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Редакторы
Ж.Т.Нурмуханова
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100012, Қазақстан, Қарағанды қ., Гоголь к-сі, 38. Тел. 51-38-20. E-mail: izd_kargu@mail.ru

Адрес редакции: 100024, Казахстан, г. Караганда, ул. Университетская, 28
Тел.: (7212) 77-03-69 (внутр. 1026); факс: (7212) 77-03-84.
E-mail: vestnick_kargu@ksu.kz. Сайт: vestnik.ksu.kz

Редактор
Ж.Т.Нурмуханова
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Postal address: 28, University Str., 100024, Kazakhstan, Karaganda
Tel.: (7212) 77-03-69 (add. 1026); fax: (7212) 77-03-84.
E-mail: vestnick_kargu@ksu.kz. Web-site: vestnik.ksu.kz

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D.M. Akhmanova, K.E. Kervenev, A.M. Baltabayeva

Ye.A. Buketov Karaganda State University, Kazakhstan
(E-mail: danna.67@ mail.ru)

On singular integral equations with variable limits of integration

The wide range of problems of mathematical physics is reduced to a special Volterra integral equation of the second kind or to integral equations with variable limits of integration. Among such problems we can include: boundary value problems for spectrally loaded differential equations [1–4], inverse problems [5, 6], nonlocal problems [7], boundary value problems for domains with moving boundaries as the domain degenerates at the time [8, 9] and others. In the study of integral equations with a variable lower limit of integration, the operational method can not be used directly, since in this case the convolution theorem is not applicable. However, the Laplace transform can be used to study this kind of integral equation by applying the method of model solutions.

Keywords: model solution, integrals operator, specter, resolvent, characteristic numbers, eigenfunctions.

1 Method of model solutions

Consider the operator equation

$$\mathcal{M}\left[y(x)\right]=f(x), \tag{1}$$

where $\mathcal{M}$ — some linear (integral) operator; $y(x)$ — sought; $f(x)$ — predetermined function [10].

Let $\mathcal{B}$ — be a certain well-known integral transformation

$$\mathcal{B}\left\{f(x)\right\}=\widehat{f}(p),$$

denote by $\psi(x,p)$ — the inverse transformation kernel $\mathcal{B}^{-1}$, which acts as follows:

$$f(x)=\mathcal{B}^{-1}\left\{\widehat{f}(p)\right\}\equiv\int\limits_a^b\widehat{f}(p)\psi(x,p)dp. \tag{2}$$

Here the limits a and b and the path of integration can lie in the complex plane.

Definition. The solution of equation (1), in which the right-hand side is the kernel of some inverse integral transformation, will be called the model solution of this equation.

Supposably $\widehat{y}(x,p)-$ the model solution of the auxiliary problem for equation (1), on the right-hand side of which there is a kernel of the inverse transformation $\mathcal{B}^{-1}$:

$$\mathcal{M}\left[\widehat{y}(x,p)\right]=\psi(x,p). \tag{3}$$

We multiply both sides of the equality (3) by $\widehat{f}(p)$, and integrate with respect to the parameter p within the same limits as in the inverse transformation (2). Since the operator $\mathcal{M}$ does not depend on p, using the equality $\mathcal{B}^{-1}\left\{\widehat{f}(p)\right\} = f(x)$, will have

$$\mathcal{M}\left\{\int\limits_a^b \widehat{y}(x,p)\widehat{f}(p)dp\right\} = f(x).$$

The last equality means that the solution of equation (1) for an arbitrary right-hand side $f(x)$ can be written in terms of the solution of the auxiliary equation (3) by the formula:

$$y(x) = \int\limits_a^b \widehat{y}(x,p)\widehat{f}(p)dp. \tag{4}$$

We apply this method to the solution of the second-kind Volterra equation with variable lower limit of integration

$$\varphi(t) - \lambda\int\limits_t^\infty k(t-\tau)\varphi(\tau)d\tau = f(t), \tag{5}$$

which can not be solved by a direct Laplace transform, since the convolution theorem is not applicable here.

We consider an auxiliary equation with exponential right-hand side

$$\varphi(t) - \lambda\int\limits_t^\infty k(t-\tau)\varphi(\tau)d\tau = e^{pt}$$

(function $e^{pt}-$ is the kernel of the inverse Laplace transform, $Re\,p > 0$).

We seek a solution of this equation in the form $\varphi(t,p) = A\cdot e^{pt}$. As a result, we get

$$\varphi(t,p) = \frac{1}{1-\lambda\widehat{k}(-p)}\cdot e^{pt}; \qquad \widehat{k}(-p) = \int\limits_0^\infty k(-z)\cdot e^{pz}dz.$$

From this, using formula (4), we obtain a solution of equation (5) for an arbitrary right-hand side in the form

$$\varphi(t) = \frac{1}{2\pi i}\cdot\int\limits_{c-i\infty}^{c+i\infty}\frac{\widehat{f}(p)}{1-\lambda\widehat{k}(-p)}\cdot e^{pt}dp,$$

where $\widehat{f}(p)-$ image of the function $f(t)$ obtained by means of the Laplace transform.

2 Solution of reference equations

The main task of this paper is to investigate the following *singular* integral equations:

$$\mathbf{K}_\lambda\mu \equiv (I-\lambda\mathbf{K})\mu \equiv \mu(t) - \lambda\int\limits_0^t \mathcal{K}(t-\tau)\mu(\tau)d\tau = f(t), \;\; t\in\mathbb{R}_+; \tag{6}$$

$$\mathbf{K}^*_\lambda\nu \equiv (I-\lambda\mathbf{K}^*)\nu \equiv \nu(t) - \overline{\lambda}\int\limits_t^\infty \mathcal{K}(\tau-t)\nu(\tau)d\tau = g(t), \;\; t\in\mathbb{R}_+, \tag{7}$$

where

$$\mathcal{K}(z) = \frac{1}{2\sqrt{\pi}z^{3/2}}\cdot\exp\left(-\frac{1}{4z}\right). \tag{8}$$

It should be noted that the kernel of the adjoint integral equation (7) — the function $\mathcal{K}(\tau - t)$ has the following property:

$$\int_t^\infty \mathcal{K}(\tau - t)d\tau = 1. \tag{9}$$

Equation (9) means that the norm of the integral operator acting in the space of summable functions and defined by the kernel $\mathcal{K}^*(\tau - t)$ is equal to one. This essentially distinguishes equation (7) from the Volterra equations of the second kind, for which the solution exists and is unique.

It is obvious that equation (7) is the union integral equation for (6).

We will solve these equations by the operational method [11]. First we investigate equation (7). As noted earlier, the Laplace transform is not directly applicable to this equation. Using the method of model solutions, we obtain

$$\widehat{\nu}(p) \cdot \left[1 - \overline{\lambda}\exp(-\sqrt{-p})\right] = \widehat{g}(p), \qquad Re\, p \leqslant 0, \tag{10}$$

where $\widehat{\nu}(p)$, $\widehat{g}(p)$ — the Laplace transform, or the functions $\nu(t)$ and $g(t)$. Function

$$\widehat{A}^*(p, \overline{\lambda}) = 1 - \overline{\lambda} \cdot \exp(-\sqrt{-p}).$$

We extend analytically to the whole complex plane with a cut along the positive real semiaxis.

We show that the homogeneous integral equation

$$\mathbf{K}^*_\lambda \nu \equiv (I - \lambda \mathbf{K}^*)\nu \equiv \nu(t) - \overline{\lambda}\int_t^\infty \mathcal{K}(\tau - t)\nu(\tau)d\tau = 0, \tag{11}$$

for some values $\lambda \in \mathbb{C}$ C has nonzero solutions. In order to find These non-trivial solutions and determine the corresponding values of λ, it is necessary clarify the picture of the zeros of the function $\widehat{A}^*(p, \overline{\lambda})$.

Assuming the parameter $\lambda \in \mathbb{C}$ to be given, we find the roots of equation

$$\widehat{\nu}(p) = 1 - \overline{\lambda}\exp(-\sqrt{-p}) = 0, \qquad p = s + i\sigma,$$

which for $|\overline{\lambda}| > 1$ have the form:

$$p_k = s_k + i\sigma_k = -\left[\ln^2|\overline{\lambda}| - (\arg\overline{\lambda} + 2k\pi)^2\right] - i2(\arg\overline{\lambda} + 2k\pi) \cdot \ln|\overline{\lambda}|, \qquad k \in \mathbb{Z}. \tag{12}$$

All the roots (12) are simple and are located on a parabola

$$s = \frac{1}{4\ln^2|\overline{\lambda}|} \cdot \sigma^2 - \ln^2|\overline{\lambda}|. \tag{13}$$

It is clear that the branches of the parabola are facing to the right, and the vertex of the parabola is located at the point $p = -\ln^2|\overline{\lambda}|$ on the real axis, and depending on the values $|\overline{\lambda}|$ is shifted left or right along the real axis of the complex plane of the variable p.

For $|\overline{\lambda}| < 1$ it is obvious that the function $\widehat{A}^*(p, \overline{\lambda})$ is not zero at any point of the complexed plane $p = s + i\sigma$ with a cut along the real positive semiaxis, since $|\exp(-\sqrt{-p})| > 1$.

But if $|\overline{\lambda}| = 1$, then the equation $|\overline{\lambda}| = |\exp(-\sqrt{-p})|$ with respect to the complex variable $\overline{\lambda}$ has a unique solution $\overline{\lambda} = 1$, which corresponds to the value $p = 0$.

The lines described by the equation $|\overline{\lambda}| = \exp(|\arg\lambda + 2k\pi|)$, divide the complex plane of the parameter λ into disjoint domains D_m, $m = 0, 1, 2, ...$, as follows:

$$D_{2n} = \left\{D_n^{(1)} \bigcap D_n^{(2)}\right\} \setminus \bigcup_{k=-1}^{2n-1} D_k, \;\; D_{-1} = \phi, \;\; D_{2n+1} = \left\{D_n^{(1)} \bigcup D_n^{(2)}\right\} \setminus \bigcup_{k=0}^{2n} D_k, \tag{14}$$

where

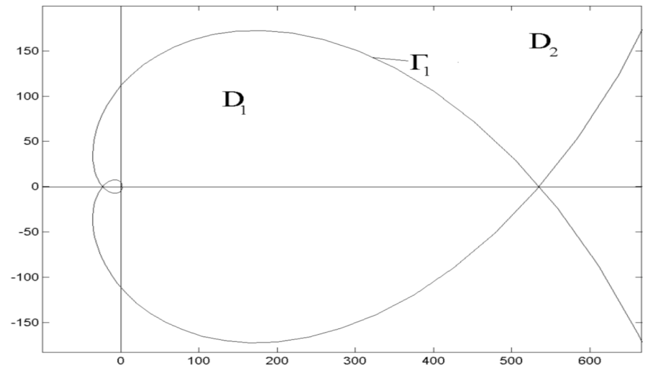
$$D_n^{(1)} = \{\overline{\lambda} : |\overline{\lambda}| < \exp[(2n+1)\pi - \arg\overline{\lambda}]\}, \;\; D_n^{(2)} = \{\overline{\lambda} : |\overline{\lambda}| < \exp[2n\pi + \arg\overline{\lambda}]\}, \;\; n = 0, 1, 2, ...$$

The outer parts of the boundaries $\partial D_m,\ \ m=0,1,2,...$, of the domains $D_m,\ \ m=0,1,2,...$, respectively are denoted by $\Gamma_m,\ \ m=0,1,2,....$ (see Picture 1).

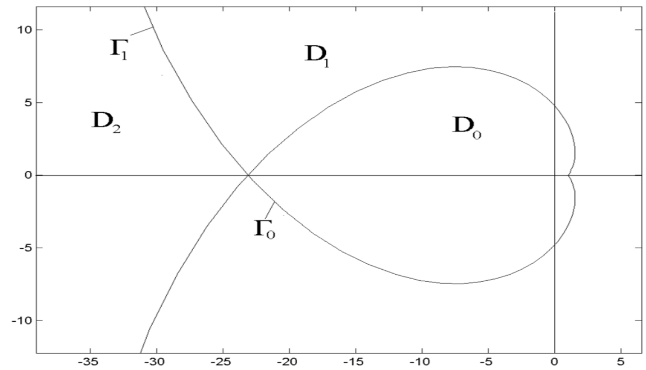
Remark 1. Note that in addition to the domain D_0 (see Picture 2) which has only the outer boundary $\Gamma_0=\partial D_0$, each of the domains D_m has the boundary ∂D_m, consisting of the outer Γ_m and the inner Γ_{m-1}:

$$\partial D_m=\Gamma_{m-1}\cup\Gamma_m \text{ and } \Gamma_{m-1}\cap\Gamma_m=(-1)^m\exp\{m\pi\},$$

i.e. the outer Γ_m and the inner Γ_{m-1} part of the boundary ∂D_m of the domain D_m have one common point lying on the real axis of the complex plane of the parameter λ.



Picture 1. Plane of the spectral parameter λ (reduced scale)



Picture 2. Plane of the spectral parameter λ (increased scale)

Thus, we get that $\overline{\lambda}\in\Gamma_m,\ \ m=0,1,2,...$, if and only if there exists at least one point p_0, for which $\widehat{A}^*(p_0,\overline{\lambda}))=0$.

Suppose that $|\overline{\lambda}|>1$. Then, according to (13) the function $\widehat{A}^*(p,\overline{\lambda})$ in the left half-plane can have only a finite number of zeros of the form (12), where

$$-N_1\le k\le N_2,\qquad N_1=\left[\frac{\ln|\overline{\lambda}|+\arg\overline{\lambda}}{2\pi}\right],\ \ N_2=\left[\frac{\ln|\overline{\lambda}|-\arg\overline{\lambda}}{2\pi}\right], \tag{15}$$

(here the symbol $[a]$ denotes the integer part of the number a, whereby the integer part of the negative number is set equal to zero). Indeed, the relations (15) follow from the condition that the real parts of the roots (12) must take negative values, that is $\operatorname{Re}\{p_k\}\leqslant 0$. Hence, from the inequality $(2\pi k+\arg)^2<\ln^2|\lambda|$ follows assertion (15).

Thus, for $|\overline{\lambda}|\geqslant 1$ the homogeneous equation (11) has a general solution of the form

$$\mu(t)=\sum_{k=-N_1}^{N_2}c_k\cdot e^{p_k t},$$

where c_k — are arbitrary constants, the numbers N_1 and N_2 are determined from the relations (15) (for given λ).

We now find a particular solution of the inhomogeneous equation (7). Suppose that the Laplace transform of $g(t)$ is analytic in the $-\varepsilon < Re\, p < \varepsilon$. Then from equation (10) for $\forall\lambda \notin \Gamma_m \ \ (m = 0, 1, 2, \ldots)$ we obtain

$$\widehat{\nu}(p) = \widehat{g}(p) + \lambda \frac{\exp(-\sqrt{-p})}{1 - \overline{\lambda}\exp(-\sqrt{-p})} \cdot \widehat{g}(p).$$

Passing in this relation to the originals we shall have

$$\nu(t) = g(t) + \overline{\lambda} \int_t^\infty r_{\lambda-}(t-\tau) g(\tau) d\tau,$$

where

$$r_{\lambda-}(y) = \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} \frac{\exp(-\sqrt{-p})}{1 - \overline{\lambda}\exp(-\sqrt{-p})} \exp(yp) dp. \tag{16}$$

If the roots of equation

$$1 - \overline{\lambda}\exp(-\sqrt{-p}) = 0$$

lie on the imaginary axis, then we will make the integration along the contour, bypassing these points on the left. The integral must be understood in the sense of Cauchy's principal value.

Since $y < 0$, we find the residue of the integrand in (16) along the right cut half-plane

$$r_{\lambda-}(y) = 2 \sum_{k=-\infty}^{-(N_1+1)} \sqrt{-p_k} \cdot \exp(p_k \cdot y) + 2 \sum_{k=N_2+1}^{\infty} \sqrt{-p_k} \cdot \exp(p_k \cdot y) +$$
$$+ \frac{1}{2\pi(-y)^{3/2}} \sum_{k=1}^{\infty} \frac{m}{\overline{\lambda}^m} \cdot \exp\left(\frac{m^2}{4y}\right), \qquad Re\, p_k > 0, \qquad |\overline{\lambda}| > 1, \qquad y \in \mathbb{R}_-, \tag{17}$$

the numbers N_1, N_2 and the roots p_k are determined from formulas (15) and (12), respectively.

But if $|\overline{\lambda}| \leqslant 1$, then

$$r_{\lambda-}(y) = \frac{1}{2\pi(-\theta)^{3/2}} \sum_{m=1}^{\infty} m \cdot \lambda^m \cdot \exp\left(\frac{m^2}{4y}\right), \qquad y \in \mathbb{R}_-. \tag{18}$$

Consequently, the general solution of the integral equation (7) for $|\lambda| > 1$ has the form

$$\nu(t) = g(t) + \overline{\lambda} \int_t^\infty r_{\lambda-}(t-\tau) g(\tau) d\tau + \sum_{k=-N_1}^{N_2} c_k \cdot \exp(p_k t), \tag{19}$$

where $r_{\lambda-}(\theta)$ — is determined from the equality (17), and if $|\lambda| < 1$, then the integral equation (7) has a unique solution

$$\nu(t) = g(t) + \overline{\lambda} \int_t^\infty r_{\lambda-}(t-\tau) g(\tau) d\tau, \tag{20}$$

where $r_{\lambda-}(\theta)$ is now found from equation (18), the numbers N_1, N_2, p_k are determined from equalities (15) and (12).

