to L_2(R)} \le C_2 < \infty, \tag{25}$$

hold, where C_0, C_1, C_1 are independent of t $(n = 0, \pm 1, \pm 2 ...)$.

Proof. (10) shows that the operator $R_0(l_t + \lambda I)$ is bounded if $\sup\limits_{j\in Z} \left\| R_0(y)\varphi_j (l_{t,j} + \lambda I)^{-1} \right\|_{L_2(\Delta_j)}$ is bounded. Therefore, we will estimate the last expression

$$\left\| R_0(y)\varphi_j(l_t+\lambda I)^{-1} \right\|^2_{L_2(R)\to L_2(R)} \le C(\lambda) \sup_{j\in Z} \left\| R_0(y)\varphi_j (l_{t,j}+\lambda I)^{-1} \right\|^2_{L_2(\Delta_j)} \le$$

$$\le C(\lambda) \sup_{j\in Z} \max_{y\in \Delta_j} |R_0(y)\varphi_j|^2 \left\| (l_{t,j}+\lambda I)^{-1} \right\|^2_{L_2(\Delta_j)} \le C(\lambda) \sup_{j\in Z} \max_{y\in \Delta_j} R_0^2(y) \left\| (l_{t,j}+\lambda I)^{-1} \right\|^2_{L_2(\Delta_j)}.$$

From the last inequality and taking into account inequality (14) and the condition ii), we obtain

$$\left\| R_0(y)(l_t+\lambda I)^{-1} \right\|^2_{L_2(R)\to L_2(R)} \le C(\lambda) \sup_{|y-t|\le 1} \frac{R_0^2(y)}{R_0^2(t)} < C(\lambda)\cdot \mu_0^2 \le C_0^2 < \infty.$$

This estimate proves inequality (23) of Lemma 2.10.

Now we prove inequality (24). By virtue of estimate (20), we have

$$\left\| R_1(y)|t|(l_t+\lambda I)^{-1} \right\|^2_{L_2(R)\to L_2(R)} \le C(\lambda) \sup_{j\in Z} \left\| R_1(y)|t|\varphi_j (l_{t,j}+\lambda I)^{-1} \right\|^2_{L_2(\Delta_j)\to L_2(\Delta_j)} \le$$

$$\le C(\lambda) \sup_{j\in Z} \max_{y\in \overline{\Delta_j}} R_1^2(y)|t|^2 \left\| (l_{t,j}+\lambda I)^{-1} \right\|^2_{L_2(\Delta_j)\to L_2(\Delta_j)}.$$

From the last inequality, Lemma 2.8 and condition ii), we obtain

$$\left\| R_1(y)|t|(l_t+\lambda I)^{-1} \right\|^2_{L_2(R)\to L_2(R)} \le C(\lambda)\cdot \mu_1^2 \le C_1^2 < \infty.$$

The inequality (24) is proved.

The inequality (25) is proved in the same way as the inequality (24). Lemma 2.10 is proved completely.

Proofs of Theorem 2.

According to Theorem 1 and equality (13), we obtain

$$R_0(y)u(x,y) = R_0(y) F^{-1}_{t\to x} (l_t+\lambda I)^{-1} \widetilde{f}(t,y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} R_0(y)(l_t+\lambda I)^{-1} \widetilde{f}(t,y)\cdot e^{itx} dt =$$

$$= F^{-1}_{t\to x} R_0(y)(l_t+\lambda I)^{-1}\widetilde{f}(t,y).$$

Hence, using the unitarity property of the operator $F^{-1}_{t\to x}$, we find

$$\|R_0(y)u(x,y)\|_2^2 = \int_{-\infty}^{\infty} (\int_{-\infty}^{\infty} R_0(y)(l_t+\lambda I)^{-1} (\widetilde{f}(t,y))^2 dy) dt =$$

$$= \int_{-\infty}^{\infty} \left\| R_0(y)(l_t+\lambda I)^{-1}\widetilde{f}(t,y) \right\|_2^2 dt \le \int_{-\infty}^{\infty} \left\| R_0(y)(l_t+\lambda I)^{-1} \right\|^2_{2\to 2} \cdot \left\| \widetilde{f}(t,y) \right\|^2 dt.$$

From the last inequality and using Parseval's equality in $L_2(R)$, we obtain

$$\|R_0(y)u(x,y)\|_2^2 \le \sup_{t\in R} \left\| R_0(y)(l_t+\lambda I)^{-1} \right\|^2_{2\to 2} \cdot \int_{-\infty}^{\infty} \left\| \widetilde{f}(t,y) \right\|^2 dt \le$$

$$\le \sup_{t\in R} \left\| R_0(y)(l_t+\lambda I)^{-1} \right\|^2_{2\to 2} \cdot \|f(x,y)\|_2^2 \, dt.$$

From the last inequality and estimate (23) it follows that

$$\|R_0(y)u(x,y)\|_2^2 \le C_0^2 \, \|f(x,y)\|_2^2,$$

i.e.
$$\|R_0(y)u(x,y)\|_2 \le C_0 \|(L+\lambda I)u\|_2^2 , \tag{26}$$
where $(L+\lambda I)u = f(x,y)$.

Further, using estimate (24) and repeating the computations and arguments that were used in the proof of (26), we have
$$\left\|R_1(y)\frac{\partial u}{\partial x}\right\|_2 \le C_1 \|(L+\lambda I)u\|_2 . \tag{27}$$
Similarly, we have
$$\left\|R_2(y)\frac{\partial^3 u}{\partial x^3}\right\|_2 \le C_2 \|(L+\lambda I)u\|_2 . \tag{28}$$
Now from inequalities (26)–(28) we have
$$\left\|\frac{\partial u}{\partial y}\right\|_2 = \left\|(L+\lambda I) - R_2(y)\frac{\partial^3 u}{\partial x^3} - R_1(y)\frac{\partial u}{\partial y} - R_0(y)u - \lambda u\right\|_2 \le$$
$$\le \|(L+\lambda I)u\|_2 + C_2 \|(L+\lambda I)u\|_2 + C_1 \|(L+\lambda I)u\|_2 + C_0 \|(L+\lambda I)u\|_2 +$$
$$+\lambda \|(L+\lambda I)u\|_2 \le C(\lambda) \|(L+\lambda I)u\|_2 . \tag{29}$$
From (26)–(29) it follows that
$$\left\|\frac{\partial u}{\partial y}\right\|_2 + \left\|R_2(y)\frac{\partial^3 u}{\partial x^3}\right\|_2 + \left\|R_1(y)\frac{\partial u}{\partial y}\right\|_2 + \|R_0(y)u\|_2 \le C(\lambda)(\|Lu\|_2 + \|u\|_2),$$
where $C > 0$ is constant number independent of $u(x,y)$. Theorem 2 is proved.

References

1 Temam R. Sur un probleme non linearie / R. Temam // Math.Pures. Apple. — 1969. — 48. — №2. — P. 159–172.

2 Лионс Ж.Л. Некоторые методы решения нелинейных краевых задач / Ж.Л. Лионс. — М.: Мир, 1972. — 586 с.

3 Дубинский Ю.А. Об одной абстрактной теореме и ее приложении к краевым задачам для неклассических уравнений / Ю.А. Дубинский // Мат. сб. — 1969. — 79 (121). — №1. — С. 91–117.

4 Юрчук Н.И. О граничных задачах для уравнений, содержащих в главной части оператор вида d^{2m+1}/dt^{2m+1} / Н.И. Юрчук // Дифф. уравн. — 1974. — 10. — № 4. — С. 759–762.

5 Пятков С.Г. Об одном линейном уравнении неклассического типа высокого порядка / С.Г. Пятков // Препринт им. СО АН СССР. — 1981. — 24 с.

6 Bona J.Y. A nonhomogeneous boundary– value problem for the Korteweg-de Vries equation posed on a finite domain / J.Y. Bona, Shu Ming Sun, Bing–Yu Zhang // Comm. Partial Differential Equations. — 2003. — 28. — № 7–8. — P. 1391–1436.

7 Тихонов А.Н. Уравнения математической физики / А.Н. Тихонов, А.А. Самарский. — М.: Наука, 1972. — 735 с.

8 Everitt W.N. Some properties of the domains of certain differential operators / W.N. Everitt, M. Giertz // Proceedings of the London Mathematical Society. — 1971 — 3. — № 23. — P. 301–324.

9 Отелбаев М. Коэрцитивные оценки и теоремы разделимости для эллиптических уравнений в R^n / М. Отелбаев // Тр. МИАН. — 1933. — 161. — С. 195–217.

10 Muratbekov M. On the existence of a resolvent and separability for a class of singular hyperbolic type differential operators on an unbounded domain / M. Muratbekov, M. Otelbaev // Eurasian Mathematical Journal. — 2011. — 7. — No. 1. — P. 50–67.

11 Muratbekov M.B. On existence of the resolvent and discreteness of the spectrum of a class of differential operators of hyperbolic type / M.B. Muratbekov, M.M. Muratbekov, A.M. Abylayeva // Electronic Journal of Theory of Differential Equations. — 2013. — 63. — P. 1–10.

12 Muratbekov M.B. Estimates of spectrum for a class of mixed type operators / M.B. Muratbekov, M.M. Muratbekov // Differential Equations. — 2007. — 43. — P. 143–146.

13 Ospanov K.N. Qualitative and approximate characteristics of solutions of Beltrami type systems / K.N. Ospanov // Complex Variables and Elliptic Equations. — 2015. — 60. — № 7. — P. 1005–1014.

14 Ospanov K.N. Discreteness and estimates of spectrum of a first order difference operator / K.N. Ospanov // Eurasian Mathematical Journal. — 2018. — 9. — No. 2. — P. 89–94.

15 Muratbekov M.M. On the existence of the resolvent and separability of a class of the Korteweg-de Vriese type linear singular operators / M.M. Muratbekov, A.O. Suleimbekova // Bulletin of the Karaganda University. — 2021. — 101. — № 1. — P. 87–97.

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Бүкіл жазықтықта берілген үшінші ретті дифференциалдық оператордың бөліктенуі туралы

Мақалада $L_2(R^2)$ кеңістігінде коэффициенттері $R(-\infty, +\infty)$-да үзіліссіз үшінші ретті дифференциалдық оператор қарастырылған. Мұнда бұл коэффициенттер шексіздікте шектеусіз функциялар болуы мүмкін. Сонымен қатар, коэффициенттерге қатысты жоғарыдағы шарттардан бөлек кей шектеу қою арқылы берілген оператордың шектеулі қайтымдылығы дәлелденген және коэрцитивті бағалау алынған, яғни бөліктену анықталған.

Кілт сөздер: резольвента, үшінші ретті дифференциалдық оператор, бөліктену.

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Разделимость дифференциального оператора третьего порядка, заданного на всей плоскости

В статье в пространстве $L_2(R^2)$ изучен дифференциальный оператор третьего порядка с непрерывными коэффициентами в $R(-\infty, +\infty)$. Здесь данные коэффициенты могут быть неограниченными функциями на бесконечности. Автором при некоторых ограничениях на коэффициенты, помимо указанных выше условий, доказана ограниченная обратимость заданного оператора и получена коэрцитивная оценка, т.е. разделимость.

Ключевые слова: резольвента, дифференциальные уравнения третьего порядка, разделимость, неограниченные функции.

References

1 Temam, R. (1969). Sur un probleme non linearie. *J. Math.Pures. Apple, 48*(2), 159–172 [in French].

2 Lyons, Zh.L. (1972). *Nekotorye metody resheniia nelineinykh kraevykh zadach [Some methods for solving nonlinear boundary value problems]*. Moscow: Mir [in Russian].

3 Dubinsky, Yu.A. (1969). Ob odnoi abstraktnoi teoreme i ee prilozhenii k kraevym zadacham dlia neklassicheskikh uravnenii [On an abstract theorem and its applications to boundary value problems for non-classical equations]. *Matematicheskii sbornik — Mathematical collection, 79*(121), 1, 91–117 [in Russian].

4 Yurchuk, N.I. (1974). O granichnykh zadachakh dlia uravnenii, soderzhashchikh v glavnoi chasti operator vida d^{2m+1}/dt^{2m+1} [On boundary value problems for equations containing operators of the form d^{2m+1}/dt^{2m+1} in the main part]. *Differentsialnye uravneniia — Differential equations, 10*(4), 759–762 [in Russian].

5 Pyatkov, S.G. (1981). Ob odnom lineinom uravnenii neklassicheskogo tipa vysokogo poriadka [On a non-classical-type high-order linear equation]. *Preprint IPM SO AN SSSR — Preprint of Institute of Mathematics, Siberian Branch of the USSR Academy of Sciences, 24* [in Russian].

6 Bona, J.Y., Shu Ming Sun, & Bing–Yu Zhang (2003). A nonhomogeneous boundary-value problem for the Korteweg-de Vries equation posed on a finite domain. *Comm. Partial Differential Equations, 28*(7, 8), 1391–1436.

7 Tikhonov, A.N., & Samarskii, A.A. (1972). *Uravneniia matematicheskoi fiziki [Equations of mathematical physics].* Moscow: Nauka, 735 [in Russian].

8 Everitt, W.N., & Giertz, M. (1971). Some properties of the domains of certain differential operators. *Proceedings of the London Mathematical Society, (3)23*, 301–324

9 Otelbaev, M. (1983). Koertsitivnye otsenki i teoremy razdelimosti dlia ellipticheskikh uravnenii v R^n [Coercive estimates and separability theorems for elliptic equations in R^n]. *Trudy Matematicheskogo instituta imeni V.A. Steklova — Proceedings of the V.A. Steklov Mathematical Institute, 161,* 195–217 [in Russian].

10 Muratbekov, M., & Otelbaev, M. (2016). On the existence of a resolvent and separability for a class of singular hyperbolic type differential operators on an unbounded domain. *Eurasian Mathematical Journal, 7*(1), 50–67.

11 Muratbekov, M.B., Muratbekov, M.M., & Abylayeva, A.M.(2013). On existence of the resolvent and discreteness of the spectrum of a class of differential operators of hyperbolic type. *Electronic Journal of Theory of Differential Equations, 63,* 1–10.

12 Muratbekov, M.B., & Muratbekov, M.M. (2007). Estimates of spectrum for a class of mixed type operators. *Differential Equations, 43*, 143–146.

13 Ospanov, K.N.(2015). Qualitative and approximate characteristics of solutions of Beltrami type systems. *Complex Variables and Elliptic Equations, 60,* 7, 1005–1014.

14 Ospanov, K.N.(2018). Discreteness and estimates of spectrum of a first order difference operator. *Eurasian Mathematical Journal, 9*(2), 89–94.

15 Muratbekov, M.M., & Suleimbekova, A.O. (2021). On the existence of the resolvent and separability of a class of the Korteweg-de Vriese type linear singular operators. *Bulletin of the Karaganda University, 1*(101), 87–97.

DOI 10.31489/2022M1/118-126

UDC 517.925, 519.21

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Construction of the differential equations system of the program motion in Lagrangian variables in the presence of random perturbations

The classification of inverse problems of dynamics in the class of ordinary differential equations is given in the Galiullin's monograph. The problem studied in this paper belongs to the main inverse problem of dynamics, but already in the class of second-order stochastic differential equations of the Ito type. Stochastic equations of the Lagrangian structure are constructed according to the given properties of motion under the assumption that the random perturbing forces belong to the class of processes with independent increments. The problem is solved as follows: First, a second-order Ito differential equation is constructed so that the properties of motion are the integral manifold of the constructed stochastic equation. At this stage, the quasi-inversion method, Erugin's method and Ito's rule of stochastic differentiation of a complex function are used. Then, by applying the constructed Ito equation, an equivalent stochastic equation of the Lagrangian structure is constructed. The necessary and sufficient conditions for the solvability of the problem of constructing the stochastic equation of the Lagrangian structure are illustrated by the example of the problem of constructing the Lagrange function from a motion property of an artificial Earth satellite under the action of gravitational forces and aerodynamic forces.

Keywords: stochastic differential equation, stochastic basic inverse problem, stochastic equation of Lagrangian structure, integral manifold, quasi-inversion method.

Introduction. Problem statement

At present, the theory of inverse problems of dynamics has been developed fully in the class of ordinary differential equations (ODE) [1–9]. This theory originates from the fundamental Erugin's work [10], in which a set of ODEs with a given integral curve is constructed. A generalization of methods for solving inverse problems of dynamics to the class of Ito stochastic differential equations is given in [11–18].

Using the given set

$$\Lambda(t) : \lambda(x, \dot{x}, t) = 0, \lambda \in R^m, \; x \in R^n, \tag{1}$$

it is required to construct stochastic equations of the Lagrangian structure

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_\nu}\right) - \frac{\partial L}{\partial x_\nu} = \sigma'_{\nu j}(x, \dot{x}, t)\dot{\xi}^j, \; \left(\nu = \overline{1, n}, \, j = \overline{1, r}\right), \tag{2}$$

so that the set $\Lambda(t)$ (1) is an integral manifold of equation (2).

Here $\{\xi_1(t, \omega), \ldots, \xi_r(t, \omega)\}$ is a system of random processes with independent increments, which, following [19], can be represented as a sum of Wiener processes and Poisson processes: $\xi = \xi_0 + \int c(y)P^0(t, dy)$. ξ_0 is a Wiener process. P^0 is a Poisson process. $P^0(t, dy)$ is a number of process P^0 jumps in the interval $[0, t]$ that fall on the set dy. $c(y)$ is the vector function mapping space R^{2n} to value space R^r of process $\xi(t)$ for any t.

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The assigned problem was considered in the class of ordinary differential equations in [20]. The stochastic Helmholtz problem (the problems of constructing stochastic equations of the Lagrangian structure equivalent to the given second-order stochastic Ito equation) was considered in [21]. In [22, 23] the above problem was considered under the assumption that the system $\{\xi_1(t,\omega),...,\xi_r(t,\omega)\}$ is a system of independent Wiener processes which are a particular case of processes with independent increments.

The scheme for solving the problem is as follows: First, the second-order Ito differential equation

$$\ddot{x} = f(x,\dot{x},t) + \sigma(x,\dot{x},t)\dot{\xi} \tag{3}$$

is constructed so that the given properties of motion are the integral manifold of the constructed stochastic equation (3). At this stage, the quasi-inversion method [3], Erugin's method [10], and Ito's rule of stochastic differentiation of a complex function in the case of processes with independent increments [19] are used. Then, using the constructed Ito equation, an equivalent stochastic equation of the Lagrangian structure is constructed.

1 Construction of Ito equation by the given properties of motion (1)

Previously, the equation of perturbed motion

$$\dot{\lambda} = \frac{\partial\lambda}{\partial t} + \frac{\partial\lambda}{\partial x}\dot{x} + \frac{\partial\lambda}{\partial\dot{x}}f + S_1 + S_2 + S_3 + \frac{\partial\lambda}{\partial\dot{x}}\sigma\dot{\xi} \tag{4}$$

is compiled according to the Ito stochastic differentiation rule, here $S_1 = \frac{1}{2}\frac{\partial^2\lambda}{\partial\dot{x}^2} : \sigma\sigma^T$, and by $\frac{\partial^2\lambda}{\partial\dot{x}^2} : D$, $D = \sigma\sigma^T$, following [19], we mean a vector whose elements are the traces of the products of the matrices of the second derivatives of the corresponding elements $\lambda_\mu(x,\dot{x},t)$ of the vector $\lambda(x,\dot{x},t)$ with respect to the components $\dot{x}$ on the matrix D $\frac{\partial^2\lambda}{\partial\dot{x}^2} : D = \left[tr\left(\frac{\partial^2\lambda_1}{\partial\dot{x}^2}D\right),\dots,tr\left(\frac{\partial^2\lambda_m}{\partial\dot{x}^2}D\right)\right]^T$; $S_2 = \int\left\{\lambda(x,\dot{x}+\sigma c(y),t) - \lambda(x,\dot{x},t) + \frac{\partial\lambda}{\partial\dot{x}}\sigma c(y)\right\}dy$; $S_3 = \int[\lambda(x,\dot{x}+\sigma c(y),t) - \lambda(x,\dot{x},t)]P^0(t,dy)$. Following Eerugin's method [1], we introduce arbitrary vector-function A and matrix B with the properties $A(0,x,\dot{x},t)\equiv 0$, $B(0,x,\dot{x},t)\equiv 0$ such that

$$\dot{\lambda} = A(\lambda,x,\dot{x},t) + B(\lambda,x,\dot{x},t)\dot{\xi}. \tag{5}$$

Comparing equations (4) and (5), we obtain the relations

$$\begin{cases} \frac{\partial\lambda}{\partial\dot{x}}f = A - \frac{\partial\lambda}{\partial t} - \frac{\partial\lambda}{\partial x}\dot{x} - S_1 - S_2 - S_3, \\ \frac{\partial\lambda}{\partial\dot{x}}\sigma = B. \end{cases} \tag{6}$$

To determine the required functions f and σ from equalities (6) we need the following statement:

Lemma 1 [4, p. 12–13]. The set of all solutions of a linear system

$$Hv = g, H = (h_{\mu k}),\ v = (v_k),\ g = (g_\mu),\ \mu = \overline{1,m},\ k = \overline{1,n},\ m \le n, \tag{7}$$

here H is the matrix of rank m, is determined by the expression

$$\nu = \alpha\nu^T + \nu^v. \tag{8}$$

Here α is scalar,

$$\nu^T = [HC] = [h_1 ... h_m c_{m+1} ... c_{n-1}] = \begin{vmatrix} e_1 & ... & e_n \\ h_{11} & ... & h_{1n} \\ ... & ... & ... \\ h_{m1} & ... & h_{mn} \\ c_{m+1,1} & ... & c_{m+1,n} \\ ... & ... & ... \\ c_{n-1,1} & ... & c_{n-1,n} \end{vmatrix}$$

is the cross product of vectors $h_\mu = (h_{\mu k})$ and arbitrary vectors $c_\rho = (c_{\rho k})$, $\rho = \overline{m+1, n-1}$; e_k are unit vectors of space R^n, $\nu^T = (\nu_k^T)$

$$\nu_k^T = \begin{vmatrix} 0 & \dots & 1 & \dots & 0 \\ h_{11} & \dots & h_{1k} & \dots & h_{1n} \\ \dots & \dots & \dots & \dots & \dots \\ h_{m1} & \dots & h_{mk} & \dots & h_{mn} \\ c_{m+1,1} & \dots & c_{m+1,n} & \dots & c_{m+1,n} \\ \dots & \dots & \dots & \dots & \dots \\ c_{n-1,1} & \dots & c_{n-1,k} & \dots & c_{n-1,n} \end{vmatrix}, \ \nu^v = H^+ g,$$

$H^+ = H^T(HH^T)^{-1}$, H^T is the matrix transposed to H.