In order for the solution $\nu(t)$, defined by (19), (20), to be summable, it is sufficient that the function $r_{\lambda-}(t-\tau)$ be bounded for any $0 < \tau \leqslant t < \infty$, so 5 as a function $g_1(t) + \sum\limits_{k=-N_1}^{N_2} c_k \exp(p_k t)$ is an integrable function of t. The function $r_{\lambda-}(t-\tau)$ will be bounded, since $r_{\lambda-}(\theta)$ (17) satisfies the estimate:

$$|\mathbf{r}_{\lambda-}(\theta)| \leqslant C_1 |\theta|^{-1/2} \exp(-\delta_0 |\theta|) + C_2 |\theta|^{-3/2} \exp(-\delta_0 |\theta|^{-1}), \ \ \forall\, \theta \in \mathbb{R}_-, \tag{21}$$

where
$$\delta_0 = \min\left\{1/4;\ [2\pi(N_1+1) + \arg\lambda]^2 - \ln^2|\lambda|;\ [2\pi(N_2+1) + \arg\lambda]^2 - \ln^2|\lambda|\right\}. \qquad (22)$$

The validity of the estimate (21) follows from the relations below. For the second we obtain from (17):

$$\left|\sum_{k=N_2+1}^{\infty}\sqrt{-p_k}\exp(p_k\theta)\right| \leqslant |\ln\lambda|\sum_{k=N_2+1}^{\infty}|\exp(p_k\theta)| \leqslant$$

$$\leqslant |\ln\lambda|\sum_{k=N_2+1}^{\infty}\exp\{\left[(2k\pi+\arg\lambda)^2 - \ln^2|\lambda|\right]\theta\} \leqslant \left\|\begin{array}{c} y = 2k\pi + \arg\lambda \\ a = 2\pi(N_2+1) + \ln|\lambda| \end{array}\right\| \leqslant$$

$$\leqslant |\ln\lambda|\int\limits_a^\infty \exp\{(y^2 - \ln^2|\lambda|)\theta\}dy = |\ln\lambda|\exp\{-\theta\ln^2|\lambda|\}\int\limits_a^\infty \exp\{\theta y^2\}dy =$$

$$= \|z = y - a\| = |\ln\lambda|\exp\{-\theta\ln^2|\lambda|\}\int\limits_0^\infty \exp\{\theta(a^2+z^2+2az)\}dz =$$

$$= |\ln\lambda|\exp\{-\theta\ln^2|\lambda| + \theta a^2\}\int\limits_0^\infty \exp\{\theta z^2 + \theta 2az\}dz \leqslant$$

$$\leqslant |\ln\lambda|(-\theta)^{-1/2}\cdot\exp\{\theta(a^2 - \ln^2|\lambda|)\}\int\limits_0^\infty \exp\{-(\sqrt{-\theta}z)^2\}d(\sqrt{-\theta}z) = |\ln\lambda|\frac{\sqrt{\pi}}{2\sqrt{-\theta}}\exp\{\delta_2\theta\},$$

where $\delta_2 = [2\pi(N_2+1) + \arg\lambda]^2 - \ln^2|\lambda| > 0$.

Similarly for the first term we have the inequality:

$$\left|\sum_{k=-\infty}^{-(N_1+1)}\sqrt{-p_k}\exp(p_k\theta)\right| \leqslant |\ln\lambda|\frac{\sqrt{\pi}}{2\sqrt{-\theta}}\exp\{\delta_1\theta\},$$

where $\delta_1 = [2\pi(N_1+1) + \arg\lambda]^2 - \ln^2|\lambda| > 0$.

The third term in (17) is estimated as follows:

$$|\theta|^{-3/2}\sum_{m=1}^{\infty}\frac{m}{\lambda^m}\exp\left(-\frac{m^2}{4|\theta|}\right) = |\theta|^{-3/2}\exp\left\{-\frac{1}{4|\theta|}\right\}\sum_{m=1}^{\infty}\frac{m}{\lambda^m}\exp\left(-\frac{m^2-1}{4|\theta|}\right) \leqslant$$

$$\leqslant C|\theta|^{-3/2}\exp\left\{-\frac{1}{4|\theta|}\right\}.$$

For the representation from (18) for $|\lambda| = 1$ we obtain the estimate:

$$|\theta|^{-3/2}\sum_{m=1}^{\infty}m\exp\left(-\frac{m^2}{4|\theta|}\right) \leqslant \frac{2}{\sqrt{|\theta|}}\int\limits_1^\infty \exp\left(-\frac{y^2}{4|\theta|}\right)d\left(-\frac{y^2}{4|\theta|}\right) = \frac{2}{\sqrt{|\theta|}}\exp\left(-\frac{1}{4|\theta|}\right);$$

and for $|\lambda| < 1$ we have the estimate:

$$|\theta|^{-3/2}\sum_{m=1}^{\infty}m\lambda^m\exp\left(-\frac{m^2}{4|\theta|}\right) \leqslant C|\theta|^{-3/2}\exp\left(-\frac{1}{4|\theta|}\right).$$

It is easy to verify that (19) is a solution of equation (7) for arbitrary coefficients c_k.ck.

We state our results in the form of the following theorems.

Theorem 1. The values $\lambda \in D_0$ in (14) are regular numbers of the operator (16).

Theorem 2. The set $\mathbb{C} \setminus D_0$ consists of the characteristic numbers of the operator $\mathbf{K}^*$ (7). Moreover, if $\lambda \in D_m \cup \Gamma_{m-1} \setminus \{(-1)^m e^{m\pi}\}$, $m = 1, 2, ...,$ then $\dim \operatorname{Ker}(\mathbf{K}^*) = m$; and the corresponding eigenfunctions have the form

$$\nu_{\lambda k}(t) = \exp(p_k t), \quad k = 1, ..., m = N_1 + N_2 + 1,$$

where the numbers p_k, N_1, N_2 are determined from the equalities (12), (15).

Now consider the integral equation (6), which is usually called the *recovery equation* [12]. This name is explained by the fact that such equations arise in the theory of recovery — the section of probability theory, which describes a wide range of phenomena associated with the failure and restoration of the elements of a system. The reconstruction equation is of great importance also in the study of both applied and theoretical problems in reliability theory, queuing theory, in reserve theory, in the theory of branching processes, and so on.

Applying the Laplace transform to (6) and using the convolution theorem in this case, we obtain

$$\widehat{\mu}(p) = \widehat{f}(p) + \frac{\lambda e^{-\sqrt{p}}}{1 - \lambda e^{-\sqrt{p}}} \widehat{f}(p), \qquad p = s + i\sigma, \qquad Re\, p = s > 0.$$

Using the inverse Laplace transform, we have:

$$\mu(p) = f(t) + \lambda \int_0^t r_{\lambda+}(t - \tau) f(\tau) d\tau, \tag{23}$$

where the resolvent $r_{\lambda+}(\theta)$ is defined in terms of the kernel of the original equation (6) by the formula

$$r_{\lambda+}(\theta) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\lambda e^{-\sqrt{p}}}{1 - \lambda e^{-\sqrt{p}}} e^{p\theta} dp, \qquad p = s + i, \tag{24}$$

the path of integration is parallel to the imaginary axis of the complex plane to the right of all the singular points of the integrand, that is, to the right of all zeros of the function

$$\widehat{\mathbf{A}}(p, \lambda) = 1 - \lambda \cdot \exp(-\sqrt{p}).$$

The zeros of the function $\widehat{\mathbf{A}}(p, \lambda)$ have the form:

$$p_n = s_n + i\sigma_n = \left[\ln^2 |\overline{\lambda}| - (\arg \lambda + 2n\pi)^2\right] + i2(\arg \lambda + 2n\pi) \cdot \ln \lambda, \qquad n \in \mathbb{Z}; \tag{25}$$

they are all simple and arranged on a parabola

$$s = -\frac{\sigma^2}{4 \ln^2 |\overline{\lambda}|} + \ln^2 |\overline{\lambda}|,$$

the branches of which are turned to the left, and the vertex is on the real axis at the point

$$s_0 = \ln^2 |\overline{\lambda}|.$$

We note that the function $\widehat{\mathbf{A}}(p, \lambda)$ in the right half-plane can have only a finite the number of zeros of p_n

$$-N_1 \leqslant n \leqslant N_2, \qquad N_1 = \left[\frac{\ln |\lambda| + \arg \lambda}{2\pi}\right], \qquad N_2 = \left[\frac{\ln |\lambda| - \arg \lambda}{2\pi}\right] \tag{26}$$

(here again the square bracket denotes the integer part of the number). Moreover, their number increases with increasing $|\lambda|$, if $\lambda \in D_m$ (14), then their number is $m = N_1 + N_2 + 1$. Note that if $\lambda \in D_0$, then the function $\widehat{\mathbf{A}}(p, \lambda)$ does not have zeros at all in the complex plane.

Let us calculate the integral (24). We continue the integrand analytically on the whole complex plane with a cut along the negative real axis. Then, according to the theory of residues, we get:

$$r_{\lambda+}(\theta) = 2 \sum_{n=-\infty}^{+\infty} \sqrt{p_n} \cdot \exp(p_n \cdot \theta) + \frac{1}{2\sqrt{\pi}\theta^{3/2}} \sum_{m=1}^{\infty} \frac{m}{\lambda_m} \cdot \exp\left(-\frac{m^2}{4\theta}\right);$$

$$Re\, p_n > 0, \quad |\lambda| > 1, \quad \theta \in \mathbf{R}_+;$$

$$r_{\lambda+}(\theta) = \frac{1}{2\sqrt{\pi}\theta^{3/2}} \sum_{m=1}^{\infty} m\lambda^m \cdot \exp\left(- \frac{m^2}{4\theta} \right), \qquad |\lambda| \leqslant 1, \qquad \theta \in \mathbf{R}_+, \tag{27}$$

where the numbers p_n are found from (25).

In order that the function $\mu(t)$ defined by (23) be essentially bounded, it is necessary and sufficient that the following conditions are satisfied:

$$\int_0^\infty f(t) \cdot \exp(-p_k t) dt = 0, \qquad -N_1 \leqslant k \leqslant N_2 \tag{28}$$

where the numbers N1, N2N_1, N_2 are determined from the equalities (26), p_k from the equality (25).

Indeed, in this case the resolvent of the integral equation (6) for $|\lambda| > 1$ will have the form (6) and at it will have the form $|\lambda| > 1$

$$r_{\lambda+}(\theta) = 2 \sum_{n=-\infty}^{-(N_1+1)} \sqrt{p_n} \cdot \exp(p_n \cdot \theta) + 2 \sum_{n=N_2+1}^{\infty} \sqrt{p_n} \cdot \exp(p_n \cdot \theta) +$$

$$+ \frac{1}{2\sqrt{\pi}\theta^{3/2}} \sum_{m=1}^{\infty} \frac{m}{\lambda^m} \cdot \exp\left(- \frac{m^2}{4\theta} \right), \; Re\, p_k > 0, \quad \theta \in \mathbf{R}_+, \tag{29}$$

and will be a summable function, since it has the estimate

$$\left| r_{\lambda+} \right| \leqslant C \cdot |\theta|^{-\frac{1}{2}} \cdot \exp(-\delta_0 |\theta|) + C_2 \cdot |\theta|^{-\frac{3}{2}} \cdot \exp(-\delta_0 |\theta|^{-1}) \qquad \forall \theta \in \mathbf{R}_+,$$

in which the constant δ_0 determines their equalities (22).

Thus it is fair

Theorem 3. If $\lambda \in D_0$, then the inhomogeneous equation (6) is unconditionally uniquely solvable; if $\lambda \in \mathbb{C} \setminus D_0$, $\lambda \in D_m$, then for the unique solvability of (6) it is necessary and sufficient that $m-$ solvability conditions (28) be satisfied. The conditions (28) mean that the free term of the integral equation (6) must be orthogonal to the solutions of the homogeneous conjugate integral equation (7).

The validity of these statements, as well as of conditions (28), can also be shown in the following way. The image of the solution of the integral equation (6) is defined by

$$\widehat{\mu}(p) = \frac{\widehat{f}(p)}{1 - \lambda e^{-\sqrt{p}}}. \tag{30}$$

The following options are possible.

1. The function $\widehat{\mathbf{A}}(p, \lambda) = 1 - \lambda \cdot \exp(-\sqrt{p})$ does not have zeros in the right half-plane (this means that $|\lambda| < 1$ and $\lambda \in D_0$ (14). In this case, the equation for any right-hand member $f(t)$ has a unique solution that is expressed in terms of the resolvent $r_{\lambda+}(\theta)$, defined by formula (27)

$$\mu(t) = f(t) + \lambda \int_0^t r_{\lambda+}(t - \tau) f(\tau) d\tau, \quad t \in \mathbf{R}_+. \tag{31}$$

2. The function $\widehat{f}(p)$ vanishes at the points p_n, $\quad N_1 \leqslant n \leqslant N_2$ from (25), that is, in the zeros of the function $\widehat{\mathbf{A}}(p, \lambda)$ located in the right half-plane. In this case, the function (30) again will not have poles in the region $Re\ p > 0$, so equation (6) also has a unique solution of the form (31), but the resolvent $r_{\lambda+}(\theta)$ is now determined from (29). The condition $\widehat{f}(p_n) = 0$, $N_1 \leqslant n \leqslant N_2$, on the inversion of the function $\widehat{f}(p)$ to zero at the points $p = p_n$ is equivalent to the following conditions

$$\int_0^\infty f(t) \cdot e^{-p_n t} dt = 0; \quad N_1 \leqslant n \leqslant N_2.$$

So we have proved the following statement.

Lemma. On the complex plane $\mathbb{C}$ there are no characteristic numbers of the operator $\mathbf{K}$ (9).

Thus, it follows from the results obtained that the solutions of the integral equations (10) and (9) are determined by expressions

$$\nu_\lambda(t) = g(t) + \overline{\lambda} \int\limits_t^\infty r_{\lambda-}(t-\tau) g(\tau) d\tau + \sum_{k=-N_1}^{N_2} c_k \cdot \exp(p_k t), \quad t \in \mathbf{R}_+,$$

where the numbers p_k, N_1, N_2 are determined from the equalities (12), (15),

$$\mu_\lambda(t) = f(t) + \lambda \int\limits_0^t r_{\lambda+}(t-\tau) f(\tau) d\tau, \qquad t \in \mathbf{R}_+,$$

and satisfy the conditions

$$\nu_\lambda(t) \in L_1(\mathbb{R}_+), \qquad \mu_\lambda(t) \in L_\infty(\mathbb{R}_+).$$

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Д.М. Ахманова, Қ.Е. Кервенев, А.М. Балтабаева

Айнымалы шекті интегралды ерекше интегралдық теңдеулер жайлы

Математикалық физика есептерінің кең аумағы Вольтерраның екінші текті арнайы интегралдық теңдеуіне, немесе айнымалы шекті интегралды интегралдық теңдеулерге, келтірілді. Мұндай есептердің ішінде келесі есептерді атап өтуге болады: спектралды жүктелген дифференциалдық теңдеулер үшін шекаралық есептер [1–4], кері есептер [5, 6], локалды емес есептер [7], жылжымалы шекаралы облыстар үшін шекаралық есептер [8, 9] т.б. Төменгі айнымалы шекті интегралданатын интегралдық теңдеулерді оқып-үйрену кезінде жұмыс әдісі, свертка теоремасын пайдалануға болмайтындықтан, бірден қолданылмайды. Алайда мұндай интегралдық теңдеулерді оқып-үйрену үшін модельдік шешу әдісін қолдана отырып, Лаплас түрлендіруін қолдануға болады.

Кілт сөздер: модельді шешу, интегралдық оператор, спектр, резольвента, сипаттамалық сандар, меншікті функциялар.

Д.М. Ахманова, К.Е. Кервенев, А.М. Балтабаева

Об особых интегральных уравнениях с переменными пределами интегрирования

Широкий спектр задач математической физики сводится к специальному интегральному уравнению Вольтерра второго рода или к интегральным уравнениям с переменными пределами интегрирования. Среди таких задач можно выделить: краевые задачи для спектрально нагруженных дифференциальных уравнений [1–4], обратные задачи [5, 6], нелокальные задачи [7], краевые задачи для областей с движущимися границами, когда область вырождается [8, 9] и др. При изучении интегральных уравнений с переменным нижним пределом интегрирования рабочий метод не может быть использован непосредственно, так как в этом случае неприменима теорема свертки. Однако для изучения такого интегрального уравнения можно использовать преобразование Лапласа путем применения метода модельных решений.

Ключевые слова: модельное решение, интегральный оператор, спектр, резольвента, характеристические числа, собственные функции.

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M.T. Jenaliyev[1], M.I. Ramazanov[2], M.T. Kosmakova[2], A.O. Tanin[2]

[1] *Institute of Mathematics and Mathematical Modeling, Almaty, Kazakhstan;*
[2] *Ye.A. Buketov Karaganda State University, Kazakhstan*
(E-mail: Svetik_mir69@mail.ru)

To the solution of one pseudo-Volterra integral equation

In this paper, we study a homogeneous singular integral Volterra equation of the second kind (pseudo-Volterra integral equation). The singularity of the integral equation is shown. Properties of its kernel are proved. The characteristic equation is constructed. It is shown that it really is a characteristic equation for the studied integral equation. The kernel estimates of its integral operator are established. Solvability issues of the corresponding non-homogeneous integral equation are also researched. The weight class of the uniqueness for its solution is determined. A weight class is also established for the right side of the nonhomogeneous equation under study. The weight class of the uniqueness for its solution is defined on the basis of estimates for the kernel of the integral operator of the equation.

Keywords: characteristic equation, kernel, integral operator, class of essentially bounded functions.

Introduction

We study the solvability of a pseudo-Volterra integral equation:

$$\varphi(t) + \int_0^t K_\omega(t,\tau)\varphi(\tau)d\tau = 0, \tag{1}$$

where the kernel $K_\omega(t,\tau)$ is representable as a sum:

$$K_\omega(t,\tau) = \sum_{i=1}^{4} K_\omega^{(i)}(t,\tau),$$

and

$$K_\omega^{(1)} = \frac{1}{2a\sqrt{\pi}} \cdot \frac{t^\omega + \tau^\omega}{(t-\tau)^{3/2}} \cdot \exp\left\{-\frac{(t^\omega + \tau^\omega)^2}{4a^2(t-\tau)}\right\};$$

$$K_\omega^{(2)} = -\frac{1}{2a\sqrt{\pi}} \cdot \frac{t^\omega - \tau^\omega}{(t-\tau)^{3/2}} \cdot \exp\left\{-\frac{(t^\omega - \tau^\omega)^2}{4a^2(t-\tau)}\right\};$$

$$K_\omega^{(3)} = -\frac{1}{a\sqrt{\pi}} \cdot \frac{1+\omega t^{\omega-1}}{(t-\tau)^{1/2}} \exp\left\{-\frac{(t^\omega + \tau^\omega)^2}{4a^2(t-\tau)}\right\};$$

$$K_\omega^{(4)} = \frac{1}{a\sqrt{\pi}} \cdot \frac{1+\omega t^{\omega-1}}{(t-\tau)^{1/2}} \exp\left\{-\frac{(t^\omega - \tau^\omega)^2}{4a^2(t-\tau)}\right\}.$$

This kind of integral equations arise in solving the following boundary value problem:

$$\frac{\partial u(x,t)}{\partial t} - a^2\frac{\partial^2 u(x,t)}{\partial x^2} = 0, \quad \{(x,t)\,|\,0 < x < t^\omega, t > 0\};$$

$$\frac{\partial u}{\partial x}\Big|_{x=0} = 0, \quad \frac{d\tilde{u}(t)}{dt} + \frac{\partial u}{\partial x}\Big|_{x=t^\omega} = 0,$$

where $\tilde{u}(t) = u(t^\omega, t), \omega > \frac{1}{2}$.

We will search for the solution of the integral equation (1) in the class of functions

$$t^{\frac{3}{2}-\omega} \cdot \varphi(t) \in L_\infty(0,\infty);$$

i.e.

$$\varphi(t) \in L_\infty(0,\infty; t^{\frac{3}{2}-\omega}).$$

The equation (1) can be written as:

$$\varphi(t) + \int_0^t \left(\frac{t}{\tau}\right)^{\frac{3}{2}-\omega} K_\omega(t,\tau)\varphi(\tau)d\tau = 0. \qquad (2)$$

Volterra integral equations of this kind were considered in papers [1–3].

1 About properties of the kernel $K_\omega(t,\tau)$

We note that the kernel $K_\omega(t,\tau)$ has the properties:
1)$K_\omega(t,\tau) \geq 0$ and is continuous when $0 < \tau \leq t \leq \infty$;
2) $\lim_{t\to t_0} \int_{t_0}^t K_\omega(t,\tau)d\tau = 0,\ t_0 \geq \varepsilon > 0$;
3) $\lim_{t\to 0+} \int_0^t K_\omega(t,\tau)d\tau = 1$.
The singularity of the integral equation (1) is property 3 of kernel $K_\omega(t,\tau)$. We prove this property.
Lemma 1. If $\omega > \frac{1}{2}$, then

$$\lim_{t\to 0+} \int_0^t K_\omega^{(1)}(t,\tau)d\tau = 1.$$

Proof. We have:

$$\int_0^t K_\omega^{(1)}(t,\tau)d\tau = \frac{1}{2a\sqrt{\pi}} \cdot \int_0^t \frac{t^\omega + \tau^\omega}{(t-\tau)^{3/2}} \exp\left\{-\frac{(t^\omega+\tau^\omega)^2}{4a^2(t-\tau)}\right\} d\tau =$$

$$= \frac{2}{\sqrt{\pi}} \cdot \int_0^t \left[\frac{t^\omega + \tau^\omega}{4a(t-\tau)^{3/2}} + \frac{\omega\cdot\tau^{\omega-1}}{2a\sqrt{t-\tau}}\right] \cdot \exp\left\{-\frac{(t^\omega+\tau^\omega)^2}{4a^2(t-\tau)}\right\} d\tau-$$

$$-\frac{1}{a\sqrt{\pi}} \cdot \int_0^t \frac{\omega\cdot\tau^{\omega-1}}{\sqrt{t-\tau}} \cdot \exp\left\{-\frac{(t^\omega+\tau^\omega)^2}{4a^2(t-\tau)}\right\} d\tau = J_1 - J_2.$$

In the integral J_1 we make a replacement

$$\frac{t^\omega + \tau^\omega}{2a(t-\tau)^{1/2}} = x.$$

As a result, we get

$$J_1 = \frac{2}{\sqrt{\pi}} \cdot \int_{\frac{t^{\omega-\frac{1}{2}}}{2a}}^\infty \exp(-x^2)dx = 1 - erf\left\{\frac{t^{\omega-\frac{1}{2}}}{2a}\right\}.$$

We estimate the second integral J_2:

$$J_2 = \frac{1}{a\sqrt{\pi}} \cdot \int_0^t \frac{\omega\cdot\tau^{\omega-1}}{\sqrt{t-\tau}} \cdot \exp\left\{-\frac{(t^\omega+\tau^\omega)^2}{4a^2\,(t-\tau)}\right\} d\tau \leq$$

$$\leq \frac{\omega\cdot\tau^{\omega-\frac{1}{2}}}{a\sqrt{\pi}} \cdot \int_0^t \frac{d\tau}{\sqrt{\tau\cdot(t-\tau)}} = \frac{\omega\cdot\sqrt{\pi}}{a} t^{\omega-\frac{1}{2}}.$$

As from conditions $\omega > \dfrac{1}{2}$ and $t \to 0$ it follows $erf\left\{\dfrac{t^{\omega-\frac{1}{2}}}{2a}\right\} \to 0$, then from here we get the required ratio.

Lemma 2. If $\omega > \dfrac{1}{2}$, then

$$\lim_{t\to 0} \int_0^t K_\omega^{(i)}(t,\tau)d\tau = 0, \quad i = 2,3,4.$$

Proof. Let be $\dfrac{1}{2} < \omega \leq 1$. In this case, we will have at $t \to 0$:

$$0 \leq \int_0^t K_\omega^2(t,\tau)d\tau \leq \frac{\omega}{2a\sqrt{\pi}} \cdot \int_0^t \frac{\tau^{\omega-1}}{\sqrt{t-\tau}} d\tau = \frac{\omega}{2a\sqrt{\pi}} \cdot B\left(\omega, \frac{1}{2}\right) \cdot t^{\omega-\frac{1}{2}} \to 0.$$

In this case, we have used the following double inequality [4; 55] for all $0 < \tau \le t \le \infty$:

$$\omega \cdot t^{\omega-1}(t-\tau) \le t^{\omega} - \tau^{\omega} \le \omega \cdot \tau^{\omega-1}(t-\tau).$$

at $0 < \omega \le 1$.

Now let be $\omega > 1$. Then we have:

$$\int_0^t K_\omega^2(t,\tau)d\tau \le \frac{1}{2a\sqrt{\pi}} \cdot \int_0^t \frac{t^\omega - \tau^\omega}{(t-\tau)^{3/2}}d\tau =$$

$$= \left\| \tau = t \cdot \sin^2\alpha,\ d\tau = 2t \cdot \sin\alpha \cdot \cos\alpha d\alpha \right\| =$$

$$= \frac{2 \cdot t^{\omega-\frac{1}{2}}}{2a\sqrt{\pi}} \cdot \int_0^{\frac{\pi}{2}} \frac{\sin\alpha(1-\sin^{2\omega}\alpha)}{\cos^{\frac{1}{2}}\alpha}d\alpha = C_1(\omega)\frac{2 \cdot t^{\omega-\frac{1}{2}}}{2a\sqrt{\pi}}.$$

The last integral is bounded (i.e. a number $C_1(\omega)$ is bounded) due to the existence of a finite limit for the integrand at a unique singular point $\alpha = \frac{\pi}{2}$. Indeed, calculating the next limit:

$$\lim_{\alpha\to\frac{\pi}{2}} \frac{\sin\alpha(1-\sin^{2\omega}\alpha)}{\cos^{\frac{1}{2}}\alpha} = \lim_{\alpha\to\frac{\pi}{2}} \frac{(1-\sin^{2\omega}\alpha)}{\cos^{\frac{1}{2}}\alpha} = 4\omega \cdot \lim_{\alpha\to\frac{\pi}{2}} \sin^{2\omega-1}\alpha \cdot \cos^{\frac{3}{2}}\alpha = 0,$$

we obtain at $\omega > 1$ required limit

$$\lim_{t\to 0}\int_0^t K_\omega^2(t,\tau)d\tau = 0.$$

For the kernels $K_\omega^3(t,\ \tau)$, $K_\omega^4(t,\tau)$ the proof is obvious.

Lemma 2 is proved.

In the sequel, we need the following lemmas.

Lemma 3. If $\omega > \frac{1}{2}$, then

$$t^{\frac{3}{2}-\omega}\int_0^t \frac{K_\omega^{(1)}(t,\tau)}{\tau^{\frac{3}{2}-\omega}}d\tau < C = const.\ 0 < t < \infty.$$

Proof. Indeed, we have:

$$t^{\frac{3}{2}-\omega}\int_0^t \frac{K_\omega^{(1)}(t,\tau)}{\tau^{\frac{3}{2}-\omega}}d\tau = \frac{1}{2a\sqrt{\pi}} \cdot \int_0^t \frac{t^{\frac{3}{2}-\omega}}{\tau^{\frac{3}{2}-\omega}} \frac{t^\omega + \tau^\omega}{(t-\tau)^{\frac{3}{2}}} \cdot \exp\left\{-\frac{(t^\omega+\tau^\omega)^2}{4a^2(t-\tau)}\right\} d\tau \le$$

$$\le \frac{1}{2a\sqrt{\pi}} \cdot \int_0^t \frac{t^{\frac{3}{2}-\omega}}{\tau^{\frac{3}{2}-\omega}} \frac{2t^\omega}{(t-\tau)^{\frac{3}{2}}} \cdot \exp\left\{-\frac{t^{2\omega}}{4a^2(t-\tau)}\right\} d\tau \le$$

$$\le \left\| \frac{\tau}{t} = 1-x;\quad \frac{t^\omega}{2a\sqrt{t-\tau}} = \frac{t^{\omega-\frac{1}{2}}}{2a\sqrt{x}};\ \frac{t^\omega dx}{4a(t-\tau)^{\frac{3}{2}}} = -\frac{t^{\omega-\frac{1}{2}}dx}{4ax^{\frac{3}{2}}} \right\| =$$

$$= \frac{1}{a\sqrt{\pi}}\int_0^1 t^{\omega-\frac{1}{2}} \cdot \frac{(1-x)^{\omega-\frac{3}{2}}}{x^{\frac{3}{2}}} \cdot \exp\left\{-\frac{t^{2\omega-1}}{4a^2x}\right\} dx =$$

$$= \frac{t^{\omega-\frac{1}{2}}}{a\sqrt{\pi}}\left[\int_0^{\frac{1}{2}} \frac{(1-x)^{\omega-\frac{3}{2}}}{x^{\frac{3}{2}}} \cdot \exp\left\{-\frac{t^{2\omega-1}}{4a^2x}\right\} dx + \int_{\frac{1}{2}}^1 \frac{(1-x)^{\omega-\frac{3}{2}}}{x^{\frac{3}{2}}} \cdot \exp\left\{-\frac{t^{2\omega-1}}{4a^2x}\right\} dx\right] \le$$

$$\le \frac{C(\omega)}{a\sqrt{\pi}}\int_0^{\frac{1}{2}} \frac{t^{\omega-\frac{1}{2}}}{x^{\frac{3}{2}}} \cdot \exp\left\{-\frac{t^{2\omega-1}}{4a^2x}\right\} dx + \frac{2^{\frac{3}{2}}t^{\omega-\frac{1}{2}}}{a\sqrt{\pi}}\int_{\frac{1}{2}}^1 (1-x)^{\omega-\frac{3}{2}} \cdot \exp\left\{-\frac{t^{2\omega-1}}{4a^2}\right\} dx \equiv J,$$

where

$$C(\omega) = \begin{cases} 1, & \omega \ge \frac{3}{2}, \\ 2^{\frac{3}{2}-\omega}, & \frac{1}{2} \le \omega < \frac{3}{2}, \end{cases} \quad \max_{\frac{1}{2}<\omega<\infty} C(\omega) = 2.$$

Further, if we make the following replacement in the first integral:

$$\xi = \frac{t^{\omega-\frac{1}{2}}}{2a\sqrt{x}}, \quad d\xi = -\frac{t^{\omega-\frac{1}{2}}}{4ax^{\frac{3}{2}}}dx$$

and leave the second integral unchanged, then we will have

$$J = \frac{8}{\pi}\int_{\frac{t^{\omega-\frac{1}{2}}}{a\sqrt{2}}}^{\infty} \exp\left\{-\xi^2\right\}dx - \frac{2^{\frac{3}{2}}t^{\omega-\frac{1}{2}}}{a\sqrt{\pi}} \cdot \exp\left\{-\frac{t^{2\omega-1}}{4a^2}\right\} \cdot \left.\frac{(1-x)^{\omega-\frac{1}{2}}}{\omega-\frac{1}{2}}\right|_{\frac{1}{2}}^{1} =$$

$$= 4\,erfc\left(\frac{t^{\omega-\frac{1}{2}}}{a\sqrt{2}}\right) + \frac{2^{4-\omega}}{a(2\omega-1)} \cdot \frac{t^{\omega-\frac{1}{2}}}{2a} \cdot \exp\left\{-\frac{t^{2\omega-1}}{4a^2}\right\} \le C < \infty.$$

Hence the assertion of Lemma 3 follows.

Lemma 4. If $\omega > \frac{1}{2}$, then

$$t^{\frac{3}{2}-\omega}\int_0^t \frac{K_\omega^{(2)}(t,\tau)}{\tau^{\frac{3}{2}-\omega}}d\tau < C(\omega) = const. \quad 0 < t < \infty.$$

Proof. We have:

$$t^{\frac{3}{2}-\omega}\int_0^t \frac{K_\omega^{(2)}(t,\tau)}{\tau^{\frac{3}{2}-\omega}}d\tau = \frac{1}{2a\sqrt{\pi}} \cdot \int_0^t \left(\frac{t}{\tau}\right)^{\frac{3}{2}-\omega} \frac{t^\omega - \tau^\omega}{(t-\tau)^{\frac{3}{2}}} \cdot \exp\left\{-\frac{(t^\omega-\tau^\omega)^2}{4a^2(t-\tau)}\right\}d\tau \le$$

$$\le \frac{1}{\sqrt{\pi}}\left\{\frac{(1-x^\omega)|_{x=0}}{(1-x)^{\frac{3}{2}}\Big|_{x=1}}\left[\frac{t^{\omega-\frac{1}{2}}}{2a} \cdot \exp\left\{-\frac{t^{2\omega-1}}{4a^2} \cdot \frac{(1-x^\omega)^2\big|_{x=\frac{1}{2}}}{(1-x)|_{x=0}}\right\}\right]\int_0^{\frac{1}{2}} x^{\omega-\frac{3}{2}}dx + \right.$$

$$\left. + \sup_{\frac{1}{2}<x<1}\left\{x^{\omega-\frac{3}{2}}\right\}\left[\frac{t^{\omega-\frac{1}{2}}}{2a} \cdot \exp\left\{-\frac{t^{2\omega-1}}{4a^2} \cdot \frac{(1-\overline{x}^\omega)^2}{1-\overline{x}}\right\}\right]\int_{\frac{1}{2}}^1 \frac{1-x^\omega}{(1-x)^{\frac{3}{2}}}d\,x\right\} \le \ C(\omega) = const.$$

For the kernels $K_\omega^3(t,\,\tau)$, $K_\omega^4(t,\tau)$ the proof is obvious.

Lemma 4 is proved.

Remark 1. Since, according to Lemma 1, there is the singularity of the kernel $K_\omega^{(1)}(t,\tau)$, for that the statement of Lemma 3 holds, then for the kernel $K_\omega^{(2)}(t,\tau)$ with a weak singularity (Lemma 2), the statement of the Lemma 4 becomes obvious. This has showed the above proof of the Lemma 4.

2 Characteristic integral equation. Estimates for the kernels of integral operators

For the integral equation (2) we will construct a characteristic equation

$$\varphi(t) + \int_0^t \left(\frac{t}{\tau}\right)^{\frac{3}{2}-\omega} K_h(t,\tau) \cdot \varphi(\tau)\,d\tau = g(t)\,, \tag{3}$$

where

$$K_h(t,\tau) = \sum_{i=1}^4 K_h^{(i)}(t,\tau)\,;$$

$$K_h^{(1)}(t,\tau) = \frac{1}{2a\sqrt{\pi}} \cdot \frac{(2\omega-1)^{3/2}\left(\tau^{2\omega-1}\cdot t^{2\omega-2} + t^{4\omega-3}\right)}{\left(t^{2\omega-1}-\tau^{2\omega-1}\right)^{3/2}} \cdot \exp\left\{-\frac{(2\omega-1)\left(t^{2\omega-1}+\tau^{2\omega-1}\right)^2}{4a^2\left(t^{2\omega-1}-\tau^{2\omega-1}\right)}\right\};$$

$$K_h^{(2)}(t,\tau) = -\frac{1}{2a\sqrt{\pi}} \cdot \frac{(2\omega-1)^{3/2}\cdot t^{2\omega-2}}{\left(t^{2\omega-1}-\tau^{2\omega-1}\right)^{1/2}} \cdot \exp\left\{-\frac{(2\omega-1)\left(t^{2\omega-1}-\tau^{2\omega-1}\right)^2}{4a^2\left(t^{2\omega-1}-\tau^{2\omega-1}\right)}\right\};$$

$$K_h^{(3)}(t,\tau) = -\frac{2}{2a\sqrt{\pi}} \cdot \frac{(2\omega-1)^{3/2} \cdot t^{2\omega-2}}{\left(t^{2\omega-1}-\tau^{2\omega-1}\right)^{1/2}} \cdot \exp\left\{-\frac{(2\omega-1)\left(t^{2\omega-1}+\tau^{2\omega-1}\right)^2}{4a^2\left(t^{2\omega-1}-\tau^{2\omega-1}\right)}\right\};$$

$$K_h^{(4)}(t,\tau) = \frac{2}{a\sqrt{\pi}} \cdot \frac{(2\omega-1)^{3/2} \cdot t^{2\omega-2}}{\left(t^{2\omega-1}-\tau^{2\omega-1}\right)^{1/2}} \cdot \exp\left\{-\frac{(2\omega-1)\left(t^{2\omega-1}-\tau^{2\omega-1}\right)^2}{4a^2\left(t^{2\omega-1}-\tau^{2\omega-1}\right)}\right\}.$$

Let us show that it really is a characteristic equation for the equation (1). First, we note that the kernel $K_h(t,\tau)$ also has a property similar to the property 3 of the kernel $K_\omega(t,\tau)$

$$\lim_{t\to 0}\int_0^t K_h^{(1)}(t,\tau)\,d\tau = 1.$$

Equation (3) with the following replacements:

$$t = \left(\frac{1}{2\omega-1}\cdot t_1\right)^{\frac{1}{2\omega-1}},\ \tau = \left(\frac{1}{2\omega-1}\cdot \tau_1\right)^{\frac{1}{2\omega-1}};$$

$$\varphi\left[\left(\frac{1}{2\omega-1}\cdot t_1\right)^{\frac{1}{2\omega-1}}\right] = \varphi_1(t_1),\ g\left[\left(\frac{1}{2\omega-1}\cdot t_1\right)^{\frac{1}{2\omega-1}}\right] = g_1(t_1),$$

is reduced to an integral equation of the form:

$$\varphi_1(t_1) + \int_0^{t_1}\sqrt{\frac{t_1}{\tau_1}}\cdot K_1(t_1,\tau_1)\cdot\varphi_1(\tau_1)\,d\tau_1 = g_1(t_1). \tag{4}$$