By Lemma 1, using (7), (8) we define the vector function f and the matrix σ in the form

$$f = s_1 \left[\frac{\partial\lambda}{\partial\dot{x}} C\right] + \left(\frac{\partial\lambda}{\partial\dot{x}}\right)^+ \left(A - \frac{\partial\lambda}{\partial t} - \frac{\partial\lambda}{\partial x}\dot{x} - S_1 - S_2 - S_3\right) \tag{9}$$

$$\sigma_i = s_{2i} \left[\frac{\partial\lambda}{\partial\dot{x}} C\right] + \left(\frac{\partial\lambda}{\partial\dot{x}}\right)^+ B_i, \tag{10}$$

here $\sigma_i = (\sigma_{1i},\ \sigma_{2i},\ \dots,\ \sigma_{ni})^T$ is the i-th column of the matrix $\sigma = (\sigma_{\nu j})$, $\left(\nu = \overline{1,n},\ j = \overline{1,r}\right)$; $B_i = (B_{1i}, B_{2i}, \dots, B_{mi})^T$ is the i-th column of the matrix $B = (B_{\mu j})$, $\left(\mu = \overline{1,m},\ j = \overline{1,r}\right)$, s_1, s_2 are the arbitrary scalars.

Consequently, it follows from (9), (10) that the set of the second-order Ito differential equations containing a given integral manifold (1) has the form

$$\ddot{x} = s_1 \left[\frac{\partial\lambda}{\partial\dot{x}} C\right] + \left(\frac{\partial\lambda}{\partial\dot{x}}\right)^+ \left(A - \frac{\partial\lambda}{\partial t} - \frac{\partial\lambda}{\partial x}\dot{x} - S_1 - S_2 - S_3\right) +$$

$$+\left(s_{21}\left[\frac{\partial\lambda}{\partial\dot{x}} C\right] + \left(\frac{\partial\lambda}{\partial\dot{x}}\right)^+ B_1, \dots\ s_{2r}\left[\frac{\partial\lambda}{\partial\dot{x}} C\right] + \left(\frac{\partial\lambda}{\partial\dot{x}}\right)^+ B_r\right)\dot{\xi}.$$

2 Construction of the Lagrangian structure equation (2) according to the Ito equation (3)

We expand the expression $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_\nu}\right)$ according to the Ito stochastic differentiation rule in the case of processes with independent increments [19]:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_\nu}\right) = \frac{\partial^2 L}{\partial \dot{x}_\nu \partial t} + \frac{\partial^2 L}{\partial \dot{x}_\nu \partial x_k}\dot{x}_k + \frac{\partial^2 L}{\partial \dot{x}_\nu \partial \dot{x}_k}\ddot{x}_k + \tilde{S}_{1\nu} + \tilde{S}_{2\nu} + \tilde{S}_{3\nu}, \tag{11}$$

here $\tilde{S}_{1\nu} = \frac{1}{2}\frac{\partial^3 L}{\partial \dot{x}_\nu \partial \dot{x}_i \partial \dot{x}_k}\sigma_{ij}\sigma_{kj}$, $\tilde{S}_{2\nu} = \int \left\{\frac{\partial L(x, \dot{x} + \sigma c(y), t)}{\partial \dot{x}_\nu} - \frac{\partial L(x, \dot{x}, t)}{\partial \dot{x}_\nu}\right\} dy$,
$\tilde{S}_{3\nu} = \int [\frac{\partial L(x, \dot{x} + \sigma c(y), t)}{\partial \dot{x}_\nu} - \frac{\partial L(x, \dot{x}, t)}{\partial \dot{x}_\nu}] P^0(t, y)$.

Therefore, equation (2), taking into account (11), can be written in the following form:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_\nu}\right) - \frac{\partial L}{\partial x_\nu} - \sigma'_{\nu j}(x, \dot{x}, t)\dot{\xi}^j = \frac{\partial^2 L}{\partial \dot{x}_\nu \partial t} + \frac{\partial^2 L}{\partial \dot{x}_\nu \partial x_k}\dot{x}_k + \frac{\partial^2 L}{\partial \dot{x}_\nu \partial \dot{x}_k}\ddot{x}_k +$$

$$+\tilde{S}_{1\nu} + \tilde{S}_{2\nu} + \tilde{S}_{3\nu} - \frac{\partial L}{\partial x_\nu} - \sigma'_{\nu j}(x, \dot{x}, t)\dot{\xi}^j. \tag{12}$$

Or, taking into account (12) and equation (3), we have

$$\frac{\partial^2 L}{\partial \dot{x}_\nu \partial t} + \frac{\partial^2 L}{\partial \dot{x}_\nu \partial x_k}\dot{x}_k + \frac{\partial^2 L}{\partial \dot{x}_\nu \partial \dot{x}_k}\ddot{x}_k + \tilde{S}_{1\nu} + \tilde{S}_{2\nu} + \tilde{S}_{3\nu} - \frac{\partial L}{\partial x_\nu} - \sigma'_{\nu j}(x, \dot{x}, t)\dot{\xi}^j \equiv$$

$$\equiv \ddot{x}_\nu - f_\nu(x,\dot{x},t) + \sigma_{\nu j}(x,\dot{x},t)\dot{\xi}^j. \tag{13}$$

Relation (13) implies the equalities

$$\frac{\partial^2 L}{\partial \dot{x}_\nu \partial \dot{x}_k} = \delta_\nu^k;\ \frac{\partial^2 L}{\partial \dot{x}_\nu \partial t} + \frac{\partial^2 L}{\partial \dot{x}_\nu \partial x_k}\dot{x}_k + \tilde{S}_{1\nu} + \tilde{S}_{2\nu} + \tilde{S}_{3\nu} - \frac{\partial L}{\partial x_\nu} = f_\nu,\ \sigma'_{\nu j}(x,\dot{x},t) = \sigma_{\nu j}. \tag{14}$$

Thus, the following theorem is proven.

Theorem 1. To construct the stochastic equation of the Lagrangian structure (2) by the given set (1), so that set (1) is the integral manifold of the constructed equation, it is necessary and sufficient to satisfy conditions (14).

3 An example

Let us consider the stochastic problem of constructing a Lagrange function for a given property of motion by the example of the motion of an artificial Earth satellite under the action of gravitational and aerodynamic forces [24].

Consider the properties of motion in the following form:

$$\Delta(t):\ \lambda = \theta^2 + \alpha_1\dot{\theta}^2 + \alpha_2 = 0,\ \lambda \in R^1. \tag{15}$$

Then the perturbed motion equation (4) takes the form

$$\dot{\lambda} = 2\theta\dot{\theta} + 2\alpha_1\dot{\theta}\ddot{\theta} + S_1 + S_2 + S_3 = 2\theta\dot{\theta} + 2\alpha_1\dot{\theta}f + S_1 + S_2 + S_3 + 2\alpha_1\dot{\theta}\sigma\dot{\xi}, \tag{16}$$

here $S_1 = \alpha_1\sigma^2$, $S_2 = \int\left\{2\alpha_1\sigma c(y)[4\dot{\theta} + \sigma c(y)]\right\}dy$, $S_3 = \int\left\{2\alpha_1\sigma c(y)[4\dot{\theta} + \sigma c(y)]\right\}P^0(t,dy)$.

Let us introduce Erugin's functions $a = a(\lambda,\ \theta,\ \dot{\theta},\ t)$, $b = b(\lambda,\theta,\dot{\theta},t)$ with the property $a(0,\theta,\dot{\theta},t) = b(0,\theta,\dot{\theta},t) \equiv 0$ and such that the relation

$$\dot{\lambda} = a\lambda(\theta,\dot{\theta},t) + b\lambda(\theta,\dot{\theta},t)\dot{\xi} \tag{17}$$

takes place. From relations (16), (17) it follows that the set of equations (3), in our example is having the form

$$\ddot{\theta} = f(\theta,\dot{\theta},t) + \sigma(\theta,\dot{\theta},t)\dot{\xi},$$

possesses the integral manifold (15) if f and σ have, respectively, the forms

$$f = \frac{a(\theta^2 + \alpha_1\dot{\theta}^2 + \alpha_2) - 2\theta\dot{\theta} - S_1 - S_2 - S_3}{2\alpha_1\dot{\theta}},\quad \sigma = \frac{b(\theta^2 + \alpha_1\dot{\theta}^2 + \alpha_2)}{2\alpha_1\dot{\theta}}. \tag{18}$$

The equation of motion of an artificial Earth satellite under the action of gravitational and aerodynamic forces, following [24], can be written in the form

$$\ddot{\theta} = \tilde{f}(\theta,\dot{\theta}) + \tilde{\sigma}(\theta,\dot{\theta})\dot{\xi}, \tag{19}$$

here θ is a pitch angle, functions $\tilde{f}$, $\tilde{\sigma}$ have the forms

$$\tilde{f} = Ql\sin 2\theta - Q[g(\theta) + \eta\dot{\theta}],\ \tilde{\sigma} = Q\delta[g(\theta) + \eta\dot{\theta}]. \tag{20}$$

Let us construct the Lagrangian using equation (19). In equation (19), we take into account relations (18). These relations give the integrality of the given set (15). It follows from the equalities $f = \tilde{f}$, $\sigma = \tilde{\sigma}$ that four parameters Q,δ,η,l, determining the dynamics of the satellite motion (20), must satisfy the following relations

$$\begin{cases} a(\theta^2 + \alpha_1\dot{\theta}^2 + \alpha_2) - 2\theta\dot{\theta} - S_1 - S_2 - S_3 = 2\alpha_1\dot{\theta}\left\{Ql\sin 2\theta - Q[g(\theta) + \eta\dot{\theta}]\right\}, \\ b(\theta^2 + \alpha_1\dot{\theta}^2 + \alpha_2) = 2\alpha_1\dot{\theta}Q\delta[g(\theta) + \eta\dot{\theta}]. \end{cases}$$

Then, by definition from [25], (19) admits an indirect analytical representation in terms of the stochastic Lagrangian equation if there exists a function h such that the identity

$$d\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} - \sigma'(\theta,\dot{\theta},t)\dot{\xi} \equiv h[\ddot{\theta} - f - \sigma\dot{\xi}]$$

takes place.

Let us find the function $h = h(t)$ so that the necessary and sufficient Helmholtz conditions [25, p.107] for the existence of the Lagrangian are satisfied for the scalar equation $l_1(\theta,\dot\theta,t)\ddot\theta + l_2(\theta,\dot\theta,t) = 0$:

$$\frac{\partial l_2}{\partial\dot\theta} = \frac{\partial l_1}{\partial t} + \dot\theta\frac{\partial l_1}{\partial\theta}.$$

In particular, a function $h = e^{-Q\eta t}$ satisfies this condition. Substituting h in (19), we obtain

$$e^{-Q\eta t}[\ddot\theta - f - \sigma\dot\xi] = \frac{\partial^2 L}{\partial\dot\theta^2}\ddot\theta + \frac{\partial^2 L}{\partial\dot\theta\partial\theta}\dot\theta + \frac{\partial^2 L}{\partial\theta\partial t} - \frac{\partial L}{\partial t}\sigma'\dot\xi.$$

Thus, the required Lagrangian is constructed in the form

$$L = e^{-Q\eta t}[\frac{1}{2}\dot\theta^2 - Q(\frac{1}{2}l\cos 2\theta + G)], \qquad \text{here} \quad G = \int g(\theta)d\theta,$$

which provides a representation of the equation (19) in the form of the Lagrangian structure equation:

$$\frac{d}{dt}\frac{\partial L}{\partial\dot\theta} - \frac{\partial L}{\partial\theta} = e^{-Q\eta t}\sigma(\theta,\dot\theta)\dot\xi.$$

Acknowledgements

This research has been funded by the Science Committee of the Ministry of Education and Science of the Republic of Kazakhstan (Grant No. AP 09258966).

References

1 Еругин Н.П. Построение всего множества систем дифференциальных уравнений, имеющих заданную интегральную кривую / Н.П. Еругин // Прикладная математика и механика. — 1952. — Т. 10. — Вып. 6. — С. 659–670.

2 Галиуллин А.С. Методы решения обратных задач динамики / А.С. Галиуллин. — М.: РУДН, 1986. — 224 с.

3 Галиуллин А.С. Избранные труды: [В 2-х т.]. Т.II / А.С. Галиуллин. — М.: РУДН, 2009. — 462 с.

4 Мухаметзянов И.А. Уравнения программных движений / И.А. Мухаметзянов, Р.Г. Мухарлямов. — М.: РУДН, 1986. — 88 с.

5 Mukharlyamov R.G. Differential-algebraic equations of programmed motions of Lagrangian dynamical systems / R.G. Mukharlyamov // Mechanics of Solids. — 2011. — 46. — No. 4. — P. 534–543.

6 Mukharlyamov R.G. Control of system dynamics and constrains stabilization / R.G. Mukharlyamov, M.I. Tleubergenov // Communications in Computer and Information Science. — 2017. — No. 700. — P. 431–442. DOI:10.1007/978-3-319-66836-936.

7 Llibre J. Inverse Problems in Ordinary Differential Equations and Applications / J. Llibre, R. Ramirez. — Switzerland: Springer International Publishing, 2016. — 266 p.

8 Zhumatov S.S. Asymptotic Stability of Implicit Differential Systems in the Vicinity of Program Manifold / S.S. Zhumatov // Ukrainian Mathematical Journal. — 2014. — 66. — No. 4. — P. 625–632. DOI: 10.1007/s11253-014-0959-y.

9 Zhumatov S.S. Exponential Stability of a Program Manifold of Indirect Control Systems / S.S. Zhumatov // Ukrainian Mathematical Journal. — 2010. — 62. — No. 6. — P. 907–915. DOI: 10.1007/s11253-010-0399-2.

10 Zhumatov S.S. Stability of a Program Manifold of Control Systems with Locally Quadratic Relations / S.S. Zhumatov // Ukrainian Mathematical Journal. — 2009. — 61. — No. 3. — P. 500–509. DOI: 10.1007/s11253-009-0224-y.

11 Ibraeva G.T. Main Inverse Problem for Differential Systems With Degenerate Diffusion / G.T. Ibraeva, M.I. Tleubergenov // Ukrainian Mathematical Journal. — 2013. — 65. — No. 5. — P. 787–792.

12 Tleubergenov M.I. On the inverse stochastic reconstruction problem / M.I. Tleubergenov // Differential Equations. — 2014. — 50. — Issue 2. — P. 274–278. DOI:10.1134/S0012266114020165.

13 Tleubergenov M.I. Stochastic Inverse Problem with Indirect Control / M.I. Tleubergenov, G.T. Ibraeva // Differential Equations. — 2017. — 53. — Issue 10. — P. 1387–1391. DOI:10.1134/S0012266117100172.

14 Vassilina G.K. Solution of the Problem of Stochastic Stability of an Integral Manifold by the Second Lyapunov Method / G.K. Vassilina, M.I. Tleubergenov // Ukrainian Mathematical Journal. — 2016. — 68. — No. 1. — P. 14–28. DOI:10.1007/s11253-016-1205-6.

15 Tleubergenov M.I. On the Solvability of the Main Inverse Problem for Stochastic Differential Systems / M.I. Tleubergenov, G.T. Ibraeva // Ukrainian Mathematical Journal. — 2019. — 71. — No. 1. — P. 157–165. DOI:10.1007/s11253-019-01631-w.

16 Tleubergenov M.I. On Inverse Problem of Closure of Differential Systems with Degenerate Diffusion / M.I. Tleubergenov, G.T. Ibraeva // Eurasian Mathematical Journal. — 2019. — 10. — No. 2. — P. 93–102.

17 Tleubergenov M.I. An Inverse Problem for Stochastic Differential Systems / M.I. Tleubergenov // Differential equations. — 2001. — 37. — Issue 5. — P. 751–753. https://doi.org/10.1023/A:1019285119532.

18 Tleubergenov M.I. On Inverse Problem of Closure of Differential Systems with Degenerate Diffusion / M.I. Tleubergenov, G.K. Vassilina, A.T. Sarypbek // News of the NAS RK. Physico-mathematical series. — 2021. — 1. — No. 335. — P. 6–13.

19 Пугачев В.С. Стохастические дифференциальные системы. Анализ и фильтрация / В.С. Пугачев, И.Н. Синицын. — М.: Наука, 1990. — 632 с.

20 Туладхар Б.М. Построение уравнений в форме Лагранжа, Гамильтона и Биркгофа по заданным свойствам движения: автореф. ... канд. физ.-мат. наук / Б.М. Туладхар. — М.: РУДН, 1983. — 11 с.

21 Тлеубергенов М.И. Обратные задачи стохастических дифференциальных систем: автореф. ... канд. физ.-мат. наук / М.И. Тлеубергенов. — Алматы: Ин-т математики, 1999. — 33 с.

22 Тлеубергенов М.И. О построении дифференциального уравнения по заданным свойствам движения при наличии случайных возмущений / М.И. Тлеубергенов, Д.Т. Ажымбаев // Изв. НАН РК. Сер. физ.-мат. — 2007. — № 3. — С. 15–20.

23 Tleubergenov M.I. On the construction of a set of stochastic differential equations on the basis of a given integral manifold independent of velocities / M.I. Tleubergenov, D.T. Azhymbaev // Ukrainian Mathematical Journal. — 2010. — 62. — No. 7. — P. 1163–1173. DOI:10.1007/s11253-010-0421-8

24 Сагиров П. Стохастические методы в динамике спутников / П. Сагиров // Механика. Период. сб. перев. ин. ст. — 1974. — № 5(147). — С. 28–47. — № 6(148). — С. 3–38.

25 Santilli R.M. Foundations of Theoretical Mechanics. 1. The Inverse Problem in Newtonian Mechanics / R.M. Santilli. — New-York: Springer-Verlag, 1978. — 266 p.

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Кездейсоқ түрткілер болғанда Лагранждың айнымалыларындағы бағдарламалық қозғалыстың дифференциалдық теңдеулер жүйесін құру

Мақалада Ито типті екінші ретті стохастикалық дифференциалдық теңдеулер класындағы динамиканың негізгі (А.С. Галиуллин жіктемесі бойынша) кері есептерінің нұсқаларының бірі қарастырылған. Ито типіндегі стохастикалық дифференциалдық теңдеулер класында Лагранж теңдеулері берілген қозғалыс қасиеттеріне сәйкес құрылады. Бұл жағдайда күштің кездейсоқ түрткілері тәуелсіз өсімшелі үдерістер класынан деп болжалынады. Алдымен есепті шешу үшін квазиқайтару әдісі бойынша қозғалыстың берілген қасиеттеріне сәйкес Еругин әдісімен және тәуелсіз өсімшелі үдерістер жағдайында күрделі функцияның стохастикалық дифференциалдау формуласымен екінші ретті Ито дифференциалдық теңдеуі берілген қозғалыс қасиеттері салынған стохастикалық теңдеудің интегралдық көпбейнесі болатындай етіп құрылады. Екінші кезеңде, алынған Ито теңдеуіне сәйкес, оған эквивалентті Лагранж құрылымының стохастикалық теңдеулері құрылады. Осылайша, тәуелсіз өсімшелі үдерістер класында кездейсоқ түрткілер болған кезде берілген қозғалыс қасиеттерінен Лагранж құрылымының теңдеуін құру мәселесінің шешімділігі үшін қажетті және жеткілікті шарттары алынған. Алынған нәтижелер Жердің жасанды серігінің тартылыс күштері мен аэродинамикалық күштердің әсерінен берілген қозғалыс қасиетіне сәйкес Лагранж функциясын құрудың стохастикалық есебінің мысалында көрсетілген.

Кілт сөздер: Ито типті стохастикалық дифференциалдық теңдеуі, стохастикалық негізгі кері есебі, Лагранж құрылымының стохастикалық теңдеуі, интегралдық көпбейне, квазиқайтару әдісі.