The kernel $K_1(t,\tau)$ has the form:

$$K_1(t,\tau) = \sum_{i=1}^{4} K_1^{(i)}(t,\tau),$$

where

$$K_1^{(1)}(t,\tau) = \frac{1}{2a\sqrt{\pi}}\cdot\frac{t+\tau}{(t-\tau)^{3/2}}\cdot\exp\left\{-\frac{(t+\tau)^2}{4a^2(t-\tau)}\right\};$$

$$K_1^{(2)}(t,\tau) = -\frac{1}{2a\sqrt{\pi}}\cdot\frac{t-\tau}{(t-\tau)^{3/2}}\cdot\exp\left\{-\frac{(t-\tau)^2}{4a^2(t-\tau)}\right\};$$

$$K_1^{(3)}(t,\tau) = -\frac{2}{a\sqrt{\pi}}\cdot\frac{1}{(t-\tau)^{1/2}}\cdot\exp\left\{-\frac{(t+\tau)^2}{4a^2(t-\tau)}\right\};$$

$$K_1^{(4)}(t,\tau) = \frac{2}{a\sqrt{\pi}}\cdot\frac{1}{(t-\tau)^{1/2}}\cdot\exp\left\{-\frac{(t-\tau)^2}{4a^2(t-\tau)}\right\}. \tag{5}$$

We found a solution to the equation (4) with kernel (5) and it has the form [5]:

$$\varphi(t) = g(t) + \int_0^t\sqrt{\frac{t}{\tau}}\cdot R(t,\tau)\cdot g(t)\,d\tau_1 + C\cdot\varphi_0(t),$$

where the resolvent $R(t,\tau) = R_1(t,\tau) + R_2(t,\tau)$ has the form:

$$R_1(t,\tau) = \frac{1}{a\sqrt{\pi}\,(\tau-t)^{\frac{3}{2}}\,\tau}\cdot\sum_{n=0}^{\infty}(-1)^n B_n\cdot\left[n\cdot\exp\left\{-\frac{n^2}{a^2(\tau-t)}\right\} + 3(n+1)\cdot\exp\left\{-\frac{(n+1)^2}{a^2(\tau-t_1)}\right\}+\right.$$

$$+3\,(n+2)\cdot\exp\left\{-\frac{(n+2)^2}{a^2\,(\tau-t)}\right\}+(n+3)\cdot\exp\left\{-\frac{(n+3)^2}{a^2\,(\tau-t_1)}\right\}\Bigg];$$

$$R_2(t,\tau)=\frac{3}{2a\sqrt{\pi}}\frac{1}{\tau\sqrt{\tau-t}\,(2\tau-t)}+$$

$$+\frac{3}{2a^2\pi\tau}\sum_{n=1}^{\infty}(-1)^n B_n\cdot\left[r_n\,(t,\tau)+r_{n+1}\,(t,\tau)-r_{n+2}\,(t,\tau)-r_{n+3}\,(t,\tau)\right];$$

$$\varphi_0\,(t)=\frac{C}{2\sqrt{\pi}}\cdot\sum_{n=1}^{\infty}(-1)^n\,(2n+3)\,B_n\cdot\exp\left\{-\frac{(2n+3)^2}{4a^2}\cdot t\right\}.$$

For the resolvent the following estimate is valid:

$$|R(t,\tau)|\le\frac{1}{t\cdot\tau^2}\cdot\left[\frac{\tau\cdot\tau^{\frac{3}{2}}\cdot t^{\frac{3}{2}}}{(t-\tau)^{3/2}}\exp\left\{-\frac{t\tau}{a^2\,(t-\tau)}\right\}+\frac{\tau\cdot\sqrt{\tau}\cdot\sqrt{t}\cdot t\cdot\tau}{\sqrt{t-\tau}\,(2t-\tau)}\right]\exp\left\{-\frac{t-\tau}{4a^2}\right\}\le$$

$$\le C_3\left[\frac{\sqrt{\tau}\cdot\sqrt{t}}{(t-\tau)^{3/2}}\exp\left\{-\frac{t\tau}{a^2\,(t-\tau)}\right\}+\frac{\sqrt{\tau}\cdot\sqrt{t}}{\sqrt{t-\tau}\,(2t-\tau)}\right]\exp\left\{-\frac{t-\tau}{4a^2}\right\}.$$

Returning to the old variables, i.e. making replacements

$$\tau_1=(2\omega-1)\cdot\tau^{2\omega-1},\quad \varphi_1\left[(2\omega-1)\cdot t^{2\omega-1}\right]=\varphi\,(t)\,;$$

$$t_1=(2\omega-1)\cdot t^{2\omega-1},\quad g_1\left[(2\omega-1)\cdot t^{2\omega-1}\right]=g\,(t)\,,$$

we get the solution of the characteristic equation (3):

$$\varphi\,(t)=g(t)+\int_0^t\left(\frac{t}{\tau}\right)^{\frac{3}{2}-\omega}\cdot R_h(t,\tau)\cdot g(t)d\tau+C\cdot\varphi_0((2\omega-1)\cdot t^{2\omega-1}),$$

where the resolvent $R_h\,(t,\tau)$ will have the following estimate

$$R_h(t,\tau)\le C_2(\omega)\left[\left(\frac{1}{2\omega-1}\right)^{\frac{3}{2}}\cdot\frac{t^{\omega-\frac{1}{2}}\tau^{\omega-\frac{1}{2}}\tau^{2\omega-2}}{(t^{2\omega-1}-\tau^{2\omega-1})}\exp\left\{-(2\omega-1)\frac{t^{2\omega-1}\cdot\tau^{2\omega-1}}{a^2\,(t^{2\omega-1}-\tau^{2\omega-1})}\right\}+\right.$$

$$\left.+\,(2\omega-1)^{\frac{3}{2}}\cdot\frac{t^{\omega-\frac{1}{2}}\tau^{\omega-\frac{1}{2}}\tau^{2\omega-2}}{(t^{2\omega-1}-\tau^{2\omega-1})^{\frac{1}{2}}\,(2t^{2\omega-1}-\tau^{2\omega-1})}\right]\le$$

$$\le C_3(\omega)\,(2\omega-1)^{\frac{3}{2}}\left[\left(\frac{1}{2\omega-1}\right)^{\frac{3}{2}}\cdot\frac{t^{\omega-\frac{1}{2}}\cdot\tau^{3\omega-\frac{5}{2}}}{\tau^{3\omega-3}(t-\tau)^{\frac{3}{2}}}\exp\left\{\frac{t^{2\omega-1}\cdot\tau}{a^2\,(t-\tau)}\right\}+\frac{t^{\omega-\frac{1}{2}}\cdot\tau^{3\omega-\frac{5}{2}}}{(2\omega-1)^{\frac{1}{2}}\,(t-\tau)^{\frac{1}{2}}\tau^{2\omega-\frac{3}{2}}}\right]\le$$

$$\le C_3(\omega)\frac{\sqrt{\tau}\cdot t^{\omega-\frac{1}{2}}}{(t-\tau)^{\frac{3}{2}}}\cdot\exp\left\{-\frac{t^{2\omega-1}\cdot\tau}{a^2(t-\tau)}\right\}+C_4(\omega)\frac{\tau^{2\omega-\frac{1}{2}}}{t^{\omega+\frac{1}{2}}\cdot\sqrt{t-\tau}}.$$

Indeed:

$$\frac{t^{\omega-\frac{1}{2}}\cdot\tau^{3\omega-\frac{5}{2}}}{(2\omega-1)^{\frac{1}{2}}\,(t-\tau)^{\frac{1}{2}}\tau^{2\omega-\frac{3}{2}}}\le C_4(\omega)\frac{\tau^{2\omega-\frac{1}{2}}}{t^{\omega+\frac{1}{2}}\cdot\sqrt{t-\tau}}.$$

Theorem 1. The general solution to the characteristic integral equation has the form

$$\varphi\,(t)=g(t)+\int_0^t\left(\frac{t}{\tau}\right)^{\frac{3}{2}-\omega}\cdot R_h(t,\tau)\cdot g(t)d\tau+C\cdot\varphi_0((2\omega-1)\cdot t^{2\omega-1}).$$

3 Solution of an integral equation.
Regularization method by the solution of the characteristic equation

Remark 2 [6; 183]. If the (particular) solution of the integral equation

$$y(x)+\int_a^x K(x,t)\,y(t)\,dt=f(x)$$

is given by the formula

$$y(x)=f(x)+\int_a^x R(x,t)\,f(t)\,dt$$

then the (particular) solution of the integral equation (with a modified kernel)

$$y(x)+\int_a^x K(x,t)\,\frac{g(x)}{g(t)}y(t)\,dt=f(x)$$

is given by the formula

$$y(x)=f(x)+\int_a^x R(x,t)\,\frac{g(x)}{g(t)}y(t)\,dt.$$

The same is true for solutions of the corresponding homogeneous equations.

Using Remark 2, we consider equation (1), which we represent in the form

$$\varphi(t)+\int_0^t K_h(t,\tau)\cdot\varphi(\tau)d\tau=\int_0^t[K_h(t,\tau)-K_\omega(t,\tau)]\cdot\varphi(\tau)d\tau. \tag{6}$$

Assuming the right side of the equation (6) is temporarily known, we write its solution:

$$\varphi(t)=\int_0^t[K_h(t,\tau)-K_\omega(t,\tau)]\cdot\varphi(\tau)d\tau+\int_0^t R_\omega(t,\tau)\cdot\left\{\int_0^\tau[K_h(\tau,\tau_1)-K_\omega(\tau,\tau_1)]\cdot\varphi(\tau_1)d\tau_1\right\}d\tau+$$
$$+C_0\cdot\varphi_0\left((2\omega-1)\cdot t^{2\omega-1}\right).$$

In the repeated integral, we change the order of integration and change roles by variables τ and τ_1, we obtain

$$\varphi(t)+\int_0^t\bar{\tilde{K}}(t,\tau)\cdot\varphi(\tau)d\tau=C\cdot\varphi_0\left((2\omega-1)\cdot t^{2\omega-1}\right), \tag{7}$$

the kernel $\bar{\tilde{K}}(t,\tau)$ has the form

$$\bar{\tilde{K}}(t,\tau)=\tilde{K}(t,\tau)+\bar{K}(t,\tau), \tag{8}$$

where

$$\tilde{K}(t,\tau)=K_h(t,\tau)-K_\omega(t,\tau),\ \bar{K}(t,\tau)=\int_\tau^t R(t,\tau_1)\cdot[K_h(\tau_1,\tau)-K_\omega(\tau_1,\tau)]d\tau_1.$$

First we estimate the function $\tilde{K}(t,\tau)$, that is the first term of (8). For this we introduce the following notation:

$$K_h^{(i)}(t,\tau)=P_h^{(i)}e^{-Q_h^{(i)}};\ K_\omega^{(i)}(t,\tau)=P_\omega^{(i)}e^{-Q_\omega^{(i)}},\quad i=1,2,3,4,$$

where

$$P_h^1(t,\tau)=\frac{1}{2a\sqrt{\pi}}\cdot\frac{(2\omega-1)^{\frac{3}{2}}(\tau^{2\omega-1}\cdot t^{2\omega-2}+t^{4\omega-3})}{(t^{2\omega-1}-\tau^{2\omega-1})^{\frac{3}{2}}};$$

$$P_\omega^{(1)}(t,\tau)=\frac{1}{2a\sqrt{\pi}}\cdot\frac{t^\omega+\tau^\omega}{(t-\tau)^{\frac{3}{2}}};$$

$$Q_h^{(1)}(t,\tau)=\exp\left\{-\frac{(2\omega-1)(t^{2\omega-1}+\tau^{2\omega-1})^2}{4a^2(t^{2\omega-1}-\tau^{2\omega-1})}\right\};$$

$$Q_\omega^{(1)}(t,\tau)=\exp\left\{-\frac{(t^\omega+\tau^\omega)^2}{4a^2(t-\tau)}\right\}.$$

The following statement is true.

Lemma 5. If $\omega > \frac{1}{2}$, then

$$\lim_{t\to 0}\int_0^t \tilde{K}(t,\tau)d\tau = 0,\ \ 0<\tau<t<\infty \tag{9}$$

and the following estimate is correct

$$\left|\tilde{K}(t,\tau)\right| \le C_1(\omega)\frac{t^{\omega-1}}{\sqrt{t-\tau}}e^{-\tilde{Q}(t,\tau)} + C_2(\omega)\frac{1}{\sqrt{t-\tau}}, \tag{10}$$

where $\tilde{Q}(t,\tau) = \min\left\{Q_h(t,\tau); \frac{1}{2}Q^1_\omega(t,\tau)\right\}$.

Proof. We note that from the estimate (10) immediately follows the ratio (9). Note that the estimate (10) obvious for summands

$$\left|K_h^{(i)}(t,\tau) - K_\omega^{(i)}(t,\tau)\right| \ \text{ at } \ i=2,3,4.$$

We prove the estimate (10) for the first summand $\left|K^1_h(t,\tau) - K^1_\omega(t,\tau)\right|$.

We have:

$$\left|K_h^{(1)}(t,\tau) - K_\omega^{(1)}(t,\tau)\right| =$$

$$= \left|P_h^{(1)}(t,\tau)\exp\left\{-Q_h^{(1)}(t,\tau)\right\} - P_\omega^{(1)}(t,\tau)\exp\left\{-Q_\omega^{(1)}(t,\tau)\right\}\right| \le$$

$$\le \left|P_h^{(1)}(t,\tau) - P_\omega^{(1)}(t,\tau)\right|\exp\left\{-Q_h^{(1)}(t,\tau)\right\} +$$

$$+P_\omega^{(1)}(t,\tau)\exp\left\{-Q_h^{(1)}(t,\tau)\right\}\ \left|1-\exp\left\{-Q_h^{(1)}(t,\tau) - Q_\omega^{(1)}(t,\tau)\right\}\right| \le$$

$$\le \left|P_h^{(1)}(t,\tau) - P_\omega^{(1)}(t,\tau)\right|\exp\left\{-Q_h^{(1)}(t,\tau)\right\} +$$

$$+P_\omega^{(1)}(t,\tau)\left|Q_h^{(1)}(t,\tau) - Q_\omega^{(1)}(t,\tau)\right|\exp\left\{-Q_\omega^{(1)}(t,\tau)\right\}.$$

For further calculations, we first prove the following lemma.

Lemma 6. There are relationships:

$$\left|P_h^{(1)}(t,\tau) - P_\omega^{(1)}(t,\tau)\right| \le C_2(\omega)\frac{t^{\omega-1}}{\sqrt{t-\tau}};$$

$$P_\omega^{(1)}(t,\tau)\left|Q_h^{(1)}(t,\tau) - Q_\omega^{(1)}(t,\tau)\right|\exp\left\{-Q_\omega^{(1)}(t,\tau)\right\} \le$$

$$\le C_3(\omega)\frac{t^{\omega-1}}{\sqrt{t-\tau}}\exp\left\{-\frac{Q_\omega^{(1)}(t,\tau)}{2}\right\}. \tag{11}$$

Proof. We introduce the notation:

$$\tilde{P}^1(t,\tau) = \left[P_h^{(1)}(t,\tau) - P_\omega^{(1)}(t,\tau)\right]\cdot\frac{\sqrt{t-\tau}}{t^{\omega-1}}.$$

Then, making a replacement $\tau = tx$, $(0<x<1)$, we obtain

$$\tilde{P}(t,tx) = \frac{1}{2a\sqrt{\pi}}\left((2\omega-1)^{\frac{3}{2}}\cdot\frac{t^{4\omega-3}+t^{2\omega-2}\tau^{2\omega-1}}{(t^{2\omega-1}-\tau^{2\omega-1})^{\frac{3}{2}}} - \frac{t^\omega+\tau^\omega}{(t-\tau)^{\frac{3}{2}}}\right)\cdot\frac{\sqrt{t-\tau}}{t^{\omega-1}} =$$

$$= \frac{1}{2a\sqrt{\pi}}\left((2\omega-1)^{\frac{3}{2}}\cdot\frac{1+x^{2\omega-1}}{(1-x^{2\omega-1})^{\frac{3}{2}}} - \frac{1+x^\omega}{(1-x)^{\frac{3}{2}}}\right)\cdot\sqrt{1-x} =$$

$$= \frac{1}{2a\sqrt{\pi}} \cdot \frac{(2\omega-1)^{\frac{3}{2}}\,(1+x)^{2\omega-1}\,(1-x)^{\frac{3}{2}} - (1+x^{\omega})(1-x^{2\omega-1})^{\frac{3}{2}}}{(1-x)(1-x^{2\omega-1})^{\frac{3}{2}}}.$$

$$\lim_{x\to 1} \frac{1}{2a\sqrt{\pi}} \cdot \frac{1+x^{2\omega-1}-1-x^{\omega}}{1-x} = \frac{1}{2a\sqrt{\pi}} \cdot \frac{(2\omega-1)\cdot x^{2\omega-1} - \omega\cdot x^{\omega-1}}{-1} = \frac{\omega-1}{2a\sqrt{\pi}}.$$

Let us show the validity of the second statement (11) of Lemma 2. First of all, we note that

$$\left|Q_h^{(1)}(t,\tau) - Q_\omega^{(1)}(t,\tau)\right| \le C_4(\omega)\,|1-\omega|\,t^{2\omega-1}. \tag{12}$$

Indeed, we have

$$t^{1-2\omega}\,|Q_h(t,\tau) - Q_\omega(t,\tau)| = \frac{t^{1-2\omega}}{4a^2} \cdot \left\{(2\omega-1)\frac{(t^{2\omega-1}+\tau^{2\omega-1})^2}{t^{2\omega-1}-\tau^{2\omega-1}} - \frac{(t^{\omega}+\tau^{\omega})^2}{t-\tau}\right\} =$$

$$= \frac{t^{1-2\omega}}{4a^2} \cdot \left\{(2\omega-1)(t^{2\omega-1}-\tau^{2\omega-1}) - \frac{(t^{\omega}-\tau^{\omega})^2}{t-\tau} + \frac{4(2\omega-1)t^{2\omega-1}\tau^{2\omega-1}}{t^{2\omega-1}-\tau^{2\omega-1}} - \frac{4t^{\omega}\tau^{\omega}}{t-\tau}\right\} =$$

$$= \|\tau = tx\| = \frac{1}{4a^2}\left\{(2\omega-1)(1-x^{2\omega-1}) - \frac{(1-x^{\omega})^2}{1-x} + \frac{4(2\omega-1)x^{2\omega-1}}{1-x^{2\omega-1}} - \frac{4x^{\omega}}{1-x}\right\}.$$

$$\lim_{x\to 1} \frac{1}{a^2}\left\{\frac{(2\omega-1)x^{2\omega-1}(1-x) - x^{\omega}(1-x^{2\omega-1})}{(1-x)(1-x^{2\omega-1})}\right\} =$$

$$= \left\|1-x^{2\omega-1} \approx (2\omega-1)(1-x)\,if\,x\to 1\right\| = \frac{1}{a^2}\frac{(2\omega-1)[x^{2\omega-1}-x^{\omega}]}{1-x^{2\omega-1}}.$$

So,

$$t^{1-2\omega}\,|Q_h(t,\tau) - Q_\omega(t,\tau)| =$$

$$= \begin{cases} \dfrac{2\omega-1}{a^2}\cdot x^{2\omega-1}\dfrac{1-x^{1-\omega}}{1-x^{2\omega-1}} \approx \dfrac{1-\omega}{a^2},\ \dfrac{1}{2} < \omega < 1; \\ \dfrac{2\omega-1}{a^2}\cdot x^{2\omega-1}\dfrac{1-x^{1-\omega}}{1-x^{2\omega-1}} \approx -\dfrac{\omega-1}{a^2},\ \omega > 1. \end{cases}$$

This directly implies the inequality (12).