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Построение системы дифференциальных уравнений программного движения в лагранжевых переменных при наличии случайных возмущений

В статье рассмотрен один из вариантов основной (по классификации А.С. Галиуллина) обратной задачи динамики в классе стохастических дифференциальных уравнений второго порядка типа Ито. Построены уравнения Лагранжа по заданным свойствам движения в классе стохастических дифференциальных уравнений типа Ито. При этом случайные возмущающие силы предполагаются из класса процессов с независимыми приращениями. Для решения поставленной задачи на первом этапе по заданным свойствам движения методом квазиобращения в сочетании с методом Еругина и в силу стохастического дифференцирования сложной функции в случае процессов с независимыми приращениями построено дифференциальное уравнение Ито второго порядка так, чтобы заданные свойства движения являлись интегральным многообразием построенного стохастического уравнения. И, далее, на втором этапе по построенному уравнению Ито строятся эквивалентные ему стохастические уравнения лагранжевой структуры. Таким образом, получены необходимые и достаточные условия разрешимости задачи построения уравнения лагранжевой структуры по заданным свойствам движения при наличии случайных возмущений из класса процессов с независимыми приращениями. Полученные результаты проиллюстрированы на примере стохастической задачи построения функции Лагранжа по заданному свойству движения искусственного спутника Земли под действием сил тяготения и аэродинамических сил.

Ключевые слова: стохастическое дифференциальное уравнение Ито, стохастическая основная обратная задача, стохастическое уравнение лагранжевой структуры, интегральное многообразие, метод квазиобращения.

References

1 Erugin, N.P. (1952). Postroenie vsego mnozhestva sistem differentsialnykh uravnenii, imeiushchikh zadannuiu integralnuiu krivuiu [Construction of the entire set of systems of differential equations with a given integral curve]. *Prikladnaia matematika i mekhanika – Applied Mathematics and Mechanics, 10*(6), 659–670 [in Russian].

2 Galiullin, A.S. (1986). *Metody resheniia obratnykh zadach dinamiki [Methods for solving inverse problems of dynamics].* Moscow: RUDN [in Russian].

3 Galiullin, A.S. (2009). *Izbrannye trudy [Selected works in two volumes].* Vols. 1, 2. Vol. 2. Moscow: RUDN [in Russian].

4 Mukhametzyanov, I.A., & Mukharlyamov, R.G. (1986). *Uravneniia programmnykh dvizhenii [Equations of program motions].* Moscow: RUDN [in Russian].

5 Mukharlyamov, R.G. (2011). Differential-algebraic equations of programmed motions of Lagrangian dynamical systems. *Mechanics of Solids, 46*(4), 534–543.

6 Mukharlyamov, R.G., & Tleubergenov, M.I. (2017). Control of system dynamics and constrains stabilization. *Communications in Computer and Information Science, 700,* 431–442. DOI:10.1007/978-3-319-66836-936.

7 Llibre, J. (2016). *Inverse Problems in Ordinary Differential Equations and Applications.* Switzerland: Springer International Publishing.

8 Zhumatov, S.S. (2014). Asymptotic Stability of Implicit Differential Systems in the Vicinity of Program Manifold. *Ukrainian Mathematical Journal, 66*(4), 625–632. DOI: 10.1007/s11253-014-0959-y.

9 Zhumatov, S.S. (2010). Exponential Stability of a Program Manifold of Indirect Control Systems. *Ukrainian Mathematical Journal, 62*(6), 907–915. DOI: 10.1007/s11253-010-0399-2.

10 Zhumatov, S.S. (2009). Stability of a Program Manifold of Control Systems with Locally Quadratic Relations. *Ukrainian Mathematical Journal, 61*(3), 500–509. DOI: 10.1007/s11253-009-0224-y.

11 Ibraeva, G.T., Tleubergenov, M.I. (2013). Main Inverse Problem for Differential Systems With Degenerate Diffusion. *Ukrainian Mathematical Journal, 65*(5), 787–792.

12 Tleubergenov, M.I. (2014). On the inverse stochastic reconstruction problem. *Differential Equations, 50*(2), 274–278. DOI:10.1134/S0012266114020165.

13 Tleubergenov, M.I., & Ibraeva, G.T. (2017). Stochastic Inverse Problem with Indirect Control. *Differential Equations, 53*(10), 1387–1391. DOI:10.1134/S0012266117100172.

14 Vassilina, G.K., & Tleubergenov, M.I. (2016). Solution of the Problem of Stochastic Stability of an Integral Manifold by the Second Lyapunov Method. *Ukrainian Mathematical Journal, 68*(1), 14–28. DOI:10.1007/s11253-016-1205-6.

15 Tleubergenov, M.I., & Ibraeva, G.T. (2019). On the Solvability of the Main Inverse Problem for Stochastic Differential Systems. *Ukrainian Mathematical Journal, 71*(1), 157–165. DOI:10.1007/s11253-019-01631-w.

16 Tleubergenov, M.I., & Ibraeva, G.T. (2019). On Inverse Problem of Closure of Differential Systems with Degenerate Diffusion. *Eurasian Mathematical Journal, 10*(2), 93–102.

17 Tleubergenov, M.I. (2001). An Inverse Problem for Stochastic Differential Systems. *Differential Equations, 37*(5), 751–753. https://doi.org/10.1023/A:1019285119532.

18 Tleubergenov, M.I., Vassilina, G.K., & Sarypbek, A.T. (2021). On Inverse Problem of Closure of Differential Systems with Degenerate Diffusion. *News of the NAS RK. Physico-mathematical series, 1*(335), 6–13.

19 Pugachev, V.S., & Sinitsyn, I.N. (1990). *Stokhasticheskie differentsialnye sistemy. Analiz i filtratsiia [Stochastic differential systems. Analysis and filtering].* Moscow: Nauka [in Russian].

20 Tuladkhar, B.M. (1983). Postroenie uravnenii v forme Lagranzha, Gamiltona i Birkgofa po zadannym svoistvam dvizheniia [Construction of Equations of Lagrange, Hamilton, and Birkhoff Types on the Basis of Given Properties of Motion]. *Extended abstract of candidate's thesis.* Moscow: RUDN [in Russian].

21 Tleubergenov, M.I. (1999). Obratnye zadachi stokhasticheskikh differentsialnykh sistem [Inverse problems of stochastic differential systems]. *Extended abstract of candidate's thesis.* Almaty [in Russian].

22 Tleubergenov, M.I., & Azhymbaev, D.T. (2007). O postroenii differentsialnogo uravneniia po zadannym svoistvam dvizheniia pri nalichii sluchainykh vozmushchenii [On construction of a differential equation on the basis of given properties of motion under random perturbations]. *Izvestiia NAN RK. Seriia fiziko-matematicheskaia — News of the NAS RK. Physico-Mathematical Series, 3,* 15–20 [in Russian].

23 Tleubergenov, M.I., & Azhymbaev D.T. (2010). On the construction of a set of stochastic differential equations on the basis of a given integral manifold independent of velocities. *Ukrainian Mathematical Journal, 62*(7), 1163–1173. DOI:10.1007/s11253-010-0421-8

24 Sagirov, P. (1974). Stokhasticheskie metody v dinamike sputnikov [Stochastic methods in dynamics of satellites]. *Mekhanika. Periodicheskii sbornik perevodov inostrannykh statei – Mechanics. Periodic collection of foreign translated articles, 5* (147), 28–47, *6* (148), 3–38 [in Russian].

25 Santilli, R.M. (1978). *Foundations of Theoretical Mechanics. 1. The Inverse Problem in Newtonian Mechanics.* New-York: Springer-Verlag.

DOI 10.31489/2022M1/127-135

UDC 510.67

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Connection between the amalgam and joint embedding properties

The paper aims to study the model-theoretic properties of differentially closed fields of zero and positive characteristics in framework of study of Jonsson theories. The main attention is paid to the amalgam and joint embedding properties of DCF theory as specific features of Jonsson theories, namely, the implication of JEP from AP. The necessity is identified and justified by importance of information about the mentioned properties for certain theories to obtain their detailed model-theoretic description. At the same time, the current apparatus for studying incomplete theories (Jonsson theories are generally incomplete) is not sufficiently developed. The following results have been obtained: The subclasses of Jonsson theories are determined from the point of view of joint embedding and amalgam properties. Within the exploration of one of these classes, namely the AP-theories, that the theories of differential and differentially closed fields of characteristic 0, differentially perfect and differentially closed fields of fixed positive characteristic are shown to be Jonsson and perfect. Along with this, the theory of differential fields of positive characteristic is considered as an example of an AP-theory that is not Jonsson, but has the model companion, which is perfect Jonsson theory, and the sufficient condition for the theory of differential fields is formulated in the context of being Jonsson.

Keywords: Jonsson theory, perfect Jonsson theory, differential field, differential closed field, differentially perfect field, amalgam property, joint embedding property, AP-theory, JEP-theory, strongly convex theory.

In Model Theory, when studying various examples of theories, information about the amalgam and joint embedding properties for considered theories is useful. The amalgam property and the joint embedding property are independent of each other. There are many examples of this fact. In particular, one can find some of them in [1; 270].

In this article, we examine the case when these two cases are dependent on each other. We call a theory AP-theory if the joint embedding property for this theory is a consequence of the amalgam property of this theory, i.e. when JEP follows from AP. At the same time, the amalgam property and the joint embedding property are necessary attributes of a class of Jonsson theories.

We consider a classic example of differentially closed fields of zero and positive characteristic within the study of AP-Jonsson theories.

As for differential algebra, the first works where differential algebra was separated into an independent branch of mathematics are the books of Ritt [2–3]. There are formulated many significant problems, many of them have not yet been solved. At the Moscow Congress of Mathematicians in 1966, Kolchin presented a report where the author formulated open problems that have determined the direction of differential algebra in recent years. The monograph [4] details the state of most of these sections. As Kaplansky wrote in his monograph [5], “differential algebra consists mostly of the works of Kolchin and Ritt”.

We begin by presenting the basic facts about differential rings, whose special case is namely differential fields, that will help us to reveal the algebraic essence of the theories and classes of their models considered in this article.

The differentiation of the ring R is a map

$$D : R \to R, \tag{1}$$

that satisfies the following conditions:

1) the mapping D is additive;

2) for any two elements x, y of the ring R, $D(xy) = xDy + yDx$ is executed.

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The element $D(x)$ will be called the derivative of the element a, whereas x itself is called the integral element $D(x)$. For derivatives $D^2(x), D^3(x), ..., D^n(x)$, the Leibniz rule can be written as

$$D^n(xy) = D^n(x)y + ... + C_n^i D^{n-i}(x)D^i(y) + ... + xD^n(y).$$

If the commutation property is observed for the element x and the derivative $D(x)$, we have $D^n(x) = nx^{n-1}D(x)$. In the case when the ring R has the unit and an inverse element x^{-1} for x,

$$D(x^{-1}) = -x^{-1}D(x)x^{-1}$$

holds. Moreover, $D(1) = 0$.

The following theorem is known:

Theorem 1 [5; 7]. For any differentiation in an arbitrary domain of integrity, there is a single extension to the corresponding field of relations.

Let a commutative ring with the unit be given and the differentiation D be introduced on it. Such a ring is called a differential ring. Here are some examples that reflect the essence of the differential ring.

1) Any commutative ring with the unit can be represented as differential by considering zero differentiation on it ($\forall x \in R \;\; D(x) = 0$). We can conclude by this that the rings theory is a special case of the differential rings theory. It is worth mentioning that on the ring of integers and the field of rational numbers, it is impossible to introduce any differentiation other than zero.

2) The usual differentiation on the ring of infinitely differentiable functions on the real axis is also an example of the map (1). Moreover, infinitely differentiable functions form a ring closed with respect to differentiation.

3) On the ring of integer functions, it is also possible to introduce differentiation in the usual sense. There are no zero divisors available in the ring of infinitely differentiable functions, which makes it possible to form a field of relations.

4) If R is a differential ring, then there exists a ring of $R[x]$ polynomials formed with coefficients of R in variable x. If R is a field, then $R(x)$ denotes the field of rational functions of x. Using Theorem 1, we can continue differentiating the ring (field) A into the ring of polynomials $R[x]$ and the field $R(x)$. At the same time, we assume $D(x^n) = nx^{n-1}D(x)$ and then continue this mapping linearly.

5) If R is a differential ring, then in under $R[x_i]$ we mean the ring of polynomials in infinite number of variables $x_0, x_1, ...$, and each subsequent element x_{i+1} is a derivative of the previous x_i. Thus, some differentiation in the ring $R\{x_i\}$ is uniquely determined. Let us replace the designations with more suitable ones

$$x_0 = x, \;\; x_n = D^n(x).$$

The described process is called the adjunction of a differential indeterminate and gives us, as a result, a differential ring, the elements of which we call differential polynomials. These are ordinary polynomials from x and its derivatives.

In the case when R is a field, then the ring $R\{x\}$ is a differential domain of integrity, and Theorem 1 gives us the opportunity in the only way to continue differentiating into the corresponding field of relations $R\langle x\rangle$, whose elements are called differential rational functions of x.

In any differential ring R, the elements whose derivative is zero form a subring C called the ring of constants. Moreover, if R is a field, then C, respectively, is also a field. In addition, the constant field C contains within itself a subfield generated by the unit element R.

The characteristic of differential rings is of considerable importance. As the structure of the ring becomes more complex and gradually turns into a field, the characteristic plays an increasingly significant role. Differential fields with zero characteristic are well-studied, while the case with positive characteristic remains more sophisticated. One of them is described below.

Next, we consider the fields of the characteristic $p = 0$ and $p > 0$. We present important information about differential fields of characteristic 0 and consider some of their model-theoretic properties.

D. Marker [6] described differential and differentially closed fields as follows.

Definition 1 [6]. A differential field is a field K with the given differentiation operator $D : K \to K$, such that

$$\forall x \forall y D(x + y) = D(x) + D(y), \tag{2}$$

$$\forall x \forall y D(x, y) = xDy + yDx,$$

where $x, y \in K$. The language used to study differential fields is the language $L = \{+, -, \cdot, D, 0, 1\}$. Here the differentiation operator D plays the role of a single functional symbol.

Thus, the theory DF_0 of differential fields of characteristic 0 is given by the axioms of field theory and axioms (2).

As mentioned before, each differential field has a so-called subfield of constants C consisting of all elements x of the field for which $D(x) = 0$.

Let K be a differential field. Then, over K, the ring $K\{X_1, ..., X_n\}$ of differential polynomials can be defined as the ring of polynomials in infinite number of variables as follows:

$$K[X_1, ..., X_n, D(X_1), ..., D(X_n), ..., D^m(X_1), ..., D^m(X_n), ...],$$

where $D(D^n(X_i)) = D^{n+1}(X_i)$.

If $f \int\{X_1, ..., X_n\}\backslash K$, the order of f is the largest m such that $D^m(X^i)$ occurs in f for some i. If f is a constant, we say that f has order -1.

Definition 2 [6]. A differential field K is called differentially closed if whenever $f, g \in K\{X\}$, g has a nonzero value and the order of f is greater than the order of g, there exists $a \in K$ such that $f(a) = 0$ and $g(a) \neq 0$.

In 1959, A.Robinson [7] showed that the theory of differential fields has a model completion. Robinson also introduced the concept of a differentially closed field. However, as noted by Mikhalev A.V. and Pankratiev E.V. in their review [8], in Robinson's works the theory DCF_0 received a specific description in the sense of axiomatization, which was corrected by L. Blum. In her PhD thesis [9], she formulates two missing axioms (in addition to the DF_0 axioms) for the theory of differentially closed fields of characteristic zero as follows:
1) Each nonconstant polynomial in one variable has a solution.
2) If $f(x)$ and $g(x)$ are differential equations, such that the order of $f(x)$ is higher in the order of $g(x)$, then $f(x)$ has a solution, not a solution of $g(x)$. B. Poizat, in his work [10], proved that DCF_0 is complete and is the model completion of the DF_0.

To study the model-theoretic properties of the theories DF_0 and DCF_0, we need the following definitions:

Definition 3 [11; 99]. The theory of T has the joint embedding property(JEP), if for any models U, B of the theory T there exists a model M of the theory T and isomorphic embeddings $f : U \to M$, $g : B \to M$.

Definition 4 [11; 99]. The theory of T has the amalgam property (AP), if for any models U, B_1, B_2 of the theory T and isomorphic embeddings $f_1 : U \to B_1$, $f_2 : U \to B_2$ there are $M \models T$ and isomorphic embeddings $g_z : B_1 \to M$, $B_2 \to M$, such that $g_1 \circ f_1 = g_2 \circ f_2$.

We will consider the theories of DF_0 and DCF_0 from the point of view of Jonssonness.

To begin with, let us recall the definitions of the Jonsson theory and some related concepts.

Definition 5 [11; 144]. The theory of T is called a Jonsson theory if:
1. The theory T has infinite models;
2. T is an inductive theory;
3. The theory T has the amalgam property (AP).
4. The theory T has the joint embedding property (JEP).

Examples of Jonsson theories are:
1) group theory;
2) abelian groups theory;
3) boolean algebras theory;
4) linear order theory;
5) the theory of fields of characteristic p, where p is zero or a prime number;
6) ordered fields theory;
7) modules theory.

One can found the proofs in [12-13].

Definition 6. [14] It is said that C_T is a semantic model of the Jonsson theory of T if C_T is a ω^+-homogeneous ω^+-universal model of the theory T.

Theorem 2 [11; 152]. The theory T is Jonsson if and only if it has the semantic model C_T. Many facts concerning semantic models and related concepts of cosemanticity and similarity of the Jonsson theories are described in [15].

Definition 7 [16]. A Jonsson theory T is called perfect if its semantic model C_T is saturated.

Definition 8 [16]. An elementary theory of the semantic model of Jonsson theory T is called to be the center of this theory. Denoted through T^*, i.e. $Th(C) = T^*$.

Theorem 3 [11; 155]. Let T be an arbitrary Jonsson theory. Then the following conditions are equivalent:

1) The theory T is perfect;

2) $T^* = Th(C)$ is a model companion of the theory T. More information about the concept of Jonsson perfection can be found in [17].

We define the following subclasses of Jonsson theories. Focusing on AP and JEP properties for certain theories, we distinguish the following four types of theories:

Definition 9. A theory T is called to be

1) AP-theory if in theory T amalgam property entails joint embedding property;

2) JEP-theories if in theory T joint embedding property entails amalgam property;

3) AJ-theories if in theory T both properties are equivalent.

Otherwise, we say that for the theory of T, the properties of AP and JEP are independent of each other.

The described types form corresponding subclasses in the class of Jonsson theories, on which our interest is focused. However, there are theories relating to some of the types 1–3, which are not Jonsson. An example of such a theory will be discussed later in this paper.

We need the following definition.

Definition 10 [18]. A theory T is called convex if for any model of A and any family $\{A_i | i \in I\}$ submodels A which are models of the theory T the intersection $\bigcap_{i \in I}$ is also a submodel of T, if it is nonempty. If the intersection is never empty, T is said to be strongly convex.

It is important to note that, according to this definition, the theory of differential fields of any characteristic is strongly convex. Based on this fact the theory can be attributed to AP-theories.

As for Definitions 3 and 4, B. Jonsson [19] was engaged in the study of "amalgam properties", who cited DF_0 as examples of theories with these properties. The proof in [5] was presented by I. Kaplansky. Robinson [7] noted that this is the result of the existence of a model companion for the considered theory:

Theorem 4 [20; 157]. The theory of T admits the amalgam property if and only if it has a model completion.

Property 1. [9; 130] DCF_0 allows quantifier elimination.

Property 2. [9; 131] DF_0 has the joint embedding and the amalgam properties.

Note that originally [9] in the formulation and proof of these properties, L. Blum refers to Theorem (0.3.7), which states that if a universal theory T has a model companion, the theory T has "amalgam properties", which mean both the amalgam property and the joint embedding property in our sense.

Theorem 5 [9; 128]. The DCF_0 theory is a model completion of the DF_0 theory.

Finally, we proceed to consider the model-theoretic properties of the described fields from the point of view of Jonssonness. Let DF_0 be the theory of differential fields of characteristic 0.

Theorem 6. DF_0 is a Jonsson theory.

Proof. (1) It is easy to see that DF_0 has infinite models.

(2) Since DF_0 is a $\forall$-axiomatizable theory, it is also $\forall\exists$-theory. Hence, it is inductive.

(3),(4) As already noted in [9] Blum, DF_0 has the amalgam property (AP) and the joint embedding property (JEP) due to the presence of a model replenishment of DCF_0. Moreover, in the case under consideration, the property JEP follows from AP: Two differential fields F_1 and F_2 always have a nonempty intersection, which will also be a differential field, isomorphically embedded in both of these fields. Then, by virtue of AP, there are isomorphic embeddings of F_1 and F_2 in some differential field F. The role of F can be played, for example, by a composite of fields F_1 and F_2 – the intersection of all differential fields of characteristic 0 containing F_1 and F_2, on which differentiation is continued accordingly. Thus, the result of having JEP follows from posession of AP in the theory of differential fields of characteristic 0. This is a consequence of the fact that DF_0 is a strongly convex theory.

Theorem 7. DF_0 is a perfect Jonsson theory.

Proof. The proof follows from the fact that DF_0 has a model completion, which is DCF_0. Let us conduct it in detail. According to Theorem 5, the theory of differential fields of characteristic 0 has a model completion – the theory of differentially closed fields of characteristic 0, which is also its model companion. In addition, as Theorem 2 states, DF_0 due to its Jonssonness, there must be the semantic model C_T and, accordingly, the center $DF_0^* = Th(C_T)$. If $DF_0^* = DCF_0$, then, by virtue of Theorem 3, DF_0 will be perfect. Let us show it.

The proof will be carried out from the opposite: let us say $DF_0^* \neq DCF_0$. In this case, since DF_0^* is complete and DCF_0 is model complete, for any sentence ψ of the signature in question, either

$$\psi \in DF_0 \text{ and } \neg\psi \in DCF_0, \tag{3}$$

or

$$\psi \in DF_0 \text{ and } \psi \notin DCF_0,\ \neg\psi \notin DCF_0. \tag{4}$$

However, DF_0, DCF_0, and DF_0^* are obviously model consistent, and, at the same time, are be embedded into the semantic model of the theory of DF_0. Using this fact, we can easily get a contradiction for both cases (3) and (4), which means that $DF_0^* = DCF_0$. Therefore, DF_0 is a perfect Jonsson theory.

Theorem 8. DCF_0 is a perfect Jonsson theory.

Proof. To begin with, let us show the Jonssonness of DCF_0 theory.

(1)DCF_0 has infinite models;

(2) DCF_0 is a $\forall\exists$-theory, and therefore it is inductive;

(3) DCF_0 is a model-complete theory, which means that by the Theorem 4 has AP;

(4) From (3) it follows JEP, since the nonempty intersection of two differentially closed fields of characteristic 0 always exists and is their submodel embedded into both fields. This fact is confirmed by the well-known Robinson criterion (Theorem 8).

The perfectness of DCF_0 follows again from the fact that the theory in question is model complete, which means it represents a model companion for itself.

Here again, it is important to note that due to the strong convexity of DCF_0, we have the result obtained, namely, that DCF_0 (along with the DF_0 described above) is an AP-Jonsson theory.

Now consider the differential fields of characteristic $p > 0$. To define such a field, we, similarly, add axiom (2) to the axiomatics of the field theory of characteristic p again, thus obtaining the theory of DF_p.

For differential fields of characteristic p, the relation $F^p \subseteq C$ is fulfilled, where F^p are all elements of the field raised to the power of p, C is a subfield of constants. The relation is true because $D(a^p) = pa^{p-1}D(a)$ for any $a \in F$.

In the works [21, 22], C. Wood obtained the following results regarding differential fields of characteristic p:

Theorem 9 [21]. The theory DF_p of differential fields of characteristic p does not admit the amalgam property.

The author notes that the main reason is the absence of the p-th roots for some constant elements of the field.

The consequence of the absence of AP C. Wood also highlights the following important theorem: *Theorem 10* [21]. The DF_p theory has no model completion.

In fact, to prove it, it is enough to refer to the Theorem 9.

To obtain a theory that allows the elimination of quantifiers, which has the amalgam property and model completion, C. Wood [21, 22] modifies the theory of DF_p, supplementing it with the axiom

$$\forall x \exists y\ (D(x) = 0 \rightarrow y^p = x),$$

and obtains the so-called theory DPF of differentially perfect fields:

Definition 11. A differentially perfect field F is a differential field such that $F^p = C$.

The DPF models are the DF_p models in which the fields of constants are closed with respect to the operation of extracting p-th root. Thus, the following theorem holds:

Theorem 11 [21]. The theory of differentially perfect fields of characteristic p admits the amalgam property.

Let us now define the theory of DCF_p differentially closed fields of characteristic p: To the axioms of DF_p we will add the following definition of a differentially closed field.

Definition 12. A differential field of characteristic p is called differentially closed if for each positive integer n in the language L we can determine the sentence φ_n stating that there is a solution for $f(x) = 0, g(x) \neq 0$ for each pair of differential polynomials in one differential variable such that f and g have order and total degree at most n, and the order of f is higher than the order of g.

The most important model-theoretic properties of DCF_p are completeness and model completeness. The key place is occupied by the following statement:

Theorem 12 [21]. DCF_p is a model companion for DF_p and a model completion for DPF.

The following results demonstrate the behavior of the theories DF_p, DPF and DCF_p from the point of view of studying the Jonsson theories.

Theorem 13. DF_p is not a Jonsson theory.

Proof. According to Theorem 9, since, DF_p does not have the amalgam property, it, following the Definition 5, is not a Jonsson. In addition, DF_p does not have JEP since it is AP-theory.

This fact is noteworthy because, as mentioned earlier, the fields theory F_p in case of characteristic p is a Jonsson, whereas the introduction of the functional symbol D into the signature F_p deprives DF_p of this property. At the same time, the Jonssonness appears when the differential field of characteristic 0 is transformed into a differentially perfect:

Theorem 14. DPF is a Jonsson theory.

Proof. Again, we will carry out the proof following the definition of Jonsson theory.

(1) DPF has infinite models;

(2) DPF is $\forall\exists$-axiomatizable, which means it is inductive;

(3) DPF has model completion, hence has AP;

(4) As mentioned before, any field of constants in a differential field contains a subfield generated by a unit element. Such a field is differentially perfect by definition and can serve as a DPF model that is embedded in any two differentially perfect fields F_1 and F_2. Further, by property (3), there is a model DPF in which F_1 and F_2 are embedded.

Here again we see the manifestation of the property of being AP-theory: Possession of the amalgam property allowed DPF to also have the joint embedding property .

Moreover,

Theorem 15. DPF is a perfect Jonsson theory.

Proof. The proof is similar to the proof of Theorem 7 and follows from the fact that DPF has a model complement (and, accordingly, a model companion), which is the theory of DCF_p, as stated by Theorem 11.

Theorem 16. DCF_p is a perfect Jonsson theory.

Proof. Let us show the Jonssonness of the DCF_p theory.

(1) DCF_p has infinite models;

(2) DCF_p is $\forall\exists$-axiomatizable, hence inductive;

(3) DCF_p, by Theorem 13, the model complete and, therefore, has AP.

(4) From [21] we can find out that theory DF_p has the prime model F_p which is unique. Since every differentially closed field is differential, this means that for any two models F_1 and F_2 of DCF_p, there exists a model F that can be embedded into F_1 and F_2, and, further by (3), there is a model F' such that F_1 and F_2 are embedded into F'.

Although DF_p is not a Jonsson theory, note that, it has a Jonsson model completion DCF_p (which is perfect in Jonsson sense). At the same time, another important remark that we can make based on the results obtained is the following fact: The perfectness (in the field sence) based on the differential field is a sufficient condition for the theory of differential fields of characteristic p to be perfect Jonsson theory.

Acknowledgments

This research was funded by the Science Committee of the Ministry of Education and Science of the Republic of Kazakhstan (Grant No. AP09260237).

References

1 Forrest W.K. Model theory for universal classes with the amalgamation property: A study in the foundations of Model Theory and algebra / W.K. Forrest // Annals of Mathematical Logic. — 1977. — 11(3). — P. 263–366.

2 Ritt J.F. Integration in finite terms / J.F. Ritt. — Ann Arbor (MI): UMI, 1995.

3 Ritt J.F. Differential algebra / J.F. Ritt. — New York: Publ. by American Mathematical Society. — 1950.

4 Kolchin E.R. Differential algebra and algebraic groups / E.R. Kolchin. — New York: Academic Press. — 1973.

5 Kaplansky I. An introduction to differential algebra / I. Kaplansky. — Paris: Hermann. — 1957. — 85 p.

6 Marker D. Model Theory of Differential Fields / D. Marker // Model Theory, Algebra, and Geometry. — MSRI Publications. — 2000. — V.39. — P. 53–63.

7 Robinson A. Introduction to Model Theory and to the metamathematics of algebra / A. Robinson. — Amsterdam: North-Holland. — 1986.

8 Михалёв А.В. Дифференциальная и разностная алгебра / А.В Михалёв, Е.В. Панкратьев // Итоги науки и техн. Сер. Алгебра. Тополгия Геометрия. — 1987. — Т. 25. — С. 67–139.

9 Blum L.C. Generalized algebraic theories: A model theoretic approach: PhD thesis / Lenore Carol Blum. — Massachusetts: MIT. — 1968. — 180 p.

10 Poizat B. Rangs des types dans les corps differentiels. Group etude theor. stables / B. Poizat // Univ Pierre et Marie Curie. — 1977-1978. — 1. — P. 6/01-6/13.

11 Ешкеев А.Р. Йонсоновские теории и их классы моделей: моногр. / А.Р. Ешкеев, М.Т. Касыметова. — Караганда: Изд-во КарГУ, 2016. — 370 с.

12 Ешкеев А.Р. JSp-косемантичность и JSB-свойство абелевых групп / А.Р.Ешкеев, О.И. Ульбрихт // Сиб. электрон. мат. изв. — 2016. — Т. 13. — С. 861–864.

13 Ешкеев А.Р. JSp-косемантичность R-модулей / А.Р.Ешкеев, О.И. Ульбрихт // Сиб. электрон. мат. изв. — 2019. — Т. 16. — С. 1233–1244.

14 Yeshkeyev A. R. Properties of hybrids of Jonsson theories / A. Yeshkeyev, N.Mussina // Bulletin of the Karaganda University-Mathematics. — 2018. — № 4(92). — P. 99–104.

15 Yeshkeyev A. R. The Properties of Similarity for Jonsson's Theories and Their Models / A. R. Yeshkeyev // Bullutin of the Karaganda University-Mathematics. — 2015. — № 4(80). — P. 52–59.

16 Yeshkeyev A. R. The atomic definable subsets of semantic model / A. Yeshkeyev, A. Issaeva, N. Mussina // Bulletin of the Karaganda University-Mathematics. — 2019. — № 2(94). — P. 84–91.

17 Yeshkeyev A. R. Companions of the fragments in the Jonsson enrichment / A. R. Yeshkeyev // Bulletin of the Karaganda University-Mathematics. — 2017. — № 1(85). — P. 41–45.

18 Yeshkeyev A. R. Model-theoretical questions of the Jonsson spectrum / A. R. Yeshkeyev // Bulletin of the Karaganda University-Mathematics. — 2020. — № 2(98). — P. 165–173.

19 Jonsson B. Extensions of relational structures / B. Jonsson. — The Theory of Models, North Holland, Amsterdam. — 1965. — P. 146–157.

20 Barwise J. Handbook of mathematical logic / J. Barwise, H. J. Keisler. — Amsterdam: Elsevier. — 2006.

21 Wood C. The Model Theory of differential fields of characteristic $p \neq 0$ / C. Wood // Proceedings of the American. Mathematical Society. — 1973. — 40(2). — P. 577–584.

22 Wood C. Prime model extensions for differential fields of characteristic $p \neq 0$ / C. Wood // Journal of Symbolic Logic. — 1974. — 39(3). — P. 469–477.

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Амальгама мен үйлесімді енгізу қасиеттерінің байланысы

Зерттеудің мақсаты – йонсондық теорияларды зерттеу аясында нөлдік және оң сипаттамамен дифференциалдық тұйық өрістер теориясының модельді-теоретикалық қасиеттерін анықтау. Негізгі назар амальгама мен үйлесімді енгізу қасиеттерін зерттеуге және осы теорияны йонсондық теорияларының маңызды белгілері ретінде біріктіруге, атап айтқанда АР-тен JEP қасиетінің болуына байланысты. Қажеттілік белгілі бір теориялардың жоғарыда аталған қасиеттері туралы ақпаратты неғұрлым толық модельді-теоретикалық сипаттаудың маңыздылығына байланысты. Сонымен қатар, бүгінгі таңда жалпы жағдайда йонсондық болып табылатын толық емес теорияларды зерттеу аппараты жеткіліксіз дамыған. Мына нәтижелер алынды: үйлесімді енгізу мен амальгама қасиеттерінің болуы тұрғысынан йонсондық теориялардың ішкі кластары анықталды. Осы кластардың бірін, атап айтқанда АР-теориялардың класын зерттеу аясында 0 сипаттамамен дифференциалды тұйық және дифференциалдық өрістерінің, бекітілген оң сипаттамамен дифференциалдық тұйық және дифференциалдық кемел өрістерінің теорияларының йонсондылығы мен кемелділігі көрсетілген. Сонымен қатар, йонсондық емес, бірақ кемел йонсондық модельді компаньоны бар АР теориясының мысалы ретінде оң сипаттамамен дифференциалдық өрістер теориясы қарастырылды, сондай-ақ, йонсондық болу қасиеті тұрғысынан дифференциалдық өрістер теориясы үшін жеткілікті шарт тұжырымдалды.

Кілт сөздер: йонсондық теория, йонсондық кемел теория, дифференциалдық өріс, дифференциалдық тұйық өріс, дифференциалдық кемел өріс, амальгама қасиеті, үйлесімді енгізу қасиеті, AP-теориясы, JEP-теориясы, қатты дөңес теория.

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Связь свойств амальгамы и совместного вложения

Цель исследования — изучение теоретико-модельных свойств теории дифференциально замкнутых полей нулевой и положительной характеристик в рамках исследования йонсоновских теорий. Основное внимание удалено свойствам амальгамы и совместному вложению данной теории как важнейших особенностей йонсоновских теорий, а именно следствия наличия свойства JEP из AP. Необходимость обусловлена важностью владения информацией об упомянутых выше свойствах у тех или иных теорий для их более полного теоретико-модельного описания. При этом на сегодняшний день аппарат изучения неполных теорий, которыми в общем случае являются йонсоновские, развит недостаточно. Получены следующие результаты: определены подклассы йонсоновских теорий с точки зрения наличия свойств совместного вложения и амальгамы. В рамках рассмотрения класса AP-теорий показаны йонсоновость и совершенность теорий дифференциальных и дифференциально замкнутых полей характеристики 0, дифференциально совершенных и дифференциально замкнутых полей фиксированной положительной характеристики. Наряду с этим, в качестве примера AP-теории, не являющейся йонсоновской, но имеющей совершенный йонсоновский модельный компаньон, изучена теория дифференциальных полей положительной характеристики, а также сформулировано достаточное условие для теории дифференциальных полей в контексте свойства быть йонсоновской.

Ключевые слова: йонсоновская теория, совершенная йонсоновская теория, дифференциальное поле, дифференциально замкнутое поле, дифференциально совершенное поле, свойство амальгамы, свойство совместного вложения, AP-теория, JEP-теория, сильно выпуклая теория.

References

1 Forrest, W.K. (1977). Model Theory for universal classes with the amalgamation property: A study in the foundations of Model Theory and Algebra. *Annals of Mathematical Logic, 11*(3), 263–366.

2 Ritt, J.F. (1995). *Integration in finite terms.* Ann Arbor (MI): UMI.

3 Ritt, J.F. (1950). *Differential algebra.* New York: Publ. by American Mathematical Society.

4 Kolchin, E.R. (1973). *Differential algebra and algebraic groups.* New York: Academic Press.

5 Kaplansky, I. (1957). *An introduction to differential algebra.* Paris: Hermann.

6 Marker, D. (2000) Model Theory of Differential Fields. *Model Theory, Algebra, and Geometry, 39*, 53–63.

7 Robinson, A. (1986). *Introduction to Model Theory and to the metamathematics of algebra.* Amsterdam: North-Holland.

8 Mikhalev, A.V., & Pankratiev, Ye.V. (1987). Differentsialnaia i raznostnaia algebra [Differential and difference algebra]. *Itogi nauki i tekhnoiki. Seriia Algebra. Topologiia. Geometriia – Results of Science and Technology. Algebra. Topology. Geometry, 25*, 67–139 [in Russian].

9 Blum, L.C. (1968). Generalized algebraic theories: A model theoretic approach (PhD thesis, B.S., Simone College, 1963). Massachusetts: MIT.

10 Poizat, B. (1977–1978). Rangs des types dans les corps differentiels. *Group etude theor. stables, 1*(6), 6-01–6-13 [in Frech].

11 Yeshkeyev, A.R. & Kassymetova, M.T. (2016). *Ionsonovskie teorii i ikh klassy modelei [Jonsson theories and their classes of models].* Karaganda: Izdatelstvo Karagandinskogo universiteta [in Russian].

12 Yeshkeyev, A.R., & Ulbrikht, O.I. (2016). JSp-kosemantichnost i JSB-svoistvo abelevykh grupp [*JSp*-cosemanticness and JSB property of Abelian groups]. *Sibirskie elektronnye matematicheskie izvestiia — Siberian Electronic Mathematical Collections, 13*, 861–864 [in Russian].

13 Yeshkeyev, A.R., & Ulbrikht, O.I. (2019). JSp-kosemantichnost R-modulei [JSp-cosemanticness of R-modules]. *Sibirskie elektronnye matematicheskie izvestiia — Siberian Electronic Mathematical Collections, 16,* 1233–1244 [in Russian].

14 Yeshkeyev, A.R., & Mussina, N. (2018). Properties of hybrids of Jonsson theories. *Bulletin of the Karaganda University-Mathematics, 92*(4), 99–104.

15 Yeshkeyev, A. R. (2015). The Properties of Similarity for Jonsson's Theories and Their Models . Bullutin of the Karaganda University-Mathematics, 4(80), 52–59.

16 Yeshkeyev, A., Issaeva, A., & Mussina, N. (2019). The atomic definable subsets of semantic model. *Bulletin of the Karaganda University-Mathematics, 94*(2), 84–91.

17 Yeshkeyev, A. (2017). Companions of the fragments in the Jonsson enrichment. *Bulletin of the Karaganda University-Mathematics, 85*(1), 41–45.

18 Yeshkeyev, A.R. (2020). Model-theoretical questions of the Jonsson spectrum. *Bulletin of the Karaganda University-Mathematics, 98*(2), 165–173.

19 Jonsson, B. (1965). *Extensions of relational structures.* The Theory of Models, North Holland, Amsterdam, 146–157.

20 Barwise, J., & Keisler, H. J. (2006). *Handbook of mathematical logic.* Amsterdam: Elsevier.

21 Wood, C. (1973). The Model Theory of differential fields of characteristic $p \neq 0$. *Proceedings of the American Mathematical Society, 40*(2), 577–584.

22 Wood, C. (1974). Prime model extensions for differential fields of characteristic $p \neq 0$. *Journal of Symbolic Logic, 39*(3), 469–477.

DOI 10.31489/2022M1/136-149

UDC 517.968.73

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An inverse problem for Hilfer type differential equation of higher order

In three-dimensional domain, an identification problem of the source function for Hilfer type partial differential equation of the even order with a condition in an integral form and with a small positive parameter in the mixed derivative is considered. The solution of this fractional differential equation of a higher order is studied in the class of regular functions. The case, when the order of fractional operator is $0 < \alpha < 1$, is studied. The Fourier series method is used and a countable system of ordinary differential equations is obtained. The nonlocal boundary value problem is integrated as an ordinary differential equation. By the aid of given additional condition, we obtained the representation for redefinition (source) function. Using the Cauchy–Schwarz inequality and the Bessel inequality, we proved the absolute and uniform convergence of the obtained Fourier series.

Keywords: fractional order, Hilfer operator, inverse source problem, Fourier series, integral condition, unique solvability.

Introduction

The theory of the inverse boundary value problems is currently one of the most important fields of the modern theory of differential equations. Consequently, a large number of research works are devoted to study the different kind of inverse problems for differential and integro-differential equations (see, for example, [1–10]). In cases where the boundary of the flow domain of a physical process is unavailable for measurements, nonlocal conditions in an integral form can serve as additional information sufficient for unique solvability of the problem. Therefore, researches on the study of nonlocal boundary value problems for differential and integro-differential equations with integral conditions have been intensified (see, for example, [11–20]). In addition, we note that studies of many problems of gas dynamics, theory of elasticity, theory of plates and shells are described by higher-order partial differential equations.

Fractional calculus plays an important role for the mathematical modeling in many natural and engineering sciences [21]. In [22], it is considered problems of continuum and statistical mechanics. In [23] is studied the mathematical problems of Ebola epidemic model. In [24] and [25], it is studied the fractional model for the dynamics of tuberculosis infection and novel coronavirus (nCoV-2019), respectively. The construction of various models of theoretical physics by the aid of fractional calculus is described in [26, Vol. 4, 5], [27], [28]. A detailed review of the application of fractional calculus in solving problems of applied sciences is given in [29, Vol. 6-8], [30]. In [31], the unique solvability of boundary value problem for weak nonlinear partial differential equations of mixed type with fractional Hilfer operator is studied. In [32], the solvability of nonlocal problem for a mixed type fourth-order differential equation with Hilfer fractional operator is studied. In the direction of applications of fractional derivatives to solving partial differential equations the interesting results were also obtained in [33–42].

We recall some basic terms of fractional integro-differentiation, which have been used during the study. Let $(t_0; T) \subset \mathbb{R}^+ \equiv [0; \infty)$ be an interval on the set of positive real numbers, where $0 \leq t_0 < T < \infty$. The Riemann–Liouville $0 < \alpha$-order fractional integral of a function $\eta(t)$ is defined as follows:

$$I^{\alpha}_{t_0+}\eta(t) = \frac{1}{\Gamma(\alpha)} \int\limits_{t_0}^{t} (t-s)^{\alpha-1}\eta(s)ds, \quad \alpha > 0, \quad t \in (t_0; T),$$

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where $\Gamma(\alpha)$ is the Gamma function.