We now turn to the proof of inequality (11). We have:

$$P_\omega^{(1)}(t,\tau)\left|Q_h^{(1)}(t,\tau) - Q_\omega^{(1)}(t,\tau)\right|\exp\left\{-Q_\omega^{(1)}(t,\tau)\right\} \le$$

$$\le C_4(\omega)\cdot t^{2\omega-1}\frac{t^{\omega}+\tau^{\omega}}{2a\sqrt{\pi}(t-\tau)^{\frac{3}{2}}}\exp\left\{-\frac{(t^{\omega}+\tau^{\omega})^2}{4a^2(t-\tau)}\right\} \le$$

$$\le C_5(\omega)\cdot t^{2\omega-1}\frac{t^{\omega}+\tau^{\omega}}{2a\sqrt{\pi}(t-\tau)^{\frac{3}{2}}}\cdot\frac{8a^2(t-\tau)}{(t^{\omega}+\tau^{\omega})^2}\exp\left\{-\frac{(t^{\omega}+\tau^{\omega})^2}{8a^2(t-\tau)}\right\} =$$

$$= C_5(\omega)\cdot t^{2\omega-1}\frac{1}{\sqrt{\pi}(t-\tau)^{\frac{1}{2}}}\cdot\frac{4a}{t^{\omega}+\tau^{\omega}}\exp\left\{-\frac{(t^{\omega}+\tau^{\omega})^2}{8a^2(t-\tau)}\right\} \le$$

$$\le C_3(\omega)\cdot\frac{t^{\omega-1}}{(t-\tau)^{\frac{1}{2}}}\cdot\exp\left\{-\frac{Q_\omega(t,\tau)}{2}\right\}.$$

Lemma 6 is completely proved.

The proof of Lemma 5 is completed by applying the estimates from Lemma 6.

Lemma 7. If $\omega > 1/2$, *then the following estimate is correct*

$$|\overline{K}(t,\tau)| \le C\left[t^{2w-2} + t^{w-1} + \frac{1}{\sqrt{t-\tau}}\cdot\exp\left\{-\frac{t^{2w-1}\cdot\tau}{t-\tau}\right\} + \frac{t^{w-1}}{\sqrt{t-\tau}}\cdot\exp\left\{-\frac{t^{2w-1}\cdot\tau}{t-\tau}\right\}\right].$$

We now turn to an estimate of the summand $\overline{K}(t,\tau)$ from (8).

$$\overline{K}(t,\tau)=\int_\tau^t R_h(t,\tau_1)\cdot\tilde{K}(\tau_1,\tau)d\tau_1,$$

using the following inequalities:

$$R_h(t,\tau)\leq C_5(\omega)\frac{\sqrt{\tau}\cdot t^{\omega-\frac{1}{2}}}{(t-\tau)^{\frac{3}{2}}}\cdot\exp\left\{-\frac{t^{2\omega-1}\cdot\tau}{a^2(t-\tau)}\right\}+C_6(\omega)\frac{\tau^{2\omega-\frac{1}{2}}}{t^{\omega+\frac{1}{2}}\cdot\sqrt{t-\tau}};$$

$$|\tilde{K}(t,\tau)|\leq C_1(\omega)\frac{t^{\omega-1}}{\sqrt{t-\tau}}\exp\left\{-\tilde{Q}(t,\tau)\right\}+C_2(\omega)\frac{1}{\sqrt{t-\tau}}.$$

The estimate is carried out in four stages:

1) $I_{11}=\int_\tau^t\frac{\tau_1^{\omega-1}}{\sqrt{\tau_1-\tau}}\cdot\frac{t^{\omega-1/2}\sqrt{\tau_1}}{(t-\tau_1)^{3/2}}\cdot\exp\left\{-\frac{t^{2\omega-1}\cdot\tau_1}{a^2(t-\tau_1)}\right\}d\tau_1=$

$=t^{\omega-\frac{1}{2}}\int_0^\infty\frac{(1+z^2)^{\frac{1}{2}}}{(t-\tau)^{1/2}}\cdot\frac{(1+z^2)^{3/2}}{(t-\tau)^{3/2}}\left(\frac{tz^2+\tau}{1+z^2}\right)^{\omega-\frac{1}{2}}\frac{(t-\tau)2z}{(1+z^2)^2}\exp\left\{-\frac{t^{2\omega-1}\cdot(tz^2+\tau)}{t-\tau}\right\}dz=$

$=\frac{2\cdot t^{\omega-\frac{1}{2}}}{t-\tau}\exp\left\{-\frac{t^{2\omega-1}\cdot\tau}{t-\tau}\right\}\int_0^\infty t^{\omega-\frac{1}{2}}\cdot\left(\frac{z^2+\frac{\tau}{t}}{1+z^2}\right)^{\omega-\frac{1}{2}}\exp\left\{-\frac{t^{2\omega}\cdot z^2}{a^2(t-\tau)}\right\}dz\leq$

$\leq\frac{2\cdot t^{2\omega-1}}{t-\tau}\exp\left\{-\frac{t^{2\omega-1}\cdot\tau}{t-\tau}\right\}\frac{\sqrt{\pi}}{2}\frac{a\sqrt{t-\tau}}{t^\omega}=\frac{a\sqrt{\pi}t^{\omega-1}}{\sqrt{t-\tau}}\exp\left\{-\frac{t^{2\omega-1}\cdot\tau}{t-\tau}\right\}.$

Further

2) $I_{12}=\int_\tau^t\frac{1}{\sqrt{\tau_1-\tau}}\frac{t^{\omega-1/2}\sqrt{\tau_1}}{(t-\tau_1)^{3/2}}\exp\left\{-\frac{t^{2\omega-1}}{a^2(t-\tau_1)}\right\}d\tau_1=$

$=t^{\omega-\frac{1}{2}}\int_0^\infty\frac{(1+z^2)^{\frac{1}{2}}}{(t-\tau)^{1/2}}\frac{(1+z^2)^{3/2}}{(t-\tau)^{3/2}}\left(\frac{tz^2+\tau}{1+z^2}\right)^{1/2}\frac{(t-\tau)\cdot 2z}{(1+z^2)^2}\exp\left\{-\frac{t^{2\omega-1}\cdot(tz^2+\tau)}{t-\tau}\right\}dz=$

$=\frac{2\cdot t^{\omega-1/2}}{t-\tau}\exp\left\{-\frac{t^{2\omega-1}\cdot\tau}{t-\tau}\right\}\int_0^\infty\sqrt{t}\cdot\left(\frac{z^2+\tau/t}{1+z^2}\right)^{1/2}\exp\left\{-\frac{t^{2\omega}\cdot z^2}{a^2(t-\tau)}\right\}dz\leq$

$\leq\frac{2\cdot t^\omega}{t-\tau}\exp\left\{-\frac{t^{2\omega-1}\cdot\tau}{t-\tau}\right\}\frac{\sqrt{\pi}}{2}\cdot\frac{a\sqrt{t-\tau}}{t^\omega}=\frac{a\sqrt{\pi}}{\sqrt{t-\tau}}\exp\left\{-\frac{t^{2\omega-1}\cdot\tau}{t-\tau}\right\}.$

3) $I_{21}=\int_\tau^t\frac{\tau_1^{\omega-1}}{\sqrt{\tau_1-\tau}}\cdot\frac{\tau_1^{2\omega-\frac{1}{2}}}{t^{\omega+\frac{1}{2}}\cdot\sqrt{t-\tau_1}}d\tau_1=$

$$=\left\|\begin{array}{l}\dfrac{\tau_1-\tau}{t-\tau_1}=z^2;\quad\tau_1-\tau=(t-\tau_1)z^2;\ \tau_1=\dfrac{tz^2+\tau}{1+z^2};\\ \tau_1-\tau=(t-\tau)\dfrac{z^2}{1+z^2};\ t-\tau_1=\dfrac{t-\tau}{1+z^2};\ \ d\tau_1=\dfrac{2(t-\tau)zdz}{(1+z^2)^2}\end{array}\right\|=$$

$=t^{-\omega-1/2}\int_0^\infty\left(\frac{tz^2+\tau}{1+z^2}\right)^{3\omega-3/2}\frac{1+z^2}{(t-\tau)z}\cdot\frac{(t-\tau)\cdot 2z\cdot dz}{(1+z^2)^2}\leq C\cdot t^{2\omega-2}.$

4) $I_{22}=\int_\tau^t\frac{1}{\sqrt{\tau_1-\tau}}\cdot\frac{\tau_1^{2\omega-\frac{1}{2}}}{t^{\omega+\frac{1}{2}}\cdot\sqrt{t-\tau_1}}d\tau_1=t^{-\omega-\frac{1}{2}}\int_0^\infty\left(\frac{1+z^2}{tz^2+\tau}\right)^{2\omega-\frac{1}{2}}\frac{2dz}{1+z^2}\leq C\cdot t^{\omega-1}.$

Lemma is iproved.

4 Main result

Thus, the following statement is proved:

Theorem 2. If $\omega>\frac{1}{2}$, then the kernel of the integral equation (7) has the estimate

$$\left|\bar{\tilde{K}}(t,\tau)\right|\leq C\left\{t^{2\omega-2}+t^{\omega-1}+\frac{1}{\sqrt{t-\tau}}+\frac{t^{\omega-1}}{\sqrt{t-\tau}}\right\},$$

which means that the integral equation (2) for any

$$t^{\frac{3}{2}-\omega}\cdot f(t)\in L_\infty(0,\infty)$$

has a unique nonzero solution: $t^{\frac{3}{2}-\omega}\cdot\varphi(t)\in L_\infty(0,\infty)$.

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М.Т. Дженалиев, М.И. Рамазанов, М.Т. Космакова, А.О. Танин

Псевдо-Вольтерраның интегралдық теңдеуінің шешуі туралы

Мақалада біртекті сингулярлы 2-текті Вольтерра интегралды теңдеуі (псевдо-Вольтерра интегралдық теңдеуі) қарастырылған. Оның ядросының қасиеттері дәлелденген. Характеристикалық теңдеуі құрастырылып, зерттеліп отырған интегралдық теңдеудің характеристикалық теңдеуі болатыны көрсетілді. Оның интегралдық операторының ядросының бағалауы анықталды. Сонымен қатар сәйкес біртекті емес интегралдық теңдеудің шешімі туралы сұрақтар қарастырылды, оның шешімінің жалғыздық класы анықталды. Сонымен бірге зерттеліп отырған біртекті емес теңдеудің оң жағы үшін салмақтық класы тағайындалды. Оның шешімінің жалғыздығының салмақтық класы теңдеудің интегралдық операторының бағалауы негізінде орнықтырылған.

Кілт сөздер: сипаттамалық теңдеу, ядро, интегралдық оператор, елеулі шенелген функциялар кластары.

М.Т. Дженалиев, М.И. Рамазанов, М.Т. Космакова, А.О. Танин

К решению одного псевдо-Вольтеррового интегрального уравнения

В статье изучено однородное сингулярное интегральное уравнение Вольтерра второго рода (псевдо-Вольтеррово интегральное уравнение). Доказаны свойства его ядра. Построено характеристическое уравнение. Показано, что оно действительно является характеристическим уравнением исследуемого интегрального уравнения. Установлены оценки ядра его интегрального оператора. Рассмотрены также вопросы разрешимости соответствующего неоднородного интегрального уравнения. Определен весовой класс единственности для его решения. Также установлен весовой класс для правой части исследуемого неоднородного уравнения. Весовой класс единственности его решения установлен на основе оценок ядра интегрального оператора уравнения.

Ключевые слова: характеристическое уравнение, ядро, интегральный оператор, класс существенно ограниченных функций.

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M. Muhtarov, A.H. Kalidolday

S.Toraigyrov Pavlodar State University, Kazakhstan
(E-mail: kalidoldai_ a@mailnisedu.onmicrosoft.com)

On solving a linear control problem

The problem of a linear regulator is considered. There is a system of linear differential equations with a quadratic control quality criterion. The method of dynamic programming is applied to the solution of the considered linear problem. As is known, the main difficulty in applying this method is to integrate partial differential equations. In this problem, the obtained optimal control function depends on the solution of the Riccati equation. In [1], conditions were obtained under which there is a solution to such optimal control problems with a quadratic quality criterion. These conditions were obtained along with formulas for minimizing control and for optimal trajectory. But all these statements depended on the ability to solve the matrix Riccati equation under certain boundary conditions given at some time point. To construct a solution to the problem under consideration, a system of $2n$ adjoint differential equations is constructed. After splitting the transition matrix of this system into block ones, it is possible to express the state of the system at the time instant t through the state variable and the adjoint variable at the final time instant t_1. A feature of this work is that an example is given, where the solution of the Riccati equation, which determines the optimal solution of the problem, was obtained explicitly.

Keywords: dynamic programming, optimal control, evaluation of control quality, quadratic quality criterion, adjoint equations.

The system of controls expressed through the linear differential equation is given

$$\dot{x} = A(t)x(t) + B(t)u\,; \qquad x(t_0) = x_0, \tag{1}$$

where $A(t)$, $B(t)$ — continuous matrixes of dimension $(n \times n)$, $(n \times q)$ [1, 2].

The functional that evaluates the quality of control is given in the form:

$$I = \frac{1}{2}\int_{t_0}^{t_1} \left[x^T(t)Dx(t) + u^T(t)Ru(t)\right]\, dt + \frac{1}{2}x^T(t_1)Qx(t_1), \tag{2}$$

where $D(t)$, Q — is a nonnegative definite symmetric matrix of size $(n \times n)$; $R(t)$ — positive definite symmetric matrix of size $(q \times q)$. It is required to find the optimal control function $u^0(t, x)$ for problem (1), (2). To do this, we make the Bellman equation [3]

$$\min_u \left\{ \frac{\partial s(t,x)}{\partial t} + \left(\frac{\partial s(t,x)}{\partial x}\right)^T [A(t)x + B(t)u] - \frac{1}{2}\left[x^T D(t)x(t) + u^T R(t)u\right] \right\} = 0;$$

$$s(t_1, x) = -\frac{1}{2}x^T Qx\,. \tag{3}$$

Let's differentiate the expression in square brackets by u and equating the result obtained to zero, we obtain the control function.

$$u^0(t,\, x) = R^{-1}(t)B^T(t)\frac{\partial s(t,x)}{\partial x}\,.$$

The solution of equation (3) is found in the form

$$s(t, x) = \frac{1}{2}x^T Kx, \tag{4}$$

where $K(t)$– is an unknown symmetric matrix of size $(n \times n)$.

Expression (4) substituting into the relation (3) we obtain the following equation:

$$\dot{K} = -A^T(t)K(t) - K(t)A(t) - K(t)B(t)R^{-1}(t)B^T(t)K(t) + D(t);$$
$$K(t_1) = -Q.$$

As shown in this paper, the existence of many problems of the theory of optimal control is obviously related to the problem of the existence of a solution to the matrix equation of Riccati in some boundary conditions at any given time.

Now we consider one method for solution the Riccati equation. To do this, using the variable p conjugate to the variable x and using method of a variation

$$\dot{x} = A(t)x(t) - B(t)R^{-1}B^T(t)p(t),$$

we obtain a linear differential equation

$$\dot{p} = -A^T(t)p(t) - D(t)x(t).$$

Boundary conditions of this problem

$$x(t_0) = x_0,\ p(t_1) = Qx(t_1).$$

By combining these two differential equations, we make a system of $(2n)$ equations

$$\begin{pmatrix} \dot{x}(t) \\ \dot{p}(t) \end{pmatrix} = \begin{pmatrix} A(t) & -B(t)R^{-1}B^T(t) \\ -D(t) & -A^T(t) \end{pmatrix} \begin{pmatrix} x(t) \\ p(t) \end{pmatrix}.$$

The fundamental matrix (transfer) of this system has the form

$$\Phi(t,\, t_0) = \begin{pmatrix} \Phi_{11}(t,\, t_0) & \Phi_{12}(t,\, t_0) \\ \Phi_{21}(t,\, t_0) & \Phi_{22}(t,\, t_0) \end{pmatrix}.$$

When you partition the transition matrices into block matrices it is possible to express the state system in timepoint t through the state variable and the conjugate variable at the final timepoint 2 as follows: final timepoint t_1 as follows:

$$x(t) = \Phi_{11}(t, t_1)x(t_1) + \Phi_{12}(t, t_1)p(t_1).$$

Further from a condition $p(t_1) = Qx(t_1)$ receive

$$x(t) = [\Phi_{11}(t, t_1) + \Phi_{12}(t, t_1)Q]x(t_1). \tag{5}$$

Thus, for the conjugate variable we obtain

$$p(t) = \Phi_{21}(t, t_1)x(t_1) + \Phi_{22}(t, t_1)p(t_1) = [\Phi_{21}(t, t_1) + \Phi_{22}(t, t_1)Q]x(t_1). \tag{6}$$

Excluding $x(t_1)$ in the expressions (5) and (6) we obtain the following ratio:

$$p(t) = [\Phi_{21}(t, t_1) + \Phi_{22}(t, t_1)Q] \cdot [\Phi_{11}(t, t_1) + \Phi_{12}(t, t_1)Q]^{-1}x(t). \tag{7}$$

From expression (7) it is visible that between functions $p(t)$ and $x(t_1)$ there is a linear relationship.

$$p(t) = K(t)x(t),$$

here

$$K(t) = [\Phi_{21}(t, t_1) + \Phi_{22}(t, t_1)Q] \cdot [\Phi_{11}(t, t_1) + \Phi_{12}(t, t_1)Q]^{-1}. \tag{8}$$

So, if we consider such a problem of a linear regulator, then the optimal control function will take the form

$$u^0(t,\, x) = -R^{-1}(t)B^T(t)K\,x(t),$$

where $K(t)$ (8) — is the solution of the Riccati equation [4].