Let $n-1<\alpha\le n,\ \ n\in\mathbb{N}$. The Riemann–Liouville α-order fractional derivative of a function $\eta(t)$ is defined as follows:

$$D_{t_0+}^{\alpha}\eta(t)=\frac{d^n}{dt^n}I_{t_0+}^{n-\alpha}\eta(t),\quad t\in\big(t_0;\,T\big).$$

The Gerasimov-Caputo α-order fractional derivative of a function $\eta(t)$ is defined by

$${}_*D_{t_0+}^{\alpha}\eta(t)=I_{t_0+}^{n-\alpha}\eta^{(n)}(t)=\frac{1}{\Gamma(n-\alpha)}\int\limits_{t_0}^{t}\frac{\eta^{(n)}(s)\,d\,s}{(t-s)^{\alpha-n+1}},\ \ t\in\big(t_0;\,T\big).$$

These derivatives are reduced to the n-th order derivatives for $\alpha=n\in\mathbb{N}$:

$$D_{t_0+}^{n}\eta(t)={}_*\,D_{t_0+}^{n}\eta(t)=\frac{d^n}{dt^n}\eta(t),\quad t\in\big(t_0;\,T\big).$$

The Hilfer fractional derivatives of α-order $(n-1<\alpha\le n,\ \ n\in\mathbb{N})$ and β-type $(0\le\beta\le 1)$ are defined by the following composition of three operators:

$$D_{t_0+}^{\alpha,\,\beta}\eta(t)=I_{t_0+}^{\beta(n-\alpha)}\frac{d^n}{dt^n}I_{t_0+}^{(1-\beta)(n-\alpha)}\eta(t),\quad t\in\big(t_0;\,T\big).$$

For $\beta=0$, this operator is reduced to the Riemann–Liouville fractional derivative $D_{t_0+}^{\alpha,\,0}=D_{t_0+}^{\alpha}$and the case $\beta=1$ corresponds to the Gerasimov–Caputo fractional derivative $D_{t_0+}^{\alpha,\,1}={}_*\,D_{t_0+}^{\alpha}$. Let $\gamma=\alpha+\beta\,n-\alpha\,\beta$. It is easy to see, that $\alpha\le\gamma\le n$. Then it is convenient to use another designation for the operator $D^{\,\alpha,\,\gamma}\eta(t)=D_{t_0+}^{\alpha,\,\beta}\eta(t)$. The generalized Riemann–Liouville operator was introduced by R. Hilfer based on time evolutions that arise during the transition from the microscopic scale to the macroscopic time scale (see [26]).

In this paper, for $0<\alpha<\gamma\le 1$ we study the regular solvability of an inverse boundary value problem for a Hilfer type partial differential equation of even order with positive small parameter. The source function is in the integral condition containing the Riemann–Liouville $0<\alpha<1$-order fractional integral. The stability of the solution from the given functions is proved.

In the three-dimensional domain $\Omega=\{(t,\,x,\,y)\,|\,0<t<T,\ 0<x,\,y<l\}$ a partial differential equation of the following form is considered

$$D_{\varepsilon}^{\alpha,\,\gamma}\,[U]=a\,(t)\,b\,(x,\,y) \tag{1}$$

with a nonlocal condition on the integral form containing the Riemann–Liouville $0<\alpha<1$-order fractional integral

$$U\,(T,\,x,\,y)+\big(I_{0+}^{\rho}U\,(t,\,x,\,y)\big)\,|_{t=T}\,=\varphi(x,\,y),\quad 0\le x\,,\,y\le l, \tag{2}$$

where ρ, T and l are given positive real numbers,

$$D_{\varepsilon}^{\alpha,\,\gamma}\,[U]=\left[D^{\alpha,\,\gamma}+\varepsilon D^{\alpha,\,\gamma}\left(\frac{\partial^{\,4k}}{\partial\,x^{\,4k}}+\frac{\partial^{\,4k}}{\partial\,y^{\,4k}}\right)+\omega\left(\frac{\partial^{\,4k}}{\partial\,x^{\,4k}}+\frac{\partial^{\,4k}}{\partial\,y^{\,4k}}\right)\right]U\,(t,\,x,\,y),$$

ω is a positive parameter, ε is a positive small parameter, $0<\alpha<\gamma\le 1$, k is a given positive integer, $a\,(t\,)\in C\,(\Omega_T)$, $\Omega_{\,T}\equiv[0;\,T]$, $\Omega_{\,l}\equiv[0;\,l]$, $b\,(x,\,y)\in C\,\left(\Omega_{\,l}^{\,2}\right)$ is a known function, $\varphi(x,\,y)$ is a source (redefinition) function, $\Omega_{\,l}^{\,2}\equiv\Omega_l\times\Omega_l$. We assume that for given functions are true the following boundary conditions

$$\varphi(0,\,y)=\varphi(l,\,y)=\varphi(x,\,0)=\varphi\,(x,\,l)=0,$$

$$b\,(0,\,y)=b\,(l,\,y)=b\,(x,\,0)=b\,(x,\,l)=0.$$

Problem Statement. We find the pair of unknown functions $\{U\,(t,\,x,\,y);\,\varphi\,(x,\,y)\}$, first of them satisfies differential equation (1), nonlocal integral condition (2), zero boundary value conditions

$$U\,(t,\,0,\,y)=U\,(t,\,l,\,y)=U\,(t,\,x,\,0)=U\,(t,\,x,\,l)=$$

$$=\frac{\partial^{\,2}}{\partial\,x^2}U\,(t,\,0,\,y)=\frac{\partial^{\,2}}{\partial\,x^2}U\,(t,\,l,\,y)=\frac{\partial^{\,2}}{\partial\,x^2}U\,(t,\,x,\,0)=\frac{\partial^{\,2}}{\partial\,x^2}U\,(t,\,x,\,l)=$$

$$= \frac{\partial^2}{\partial y^2} U(t, 0, y) = \frac{\partial^2}{\partial y^2} U(t, l, y) = \frac{\partial^2}{\partial y^2} U(t, x, 0) = \frac{\partial^2}{\partial y^2} U(t, x, l) = \ldots =$$

$$= \frac{\partial^{4k-2}}{\partial x^{4k-2}} U(t, 0, y) = \frac{\partial^{4k-2}}{\partial x^{4k-2}} U(t, l, y) = \frac{\partial^{4k-2}}{\partial x^{4k-2}} U(t, x, 0) = \frac{\partial^{4k-2}}{\partial x^{4k-2}} U(t, x, l) =$$

$$= \frac{\partial^{4k-2}}{\partial y^{4k-2}} U(t, 0, y) = \frac{\partial^{4k-2}}{\partial y^{4k-2}} U(t, l, y) = \frac{\partial^{4k-2}}{\partial y^{4k-2}} U(t, x, 0) = \frac{\partial^{4k-2}}{\partial y^{4k-2}} U(t, x, l) = 0, \qquad (3)$$

properties of the class of functions

$$\left[\begin{array}{l} t^{1-\gamma} U(t, x, y) \in C(\overline{\Omega}), \\ D^{\alpha, \gamma} U(t, x, y) \in C^{4k, 4k}_{x, y}(\Omega) \cap C^{4k+0}_{x, y}(\Omega) \cap C^{0+4k}_{x, y}(\Omega) \end{array} \right. \qquad (4)$$

and the additional condition

$$U\big(t_1, x, y\big) = \psi(x, y), \quad 0 < t_1 < T, \ \ 0 \le x, y \le l, \qquad (5)$$

$\varphi(x) \in C[0; l]$, where $\psi(x, y)$ are given smooth function and

$$\psi(0, y) = \psi(l, y) = \psi(x, 0) = \psi(x, l) = 0,$$

$C^{4k+0}_{x, y}(\Omega)$ is the class of continuous functions $\frac{\partial^{4k} U(t, x, y)}{\partial x^{4k}}$ on Ω, while $C^{0+4k}_{x, y}(\Omega)$ is the class of continuous functions $\frac{\partial^{4k} U(t, x, y)}{\partial y^{4k}}$ on Ω, $\frac{\partial^{4k-2}}{\partial y^{4k-2}} U(t, x, l)$ we understand as $\frac{\partial^{4k-2}}{\partial y^{4k-2}} U(t, x, y)\Big|_{y=l}$, $\overline{\Omega} = \{(t, x, y) | 0 \le t \le T,$ $0 \le x, y \le l\}$.

1 Expansion of the solution in a Fourier series

We seek nontrivial solutions of the problem in the form of Fourier series

$$U(t, x, y) = \sum_{n, m=1}^{\infty} u_{n, m}(t) \, \vartheta_{n, m}(x, y), \qquad (6)$$

where

$$u_{n, m}(t) = \int_0^l \int_0^l U(t, x, y) \, \vartheta_{n, m}(x, y) \, dx \, dy, \qquad (7)$$

$$\vartheta_{n, m}(x, y) = \frac{2}{l} \sin \frac{\pi n}{l} x \sin \frac{\pi m}{l} y, n, m = 1, 2, \ldots$$

We also suppose that the following function is expand in a Fourier series

$$b(x, y) = \sum_{n, m=1}^{\infty} b_{n, m} \vartheta_{n, m}(x, y), \qquad (8)$$

where

$$b_{n, m} = \int_0^l \int_0^l b(x, y) \, \vartheta_{n, m}(x, y) \, dx \, dy. \qquad (9)$$

Substituting Fourier series (6) and (8) into given partial differential equation (1), we obtain the countable system of ordinary differential equations of a fractional $0 < \alpha, \gamma < 1$-order

$$D^{\alpha, \gamma} u_{n, m}(t) + \lambda^{2k}_{n, m}(\varepsilon) \, \omega \, u_{n, m}(t) = \frac{a(t) \, b_{n, m}}{1 + \varepsilon \, \mu^{4k}_{n, m}}, \qquad (10)$$

where

$$\lambda^{2k}_{n, m}(\varepsilon) = \frac{\mu^{4k}_{n, m}}{1 + \varepsilon \, \mu^{4k}_{n, m}}, \ \ \mu^{k}_{n, m} = \left(\frac{\pi}{l}\right)^k \sqrt{n^{2k} + m^{2k}}.$$

The general solution of the countable system of differential equations (10) has the form [30]

$$u_{n,m}(t) = C_{n,m} t^{\gamma-1} E_{\alpha,\gamma}\left(-\lambda_{n,m}^{2k}(\varepsilon)\,\omega\, t^{\alpha}\right) + b_{n,m} h_{n,m}(t), \tag{11}$$

where

$$E_{\alpha,\gamma}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \gamma)}, \quad z, \alpha, \gamma \in \mathbb{C}, \quad Re(\alpha) > 0$$

is the Mittag–Leffler function [26, 269–295] and

$$h_{n,m}(t) = \frac{1}{1 + \varepsilon \mu_{n,m}^{4k}} \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}\left(-\lambda_{n,m}^{2k}(\varepsilon)\,\omega\,(t-s)^{\alpha}\right) a(s)\, ds,$$

$C_{n,m}$ is an arbitrary constant.

By Fourier coefficients (7), we rewrite integral condition (2) for the countable system (10)

$$u_{n,m}(T) + \left(I_{0+}^{\rho} u_{n,m}(t)\right)|_{t=T} = \int_0^l \int_0^l \left(U(T, x, y) + \left(I_{0+}^{\rho} U(t, x, y)\right)|_{t=T}\right) \vartheta_{n,m}(x, y)\, dx\, dy =$$

$$= \int_0^l \int_0^l \varphi(x, y)\, \vartheta_{n,m}(x, y)\, dx\, dy = \varphi_{n,m}. \tag{12}$$

To find the unknown coefficients $C_{n,m}$ in (11), we use condition (12) and from (11) we have

$$C_{n,m} = \frac{1}{\sigma_{0\,n,m}} \left[\varphi_{n,m} - b_{n,m} \sigma_{1\,n,m}\right], \tag{13}$$

where

$$\sigma_{0\,n,m} = T^{\gamma-1}\left[E_{\alpha,\gamma}\left(-\lambda_{n,m}^{2k}(\varepsilon)\,\omega\, T^{\alpha}\right) + T^{\rho} E_{\alpha,\rho+\gamma}\left(-\lambda_{n,m}^{2k}(\varepsilon)\,\omega\, T^{\alpha}\right)\right],$$

$$\sigma_{1\,n,m} = h_{n,m}(T) + \left(I_{0+}^{\rho} h_{n,m}(t)\right)_{|\,t=T}.$$

Hereinafter, we use the following properties of the Mittag–Leffler function:

1) The function $E_{\alpha,\beta}(-t)$ with $\alpha \in (0; 1]$, $\beta \geq \alpha$ is completely monotonic for $t > 0$, i.e.

$$(-1)^n \left[E_{\alpha,\beta}(-t)\right]^{(n)} \geq 0, \quad n = 0, 1, 2, ...$$

2) For all $\alpha \in (0; 2)$, $\beta \in \mathbb{R}$ and $\arg z = \pi$ there takes place the following estimate

$$|\,E_{\alpha,\gamma}(z)\,| \leq \frac{M_1}{1 + |z|},$$

where $0 < M_1 = \text{const}$ does not depend on z.

Then, from here follows that there exists numbers $M_2, M_3 > 0$ such that $0 < M_2 \leq \sigma_{0\,n,m} \leq M_3$.

Further, substituting the defined coefficients (13) into representation (11), we derived that

$$u_{n,m}(t) = \varphi_{n,m} A_{n,m}(t) + b_{n,m} B_{n,m}(t), \tag{14}$$

where

$$A_{n,m}(t) = \frac{1}{\sigma_{0\,n,m}} t^{\gamma-1} E_{\alpha,\gamma}\left(-\lambda_{n,m}^{2k}(\varepsilon)\,\omega\, t^{\alpha}\right), \ B_{n,m}(t) = h_{n,m}(t) - \frac{\sigma_{1\,n,m}}{\sigma_{0\,n,m}} t^{\gamma-1} E_{\alpha,\gamma}\left(-\lambda_{n,m}^{2k}(\varepsilon)\,\omega\, t^{\alpha}\right).$$

Substituting the representation of Fourier coefficients (14) of main unknown function into Fourier series (6), we obtain

$$U(t, x, y) = \sum_{n,m=1}^{\infty} \vartheta_{n,m}(x, y)\left[\varphi_{n,m} A_{n,m}(t) + b_{n,m} B_{n,m}(t)\right]. \tag{15}$$

Fourier series (15) is a formal solution of the direct problem (1)–(4).

2 Determination of source function

Using additional condition (5) and taking into account (12), we obtain from Fourier series (15) following countable system for Fourier coefficients of the source function

$$\varphi_{n,m} A_{n,m}(t_1) + b_{n,m} B_{n,m}(t_1) = \psi_{n,m}, \tag{16}$$

where

$$\psi_{n,m} = \int_0^l \int_0^l \psi(x,y)\, \vartheta_{n,m}(x,y)\, dx\, dy. \tag{17}$$

From relation (16) we find the source function as

$$\varphi_{n,m} = \psi_{n,m} \chi_{1\,n,m} + b_{n,m} \chi_{2\,n,m}, \tag{18}$$

where

$$\chi_{1\,n,m} = \frac{1}{A_{n,m}(t_1)}, \quad \chi_{2\,n,m} = -\frac{B_{n,m}(t_1)}{A_{n,m}(t_1)}.$$

$$A_{n,m}(t_1) = \frac{1}{\sigma_{0\,n,m}} t_1^{\gamma-1} E_{\alpha,\gamma}\left(-\lambda_{n,m}^{2k}(\varepsilon)\,\omega\, t_1^{\alpha}\right) \neq 0, \quad 0 < t_1 < T.$$

Since $\varphi_{n,m}$ are Fourier coefficients (see (12)), we substitute representation (18) into the Fourier series

$$\varphi(x,y) = \sum_{n,m=1}^{\infty} \vartheta_{n,m}(x,y) \left[\psi_{n,m}\, \chi_{1\,n,m} + b_{n,m}\, \chi_{2\,n,m}\right]. \tag{19}$$

We prove absolutely and uniformly convergence of Fourier series (19) for the source function. We need to use the concepts of the following Banach spaces:
the Hilbert coordinate space ℓ_2 of number sequences $\{\varphi_{n,m}\}_{n,m=1}^{\infty}$ with norm

$$\|\varphi\|_{\ell_2} = \sqrt{\sum_{n,m=1}^{\infty} |\varphi_{n,m}|^2} < \infty;$$

the space $L_2(\Omega_l^2)$ of square-summable functions on the domain $\Omega_l^2 = \Omega_l \times \Omega_l$ with norm

$$\|\vartheta(x,y)\|_{L_2(\Omega_l^2)} = \sqrt{\int_0^l \int_0^l |\vartheta(x,y)|^2\, dx\, dy} < \infty.$$

Conditions of smoothness. Let for functions

$$\psi(x,y),\ b(x,y) \in C^{4k}(\Omega_l^2)$$

there exist piecewise continuous $4k+1$ order derivatives. Then by integrating in parts the functions (9) and (17) $4k+1$ times over every variable x, y, we obtain the following relations

$$|\psi_{n,m}| = \left(\frac{l}{\pi}\right)^{8k+2} \frac{\left|\psi_{n,m}^{(8k+2)}\right|}{n^{4k+1} m^{4k+1}}, \quad |b_{n,m}| = \left(\frac{l}{\pi}\right)^{8k+2} \frac{\left|b_{n,m}^{(8k+2)}\right|}{n^{4k+1} m^{4k+1}}, \tag{20}$$

$$\left\|\psi_{n,m}^{(8k+2)}\right\|_{\ell_2} \leq \frac{2}{l} \left\|\frac{\partial^{8k+2}\psi(x,y)}{\partial x^{4k+1}\partial y^{4k+1}}\right\|_{L_2(\Omega_l^2)}, \tag{21}$$

$$\left\|b_{n,m}^{(8k+2)}\right\|_{\ell_2} \leq \frac{2}{l} \left\|\frac{\partial^{8k+2} b(x,y)}{\partial x^{4k+1}\partial y^{4k+1}}\right\|_{L_2(\Omega_l^2)}, \tag{22}$$

where

$$\psi_{n,m}^{(8k+2)} = \int_0^l \int_0^l \frac{\partial^{8k+2}\psi(x,y)}{\partial x^{4k+1}\partial y^{4k+1}} \vartheta_{n,m}(x,y)\,dx\,dy,$$

$$b_{n,m}^{(8k+2)} = \int_0^l \int_0^l \frac{\partial^{8k+2} b(x,y)}{\partial x^{4k+1}\partial y^{4k+1}} \vartheta_{n,m}(x,y)\,dx\,dy.$$

In obtaining estimates for the solution, we have used these formulas (20)–(22) and the above indicated properties of the Mittag–Leffler function. Then it is easy to see that

$$\sigma_2 = \max_{n,m} \{|\chi_{1\,n,m}|;\ |\chi_{2\,n,m}|\} < \infty, \tag{23}$$

where

$$\chi_{1\,n,m} = \frac{1}{A_{n,m}(t_1)}, \quad \chi_{2\,n,m} = -\frac{B_{n,m}(t_1)}{A_{n,m}(t_1)}, \quad 0 < t_1 < T,$$

$$A_{n,m}(t) = \frac{1}{\sigma_{0\,n,m}} t^{\gamma-1} E_{\alpha,\gamma}\left(-\lambda_{n,m}^{2k}(\varepsilon)\,\omega\, t^{\alpha}\right),\ B_{n,m}(t) = h_{n,m}(t) - \frac{\sigma_{1\,n,m}}{\sigma_{0\,n,m}} t^{\gamma-1} E_{\alpha,\gamma}\left(-\lambda_{n,m}^{2k}(\varepsilon)\,\omega\, t^{\alpha}\right).$$

Theorem 1. Suppose that the conditions of smoothness and (23) are fulfilled. Then Fourier series (19) convergence absolutely and uniformly in the domain Ω_l^2.

Proof. We use formulas (20)–(22) and estimate (23). Using the Cauchy–Schwartz inequality for series (19), we obtain the estimate

$$|\varphi(x,y)| \le \sum_{n,m=1}^{\infty} |\vartheta_{n,m}(x,y)| \cdot |\psi_{n,m}\chi_{1\,n,m} + b_{n,m}\chi_{2\,n,m}| \le$$

$$\le \frac{2}{l}\sigma_2 \left[\sum_{n,m=1}^{\infty} |\psi_{n,m}| + \sum_{n,m=1}^{\infty} |b_{n,m}|\right] \le$$

$$\le \frac{2}{l}\left(\frac{l}{\pi}\right)^{8k+2} \sigma_2 \left[\sum_{n,m=1}^{\infty} \frac{\left|\psi_{n,m}^{(8k+2)}\right|}{n^{4k+1} m^{4k+1}} + \sum_{n,m=1}^{\infty} \frac{\left|b_{n,m}^{(8k+2)}\right|}{n^{4k+1} m^{4k+1}}\right] \le$$

$$\le \frac{2}{l}\left(\frac{l}{\pi}\right)^{8k+2} \sigma_2 C_{01} \left[\left\|\psi_{n,m}^{(8k+2)}\right\|_{\ell_2} + \left\|b_{n,m}^{(8k+2)}\right\|_{\ell_2}\right] \le$$

$$\le \gamma_1 \left[\left\|\frac{\partial^{8k+2}\psi(x,y)}{\partial x^{4k+1}\partial y^{4k+1}}\right\|_{L_2(\Omega_l^2)} + \left\|\frac{\partial^{8k+2} b(x,y)}{\partial x^{4k+1}\partial y^{4k+1}}\right\|_{L_2(\Omega_l^2)}\right] < \infty, \tag{24}$$

where

$$\gamma_1 = \sigma_2 C_{01}\left(\frac{2}{l}\right)^2 \left(\frac{l}{\pi}\right)^{8k+2}, \quad C_{01} = \sqrt{\sum_{n,m=1}^{\infty} \frac{1}{n^{8k+2} m^{8k+2}}} < \infty.$$

From estimate (24) the absolutely and uniformly convergence of Fourier series (19) implies. The Theorem 1 is proved.