Consider an example that can be used to determine the exact solution of the Riccati equation using these results.

Let the following control problem be given:

$$\begin{aligned} \dot{x}_1 &= x_2; \quad x_1(0) = x_{10};\\ \dot{x}_2 &= u; \quad x_2(0) = x_{20}; \end{aligned} \tag{9}$$

$$I = \frac{1}{2}\,x_1^2(t) + \frac{1}{2}\int_0^t (x_2^2 + u^2)dt \ \to \ \min .$$

The solution of this problem is looking for the method of variation. Equality

$$u(t) = p(t),$$

substituting into the equation (9) we obtain the following system of the equations

$$\begin{aligned} \dot{x}_1 &= x_2; \quad x_1(0) = x_{10};\\ \dot{x}_2 &= u; \quad x_2(0) = x_{20}; \end{aligned}$$

$$\begin{aligned} \dot{p}_1 &= 0\,;\\ \dot{p}_2 &= -p_1 + x_2. \end{aligned}$$

To find the control function that determines the optimal solution of this problem, it is necessary to find a solution to the corresponding Riccati equation. Now consider this in practice. Let's assume that the transition matrix connected with a type of the solution of system (9) is defined.

$$\Phi(t,t_1) = \begin{pmatrix} 1 & sh(t-t_1) & t-t_1-sh(t-t_1) & ch(t-t_1)-1\\ 0 & ch(t-t_1) & 1-ch(t-t_1) & sh(t-t_1)\\ 0 & 0 & 1 & 0\\ 0 & sh(t-t_1) & -sh(t-t_1) & ch(t-t_1) \end{pmatrix}.$$

Let the matrix be divided into blocks

$$\Phi_{11} = \begin{pmatrix} 1 & sh(t-t_1)\\ 0 & ch(t-t_1) \end{pmatrix}; \ \Phi_{12} = \begin{pmatrix} t-t_1-sh(t-t_1) & ch(t-t_1)-1\\ 1-ch(t-t_1) & sh(t-t_1) \end{pmatrix};$$

$$\Phi_{21} = \begin{pmatrix} 0 & 0\\ 0 & sh(t-t_1) \end{pmatrix}; \ \Phi_{22} = \begin{pmatrix} 1 & 0\\ -sh(t-t_1) & ch(t-t_1) \end{pmatrix}.$$

In this case, the solution of the Riccati equation, determining the solution of this problem will take the form (8).

$$K(t) = [\Phi_{21}(t,t_1) + \Phi_{22}(t,t_1)Q]\, \cdot [_{11}(t,t_1) + \Phi_{12}(t,t_1)Q]^{-1}.$$

Substituting the given matrices here, after simple transformations, this matrix will take the form

$$\begin{aligned} K(t) &= \begin{pmatrix} 1 & 0\\ -sh(t-t_1) & sh(t-t_1) \end{pmatrix} \cdot \tfrac{1}{\varphi} \begin{pmatrix} ch(t-t_1) & -sh(t-t_1)\\ ch(t-t_1)-1 & 1+t-t_1-sh(t-t_1) \end{pmatrix} =\\ &= \tfrac{1}{\varphi} \begin{pmatrix} ch(t-t_1) & -sh(t-t_1)\\ -sh(t-t_1) & (1+t-t_1)sh(t-t_1) \end{pmatrix}. \end{aligned} \tag{10}$$

Here designations $\varphi = (1+t-t_1)ch(t-t_1) - sh(t-t_1)$ are entered. From this formula, the elements of the matrix are written in the following form:

$$K_{11} = \tfrac{1}{\varphi}\cdot ch(t-t_1); \qquad K_{12} = \tfrac{1}{\varphi}\cdot\,(-sh(t-t_1));$$

$$K_{21} = \tfrac{1}{\varphi}\cdot\,(-sh(t-t_1)); \quad K_{22} = \tfrac{1}{\varphi}\cdot\, sh(t-t_1)\cdot(1+t-t_1).$$

Let's check that the matrix $K(t)$ (8) satisfies to Riccati equation.

$$A = \begin{pmatrix} 0 & 1\\ 0 & 0 \end{pmatrix}, \ B = \begin{pmatrix} 0\\ 1 \end{pmatrix}, \ Q = \begin{pmatrix} 1 & 0\\ 0 & 0 \end{pmatrix}, \ D = \begin{pmatrix} 0 & 0\\ 0 & 1 \end{pmatrix}, \ R = 1\,.$$

The Riccati equation can be reduced to the form:

$$\dot{K} = -A^T(t)K(t) - K(t)A(t) - K(t)B(t)R^{-1}(t)B^T(t)K(t) + D(t). \tag{11}$$

Let's calculate the derivative of the function on the left side of the Riccati equation

$$\dot{K} = \frac{1}{\varphi^2}\begin{pmatrix} \dot{K}_{11} & \dot{K}_{12} \\ \dot{K}_{21} & \dot{K}_{22} \end{pmatrix} = \frac{1}{\varphi^2}\begin{pmatrix} -sh^2(t-t_1) & sh(t-t_1)cht-t_1) - (1+t-t_1) \\ sh(t-t_1)cht-t_1) - (1+t-t_1) & (1+t-t_1)^2 - sh^2(t-t_1) \end{pmatrix},$$

where

$$\varphi = (1+t-t_1)ch(t-t_1) - sh(t-t_1);$$

$$\dot{\varphi} = (1+t-t_1)sh(t-t_1).$$

At first we will check that the left part of the equations are equal to these values.

$$\dot{K} = -A^T(t)K(t) - K(t)A(t) - K(t)B(t)R^{-1}(t)B^T(t)K(t) + D(t) =$$

$$= -\begin{pmatrix} 0 & 0 \\ K_{11} & K_{12} \end{pmatrix} - \begin{pmatrix} 0 & K_{11} \\ 0 & K_{21} \end{pmatrix} - \begin{pmatrix} K_{12}K_{21} & K_{12}K_{22} \\ K_{22}K_{21} & K_{22}^2 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} =$$

$$= -\begin{pmatrix} 0 & K_{11} \\ K_{11} & K_{12}+K_{21} \end{pmatrix} - \begin{pmatrix} K_{12}K_{21} & K_{12}K_{22} \\ K_{22}K_{21} & K_{22}^2 - 1 \end{pmatrix} = -\begin{pmatrix} K_{12}K_{21} & K_{11}+K_{12}K_{22} \\ K_{11}+K_{22}K_{21} & K_{12}+K_{21}+K_{22}^2 - 1 \end{pmatrix} =$$

$$= \frac{1}{\varphi^2}\begin{pmatrix} -sh^2(t-t_1) & sh(t-t_1)ch(t-t_1) - (1+t-t_1) \\ sh(t-t_1)ch(t-t_1) - (1+t-t_1) & (1+t-t_1)^2 - sh^2(t-t_1) \end{pmatrix},$$

where $\varphi = (1+t-t_1)ch(t-t_1) - sh(t-t_1)$, $\dot{\varphi} = (1+t-t_1)sh(t-t_1)$.

After simple transformations, we obtain that the expression on the right side of the equation is equal to the value of the derivative expression on the left. So the function (10) is the solution of the Riccati equation (11).

$$\dot{K} = -A^T(t)K(t) - K(t)A(t) - K(t)B(t)R^{-1}(t)B^T(t)K(t) + D(t);$$

$$K(t_1) = -Q.$$

Using the solution of the obtained Riccati equation, we determine the optimal control function, which is the solution of the considered task.

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М. Мұхтаров, А.Х. Қалидолдай

Басқарудын бір сызықтық теңдеуінің шешуі жайлы

Мақалада сызықты реттегіш есебі қарастырылды. Сызықты дифференциалдық теңдеулер жүйесі және басқару сапасын бағалайтын квадраттық белгі (критерий) берілген болсын. Есептің шешімін табу үшін динамикалық бағдарлау әдісі қолданылды. Ал осы әдісті қолданудың негізгі қиыншылығы дербес туындылы теңдеуді интегралдау екені белгілі. Осы қарастырылып отырған есепте алынган тиімді басқару функциясы Риккати теңдеуінің шешіміне тәуелді. Қолданыстағы [1] әдебиетте осындай критерийі квадраттық болатын тиімді басқару есептерінін шешімдерінің бар болуы шарттары келтірілген. Бұл шарттар минимумдалатын баскару функциялары үшін және тиімді траекториялардың формулаларымен бірге алынған. Бірақ та бұл тұжырымдар матрицалық Риккати теңдеуінің қандайда бір уақыт мезгілінде, белгілі бір шекаралық шарттары орындалғанда шешімінің бар болуы мүмкіншілігімен ғана байланысты алынған. Қарастырылып отырған есептің шешімін құру үшін түйіндес $2n$ дифференциалдық теңдеулер жүйесі құрылады. Осы жүйенің көшу матрицасын блок матрицаларға бөлгеннен кейін жүйенің t уақыт мезгіліндегі күйін жүйенің айнымалысының күйі мен түйіндес айнымалының t_1 уақыт мезгіліндегі күйімен байланыстыруға болады. Бұл жұмыстың ерекшелігі есептің тиімді басқару функциясын анықтайтын Риккати теңдеуінің шешімі айқын түрде алынған мысал қарастырылған.

Кілт сөздер: динамикалық бағдарламалау, оңтайлы басқару, сапа басқаруды бағалау, сапаның квадраттық көрсеткіші, қосарланған теңдеулер.

М. Мухтаров, А.Х. Калидолдай

О решении одной линейной задачи управления

Рассмотрена задача линейного регулятора. Имеется система линейных дифференциальных уравнений с квадратичным критерием качества управления. Для решения задачи применен метод динамического программирования. Как известно, основная трудность применения этого метода состоит в интегрировании уравнения с частными производными. В данной задаче полученная оптимальная управляющая функция зависит от решения уравнения Риккати. В [1] были получены условия, при которых существует решение таких задач оптимального управления с квадратичным критерием качества. Эти условии были получены вместе с формулами для минимизирующего управления и для оптимальной траектории. Но все эти утверждения зависели от возможности решить матричное уравнение Риккати при определенных граничных условиях, заданных в некоторый момент времени. Для построения решения рассматриваемой задачи составляется система $2n$ сопряженных дифференциальных уравнений. После разбиения переходной матрицы этой системы на блочные можно выразить состояние системы в момент времени t через переменную состояния и сопряженную переменную в конечный момент времени t_1. Особенностью этой работы является то, что приводится пример, где решение уравнения Риккати, определяющее оптимальное решение задачи, получено в явном виде.

Ключевые слова: динамическое программирование, оптимальное управление, оценка качества управления, квадратичный критерий качества, присоединенные уравнения.

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M.B. Muratbekov[1], M.M. Muratbekov[2]

[1] *Taraz State Pedagogical University, Kazakhstan;*
[2] *Kazakh University of Economics, Finance and International Trade, Nur-Sultan, Kazakhstan*
(E-mail: mmuratbekov@kuef.kz)

On the bounded invertibility of a Schrödinger operator with a negative parameter in the space $L_2(R^n)$

The Schrödinger operator $L = -\Delta + q(x),\ x \in R^n$, is one of the main operators of modern quantum mechanics and theoretical physics. It is known that many fundamental results have been obtained for the Schrödinger operator L. Among them, for example, are questions about the existence of a resolvent, separability (coercive estimate), various weight estimates, estimates of intermediate derivatives of functions from the domain of definition of an operator, estimates of eigenvalues and singular numbers (s-numbers). At present, there are various generalizations of the above results for elliptic operators. For general differential operators, the solution of such problem as a whole is far from complete. In particular, as far as we know, there was no result until now showing the existence of the resolvent and coercivity, as well as the discreteness of the spectrum of a hyperbolic type operator in an infinite domain with increasing and oscillating coefficients. It is easy to see that the study of some classes of differential operators of hyperbolic type defined in the space $L_2(R^{n+1})$, using the Fourier method, can be reduced to the study of the Schrödinger operator with a negative parameter : $L_t = -\Delta + \left(-t^2 + itb(x) + q(x)\right)$, where t is a parameter $(-\infty < t < \infty)$, $i^2 = -1$. Hence, it is easy to see that we get $-t^2 \to -\infty$ when $|t| \to \infty$ for the operator L_t. Consequently, a completely different situation arises here compared to the Schrödinger operator $L = -\Delta + q(x)$, and in particular, the methods worked out for the Schrödinger operator L turn out to be little adapted when studying the Schrödinger operator L_t with a negative parameter. All these questions indicate the relevance and novelty of this work. In the paper we study the problems of the existence of the resolvent and the coercivity of the Schrödinger operator with a negative parameter.

Keywords: Schrödinger operator, singular differential operator, hyperbolic type, negative parameter, coercive estimates, resolvent.

1 Formulation of the main results

In the paper, the Schrödinger operator with negative parameter

$$L_t = -\Delta + \left(-t^2 + itb\,(x) + q\,(x)\right)$$

is studied in the space $L_2\,(R^n)$. Here, $-\infty < t < \infty$, $i^2 = -1$, Δ is the Laplace operator, $x = (x_1, x_2, ..., x_n) \in R^n$.

The operator L_t, as is easily seen, arises naturally in the study of singular differential operators of hyperbolic type in the space $L_2\left(R^{n+1}\right)$.

As is well known the existence of the resolvent (self-adjointness) of the Schrödinger operator $\Delta + q\,(x)$ for $t = 0$ is sufficiently well studied in research of T. Kato [1], M. Reed and B. Simon [2], B.M. Levitan, M. Otelbaev [3], M. Otelbaev [4, 5], R.S. Ismagilov [6], F.A. Berezin, M.A. Shubin [7], K.Kh. Boymatov [8], C.F. Yang [9], A. Zettl [10], T. Iwabuchi, T. Matsuyama, K. Tanigichi [11] and others.

The existence of the resolvent for $q\,(x) = q_1\,(x) + iq_2\,(x)\ \ q_1\,(x) \geq 0,\ q_2\,(x) \geq 0$ is investigated in the papers of V.B. Lidskii [12], M. Otelbaev [5] and others.

Naturally, a similar question should be investigated for the operator L_t, i.e. should be studied the question: does there exist a bounded inverse operator L_t^{-1} for all $t \in (-\infty; \infty)$? Also the question of coercive estimates for the operator L_t has been studied in the paper.

We denote by c, c_0, $c_1, ...$ different constants (different in different places), which exact values are not important to us; R^n is n-dimensional real Euclidean space; $x = (x_1, x_2, ..., x_n)$ point in R^n; $\|\cdot\|_2$ is the norm in $L_2\,(R^n)$; $D\,(L_t)$ is a definition domain of the operator L_t; $C_0^\infty\,(R^n)$ is a set of infinitely differentiable and compactly supported functions. Other notation will be introduced along the course of the exposition.

Here are the formulations of the main results.

Consider the operator

$$(L_t + \mu I)\, u = -\Delta u + \left(-t^2 + itb\,(x) + q\,(x) + \mu\right) u$$

originally defined on the set $C_0^\infty(R^n)$, where $\mu \ge 0$.

Further, we assume that the coefficients $b\,(x)$, $q\,(x)$ satisfy the conditions:

i) $|b\,(x)| \ge \delta_0 > 0,\;\; q\,(x) \ge \delta > 0$ are continuous functions in R^n.

The operator $L_t + \mu I$ admits closure in the space $L_2(R^n)$, which is also denoted by $L_t + \mu I$.

Theorem 1.1. Let the condition *i)* be fulfilled. Then the operator $L + \mu E$ is boundedly invertible for $\mu \ge 0$ in the space $L_2(R^n)$.

Theorem 1.2. Let the condition *i)* be fulfilled and $\mu \ge 0$. Then the following estimates

a) $\sum_{i=1}^n \left\| \frac{\partial u}{\partial x_i} \right\|_2 + \left\| \sqrt{q\,(x)}u \right\|_2 \le c \cdot \|(L_t + \mu I)\, u\|_2$;

b) $\sum_{i=1}^n \left\| \frac{\partial u}{\partial x_i} \right\|_2 + \left\| \sqrt{q\,(x)}u \right\|_2 + \left\| \sqrt{|t| \cdot b\,(x)}u \right\|_2 \le c \cdot \|(L_t + \mu I)\, u\|_2$

hold for all $u \in D\,(L_t)$ and all $|t| \ge \beta > 0$, where $c > 0$ is a constant.

2 Auxiliary estimates and lemmas

Lemma 2.1. Let the condition *i)* be fulfilled and $\mu \ge 0$. Then the estimate

$$c\,(\delta) \cdot \|(L_t + \mu I)\, u\|_2 \ge (\delta + \mu)^{1/2} \|u\|_2 \tag{1}$$

holds for all $u \in D\,(L_t)$, where $c\,(\delta) > 0$.

Proof. Let $u \in C_0^\infty(R^n)$. Then the equality

$$\langle (L_t + \mu I)\, u, u \rangle = \int_{R^n} \left(-\Delta u + \left(-t^2 + itb\,(x) + q\,(x) + \mu\right) u\right) \cdot \bar{u} dx =$$

$$= \int_{R^n} \sum_{i=1}^n \left| \frac{\partial u}{\partial x_i} \right|^2 dx + \int_{R^n} \left(-t^2 + itb\,(x) + q\,(x) + \mu\right) |u|^2 dx,$$

holds, where $\langle \cdot, \cdot \rangle$ is a scalar product in $L_2(R^n)$.