3 Determination of main unknown function

We determined the source function as a Fourier series (19). So, the source function is known. Using representation (16), Fourier series (15), we can present the main unknown function as

$$U(t,x,y) = \sum_{n,m=1}^{\infty} \vartheta_{n,m}(x,y)\left[\psi_{n,m} P_{n,m}(t) + b_{n,m} Q_{n,m}(t)\right], \tag{25}$$

where

$$P_{n,m}(t) = \chi_{1\,n,m} A_{n,m}(t), \quad Q_{n,m}(t) = \chi_{2\,n,m} A_{n,m}(t) + B_{n,m}(t).$$

To establish the uniqueness of the function $U(t, x, y)$ we suppose that there are two functions U_1 and U_2 satisfying given conditions (1)–(5). Then their difference $U = U_1 - U_2$ is a solution of differential equation (1), satisfying conditions (2)–(5) with the function $\psi(x, y) \equiv 0$. By virtue of relations (9) and (16), we have $\psi_{n, m} = 0$. Hence, from formulas (7) and (25) in the domain Ω we obtain the zero identity

$$\int\limits_0^l \int\limits_0^l t^{1-\gamma} U(t, x, y)\, \vartheta_{n, m}(x, y)\, dx\, dy \equiv 0.$$

By virtue of the completeness of the systems of eigenfunctions $\left\{\sqrt{\frac{2}{l}} \sin \frac{\pi n}{l} x\right\}$, $\left\{\sqrt{\frac{2}{l}} \sin \frac{\pi m}{l} y\right\}$ in $L_2\left(\Omega_l^2\right)$ we deduce that $U(t, x, y) \equiv 0$ for all $x \in \Omega_l^2 \equiv [0; l]^2$ and $t \in \Omega_T \equiv [0; T]$.

Since $t^{1-\gamma} U(t, x, y) \in C\left(\overline{\Omega}\right)$ then $t^{1-\gamma} U(t, x, y) \equiv 0$ in the domain $\overline{\Omega}$. Therefore, the solution to problem (1)–(5) is unique in the domain $\overline{\Omega}$.

Theorem 2. Let the conditions of the Theorem 1 be fulfilled. Then the series (25) converges in the domain Ω. At the same time, in this domain their term-by-term differentiation is possible.

Proof. By virtue of conditions of the theorem 1 and properties of Mittag–Leffler function, as in the case of (23), the functions $t^{1-\gamma} P_{n, m}(t)$, $t^{1-\gamma} Q_{n, m}(t)$ are uniformly bounded on the segment $[0; T]$. So, for any positive integers n, m there exists a finite constant σ_3, that there takes place the following estimate

$$\max_{n, m} \left\{ \max_{0 \le t \le T} \left| t^{1-\gamma} P_{n, m}(t) \right|; \max_{0 \le t \le T} \left| t^{1-\gamma} Q_{n, m}(t) \right| \right\} \le \sigma_3. \tag{26}$$

Using estimates (20)–(22) and (26), analogously to estimate (24), for series (25) we obtain

$$\left| t^{1-\gamma} U(t, x, y) \right| \le \sum_{n, m=1}^{\infty} \left| \vartheta_{n, m}(x, y) \right| \cdot \left| \psi_{n, m} t^{1-\gamma} P_{n, m}(t) + b_{n, m} t^{1-\gamma} Q_{n, m}(t) \right| \le$$

$$\le \gamma_2 \left[\left\| \frac{\partial^{8k+2} \psi(x, y)}{\partial x^{4k+1} \partial y^{4k+1}} \right\|_{L_2\left(\Omega_l^2\right)} + \left\| \frac{\partial^{8k+2} b(x, y)}{\partial x^{4k+1} \partial y^{4k+1}} \right\|_{L_2\left(\Omega_l^2\right)} \right] < \infty, \tag{27}$$

where $\gamma_2 = C_{01} \sigma_3 \left(\frac{2}{l}\right)^2 \left(\frac{l}{\pi}\right)^{8k+2}$.

From estimate (27) the absolutely and uniformly convergence of Fourier series (25) implies. We differentiate the required number of times function (25)

$$\frac{\partial^{4k}}{\partial x^{4k}} t^{1-\gamma} U(t, x, y) = \sum_{n, m=1}^{\infty} \left(\frac{\pi n}{l}\right)^{4k} \vartheta_{n, m}(x, y) \left[\psi_{n, m} t^{1-\gamma} P_{n, m}(t) + b_{n, m} t^{1-\gamma} Q_{n, m}(t) \right], \tag{28}$$

$$\frac{\partial^{4k}}{\partial y^{4k}} t^{1-\gamma} U(t, x, y) = \sum_{n, m=1}^{\infty} \left(\frac{\pi m}{l}\right)^{4k} \vartheta_{n, m}(x, y) \left[\psi_{n, m} t^{1-\gamma} P_{n, m}(t) + b_{n, m} t^{1-\gamma} Q_{n, m}(t) \right]. \tag{29}$$

The expansions of the following functions in a Fourier series are defined in a similar way

$$t^{1-\gamma} D^{\alpha, \gamma} U(t, x, y), \quad \frac{\partial^{4k}}{\partial x^{4k}} t^{1-\gamma} D^{\alpha, \gamma} U(t, x, y), \quad \frac{\partial^{4k}}{\partial y^{4k}} t^{1-\gamma} D^{\alpha, \gamma} U(t, x, y).$$

We show the convergence of series (28) and (29). As in the case of estimate (27), applying the Cauchy–Schwarz inequality, we obtain:

$$\left| \frac{\partial^{4k}}{\partial x^{4k}} t^{1-\gamma} U(t, x, y) \right| \le \sum_{n, m=1}^{\infty} \left(\frac{\pi n}{l}\right)^{4k} \left| t^{1-\gamma} u_{n, m}(t) \right| \cdot \left| \vartheta_{n, m}(x, y) \right| \le$$

$$\le \frac{2}{l} \left(\frac{\pi}{l}\right)^{4k} \sigma_3 \left[\sum_{n, m=1}^{\infty} n^{4k} \left| \psi_{n, m} \right| + \sum_{n, m=1}^{\infty} n^{4k} \left| b_{n, m} \right| \right] \le$$

$$\leq \frac{2}{l}\left(\frac{l}{\pi}\right)^{4k+2}\sigma_3\left[\sum_{n,\,m=1}^{\infty}\frac{\left|\psi_{n,\,m}^{(8k+2)}\right|}{n\,m^{4k+1}}+\sum_{n,\,m=1}^{\infty}\frac{\left|b_{n,\,m}^{(8k+2)}\right|}{n\,m^{4k+1}}\right]\leq$$

$$\leq \frac{2}{l}\left(\frac{l}{\pi}\right)^{4k+2}\sigma_3 C_{02}\left[\left\|\psi_{n,\,m}^{(8k+2)}\right\|_{\ell_2}+\left\|b_{n,\,m}^{(8k+2)}\right\|_{\ell_2}\right]\leq$$

$$\leq \gamma_3\left[\left\|\frac{\partial^{8k+2}\psi(x,\,y)}{\partial x^{4k+1}\partial y^{4k+1}}\right\|_{L_2(\Omega_l^2)}+\left\|\frac{\partial^{8k+2}b\,(x,\,y)}{\partial x^{4k+1}\partial y^{4k+1}}\right\|_{L_2(\Omega_l^2)}\right]<\infty, \tag{30}$$

where $\gamma_3=\left(\frac{2}{l}\right)^2\left(\frac{l}{\pi}\right)^{4k+2}\sigma_3 C_{02},\;\; C_{0\,2}=\sqrt{\sum\limits_{n,\,m=1}^{\infty}\frac{1}{n\,m^{8k+2}}}<\infty;$

$$\left|\frac{\partial^{4k}}{\partial y^{4k}}t^{1-\gamma}U\,(t,\,x,\,y)\right|\leq\sum_{n,\,m=1}^{\infty}\left(\frac{\pi\,m}{l}\right)^{4k}\left|\,t^{1-\gamma}u_{n,\,m}(t)\right|\cdot|\,\vartheta_{n,\,m}(x,\,y)\,|\leq$$

$$\leq\frac{2}{l}\left(\frac{\pi}{l}\right)^{4k}\sigma_3\left[\sum_{n,\,m=1}^{\infty}m^{4k}\,|\,\psi_{n,\,m}|+\sum_{n,\,m=1}^{\infty}m^{4k}\,|\,b_{n,\,m}\,|\right]\leq$$

$$\leq \frac{2}{l}\left(\frac{l}{\pi}\right)^{4k+2}\sigma_3\left[\sum_{n,\,m=1}^{\infty}\frac{\left|\psi_{n,\,m}^{(8k+2)}\right|}{n^{4k+1}m}+\sum_{n,\,m=1}^{\infty}\frac{\left|b_{n,\,m}^{(8k+2)}\right|}{n^{4k+1}m}\right]\leq$$

$$\leq \frac{2}{l}\left(\frac{l}{\pi}\right)^{4k+2}\sigma_3 C_{03}\left[\left\|\psi_{n,\,m}^{(8k+2)}\right\|_{\ell_2}+\left\|b_{n,\,m}^{(8k+2)}\right\|_{\ell_2}\right]\leq$$

$$\leq \gamma_4\left[\left\|\frac{\partial^{8k+2}\psi(x,\,y)}{\partial x^{4k+1}\partial y^{4k+1}}\right\|_{L_2(\Omega_l^2)}+\left\|\frac{\partial^{8k+2}b\,(x,\,y)}{\partial x^{4k+1}\partial y^{4k+1}}\right\|_{L_2(\Omega_l^2)}\right]<\infty, \tag{31}$$

where

$$\gamma_4=\left(\frac{2}{l}\right)^2\left(\frac{l}{\pi}\right)^{4k+2}\sigma_3 C_{03},\;\; C_{0\,3}=\sqrt{\sum_{n,\,m=1}^{\infty}\frac{1}{n^{8k+2}m}}<\infty.$$

It is easy to prove the convergence of Fourier series for functions

$$t^{1-\gamma}D^{\alpha,\,\gamma}U\,(t,\,x,\,y),\;\;\frac{\partial^{4k}}{\partial x^{4k}}t^{1-\gamma}D^{\alpha,\,\gamma}U\,(t,\,x,\,y),\;\;\frac{\partial^{4k}}{\partial y^{4k}}t^{1-\gamma}D^{\alpha,\,\gamma}U\,(t,\,x,\,y),$$

since the necessary estimates can be obtained by a similar way as for the cases of estimates (29), (30) and (31). Therefore, the function $U\,(t,\,x,\,y)$ belongs to the class of functions (4). Theorem 2 is proved.

4 Stability of the solution $U(t,x,y)$ with respect to the given functions and the source function

Theorem 3. Suppose that all the conditions of Theorem 2 are fulfilled. Then, the function $U\,(t,\,x,\,y)$ as a solution to problem (1)–(5) is stable with respect to a given function $\psi(x,\,y)$.

Proof. We show that the solution $U\,(t,\,x,\,y)$ of differential equation (1) is stable with respect to a given function $\psi(x,\,y)$. Let $U_1(t,\,x,\,y)$ and $U_2(t,\,x,\,y)$ be two different solutions of the inverse boundary value problem (1)–(5), corresponding to two different values of the function $\psi_1(x,\,y)$ and $\psi_2(x,\,y)$, respectively.

We put that $|\,\psi_{1\,n,\,m}-\psi_{2\,n,\,m}\,|<\delta_{n,\,m}$, where $0<\delta_{n,\,m}$ is a sufficiently small positive quantity and the series $\sum\limits_{n,\,m=1}^{\infty}|\,\delta_{n,\,m}\,|$ is convergent. Then, considering this fact by virtue of the conditions of the theorem, from Fourier series (25), it is easy to obtain that

$$\left\|\,t^{1-\gamma}\left[U_1(t,\,x,\,y)-U_2(t,\,x,\,y)\right]\right\|_{C\,(\overline{\Omega})}\leq\frac{2}{l}\sigma_3\sum_{n,\,m=1}^{\infty}|\,\psi_{1\,n,\,m}-\psi_{2\,n,\,m}\,|<\frac{2}{l}\sigma_3\sum_{n,\,m=1}^{\infty}|\,\delta_{n,\,m}\,|<\infty.$$

We put $\varepsilon = \frac{2}{l}\sigma_3 \sum\limits_{n,\, m=1}^{\infty} |\, \delta_{n,\, m} \,| < \infty$. Then, from last estimate we finally obtain assertions about the stability of the solution of the differential equation (1) with respect to a given function $\psi(x,\, y)$ in (5). The Theorem 3 is proved.

By a similar way we have have proved that there hold the following two theorems.

Theorem 4. Suppose that all conditions of Theorem 2 are fulfilled. Then, the function $U\,(t,\, x,\, y)$ as a solution to problem (1)–(5) is stable with respect to the given function $b\,(x,\, y)$ in the right-hand side of equation (1).

Theorem 5. Suppose that all conditions of Theorem 2 are fulfilled. Then, the function $U\,(t,\, x,\, y)$ as a solution to problem (1)–(5) is stable with respect to the source function $\varphi\,(x,\, y)$.

Remark. It is easy to study the stability of function $U\,(t,\, x,\, y)$ with respect to asmall parameter ε (see [43]).

Conclusions

In three-dimensional domain, an inverse problem of identification of a source function for Hilfer type partial differential equation (1) of the higher even order with integral form condition (2) and a small positive parameter in mixed derivative is considered. Suppose that the conditions of smoothness are fulfilled. Then the solution to this fractional differential equation of the higher order for $0 < \alpha < \gamma \leq 1$ is studied in the class of regular functions. The Fourier series method have been used and a countable system of ordinary differential equations has been obtained (10). The nonlocal inverse boundary value problem is integrated as an ordinary differential equation. By the aid of given additional condition, we obtained the representation for the source function. Using the Cauchy–Schwarz inequality and the Bessel inequality, we proved the absolute and uniform convergence of the obtained Fourier series (19) for the source function $\varphi\,(x,\, y)$ and (25) for the unknown function $U\,(t,\, x,\, y)$ and its derivatives. It is proved that solution of problem (1)–(5) $U\,(t,\, x,\, y)$ is stable with respect to the given functions $\psi(x,\, y),\; b\,(x,\, y)$ and the source function $\varphi\,(x,\, y)$.

References

1 Ашуров Р.Р. Обратная задача по определению плотности тепловых источников для уравнения субдиффузии / Р.Р. Ашуров, А.Т. Мухиддинова // Дифф. уравн. — 2020. — 56. — № 12. — С. 1596–1609.

2 Ashyralyyev C. Numerical solution to elliptic inverse problem with Neumann-type integral condition and overdetermination / C. Ashyralyyev, A. Cay // Bulletin of the Karaganda University. Mathematics Series. — 2020. — № 3(99). — P. 5–17.

3 Денисов А.М. Итерационный метод численного решения обратной коэффициентной задачи для системы уравнений в частных производных / А.М. Денисов, А.А. Ефимов // Дифф. уравн. — 2020. — 56. — № 7. — С. — 927–935.

4 Kabanikhin S.I. An algorithm for source reconstruction in nonlinear shallow-water equations / S.I. Kabanikhin, O.I. Krivorotko // Computational Mathematics and Mathematical Physics. — 2018. — 58. — № 8. — P. 1334–1343.

5 Kostin A.B. The inverse problem of recovering the source in a parabolic equation under a condition of nonlocal observation / A.B. Kostin // Collection Mathematics. — 2013. — 204. — № 10. — P. 1391–1434.

6 Mamedov Kh.R. Uniqueness of the solution to the inverse problem of scattering theory for the Sturm–Liouville operator with a spectral parameter in the boundary condition / Kh.R. Mamedov // Mathematical Notes. — 2003. — 74. — № 1. — P. 136–140.

7 Прилепко А.И. Обратные задачи нахождения источника и коэффициентов для эллиптических и параболических уравнений в пространствах Гёльдера и Соболева / А.И. Прилепко, А.Б. Костин, В.В. Соловьёв // Сиб. журн. чист. и прикл. мат. — 2017. — 17. — 3. — С. 67–85.

8 Romanov V.G. Inverse phaseless problem for the electrodynamic equations in an anisotropic medium / V.G. Romanov // Doklady Mathematics. — 2019. — 100. — № 2. — P. 495–500.

9 Romanov V.G. Phaseless inverse problems with interference waves / V.G. Romanov, M. Yamamoto // Journal of Inverse and Ill-Posed Problems. — 2018. — 26. — № 5. — P. 681–688.

10 Yuldashev T.K. Nonlocal inverse problem for a pseudohyperbolic-pseudoelliptic type integro-differential equations / T.K. Yuldashev // Axioms. — 2020. — 9. — № 2. — ID 45. — 21 p.

11 Амангалиева М.М. Об одной однородной задаче для уравнения теплопроводности в бесконечной угловой области / М. М. Амангалиева, М.Т. Дженалиев, М.Т. Космакова, М.И. Рамазанов // Сиб. мат. журн. — 2015. — 56. — № 6. — С. 1234–1248.

12 Assanova A.T. An integral-boundary value problem for a partial differential equation of second order / A.T. Assanova // Turkish Journal of Mathematics. — 2019. — 43. — № 4. — P. 1967–1978.

13 Assanova A.T. A nonlocal problem for loaded partial differential equations of fourth order / A.T. Assanova, A.E. Imanchiyev, Zh.M. Kadirbayeva // Bulletin of the Karaganda University. Mathematics. — 2020. — 97. — № 1. — P. 6–16.

14 Ильин В.А. Единственность обобщенных решений смешанных задач для волнового уравнения с нелокальными граничными условиями / В.А. Ильин // Дифф. уравн. — 2008. — 44. — № 5. — С. 672–680.

15 Дженалиев М.Т. О граничной задаче для спектрально-нагруженного оператора теплопроводности / М.Т. Дженалиев, М.И. Рамазанов // Сиб. мат. журн. — 2006. — 47. — № 3. — С. 527–547.

16 Jenaliyev M.T. On the solution to a two-dimensional heat conduction problem in a degenerate domain / M.T. Jenaliyev, M.I. Ramazanov, M.T. Kosmakova, Zh.M. Tuleutaeva // Eurasian Mathematical Journal. — 2020. — 11. — № 3. — P. 89–94.

17 Yuldashev T.K. Nonlocal mixed-value problem for a Boussinesq-type integro-differential equation with degenerate kernel / T.K. Yuldashev // Ukrainian Mathematical Journal. — 2016. — 68. — № 8. — P. 1278–1296.

18 Юлдашев Т.К. Смешанная задача для псевдопараболического интегро-дифференциального уравнения с вырожденным ядром / Т.К. Юлдашев // Дифф. уравн. — 2017. — 53. — № 1. — С. 101–110.

19 Юлдашев Т.К. О разрешимости одной краевой задачи для дифференциального уравнения типа Буссинеска / Т.К. Юлдашев // Дифф. уравн. — 2018. — 54. — № 10. — С. 1411–1419.

20 Yuldashev T.K. On a boundary-value problem for Boussinesq type nonlinear integro-differential equation with reflecting argument / T.K. Yuldashev // Lobachevskii Journal of Mathematics. — 2020. — 41. — № 1. — P. 111–123.

21 Samko S.G. Fractional integrals and derivatives / S.G. Samko, A.A. Kilbas, O.I. Marichev // Theory and Applications. Gordon and Breach, Yverdon, 1993.

22 Mainardi F. Fractional calculus: some basic problems in continuum and statistical mechanics / F. Mainardi // In: Carpinteri, A., Mainardi, F. (Eds.) Fractals and Fractional Calculus in Continuum Mechanics. Springer, Wien, 1997.

23 Area I. On a fractional order Ebola epidemic model / I. Area, H. Batarfi, J. Losada, J.J. Nieto, W. Shammakh, A. Torres // Advances in Difference Equations. — 2015. — 2015. — № 1. — ID 278. — P. 1–12.

24 Hussain A. Existence of solution and stability for the fractional order novel coronavirus (nCoV-2019) model / A. Hussain, D. Baleanu, M. Adeel // Advances in Difference Equations. — 2020. — 384.

25 Ullah S. A fractional model for the dynamics of tuberculosis infection using Caputo–Fabrizio derivative / S. Ullah, M.A. Khan, M. Farooq, Z. Hammouch, D. Baleanu // Discrete and Continuous Dynamical Systems. Ser. S. — 2020. — 13. — 3. — P. 975–993.

26 Handbook of fractional calculus with applications. Vols. 1-8. Tenreiro Machado J.A. (ed.). Berlin, Boston: Walter de Gruyter GmbH, 2019.

27 Kumar D. Editorial: fractional calculus and its applications in physics / D. Kumar, D. Baleanu // Frontiers Physics. — 2019. — 7. — № 6.

28 Sun H. A review on variable-order fractional differential equations: mathematical foundations, physical models, numerical methods and applications / H. Sun, A. Chang, Y. Zhang, W. Chen // Fractional Calculus and Applied Analysis. — 2019. — 22. — № 1. — P. 27–59.

29 Patnaik S. Applications of variable-order fractional operators: a review / S. Patnaik, J.P. Hollkamp, F. Semperlotti // Proceedins of the Royal Society. A. — 2020. — 476. — № 2234. — P. 1–32.

30 Yuldashev T.K. Boundary value problem for weak nonlinear partial differential equations of mixed type with fractional Hilfer operator / T.K. Yuldashev, B.J. Kadirkulov // Axioms. — 2020. — 9. — № 2. — ID 68. — P. 1–19.

31 Yuldashev T.K. Nonlocal problem for a mixed type fourth-order differential equation with Hilfer fractional operator / T.K. Yuldashev, B.J. Kadirkulov // Ural Mathematical Journal. — 2020. — 6. — № 1. — P. 153–167.

32 Yuldashev T.K. Inverse problem for a mixed type integro-differential equation with fractional order Caputo operators and spectral parameters / T.K. Yuldashev, E.T. Karimov // Axioms. — 2020. — 9. — 4. — ID 121. — P. 1–19.

33 Abdullaev O.Kh. Non-local problems with integral gluing condition for loaded mixed type equations involving the Caputo fractional derivative / O.Kh. Abdullaev, K.B. Sadarangani // Electronic Journal of Differential Equations. — 2016. — 2016. — № 164. — P. 1–10.

34 Berdyshev A.S. On a nonlocal problem for a fourth-order parabolic equation with the fractional Dzhrbashyan–Nersesyan operator / A.S. Berdyshev, J.B. Kadirkulov // Differential Equations. — 2016. — 52. — № 1. — P. 122–127.

35 Islomov B.I. On a boundary value problem for a parabolic-hyperbolic equation with fractional order Caputo operator in rectangular domain / B.I. Islomov, U.Sh. Ubaydullayev // Lobachevskii Journal of Mathematics. — 2020. — 41. — № 9. — P. 1801–1810.

36 Karimov E.T. Frankl-type problem for a mixed type equation with the Caputo fractional derivative / E.T. Karimov // Lobachevskii Journal of Mathematics. — 2020. — 41. — № 9. — P. 1829–1836.

37 Karimov E. Nonlocal initial problem for second order time-fractional and space-singular equation / E. Karimov, M. Mamchuev, M. Ruzhansky // Hokkaido Mathematical Journal. — 2020. — 49. — № 2. — P. 349–361.

38 Malik S.A. An inverse source problem for a two parameter anomalous diffusion equation with nonlocal boundary conditions / S.A. Malik, S. Aziz // Computers and Mathematics with Applications. — 2017. — 73. — № 12. — P. 2548–2560.

39 Serikbaev D. A source inverse problem for the pseudo-parabolic equation with the fractional Sturm–Liouville operator / D. Serikbaev, N. Tokmagambetov // Bulletin of the Karaganda University. Mathematics. — 2020. — № 4 (99). — P. 143–151.

40 Sadarangani K.B. A non-local problem with discontinuous matching condition for loaded mixed type equation involving the Caputo fractional derivative / K.B. Sadarangani, O.Kh. Abdullaev // Advances in Difference Equations. — 2016. — № 241. — P. 1–10.

41 Sadarangani K.B. About a problem for loaded parabolic-hyperbolic type equation with fractional derivatives / K.B. Sadarangani, O.Kh. Abdullaev // International Journal of Differerential Equations. — 2016. — 2016. — ID 9815796. — P. 1–6.

42 Yuldashev T.K. Inverse boundary value problem for a fractional differential equations of mixed type with integral redefinition conditions / T.K. Yuldashev, B.J. Kadirkulov // Lobachevskii Journal of Mathematics. — 2021. — 42. — № 3. — P. 649–662.

43 Юлдашев Т.К. Смешанная задача для нелинейного псевдопараболического уравнения высокого порядка / Т.К. Юлдашев, К.Х. Шабадиков // Итоги науки и техники. Сер. Современная математика и ее приложения. Тематические обзоры. — 2018. — 156. — С. 73–83.

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Жоғары ретті Хильфер типінің жартылай туындылы дифференциалдық теңдеудің кері есебі

Үшөлшемді облыста интегралдық формадағы және аралас туындылы кіші оң параметрі бар жұп ретті Хильфер типінің жартылай туындылы теңдеу үшін функция көзін анықтау есебі қарастырылған. Бұл жоғары ретті бөлшекті дифференциалдық теңдеудің шешімі тұрақты функциялар класында зерттелген. Бөлшекті оператордың реті $0 < \alpha < 1$ болатын жағдай қарастырылды. Фурье қатарлары әдісі қолданылды және қарапайым дифференциалдық теңдеулердің есептеу жүйесі алынды. Локалды емес шеттік есебі қарапайым дифференциалдық теңдеу ретінде интегралданады. Қосымша шарт арқылы қайта анықтау функциясы туралы түсінік берілген. Коши-Шварц теңсіздігі мен Бессель теңсіздігін қолдана отырып, алынған Фурье қатарларының абсолютті және бірқалыпты жинақтылығы дәлелденді.

Кілт сөздер: бөлшекті рет, Хилфер операторы, функция көзі туралы кері есеп, Фурье қатарлары, интегралдық шарт, бірмәнді шешілуі.

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Обратная задача для дифференциального уравнения в частных производных типа Хильфера высшего порядка

В трехмерной области рассмотрена задача идентификации функции источника для уравнения в частных производных типа Хильфера четного порядка с условием в интегральной форме и малым положительным параметром в смешанной производной. Решение этого дробного дифференциального уравнения высшего порядка получено в классе регулярных функций. Авторами изучен случай для порядка дробного оператора $0 < \alpha < 1$. Применен метод рядов Фурье, и получена счетная система обыкновенных дифференциальных уравнений. Нелокальная краевая задача интегрирована как обыкновенное дифференциальное уравнение. С помощью дополнительного условия получено представление для функции переопределения. С помощью неравенств Коши–Шварца и Бесселя доказана абсолютная и равномерная сходимость полученных рядов Фурье.

Ключевые слова: дробный порядок, оператор Хильфера, обратная задача об источнике, ряды Фурье, интегральное условие, однозначная разрешимость.

References

1 Ashurov, R.R., & Mukhiddinova, A.T. (2020). Obratnaia zadacha po opredeleniiu plotnosti teplovykh istochnikov dlia uravneniia subdiffuzii [Inverse problem of determining the heat source density for the subdiffusion equation. *Differentsialnye uravneniia — Differential Equations, 56, 12,* 1550–1563 [in Russian].

2 Ashyralyyev, C., & Cay, A. (2020). Numerical solution to elliptic inverse problem with Neumann-type integral condition and overdetermination. *Bulletin of the Karaganda University. Mathematics Series, 99, 3,* 5–17.

3 Denisov, A.M., & Efimov, A.A. (2020). Iteratsionnyi metod chislennogo resheniia obratnoi koeffitsientnoi zadachi dlia sistemy uravnenii v chastnykh proizvodnykh [Iterative method for the numerical solution of an inverse coefficient problem for a system of partial differential equations]. *Differentsialnye uravneniia — Differential Equations, 56, 7,* 900–909 [in Russian].

4 Kabanikhin, S.I., & Krivorotko, O.I. (2018). An algorithm for source reconstruction in nonlinear shallow-water equations. *Computational Mathematics and Mathematical Physics, 58, 8,* 1334–1343.

5 Kostin, A.B. (2013). The inverse problem of recovering the source in a parabolic equation under a condition of nonlocal observation. *Collection Mathematics, 204, 10,* 1391–1434.

6 Mamedov, Kh.R. (2003). Uniqueness of the solution to the inverse problem of scattering theory for the Sturm-Liouville operator with a spectral parameter in the boundary condition. *Mathematical Notes, 74, 1,* 136–140.

7 Prilepko, A.I., Kostin, A.B. & Solovev, V.V. (2017). Obratnye zadachi nakhozhdeniia istochnika i koeffitsientov dlia ellipticheskikh i parabolicheskikh uravnenii v prostranstvakh Geldera i Soboleva [Inverse problems of finding the source and coefficients for elliptic and parabolic equations in Hölder and Sobolev spaces]. *Sibirskii zhurnal chistoi i prikladnoi matematiki – Siberian Journal of Pure and Applied Mathematics, 17, 3,* 67–85 [in Russian].

8 Romanov, V.G. (2019). Inverse phaseless problem for the electrodynamic equations in an anisotropic medium. *Doklady Mathematics, 100, 2,* 495–500.

9 Romanov, V.G., & Yamamoto, M. (2018). Phaseless inverse problems with interference waves. *Journal of Inverse and Ill-Posed Problems, 26, 5,* 681–688.

10 Yuldashev, T.K. (2020). Nonlocal inverse problem for a pseudohyperbolic-pseudoelliptic type integro-differential equations. *Axioms, 9, 2, ID 45,* 1–21.

11 Amangalieva, M.M., Dzhenaliev, M.T., Kosmakova, M.T., & Ramazanov, M.I. (2015). Ob odnoi odnorodnoi zadache dlia uravneniia teploprovodnosti v beskonechnoi uglovoi oblasti [On one homogeneous problem for the heat equation in an infinite angular domain]. *Sibirskii matematicheskii zhurnal — Siberian Mathematical Journal, 56, 6,* 982–995 [in Russian].

12 Assanova, A.T. (2019). An integral-boundary value problem for a partial differential equation of second order. *Turkish Journal of Mathematics, 43, 4,* 1967–1978.

13 Assanova, A.T., Imanchiyev, A.E., & Kadirbayeva, Zh.M. (2020). A nonlocal problem for loaded partial differential equations of fourth order. *Bulletin of the Karaganda University. Mathematics Series, 97, 1,* 6–16.

14 Ilin, V.A. (2008). Edinstvennost obobshchennykh reshenii smeshannykh zadach dlia volnovogo uravneniia s nelokalnymi granichnymi usloviiami [Uniqueness of generalized solutions of mixed problems for the wave equation with nonlocal boundary conditions]. *Differentsialnye uravneniia — Differential Equations, 44, 5,* 692–700 [in Russian].

15 Dzhenaliev, M.T., & Ramazanov, M.I. (2006). O granichnoi zadache dlia spektralno-nagruzhennogo operatora teploprovodnosti [On the boundary value problem for the spectrally loaded heat conduction operator]. *Sibirskii matematicheskii zhurnal — Siberian Mathematical Journal, 47, 3,* 433–451 [in Russian].

16 Jenaliyev, M.T., Ramazanov, M.I., Kosmakova, M.T., & Tuleutaeva, Zh.M. (2020). On the solution to a two-dimensional heat conduction problem in a degenerate domain. *Eurasian Mathematical Journal, 11, 3,* 89–94.

17 Yuldashev, T.K. (2016). Nonlocal mixed-value problem for a Boussinesq-type integro-differential equation with degenerate kernel. *Ukrainian Mathematical Journal, 68, 8,* 1278–1296.

18 Yuldashev, T.K. (2017). Smeshannaia zadacha dlia psevdoparabolicheskogo integro-differentsialnogo uravneniia s vyrozhdennym iadrom [Mixed problem for pseudoparabolic integro-differential equation with degenerate kernel]. *Differentsialnoe uravneniia — Differential Equations, 53, 1,* 99–108 [in Russian].

19 Yuldashev, T.K. (2018). O razreshimosti odnoi kraevoi zadachi dlia differentsialnogo uravneniia tipa Bussineska [Solvability of a boundary value problem for a differential equation of the Boussinesq type. *Differentsialnoe uravneniia — Differential Equations, 54, 10,* 1384–1393 [in Russian].

20 Yuldashev, T.K. (2020). On a boundary-value problem for Boussinesq type nonlinear integro-differential equation with reflecting argument. *Lobachevskii Journal of Mathematics, 41, 1,* 111–123.

21 Samko, S.G., Kilbas, A.A., & Marichev, O.I. (1993). Fractional integrals and derivatives. Theory and Applications. Gordon and Breach, Yverdon, 1993.

22 Mainardi, F. (1997). *Fractional calculus: some basic problems in continuum and statistical mechanics.* In: Carpinteri, A., Mainardi, F. (Eds.) Fractals and Fractional Calculus in Continuum Mechanics. Springer, Wien, 1997.

23 Area, I., Batarfi, H., Losada, J., Nieto, J.J., Shammakh, W., & Torres A. (2015). On a fractional order Ebola epidemic model. *Advances in Difference Equations, 2015, 1, ID 278,* 1–12.

24 Hussain, A., Baleanu, D., & Adeel, M. (2020). Existence of solution and stability for the fractional order novel coronavirus (nCoV-2019) model. *Advances in Difference Equations, 2020, 384.*

25 Ullah, S., Khan, M.A., Farooq, M., Hammouch, Z., & Baleanu, D. (2020). A fractional model for the dynamics of tuberculosis infection using Caputo–Fabrizio derivative. *Discrete and Continuous Dynamical Systems, Ser. S, 13, 3,* 975–993.

26 Tenreiro Machado, J.A. (Ed.) (2019). *Handbook of fractional calculus with applications.* Vols. 1-8. Berlin, Boston: Walter de Gruyter GmbH, 2019.

27 Kumar, D., & Baleanu, D. (2019). Editorial: fractional calculus and its applications in physics. *Frontiers Physics, 7, 6.*

28 Sun, H., Chang, A., Zhang, Y., & Chen, W. (2019). A review on variable-order fractional differential equations: mathematical foundations, physical models, numerical methods and applications. *Fractional Calculus and Applied Analysis, 22, 1,* 27–59.

29 Patnaik, S., Hollkamp, J.P., & Semperlotti F. (2020). Applications of variable-order fractional operators: a review. *Proceedings of the Royal Society, A., 476, 2234,* 1–32.

30 Yuldashev, T.K., & Kadirkulov, B.J.(2020). Boundary value problem for weak nonlinear partial differential equations of mixed type with fractional Hilfer operator. *Axioms, 9, 2, ID 68,* 1–19.

31 Yuldashev, T.K., & Kadirkulov, B.J. (2020). Nonlocal problem for a mixed type fourth-order differential equation with Hilfer fractional operator. *Ural Mathematical Journal, 6, 1,* 153–167.

32 Yuldashev, T.K., & Karimov, E.T. (2020). Inverse problem for a mixed type integro-differential equation with fractional order Caputo operators and spectral parameters. *Axioms, 9, 4, ID 121,* 1–24.

33 Abdullaev, O.Kh., & Sadarangani, K.B. (2016). Nonlocal problems with integral gluing condition for loaded mixed type equations involving the Caputo fractional derivative. *Electronic Journal of Differential Equations, 2016, 164,* 1–10.

34 Berdyshev, A.S., & Kadirkulov, J.B. (2016). On a nonlocal problem for a fourth-order parabolic equation with the fractional Dzhrbashyan–Nersesyan operator. *Differential Equations, 52, 1,* 122–127.

35 Islomov, B.I., & Ubaydullayev, U.Sh. (2020). On a boundary value problem for a parabolic-hyperbolic equation with fractional order Caputo operator in rectangular domain. *Lobachevskii Journal of Mathematics, 41, 9,* 1801–1810.

36 Karimov, E.T. (2020). Frankl-type problem for a mixed type equation with the Caputo fractional derivative. *Lobachevskii Journal of Mathematics, 41, 9,* 1829–1836.

37 Karimov, E., Mamchuev, M., & Ruzhansky M. (2020). Nonlocal initial problem for second order time-fractional and space-singular equation. *Hokkaido Mathematical Journal, 49, 2,* 349–361.

38 Malik, S.A., & Aziz, S. (2017). An inverse source problem for a two parameter anomalous diffusion equation with nonlocal boundary conditions. *Computers and Mathematics with Applications, 73, 12,* 2548–2560.

39 Serikbaev, D., & Tokmagambetov, N. (2020). A source inverse problem for the pseudo-parabolic equation with the fractional Sturm-Liouville operator. *Bulletin of the Karaganda University. Mathematic Series, 4 (99),* 143–151.

40 Sadarangani, K.B., & Abdullaev, O.Kh. (2016). A nonlocal problem with discontinuous matching condition for loaded mixed type equation involving the Caputo fractional derivative. *Advances in Difference Equations, 2016, 241.* 1–12.

41 Sadarangani, K.B., & Abdullaev, O.Kh. (2016). About a Problem for Loaded Parabolic-Hyperbolic Type Equation with Fractional Derivatives. *International Journal of Differential Equations, 2016, ID 9815796,* 1–6.

42 Yuldashev, T.K., & Kadirkulov, B.J. (2021). Inverse boundary value problem for a fractional differential equations of mixed type with integral redefinition conditions. *Lobachevskii Journal of Mathematics, 42, 3,* 649–662.

43 Yuldashev, T.K., & Shabadikov, K.H. (2018). Smeshannaia zadacha dlia nelineinogo psevdoparaboliche-skogo uravneniia vysokogo poriadka [Mixed problem for a nonlinear pseudoparabolic equation of higher order]. *Itogi nauki i tekhniki. Seriia Sovremennaia matematika i ee prilozheniia. Tematicheskie obzory – Results of Science and Technology. Series: Contemporary mathematics and its applications. Thematic reviews., 156,* 73–83 [in Russian].

75th anniversary of Doctor of Physical and Mathematical Sciences, Professor M.T. Jenaliyev

On January 25, 2022, a well-known specialist in the field of the theory of partial differential equations and its applications, Chief Researcher of the Institute of Mathematics and Mathematical Modeling of the Committee of Science of the Ministry of Education and Science of the Republic of Kazakhstan, Doctor of Physical and Mathematical Sciences, Professor Muvasharkhan Tanabaevich Jenaliyev, turned 75 years old.

M.T. Jenaliyev was born in a family of rural workers in Aktobe (now the Tole-bi farm) of the Shu district of the Zhambyl region. His father, Zhienaliev Tanabai, worked for many years as a shepherd, his mother, Zhienalieva Tenge, helped her husband in this difficult shepherd business, until his death. Before retiring, she worked in various jobs. The shepherd's hard work was not alien to Muvasharkhan either, during the summer holidays he helped his parents.

In 1953, Muvasharkhan Jenaliyev entered the seven-year Kazakh school of the Aktobe district, where he graduated from the first grade with a commendable diploma. Since his parents' move to the settlement Mikhailovka (in the subsequent settlement Chatyrkul) in 1954, he again entered the first grade of the now seven-year-old Russian school, since he did not speak Russian. In a year, he manages to learn Russian and finishes the first grade with a commendable diploma. Then he continues to study in Russian and in 1965 he graduated from the 10th grade of the Gorky Secondary School in the settlement Novotroitskoye (now Tolebi). Back in the 9th grade, Muvasharkhan became interested in mathematics and this passion was instilled in him by his teacher Kutuzov Alexander Yakovlevich. In 1964-1965, he participated in the republican Olympiads of schoolchildren in Almaty. At the 3rd Kazakhstan Mathematical Olympiad, he was awarded a special prize and a diploma of the second degree.

In 1965, M.T. Jenaliyev entered the Kazakh Polytechnic Institute named after. V.I. Lenin at the Faculty of Automation and Computer Engineering and in 1971 he graduated from it with a degree in Automation and Telemechanics with the qualification of an "Electrical Engineer". In 1971-1976, he worked as an engineer, senior engineer, and head of the design team at the Kazakh branch of the SDI "Projectmontazhavtomatika"(Almaty), engaged in the design of dispatching systems for power supply facilities using telemechanical devices.

In 1976–1980, he is a full-time postgraduate student at Kazakh State University under the supervision of Professor S.A. Aisagaliyev. In 1982, M.T. Jenaliyev defended his candidate's and in 1994, his doctoral dissertations, in 1996, he was awarded the academic title of professor.

Since 1980 M.T. Jenaliyev has been working at the Institute of Mathematics and Mechanics of the Academy of Sciences of the Kazakh SSR (now the Institute of Mathematics and Mathematical Modeling of the CS MES RK). Muvasharkhan Tanabaevich goes through all the stages of the positions of an academic institution: Junior researcher, senior researcher, leading researcher, chief researcher, head of the laboratory of equations of mathematical physics, deputy director for scientific work, from January 1, 2007 - acting, and from August 2008 to 2011 director of the Institute of Mathematics.

Scientific achievements of M.T. Jenaliyev published in journals "Differential equations", "Siberian Mathematical Journal", "Boundary value problems", "Advances in difference equations", "Mathematical Journal"

(Almaty), “Proceedings of the Institute of Mathematics of the NAS of Belarus”, “Reports of NAS RK”, “Non-classical equations of mathematical physics” (S.L. Sobolev Institute of Mathematics SB RAS), “Reports of the AIAS”, “Proceedings of the NAS of the RK. Physico-mathematical series” and others. We list the results of his scientific research:

– M.T. Jenaliyev proved a theorem on sufficient optimality conditions and, on its basis, developed an algorithm for the approximate solution of the problem of optimal control of a parabolic equation. This result is a development of V.F. Krotov’s optimality principle for partial differential equations, which takes into account their solvability in the corresponding Sobolev classes (in the sense of an integral identity). An innovation was the introduction of an auxiliary functional and special constructions, which made it possible to remove the restriction on the reduction of partial differential equations to normal form, which facilitated to reduce the original problem for a conditional extremum to the problem for an unconditional extremum in Sobolev functional spaces. The results of these studies formed the basis of M.T. Jenaliyev’s candidate dissertation.

– For boundary value problems with time derivatives on the boundary for parabolic and hyperbolic equations, M.T. Jenaliyev discovered the effect of "overdetermination"at setting initial conditions in the domain and on its boundary from the class of square summable functions (which are not consistent with the trace theorem). The solvability of boundary value problems for linearly loaded equations with irregular coefficients is established. A symmetrizing operator for a loaded parabolic equation, a Hilbert space of the type of K. Friedrichs space and a quadratic functional are constructed, and the Euler equation is also posed, for which also a generalized statement of the original boundary value problem was given. Based on these results M.T. Jenaliyev defended his doctoral dissertation.