Hence we have that

$$|\langle (L_t + \mu I)\, u, u \rangle| \ge \int_{R^n} \sum_{i=1}^n \left| \frac{\partial u}{\partial x_i} \right|^2 dx + \int_{R^n} (q\,(x) + \mu)\, |u|^2 dx - \int_{R^n} t^2 |u|^2 dx.$$

The last inequality implies the following inequalities

$$|\langle (L_t + \mu I)\, u, u \rangle| \ge \int_{R^n} \sum_{i=1}^n \left| \frac{\partial u}{\partial x_i} \right|^2 dx - \int_{R^n} t^2 |u|^2 dx;$$

$$|\langle (L_t + \mu I)\, u, u \rangle| \ge \int_{R^n} (q\,(x) + \mu)\, |u|^2 dx - \int_{R^n} t^2 |u|^2 dx. \tag{2}$$

Using the Cauchy inequality with $\varepsilon > 0$, here $\varepsilon = \frac{\delta}{2}$, we find from (2)

$$\frac{1}{\delta} \|(L_t + \mu I)\, u\|_2^2 \ge \frac{1}{2} \int_{R^n} (q\,(x) + \mu)\, |u|^2 dx - \int_{R^n} t^2 |u|^2 dx. \tag{3}$$

Further, we consider the following scalar product

$$|\langle (L_t + \mu I)\, u, -itu \rangle| = \left| it \int_{R^n} \left(\sum_{i=1}^n \left| \frac{\partial u}{\partial x_i} \right|^2 + \left(-t^2 + q\,(x) + \mu\right) \cdot |u|^2 \right) dx + \int_{R^n} t^2 b\,(x)\, |u|^2 dx \right|. \tag{4}$$

From (4) we find that

$$|\langle (L_t + \mu I)\, u, -itu\rangle| \geq \int_{R^n} |t|^2 \, |b\,(x)| \, |u|^2 \, dx. \tag{5}$$

Since $|b\,(x)| \geq \delta_0 > 0$ then we obtain from (5)

$$|\langle (L_t + \mu I)\, u, -itu\rangle| \geq t^2 \cdot \delta_0 \cdot \|u\|_2^2 \, .$$

Hence

$$\|(L_t + \mu I)\, u\|_2^2 \cdot \geq |t|^2 \cdot \delta_0^2 \cdot \|u\|_2^2 \, . \tag{6}$$

Multiplying both sides (6) by a number $c_0 > 0$ we have

$$c_0 \, \|(L_t + \mu I)\, u\|_2^2 \geq c_0 \cdot |t|^2 \cdot \delta_0^2 \cdot \|u\|_2^2 \, . \tag{7}$$

Combining (7) and (3) and choosing the number $c_0 > 0$ so that $|t|^2 \, \delta_0^2 c_0 - |t|^2 \geq 0$ we obtain that

$$\left(c_0 + \frac{1}{\delta}\right) \|(L_t + \mu I)\, u\|_2^2 \geq \int_{R^n} (q\,(x) + \mu) \, |u|^2 \, dx.$$

Hence, taking into account the condition i), we find

$$\sqrt{c\,(\delta)} \cdot \|(L_t + \mu I)\, u\|_2 \geq (\delta + \mu)^{1/2} \, \|u\| \, ,$$

where $c\,(\delta) = \left(c_0 + \frac{1}{\delta}\right)$. The inequality (1) is proved.

We take the collection of non-negative functions $\{\varphi_j\}\,, \ j \geq 1$ from $C_0^\infty\,(R^n)$ such that

$$\sum_j \varphi_j^2 \equiv 1, \ \ \sup p \varphi_j \subset \Delta_j, \ \ \bigcup_{j \geq 1} \Delta_j \equiv R^n,$$

where Δ_j — is open sets which intersection multiplicity is not higher than a some number $\xi = \xi\,(n) < \infty$. The existence of such coverage follows from the results of [13–15].

Continue $b\,(x), \ q\,(x)$ from Δ_j on the whole R^n so that their continuations $b_j\,(x)$ and $q_j\,(x)$ are bounded and periodic functions of the same period.

By $L_{t,j,\alpha} + \mu I$ we denote the closure of the operator

$$(L_{t,j,\alpha} + \mu I)\, u = -\Delta u + \left(-t^2 + it\,(b_j\,(x) + \alpha) + q_j\,(x) + \mu\right) \cdot u$$

defined on $C_0^\infty\,(R^n)$, where the sign of the real number α coincides with the sign of the function $b\,(x)$, i.e. $\alpha \cdot b\,(x) > 0$ for $x \in R^n$.

Lemma 2.2. Let the condition i) be fulfilled. Then for $\mu \geq 0$ the estimate

$$c\,(\delta) \cdot \|(L_{t,j,\alpha} + \mu I)\, u\|_{L_2(R^n)} \geq (\delta + \mu)^{1/2} \, \|u\|_{L_2(R^n)} \tag{8}$$

holds for all $u \in D\,(L_{t,j,\alpha})$ in the space $L_2\,(R^n)$, where $c\,(\delta) > 0$.

Proof. Let $u \in C_0^\infty\,(R^n)$. Then we have

$$|\langle (L_{t,j,\alpha} + \mu I)\, u, u\rangle| \geq \left| \int_{R^n} \sum_{i=1}^n \left| \frac{\partial u}{\partial x_i} \right|^2 dx + \int_{R^n} [q_j\,(x) + \mu] \cdot |u|^2 \, dx \right| - \left| \int_{R^n} t^2 \, |u|^2 \, dx \right| .$$

Hence we have the following inequalities

$$\|(L_{t,j,\alpha} + \mu I)\, u\|_{L_2(R^n)} \cdot \|u\|_2 \geq \int_{R^n} \sum_{i=1}^n \left| \frac{\partial u}{\partial x_i} \right|^2 dx - \int_{R^n} t^2 \, |u|^2 \, dx; \tag{9}$$

$$\frac{1}{\delta} \, \|(L_{t,j,\alpha} + \mu I)\, u\|_2^2 \geq \frac{1}{2} \int_{R^n} [q_j\,(x) + \mu] \cdot |u|^2 \, dx - t^2 \int_{R^n} |u|^2 \, dx. \tag{10}$$

In the inequalities (10) we have used the Cauchy inequality with $\varepsilon > 0$, where $\varepsilon = \frac{\delta}{2}$.

Further, we consider the following scalar product

$$|\langle (L_{t,j,\alpha}+\mu I)\,u,-itu\rangle| =$$

$$= \left| -it\int_{R^n}\sum_{i=1}^{n}\left|\frac{\partial u}{\partial x_i}\right|^2 dx + \int_{R^n}\left[-t^2+q_j(x)+\mu\right]\cdot |u|^2\,dx + \int_{R^n} t^2\left(b_j(x)+\alpha\right)|u|^2\,dx \right|.$$

Hence, and by virtue of the condition $\alpha\cdot b(x)>0$ we have for $x\in R^n$

$$|\langle (L_{t,j,\alpha}+\mu I)\,u,-itu\rangle| \geq t^2\int_{R^n}\left(|b_j(x)|+|\alpha|\right)|u|^2\,dx.$$

From the last inequality we find that

$$\|(L_{t,j,\alpha}+\mu I)\,u\|_2^2 \geq |t|^2\left(\delta_0+|\alpha|\right)^2\cdot\|u\|_2^2. \tag{11}$$

Combining the inequalities (10) and (11) and choosing α so that $(\delta_0+|\alpha|)^2-1\geq 0$ we obtain

$$\frac{1}{\delta}\,\|(L_{t,j,\alpha}+\mu I)\,u\|_2^2+\|(L_{t,j,\alpha}+\mu I)\,u\|_2^2 \geq \frac{1}{2}\int_{R^n}\left[q_j(x)+\mu\right]\cdot|u|^2\,dx. \tag{12}$$

From the inequality (12) we find that

$$c(\delta)\,\|(L_{t,j,\alpha}+\mu I)\,u\|_2 \geq \sqrt{\delta+\mu}\,\|u\|_2\,,$$

where $c(\delta)=\sqrt{2\left(\frac{1}{\delta}+1\right)}$. Lemma 2.2 is proved.

Lemma 2.3. Let the condition $i)$ be fulfilled. Then for $\mu\geq 0$ the estimates

$$\|(L_{t,j,\alpha}+\mu I)\,u\|_2^2 \geq \sum_{i=1}^{n}\left\|\frac{\partial u}{\partial x_i}\right\|_2^2+\left\|\sqrt{q_j(x)+\mu}\,u\right\|_2^2; \tag{13}$$

$$\|(L_{t,j,\alpha}+\mu I)\,u\|_2^2 \geq \sum_{i=1}^{n}\left\|\frac{\partial u}{\partial x_i}\right\|_2^2+\left\|\sqrt{q_j(x)+\mu}\,u\right\|_2^2+\left\|\sqrt{|t|\left(b_j(x)+|\alpha|\right)}u\right\|_2^2 \tag{14}$$

hold for all $u\in D(L_{t,j,\alpha})$ and $|t|\geq\beta>0$, where $\beta>0$ is any positive number.

Proof. From the inequality (9), using the Cauchy inequality with $\varepsilon>0$, we have

$$\frac{1}{2\varepsilon}\,\|(L_{t,j,\alpha}+\mu I)\,u\|_2^2+\frac{\varepsilon}{2}\,\|u\|_2^2 \geq \int_{R^n}\sum_{i=1}^{n}\left|\frac{\partial u}{\partial x_i}\right|^2 dx - t^2\int_{R^n}|u|^2\,dx.$$

Now we take $\varepsilon=\frac{\delta}{2}$. Then from the last inequality we obtain

$$\frac{1}{\delta}\,\|(L_{t,j,\alpha}+\mu I)\,u\|_2^2 \geq \sum_{i=1}^{n}\left\|\frac{\partial u}{\partial x_i}\right\|_2^2-\frac{\delta}{4}\,\|u\|_2^2-t^2\,\|u\|_2^2. \tag{15}$$

From (11) and (15) we find that

$$\frac{1}{\delta}\,\|(L_{t,j,\alpha}+\mu I)\,u\|_2^2+\|(L_{t,j,\alpha}+\mu I)\,u\|_2^2 \geq \sum_{i=1}^{n}\left\|\frac{\partial u}{\partial x_i}\right\|_2^2-\frac{\delta}{4}\,\|u\|_2^2+\left(|t|^2\left(\delta_0+|\alpha|\right)^2-t^2\right)\cdot\|u\|_2^2.$$

Hence, choosing α so that $|t|^2\left[(\delta_0+|\alpha|)^2-1\right]\geq 0$, we have

$$\left(\frac{1}{\delta}+1\right)\|(L_{t,j,\alpha}+\mu I)\,u\|_2^2 \geq \sum_{i=1}^{n}\left\|\frac{\partial u}{\partial x_i}\right\|_2^2-\frac{\delta}{4}\,\|u\|_2^2. \tag{16}$$

From (16) and (12) we find that

$$\left(\frac{1}{\delta}+1\right)\|(L_{t,j,\alpha}+\mu I)\,u\|_2^2+\left(\frac{1}{\delta}+1\right)\|(L_{t,j,\alpha}+\mu I)\,u\|_2^2\geq$$

$$\geq\sum_{i=1}^{n}\left\|\frac{\partial u}{\partial x_i}\right\|_2^2-\frac{\delta}{4}\|u\|_2^2+\frac{1}{2}\int_{R^n}[q_j(x)+\mu]\cdot|u|^2\,dx.$$

Hence

$$2\left(\frac{1}{\delta}+1\right)\|(L_{t,j,\alpha}+\mu I)\,u\|_2^2\geq$$

$$\geq\sum_{i=1}^{n}\left\|\frac{\partial u}{\partial x_i}\right\|_2^2-\frac{\delta}{4}\|u\|_2^2+\frac{1}{4}\int_{R^n}[q_j(x)+\mu]\cdot|u|^2\,dx+\frac{1}{4}\int_{R^n}[q_j(x)+\mu]\cdot|u|^2\,dx.$$

From the last inequality we find

$$8\left(\frac{1}{\delta}+1\right)\|(L_{t,j,\alpha}+\mu I)\,u\|_2^2\geq 4\cdot\sum_{i=1}^{n}\left\|\frac{\partial u}{\partial x_i}\right\|_2^2+\left\|\sqrt{q_j(x)+\mu}\,u\right\|_2^2.$$

Finally, we have

$$c(\delta)\,\|(L_{t,j,\alpha}+\mu I)\,u\|_2^2\geq\sum_{i=1}^{n}\left\|\frac{\partial u}{\partial x_i}\right\|_2^2+\left\|\sqrt{q_j(x)+\mu}\,u\right\|_2^2,$$

where $c(\delta)=8\left(\frac{1}{\delta}+1\right)$. The inequality (13) is proved.

Prove the inequality (14). Let $u\in C_0^\infty(R^n)$. Then integrating by parts the scalar product

$$|\langle(L_{t,j,\alpha}+\mu I)\,u,u\rangle|\geq\left|\int_{R^n}it\,(b_j(x)+\alpha)\,|u|^2\,dx\right|.$$

Hence, by the condition $i)$ and also taking the property $\alpha\cdot b(x)>0$ into account, we have for $x\in R^n$

$$\frac{1}{2\varepsilon}\|(L_{t,j,\alpha}+\mu I)\,u\|_2^2\geq\frac{1}{2}\int_{R^n}|t|\,(|b_j(x)|+|\alpha|)\,|u|^2\,dx+\frac{1}{2}\int_{R^n}\left[|t|\,(|b_j(x)|+|\alpha|)-\frac{\varepsilon}{2}\right]\cdot|u|^2\,dx.$$

We have used here the Cauchy inequality with $\varepsilon>0$. Now, taking $|t|\geq\beta>0$ into account and choosing $\varepsilon>0$ so that $|t|\,(\delta_0+|\alpha|)-\frac{\varepsilon}{2}\geq 0$, we obtain the estimate

$$c(\varepsilon)\,\|(L_{t,j,\alpha}+\mu I)\,u\|_2^2\geq\left\|\sqrt{|t|\,(|b_j(x)|+|\alpha|)}\,u\right\|_2^2. \tag{17}$$

The estimate (14) follows from (13) and (17). Lemma 2.3 is completely proved.

Lemma 2.4. Operator $L_{t,j,\alpha}+\mu I$ for $\mu\geq 0$ has a continuous inverse operator $(L_{t,j,\alpha}+\mu I)^{-1}$ defined over the whole $L_2(R^n)$.

Proof. The estimate (8) implies that for the proof of Lemma 2.4 it suffices to show that the range of values $R(L_{t,j,\alpha}+\mu I)$ of the operator $L_{t,j,\alpha}+\mu I$ coincide with $L_2(R^n)$. We prove this by contradiction.

We assume that there exist an element $v\in L_2(R^n)$, $v\neq 0$, such that for any $u\in D(L_{t,j,\alpha})$

$$\langle(L_{t,j,\alpha}+\mu I)\;u,v>=0.$$

From the last equality we obtain

$$(L_{t,j,\alpha}+\mu I)^*\,v=-\Delta v+\left(-t^2-it\,(b_j(x)+\alpha)+q_j(x)+\mu\right)\;v=0 \tag{18}$$

in the sense of distributions. Since $b_j(x)\,v$, $q_j(x)\,v\in L_2(R^n)$, then (18) implies that $\Delta v\in L_2(R^n)$ for finite t, i.e. $v\in W_2^2(R^n)$. Now, if for any $v_n\in C_0^\infty(R^n)$ the inequality

$$\left\|(L_{t,j,\alpha}+\mu I)^*\,v\right\|_2\geq c\,\|v\|_2 \tag{19}$$

holds, where $c > 0$ is a constant then it also holds for the element $v \in W_2^2(R^n)$. Indeed, there exists a sequence $\{v_n\} \subset C_0^\infty(R^n)$ for $v \in W_2^2(R^n)$ converging to $v(x)$ in the norm $W_2^2(R^n)$. It is not difficult to prove that the inequality (19) holds for every $v_n(x)$. This is proved in exactly the same way as the inequality (8) in Lemma 2.2. Passing to the limit with respect to $n \to \infty$, we obtain, as is easy to see, the inequality (19) for $v(x)$ (this procedure is briefly called the closure of the inequality (19) in the norm $W_2^2(R^n)$), i.e.

$$\left\|(L_{t,j} + \mu I)^* v\right\|_2 \geq c\,\|v\|_2 .$$

Since $(L_{t,j,\alpha} + \mu I)^* v = 0$, then the last inequality implies that $v \equiv 0$. Lemma 2.4 is proved.

Lemma 2.5. Let the condition $i)$ be fulfilled and $\mu \geq 0$. Then the following inequalities

a) $\left\|(L_{t,j,\alpha} + \mu I)^{-1}\right\|_{2\to 2} \leq \frac{c}{(\delta+\mu)^{1/2}}, \;\; c = c(\delta) > 0;$

b) $\left\|D_{x_i}(L_{t,j,\alpha} + \mu I)^{-1}\right\|_{2\to 2} \leq \frac{c}{(\delta+\mu)^{1/4}}, \;\; c = c(\delta) > 0$

hold, where $D_{x_i} = \frac{\partial}{\partial x_i}$.

Proof. The estimate a) follows from the estimate (8). The estimate (8) also implies the inequality

$$\frac{c(\delta)}{(\delta+\mu)^{1/2}}\,\|(L_{t,j,\alpha} + \mu I)\,u\|_2 \geq \|u\|_2 . \tag{20}$$

From the inequality (9), using the inequality (20), we find that

$$\frac{c(\delta)}{\sqrt{(\delta+\mu)}}\,\|(L_{t,j,\alpha} + \mu I)\,u\|_2^2 \geq \int_{R^n} \sum_{i=1}^{n} \left|\frac{\partial u}{\partial x_i}\right|^2 dx - t^2 \int_{R^n} |u|^2\,dx. \tag{21}$$

Multiplying both sides of the inequality (11) by the number $\frac{1}{\sqrt{(\delta+\mu)}}$, we find the following inequality

$$\frac{1}{\sqrt{(\delta+\mu)}}\,\|(L_{t,j,\alpha} + \mu I)\,u\|_2^2 \geq \frac{|t|^2(\delta_0 + |\alpha|)^2}{\sqrt{\delta+\mu}}\,\|u\|_2^2 . \tag{22}$$

Now combining (21) and (22), we obtain

$$\frac{c(\delta)+1}{\sqrt{(\delta+\mu)}}\,\|(L_{t,j,\alpha} + \mu I)\,u\|_2^2 \geq \int_{R^n} \sum_{i=1}^{n} \left|\frac{\partial u}{\partial x_i}\right|^2 dx + |t|^2 \int_{R^n} \left(\frac{(\delta_0 + |\alpha|)^2}{\sqrt{\delta+\mu}} - 1\right)|u|^2\,dx.$$

Choosing α such that $\frac{(\delta_0+|\alpha|)^2}{\sqrt{\delta+\mu}} - 1 \geq 0$, from the last inequality we find the estimate

$$\frac{c(\delta)}{\sqrt{(\delta+\mu)}}\,\|(L_{t,j,\alpha} + \mu I)\,u\|_2^2 \geq \sum_{i=1}^{n} \left|\frac{\partial u}{\partial x_i}\right|^2, \; c(\delta) = c(\delta) + 1.$$

The estimate b) is proved. Lemma 2.5 is completely proved.

3 Some estimates in the whole R^n

We denote the closure of the differential expression

$$(L_{t,\alpha} + \mu I)\,u = -\Delta u + \left(-t^2 + it\,(b(x) + \alpha) + q(x) + \mu\right)u$$

in $L_2(R^n)$ by $L_{t,\alpha} + \mu I$, defined on the set $C_0^\infty(R^n)$.