– In terms of the (complex) spectral parameter, which is the coefficient of the loaded term, a description of the resolvent set and spectrum for a spectrally loaded parabolic operator is found, a characteristic of the multiplicity of eigenfunctions in the space of bounded and continuous functions depending on the value of the spectral parameter is given (together with M.I. Ramazanov).

– For boundary value problems for multidimensional linear and nonlinear heat conduction equations in non-cylindrical domains with a power law of degeneracy: Uniqueness classes are found; In the case of a power law of domain degeneracy, the dimensions of the kernel and cokernel of operators of multidimensional boundary value problems are determined and the solvability of the boundary value heat conduction problem with time derivatives under boundary conditions is proved; Algorithms for solving boundary value problems for a heat equation loaded by multidimensional manifolds with control functions on the boundary have been developed.

In recent years, M.T. Jenaliyev, together with his collaborators, has been researching one-dimensional and multidimensional boundary and inverse problems for nonlinear equations, including the Burgers, Boussinesq, and Navier-Stokes equations. In Sobolev classes, the solvability of boundary value problems with nonlinear Neumann-type conditions and boundary value problems with dynamic conditions for the Burgers equation in degenerating domains is proved. Theorems on the solvability of the inverse problem for a linearized two-dimensional Navier-Stokes system in a cylindrical domain with a final overdetermination are proved and a computational algorithm for solving the inverse problem using the optimization method is proposed. Also, for a circle, a solution of the generalized spectral problem for a biharmonic operator is given.

M.T. Jenaliyev is actively engaged in the training of scientific personnel. Under his scientific supervision, 3 doctoral, 11 candidate dissertations and 4 PhD dissertations were defended. Since 1980 the scientist has also been teaching special courses at the Mechanics and Mathematics Faculty of Al-Farabi Kazakh National University.

Muvasharkhan Tanabaevich Jenaliyev is distinguished by a businesslike, principled and creative attitude, diligence, professionalism and a high sense of responsibility. He enjoys well-deserved respect in the staff of the Institute of Mathematics and Mathematical Modeling.

The editorial board of the scientific journal cordially congratulates Muvasharkhan Tanabaevich on his 75th birthday and wishes him good health and creative longevity.

Editorial board of the journal
«Bulletin of the Karaganda University. Mathematics series»

110th anniversary of the outstanding scientist corresponding member of the Academy of Sciences of the Republic of Kazakhstan ENGVAN INSUGOVICH KIM (1911–1994)

In November 2021 it was the 110th anniversary of the birth of the outstanding Scientist, Doctor of Science in Physics and Mathematics, Professor E.I. Kim, who made a significant contribution to the development of mathematical science in Kazakhstan, created a school for the study of equations of mathematical physics and raised many students who continue his research.

Yengvan Insugovich Kim was born on November 12, 1911 in the village of Ust-Sidimi, Khasansky District, Primorsky Territory, into a Korean family, the family of Insug Kim (who came from peasants, then became a railway worker).

In 1929, after graduating from a seven-year school, he entered the Nikolsko-Ussuri Korean Pedagogical Technical College in the mathematical department. After graduating with honors in 1932 from the College, he, on the advice of a teacher (according to E.I., who noticed the mathematical abilities of a young man), decided to continue his studies at Moscow State University. As Yengvan Insugovich later recalled, the road to Moscow took about a month, and on the way, he ate mainly fried grain taken from home. Arriving in Moscow, Yengvan, almost not knowing Russian, but having received excellent marks in mathematics and physics, entered the Faculty of Mechanics and Mathematics of Moscow State University. Years of hard study, language learning, attendance at scientific seminars began under the guidance of scientists, well-known to all mathematicians: A.N. Tikhonov and S.L. Sobolev.

In 1937, Kim graduated from the University with honors and was sent to work at the Vladivostok Korean Pedagogical Institute, where he arrived with his wife. But unfortunately, in the same 1937, almost the entire Korean diaspora was resettled from the Far East to Central Asia (mainly Kazakhstan, Uzbekistan, Kyrgyzstan).

Yengvan Insugovich also left Vladivostok for Kazakhstan and was hired by the Kyzyl-Orda Pedagogical Institute (KPI), where from 1937 to 1945 he held both teaching and administrative positions and also spent a lot of time doing mathematical research. The first field of his scientific researches (suggested to him by S.L. Sobolev) was the determination of solvability conditions for general boundary value problems for harmonic functions.

For the results obtained, E.I. Kim, after defending his dissertation in 1942 on the topic: "The Hilbert problem for a multi-connected domain", awarded the degree of Candidate of Science in Physics and Mathematics. The defence took place in the Academic Council of the United Ukrainian University, which was evacuated in the Kyzyl-Orda at that time. In 1943, he received the academic title of Associate Professor.

In 1945, E.I. Kim went to work at the S.M. Kirov Kazakh State University in Alma-Ata, as the head of the Department of Geometry, and supervises the scientific work of graduate students. In 1951, he moved to Rostov-on-Don, where he worked as the head of the Department of Geometry and Dean of the Faculty of Physics and Mathematics of the Pedagogical Institute.

From 1953 to 1956 he was a postdoc student at the Mathematical Institute of the USSR Academy of Sciences named after V.A. Steklov in Moscow. His scientific adviser was the outstanding mathematician I.N. Vekua, later Academician of the AS USSR.

In 1956, after completing his doctoral studies, E.I. Kim moved to Ukraine, where he has maintained good scientific contacts since defending his PhD thesis. He was the head of the Department of Higher Mathematics at the Kharkov Polytechnic Institute, and still devotes a lot of time to scientific research.

In 1959, he defended his thesis for the title of Doctor of Science in Physics and Mathematics "On a class of singular integral equations and some problems for piecewise homogeneous materials". Such equations arise, in particular, when solving boundary value problems for the heat equation with piecewise constant coefficients by the method of thermal potentials. As Yengvan Insugovich himself notes in his dissertation, similar integral equations were studied earlier, in particular, in the works of A.B. Datsev and G. Mintz, where it was stated that they can be solved by the method of successive approximations, since they are similar to Volterra equations of the second kind. However, in a work published in the DAN USSR back in 1953, E.I. Kim showed, that this is a special class of singular equations for which successive approximations do not converge to a solution. In the following works, which became the basis of the dissertation, he proposed and substantiated regularization methods, and also determined the exact upper bounds for the spectrum of the main integral operator of these equations. These studies are a significant contribution to the theory of Voltaire integral equations, since they define a special class of equations that have the properties of the Fredholm's equations.

In 1960, E.I. Kim received the academic title of Professor. Being in Kharkov, he along with teaching and scientific work, maintained close ties with Kazakhstan, supervised postgraduate students from the Kazakh State University. Among his first students, who then defended Candidate of Science dissertations, were B.B. Baimukhanov, L.P. Ivanova, Sh.T. Irkegulov, K.K. Kabdykairov, S.A. Usoltsev. The next group included A.A. Askarov, L.Zh. Zhumabekov, V.H. Ni, M.O. Orynbasarov, S.N. Kharin. After returning to Kazakhstan, many of them worked as teachers in various universities and continued their scientific research.

In 1964, E.I. Kim was elected a corresponding member of the Academy of Sciences of the Kazakh SSR, moved to Alma-Ata, and from that time the Kazakhstan period in his life began, which lasted until its end. The main part of this time was devoted to scientific work, teaching and training of scientists. Basically, owing to his efforts, in 1964 the Laboratory of Equations of Mathematical Physics (EMPh) was created at the Institute of Mathematics and Mechanics of the Academy of Sciences of the Kazakh SSR, as well as Department of EMPh at the Kazakh State University. For a number of years he headed both of these divisions. Then he worked in them as a leading researcher and professor. At the same time, E.I. Kim organized a citywide weekly scientific seminar on the equations of mathematical physics, which played an important role in the training of mathematical scientists in the country. He was a brilliant lecturer. At his lectures, there were always many participants and visitors, who also reported the results of their works. These were not only graduate students but also students of Kazakh State University, and teachers from other universities in Alma-Ata and other regions of Kazakhstan, scientists.

These initiatives of E.I.Kim and the results achieved by the graduates of the EMPh department, members of the EMPh laboratory, participants of the seminar, made a significant contribution to the development of mathematical researches on partial differential equations in Kazakhstan, as well increasing their level. For instance, an member of the EMPh laboratory Dr M.T. Jenaliyev, after defending his doctoral dissertation, for a number of years was the head of the Institute of Mathematics (IM) of the National Academy of Sciences (NAS) of RK in Almaty. The graduate of the Department of EMPh, Doctor of Sci. in Physics and Mathematics E.S. Smailov headed the Institute of Applied Mathematics in Karaganda. One of the first students of E.I. Kim, who also worked at KazSU and in the laboratory of the EMPh – S.N. Kharin became an academician of the NAS RK, other students and employees of E.I. Kim worked as teachers in universities in many cities of Kazakhstan.

It can be noted that at Karaganda University, in addition to Dr. E.S. Smailov, (and are working) Doctor of Sci. in Physics and Mathematics M.I. Ramazanov, Candidates of Sci. in Physics and Mathematics – T.E. Omarov, S. Mataev, M.A. Pervertun also worked. Candidate of Sci. K.K. Kabdykairov was the vice-rector of the Semipalatinsk University (at that time – the Pedagogical Institute), etc.

The E.I.Kim's followers continued researches related to the above singular integral equations. Significant results in the theory of partial differential equations and their applications are associated with their names. These are singular integral equations, initial-boundary value problems for parabolic equations and for equations with discontinuous coefficients, nonlinear problems with free boundaries, and problems in angular and degenerate domains.

Of especial interest to E.I. Kim has always been challenged by tasks to which ordinary research methods are not applicable, which do not fit into a general theory, and for their solution it is necessary to show ingenuity and apply non-standard approaches.

The results of E.I. Kim in the theory of singular integral Volterra-Fredholm's equations were further developed and have got numerous applications in his joint work with students, while the research topics were significantly expanded. In particular, together with B.B. Baimukhanov, L.P. Ivanova, K.K. Kabdykairov, V.H. Ni, L.Zh. Zhumabekov, S.E. Bazarbaeva and others have studied various initial boundary value problems for equations and systems of equations of parabolic type; with E.M. Khairullin, T.V. Nekrasova studied boundary value problems with boundary conditions containing high-order derivatives of the sought functions; methods for solving problems for parabolic equations with discontinuous coefficients were developed, M.A. Abdrakhmanov, Sh.A. Kulakhmetova, V.H. Ni, F.G. Biryukova, K.D. Kulekeev, R.N. Kantaeva and others. then were effectively used in solving problems of conjugation of various types of equations.

A special class of singular integral equations generated by boundary problems for degenerate domains with moving boundaries. In this direction E.I. Kim and his students S.N. Kharin, G.I. Bizhanova, , M.I. Ramazanov, M.T. Jenaliyev, T.E. Omarov, A.A. Kavokin, U.K. Koylyshov, S.S. Domalevsky obtained a number of complete results on the solvability of such problems.

It should also be noted the significant results obtained by Doctor of Sci M.O. Orynbasarov in the investigations of boundary value problems for equations of parabolic type in domains with corner points or edges (in the multidimensional case).

E.I. Kim paid great attention to nonlinear problems, problems with free boundaries, in particular, the Stefan problem with the domain degenerating at the initial moment of time, for which asymptotic, as well as analytical methods of solution were developed. worked with him in this direction S.N. Kharin, A.A. Kavokin, Ya.A. Krasnov, G.I. Bizhanova.

In the laboratory of EMPh of the Institute of Mathematics of the AS of the Kazakh SSR, apart from fundamental research, another direction of research was formed – applied, which originated in the years of E.I. Kim at KhPI. It was headed by S.N. Kharin, now an academician of the NAS of RK. Developed methods for solving boundary value problems for heat equations are widely used in applied problems, in particular, in the theory of electrical contacts have been published in monographs. In this direction significant results were obtained in joint work with D.U.Kim, M.A. Perevertun, S.P. Gorodnichev, A.T. Kulakhmetova, Yu.R. Shpadi, S.S. Domalevsky and other students of E.I. Kim and S.N. Kharin.

In total E.I. Kim personally published and co-authored about 130 scientific articles, many of which were published in central mathematical journals, as well as two monographs (co-authored), made reports at International, All-Union, Republican conferences. He prepared 36 candidates of science in physics and mathematics, 7 of his students received the degree of Doctor of science in Physics and Mathematics.

E.I. Kim did a lot of additional public work. He was a member of the NAS RK Problem Council on physical and mathematical sciences, member of the Specialized Council for the defense of dissertations, the editorial council of the All-Union "Engineering-Physical Journal the journal "Proceedings of the Academy of Sciences of the Kazakh SSR. Series of Physics and Mathematics".

For great merits in the development of mathematics in Kazakhstan, as well as for fruitful public-pedagogical activity, E.I. Kim was awarded the title "Honoured Worker of Science of the Kazakh SSR awarded the Certificate of Honour of the Presidium of the Supreme Council of the Kazakh SSR, and inscribed in the "Golden Book of Honour"of the Kazakh SSR.

E.I. Kim passed away on December 14, 1994 due to a serious illness. Until the last minute, his wife Claudia Semenovna Kim and their daughter Evgenia looked after him. He lived a wonderful life filled with work, creative searches. The affair he served all his life lives and continues to develop by his students and followers.

More detailed information about the remarkable mathematician E.I. Kim can be found in publications:

1. АН КазССР. Енгван Инсугович Ким. (Материалы к библиографии ученых Казахстана / Сост. С.Н.Харин, М.А.Абдрахманов и др.). — Алма-Ата: Ғылым, 1991.

2.Бижанова Г.И. Член-корреспондент АН КазССР Енгван Инсугович Ким // Мат. журн. — 2011. — Т. 11. — No. 2 (40). — С. 12–16,

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List of the main published works of E.I. Kim

Books

1 Математические модели тепловых процессов в электрических контактах. — Алма-Ата: Наука, 1977. — 236 с. (совместно с В.Т. Омельченко и С.Н. Хариным).

2 Теория теплопроводности в однородных средах. — Алматы: Комитет науки МОН РК; Ин-т мат. и мат. моделир., 2020. — 230 с. (Совместно с С.Н. Хариным).

3 Обобщенная задача Гурса // Ученые записки КазГУ. — 1948. — Т. 12. — С. 28–37.

4 Об общей граничной задаче гармонической функции // Прикладная математика и механика. — 1952. — Т. 16. — № 2. — С. 18–24.

5 Об одном классе интегральных уравнений I рода с сингулярным ядром // ДАН СССР. — 1953. — Т. 16. — № 2. — С. 1014–1019.

6 Распространение тепла в бесконечном неоднородном теле в двух измерениях // Прикладная математика и механика. — 1953. — Т. 17. — № 2. — С. 43–47.

7 Об одной задаче теплообмена системы тел // Прикладная математика и механика. — 1957. — Т. 21. — № 5. — С. 19–25.

8 Решение одного класса сингулярных интегральных уравнений с контурным интегралом // ДАН СССР. — 1957. — Т. 113. — № 1. — С. 24–27

9 Об одном классе сингулярных интегральных уравнений // ДАН СССР. — 1957. — Т. 113. — № 2. — С. 268–271.

10 Двухмерная проблема тепло- и массообмена в процессе сушки // Изв. АН СССР. Энергетика и автоматика. — 1959. — № 3. — С. 79–85. (Совместно с Л.П. Ивановой).

11 Об условиях разрешимости одного класса интегро-дифференциальных уравнений // ДАН СССР. — 1959. — Т. 125. — № 4. — С. 723–726.

12 Смешанная граничная задача для одной системы дифференциальных уравнений параболического типа // ДАН СССР. — 1959. — Т. 126. — С. 1183–1186. (Совместно с Л.П. Ивановой).

13 Об условиях разрешимости некоторой граничной задачи для одной параболической системы // ДАН СССР. — 1961. — Т. 139. — № 4. — С. 795–798. (Совместно с Л.П. Ивановой).

14 О распределении температуры в кусочно-однородной полубесконечной пластине // ДАН СССР. — 1961. — Т. 140. — № 2. — С. 333–336. (Совместно с Б.Б. Баймухановым).

15 Решение уравнения теплопроводности с разрывными коэффициентами // ДАН СССР. — 1961. — Т. 140. — № 3. — С. 451–454. (Совместно с Б.Б. Баймухановым).

16 Об условиях разрешимости одной граничной задачи уравнения теплопроводности // ДАН СССР. — 1961. — Т. 140. — № 3. — С. 553–556. (Совместно с Б.Б. Баймухановым).

17 Решение уравнения теплопроводности с разрывным коэффициентом и его приложения к вопросу электрических контактов // ИФЖ. — 1963. — № 6. — С. 763–768.

18 Об условиях разрешимости общей граничной задачи уравнений. Тепло- и массоперенос // ИФЖ. — 1964. — № 5. — С. 122–126.

19 К вопросу распространения тепла в кусочно-однородных средах в многомерном пространстве // Изв. АН КазССР. Сер. физ.-мат. — 1965. — № 3. — С. 3–18.

20 Решение системы дифференциальных уравнений параболического типа, когда характеристическое уравнение имеет кратные корни // Изв. АН КазССР. Сер. физ.-мат. — 1965. — № 1. — С. 19–29. (Совместно с В.Х. Ни).

21 Решение уравнения теплопроводности с разрывным коэффициентом, когда начальные данные не согласуются // Изв. АН КазССР. Сер. физ.-мат. — 1967. — № 5. — С. 3–15.

22 Уравнения теплопроводности с разрывными коэффициентом, когда начальные данные не согласуются // Изв. АН КазССР. Сер. физ.-мат. — 1967. — № 5. — С. 56–61. (Совместно с Ф.Г. Бирюковой).

23 Нахождение оригинала одного изображения // Изв. АН КазССР. Сер. физ.-мат. — 1968. — № 3. — С. 83–86. (Совместно с Ф.Г. Бирюковой).

24 О некорректности пространственных задач типа Коши–Гурса // Изв. АН КазССР. Сер. физ.-мат. — 1970. — № 1. — С. 70–71. (Совместно с Ли).

25 О неразрешимости методом последовательных приближений одного класса интегральных уравнений Вольтера // Тр. ИММ АН КазССР. — 1971. — Т. 2. — С. 27–35. (Совместно с С.Н. Хариным).

26 Первая граничная задача без условий согласования для уравнения теплопроводности с разрывными коэффициентом, когда линия разрыва выходит на границу // Мат. и мех. — 1972. — Вып. 1. — С. 51–52.

27 Решение задачи теплопроводности с подвижной границей с помощью специальных функций // Мат. и мех. — 1972. — Вып. 1. — С. 128–130.

28 Об одном классе сингулярных интегральных уравнений I рода // Изв. АН КазССР. Сер. физ.-мат. — 1975. — № 1. — С. 39–45. (Совместно с Р.У. Аргынбаевой и Ф.Г. Бирюковой).

29 Решение в малом задач с нелинейным граничными условиями для уравнения теплопроводности в расширяющейся области // Изв. АН КазССР. Сер. физ.-мат. — 1977. — С. 41–46. (Совместно с А.А. Кавокиным).

30 Решение уравнения параболического типа с переменными и разрывными коэффициентами // Изв. АН КазССР. Сер. физ.-мат. — 1977. — С. 71–75. (Совместно с Р.У. Аргынбаевой).

31 Математические модели тепловых процессов в электрических контактах // Изв. вузов. Электромеханика, 1978. — № 1. — С. 5–28. (Совместно с С.Н. Хариным и др.).

32 Решение одного класса нелинейных краевых задач // Изв. АН КазССР. Сер. физ.-мат. — 1979. — № 1. — С. 23–29. (Совместно с Я.А. Красновым. — С. Н. Хариным).

33 Об одном интегральном уравнении типа Вольтерра II рода // Изв. АН КазССР. Сер. физ.-мат. — 1980. — № 1. — С. 42–48. (Совместно с М.И. Рамазановым).

34 Об одном классе интегро-дифференциальных уравнений // Вестн. АН КазССР. — 1982. — № 5, С.38–48. (Совместно с Г.И. Бижановой).

35 Построение решения одного сингулярного интегрального уравнения типа Вольтерра-Фредгольма // Изв. АН КазССР. Сер. физ.-мат. — 1983. — № 3. — С. 54–58. (Совместно с М.О. Орынбасаровым).

36 Решение задачи теории теплопроводности с разрывным коэффициентом и вырождающимися подвижным границами // Изв. АН КазССР. Сер. физ.-мат. — 1984. — № 3. — С. 35–39. (Совместно с У.К. Койлышовым).

37 Об одной линейно-неклассической задаче для двумерного уравнения теплопроводности с разрывным коэффициентом // Изв. АН КазССР. Сер. физ.-мат. — 1986. — № 3. — С. 18–22. (Совместно с М.К. Абеновым).

38 Уравнение теплопроводности с разрывным коэффициентом в области, вырождающейся в начальный момент времени // Вестн. АН КазССР. — 1986. — № 9. — С. 39–45. (Совместно с У.К. Койлышовым).

39 О задаче с косой производной для двумерного уравнения теплопроводности в полуполосе // Изв. АН КазССР. Сер. физ.-мат. — 1987. — № 5. — С. 17–22. (Совместно с М.К. Абеновым).

40 Решение одного класса сингулярных интегральных уравнений. Ч. 1 // Изв. АН КазССР. Сер. физ.-мат. — 1991. — № 5. — С. 31–34.

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