Lemma 3.1. Let the condition $i)$ be fulfilled and $\mu \geq 0$. Then the following inequalities

$$\|(L_{0,\alpha} + \mu I)\,u\|_2 \geq (\delta + \mu)\cdot\|u\|_2 ; \tag{23}$$

$$\|(L_{t,\alpha} + \mu I)\,u\|_2 \geq |t|\,(\delta_0 + |\alpha|)\cdot\|u\|_2 , \;\; t \neq 0, \tag{24}$$

hold for all $u \in D(L_{t,\alpha} + \mu I)$.

The inequalities (23) and (24) are proved by means of functionals $\langle (L_{0,\alpha} + \mu I)\, u, u\rangle$, $\langle (L_{t,\alpha} + \mu I)\, u, u\rangle$, $u \in C_0^{\infty}(R^n)$.

To prove the main theorems, in addition to the auxiliary lemmas given in sections 2 and 3, we use the following assertion [13–15].

Lemma 3.2. There exist a covering that has the following properties

a) $\sum_{\{j\}} \varphi_j^2 \equiv 1,\ \ \varphi_j \in C_0^{\infty}(\Delta_j),\ \ \bigcup_{\{j\}} \Delta_j \equiv R^n$,

where Δ_j is a cube by an edge equal to 1 and center at the point $x_j = \left(x_1^{(j)}, x_2^{(j)}, ..., x_n^{(j)}\right)$;

b) $\|D_x^{\alpha}\varphi_j(x)\|_{C(R)} \le c$, $c > 0$ is a constant, $D_x^{\alpha} = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1},...,\partial x_n^{\alpha_n}}$, $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n)$, $|\alpha| = \alpha_1 + \alpha_2 + ... + \alpha_n$;

c) each set Δ_j can intersect with not more than $\xi(n)$ sets from the family $\{\Delta_j\}$, where $\xi(n)$ is some constant.

Assume

$$K_{\mu,\alpha} f = \sum_{\{j\}} \varphi_j \left(L_{t,j,\alpha} + \mu I\right)^{-1} \varphi_j f,$$

where $f \in L_2(R^n)$, $\{\varphi_j\}$ is a set of functions from Lemma 3.2, $L_{t,j,\alpha} + \mu I$ is the operator from Lemma 2.2.

It is easy to see that

$$\left(L_{t,\alpha} + \mu I\right) K_{\mu,\alpha} f = f - B_{\mu,\alpha} f, \tag{25}$$

where

$$B_{\mu,\alpha} f = \sum_{\{j\}} \Delta\varphi_j \left(L_{t,j,\alpha} + \mu I\right)^{-1} f + 2\sum_{\{j\}}\sum_{i=1}^{n} D_{x_i}\varphi_j D_{x_i}\left(L_{t,j,\alpha} + \mu I\right)^{-1} \varphi_j f, D_{x_i} = \frac{\partial}{\partial_{x_i}}.$$

Lemma 3.3. Let the condition *i*) be fulfilled. Then there find a number $\mu_0 > 0$ such that

$$\|B_{\mu,\alpha}\|_{L_2(R^n)\to L_2(R^n)} < 1$$

for all $\mu \ge \mu_0$.

Proof. Let $f \in C_0^{\infty}(R^n)$. Now, taking the multiplicity of the covering $\{\Delta_j\}$ into account, we have

$$\|B_{\mu,\alpha} f\|_{L_2(R^n)} \le 2\cdot\left(\left\|\sum_{\{j\}} \Delta\varphi_j \left(L_{t,j,\alpha} + \mu I\right)^{-1}\varphi_j f\right\|_2^2 + \left\|2\sum_{\{j\}}\sum_{i=1}^{n} D_{x_i}\varphi_j D_{x_i}\left(L_{t,j,\alpha} + \mu I\right)^{-1}\varphi_j f\right\|_2^2\right) \le$$
$$\le c_1\cdot\left(\sum_{\{j\}}\left\|\Delta\varphi_j \left(L_{t,j,\alpha} + \mu I\right)^{-1}\varphi_j f\right\|_2^2 + \sum_{i=1}^{n}\sum_{\{j\}}\left\|D_{x_i}\varphi_j D_{x_i}\left(L_{t,j,\alpha} + \mu I\right)^{-1}\varphi_j f\right\|_2^2\right) \le$$
$$\le c_1\cdot\left(\sup_{\{j\}}\left\|\Delta\varphi_j \left(L_{t,j,\alpha} + \mu I\right)^{-1}\right\|_2^2 \cdot \sum_{\{j\}}\|\varphi_j f\|_2^2 + \sup_{1\le i\le n}\sup_{\{j\}}\left\|D_{x_i}\varphi_j D_{x_i}\left(L_{t,j,\alpha} + \mu I\right)^{-1}\varphi_j f\right\|_2^2\right).$$

From this and from Lemma 2.5, taking $\left\|D_{x_i}^{\alpha}\varphi_j\right\|_{C(R)} \le c$, $\alpha = 0, 1, 2$, $\sum_{\{j\}}\|\varphi_j f\|^2 = \|f\|_2^2$ into account, we obtain

$$\|B_{\mu,\alpha} f\|_{2\to 2}^2 \le c_1\cdot c\left[\frac{c^2(\delta)}{(\delta+\mu)} + \frac{c^2(\delta)}{(\delta+\mu)^{1/2}}\right]\cdot\|f\|_{L_2(R^n)}^2.$$

From the last inequality it follows that one can find a number $\mu_0 > 0$ such that $\|B_{\mu,\alpha}\|_{2\to 2} < 1$ for $\mu \ge \mu_0$. Lemma 3.3 is proved.

Lemma 3.4. Let the condition *i*) be fulfilled. Then the operator $L_{t,\alpha} + \mu I$ is boundedly invertible for $\mu \ge \mu_0 > 0$, and the equality

$$\left(L_{t,\alpha} + \mu I\right)^{-1} = K_{\mu,\alpha}\left(E - B_{\mu,\alpha} f\right)^{-1} \tag{26}$$

holds for the inverse operator $\left(L_{t,\alpha} + \mu I\right)^{-1}$.

The proof of the lemma follows from the representation (25) using Lemmas 3.1 and 3.3.

Now we consider the solvability of the original operator $L_t + \mu I$. To do this, consider the following equation

$$\left(L_t + \mu I\right) u = -\Delta u + \left(-t^2 + itb(x) + q(x) + \mu\right) u = f(x), \tag{27}$$

where $f(x) \in L_2(R^n)$.

Definition. The function $u \in L_2(R^n)$ is called a solution of the equation (27) if there exists a sequence $\{u_n\}_{n=1}^{\infty} \subset C_0^{\infty}(R^n)$ such that

$$\|u_n - u\|_{L_2(R^n)} \to 0, \ \ \|(L_t + \mu I)\, u_n - f\|_{L_2(R^n)} \to 0,$$

as $n \to \infty$.

This shows that the inverse operator $(L_t + \mu I)^{-1}$ coincides with the closure of the operator $L_t + \mu I$ in $L_2(R^n)$ defined on $C_0^{\infty}(R^n)$.

Lemma 3.5. Let the condition $i)$ be fulfilled. Then the operator $L_t+\mu I$ is boundedly invertible for $\mu \geq \mu_0 > 0$ and the equality

$$(L_t + \mu I)^{-1} f = (L_{t,\alpha} + \mu I)^{-1} (E - A_{\mu,\alpha})^{-1} f$$

holds for the inverse operator $(L_t + \mu I)^{-1}$, where $f \in L_2(R^n)\,,\ \|A_{\mu,\alpha}\|_{L_2(R^n)\to L_2(R^n)} < 1$.

Proof. Let $t \neq 0$. The equation

$$(L_t + \mu I)\, u = f \tag{28}$$

can be rewritten as

$$v - A_{\mu,\alpha} v = f$$

where $v = (L_{t,\alpha} + \mu I)\, u,\ A_{\mu,\alpha} = it\alpha\,(L_{t,\alpha} + \mu I)^{-1}$, where $i^2 = -1$. Lemma 3.1 implies that

$$\|A_{\mu,\alpha} v\|_{L_2(R^n)} \leq \frac{|t| \cdot |\alpha|}{|t|\,(\delta_0 + |\alpha|)}. \tag{29}$$

Hence $\|A_{\mu,\alpha}\|_{L_2(R^n)\to L_2(R^n)} < 1$.

(28), (29) implies that

$$u = (L_t + \mu I)^{-1} f = (L_{t,\alpha} + \mu I)^{-1} (I - A_{\mu,\alpha})^{-1} f$$

for $t \neq 0$.

As is known, if $t = 0$ then the operator is essentially self-adjoint [2] and the estimate

$$\|(L_0 + \mu I)\, u\|_{L_2(R^n)} \geq (\delta + \mu) \cdot \|u\|_{L_2(R^n)}$$

holds for all $u \in D(L_0 + \mu I)$. This implies that the operator $L_0+\mu I$ has a bounded inverse operator $(L_0 + \mu I)^{-1}$ defined over the whole $L_2(R^n)$. Lemma 3.5 is completely proved.

Lemma 3.6. Let the condition $i)$ be fulfilled and $\mu \geq 0$. Then the estimates

$$\|(L_0 + \mu I)\, u\|_{L_2(R^n)} \geq \delta\, \|u\|_{L_2(R^n)}\,,$$

$$\|(L_t + \mu I)\, u\|_{L_2(R^n)} \geq |t| \cdot \delta_0 \cdot \|u\|_{L_2(R^n)}\,,\ t \neq 0,$$

hold for all $u \in D(L_t)$.

Proof. Lemma 3.6 is proved in exactly the same way as Lemma 3.1.

The following Lemma is well-known [16].

Lemma 3.7. Let the operator $L_t + \mu_0 I$ $(\mu_0 \geq 0)$ is boundedly invertible in $L_2(R^n)$ and the estimate

$$\|(L_t + \mu I)\, u\|_{L_2(R^n)} \geq c \cdot \|u\|_{L_2(R^n)}$$

holds for all $u \in D(L_t + \mu I)$ when $\mu \in [0, \mu_0]$, where $c > 0$ is a constant. Then the operator $L_t : L_2(R^n) \to$ $\to L_2(R^n)$ is also boundedly invertible.

Proof of Theorem 1.1. The proof follows from Lemmas 2.1, 3.5 and 3.7.

Lemma 3.8. Let the condition $i)$ be fulfilled and $\mu \geq \mu_0$. Then the following estimates

a) $\left\|\sqrt{q(x) + \mu}\, u\right\|_{L_2)R^n} \leq c \cdot \|(L_{t,\alpha} + \mu I)\, u\|_{L_2)R^n}$;

b) $\|D_{x_i} u\|_{L_2)R^n} \leq c \cdot \|(L_{t,\alpha} + \mu I)\, u\|_{L_2)R^n}$,

hold for all $u \in D(L_{t,\alpha})$, where $D_{x_i} = \frac{\partial}{\partial x_i}$ $(i = 1, 2, ..., n)$, $c > 0$ is a constant.

Proof. Let $f \in C_0^\infty(R^n)$. Then, taking the properties of the functions $\varphi_j\ (j \in Z)$ into account, from the representation of (26) we have

$$\left\|\sqrt{q(x)+\mu}\,(L_{t,\alpha}+\mu I)^{-1} f\right\|_2^2 = \left\|\sqrt{q(x)+\mu}\sum\nolimits_{\{j\}}\varphi_j\,(L_{t,j,\alpha}+\mu I)^{-1}\varphi_j\,(E-B_{\mu,\alpha})^{-1} f\right\|_2^2 =$$
$$= \left\|\sum\nolimits_{\{j\}}\sqrt{q(x)+\mu}\cdot\varphi_j\,(L_{t,j,\alpha}+\mu I)^{-1}\varphi_j\,(E-B_{\mu,\alpha})^{-1} f\right\|_2^2 \le \tag{30}$$
$$\le \left\|\sum\nolimits_{\{j\}}\sqrt{q_j(x)+\mu}\,(L_{t,j,\alpha}+\mu I)^{-1}\varphi_j\,(E-B_{\mu,\alpha})^{-1} f\right\|_2^2.$$

Here it was taken into account that the functions $\sqrt{q(x)+\mu}\cdot\varphi_j$ and $\sqrt{q_j(x)+\mu}$ coincide on the set Δ_j. Taking the finite multiplicity of the covering $\{\Delta_j\}$ and the inequality (13) into account, we obtain from (30)

$$\left\|\sqrt{q(x)+\mu}\,(L_{t,\alpha}+\mu I)^{-1} f\right\|_2^2 \le c\cdot\sum\nolimits_{\{j\}}\left\|\sqrt{q_j(x)+\mu}\,(L_{t,j,\alpha}+\mu I)^{-1}\varphi_j\,(E-B_{\mu,\alpha})^{-1} f\right\|_2^2 \le$$
$$\le c\cdot\sup_{\{j\}}\left\|\sqrt{q_j(x)+\mu}\,(L_{t,j,\alpha}+\mu I)^{-1}\right\|_{2\to 2}^2\cdot\sum\nolimits_{\{j\}}\left\|\varphi_j\,(E-B_{\mu,\alpha})^{-1} f\right\|_2^2$$

From this and Lemmas 2.3 and 3.3 we finally have

$$\left\|\sqrt{q(x)+\mu}\,(L_{t,\alpha}+\mu I)^{-1} f\right\|_2^2 \le c\cdot\|f\|_2^2.$$

Whence

$$\left\|\sqrt{q(x)+\mu}u\right\|_{L_2(R^n)}^2 \le c\cdot\|(L_{t,\alpha}+\mu I)\,u\|_2^2,$$

where $(L_{t,\alpha}+\mu I)\,u = f$. The item a) of Lemma 3.8 is proved.

Let us prove the item b) of Lemma 3.8.

$$\|D_{x_i}(L_{t,\alpha}+\mu I)\,f\|_2 = \left\|D_{x_i}\sum\nolimits_j\varphi_j\,(L_{t,j,\alpha}+\mu I)^{-1}\varphi_j\,(E-B_{\mu,\alpha})^{-1} f\right\|_2^2 =$$
$$= \left\|\sum\nolimits_{\{j\}}\left(D_{x_i}\varphi_j\,(L_{t,j,\alpha}+\mu I)^{-1}\varphi_j\,(E-B_{\mu,\alpha})^{-1} f + \varphi_j D_{x_i}(L_{t,j,\alpha}+\mu I)^{-1}\varphi_j\,(E-B_{\mu,\alpha})^{-1} f\right)\right\|_2^2 \le$$
$$\le c\left(\sum\nolimits_{\{j\}}\left\|D_{x_i}\varphi_j\,(L_{t,j,\alpha}+\mu I)^{-1}\varphi_j\,(E-B_{\mu,\alpha})^{-1} f\right\|_2^2 + \right.$$
$$\left. + \sum\nolimits_{\{j\}}\left\|\varphi_j D_{x_i}(L_{t,j,\alpha}+\mu I)^{-1}\varphi_j\,(E-B_{\mu,\alpha})^{-1} f\right\|_2^2\right).$$

Here we took the finite multiplicity of the covering $\{\Delta_j\}$ into account.

From the last inequality we have

$$\|D_{x_i}(L_{t,\alpha}+\mu I)\,f\|_{L_2)R^n}^2 \le c\left(\sup_{\{j\}}\left\|D_{x_i}\varphi_j\,(L_{t,j,\alpha}+\mu I)^{-1}\right\|_2^2\cdot\sum\nolimits_{\{j\}}\left\|\varphi_j\,(E-B_{\mu,\alpha})^{-1} f\right\|_2^2 + \right.$$
$$\left. + \sup_{\{j\}}\left\|\varphi_j\cdot D_{x_i}(L_{t,j,\alpha}+\mu I)^{-1}\right\|_2^2\cdot\sum\nolimits_{\{j\}}\left\|\varphi_j\,(E-B_{\mu,\alpha})^{-1} f\right\|_2^2\right) \le$$
$$\le c\left(\tilde{c}_1\sup_{\{j\}}\left\|(L_{t,j,\alpha}+\mu I)^{-1}\right\|_2^2\cdot\sum\nolimits_{\{j\}}\left\|\varphi_j\,(E-B_{\mu,\alpha})^{-1} f\right\|_2^2 + \right.$$
$$\left. + \tilde{c}_0\sup_{\{j\}}\left\|D_{x_i}(L_{t,j,\alpha}+\mu I)^{-1}\right\|_2^2\cdot\sum\nolimits_{\{j\}}\left\|\varphi_j\,(E-B_{\mu,\alpha})^{-1} f\right\|_2^2\right),$$

where $\tilde{c}_1 = \sup\limits_{\{j\}}\left(\max\limits_{x\in R^n}|\varphi_j'|\right)$, $\tilde{c}_0 = \sup\limits_{\{j\}}\left(\max\limits_{x\in R^n}|\varphi_j|\right)$.

From this and from Lemma 2.5 we find that

$$\|D_{x_i}(L_{t,\alpha}+\mu I)\,f\|_{L_2)R^n}^2 \le c\left(\frac{\tilde{c}_1\cdot c(\delta)}{\delta+\mu}\cdot\left\|(E-B_{\mu,\alpha})^{-1}\right\|_{2\to 2}^2\cdot\|f\|_{L_2(R^n)}^2 + \right.$$
$$\left. + \tilde{c}_0\cdot\frac{c(\delta)}{(\delta+\mu)^{1/2}}\cdot\left\|(E-B_{\mu,\alpha})^{-1}\right\|_{2\to 2}^2\cdot\|f\|_{L_2(R^n)}^2\right).$$

Lemma 3.3 implies that the operator $(E - B_{\mu,\alpha})^{-1}$ is bounded in $L_2(R^n)$. Consequently, from the last inequality we finally have

$$\|D_{x_i}u\|^2_{L_2)R^n} \leq c^2 \|(L_{t,\alpha} + \mu I)\, u\|^2_{L_2(R^n)},$$

where $(L_{t,\alpha} + \mu I)\, u = f$, $i = 1, 2, 3, ..., n$. Lemma 3.8 is proved.

Proof of Theorem 1.2. The proof of Theorem 1.2 follows from Lemmas 2.3, 3.5 and 3.8.

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М.Б. Мұратбеков, М.М. Мұратбеков

$L_2(R^n)$ кеңістігінде теріс параметрлі Шрёдингер оператор үшін шектеулі кері операторының бар болуы жайлы

$L = -\Delta + q(x),\ x \in R^n$, Шрёдингер операторы қазіргі кванттық механиканың және теориялық физиканың негізгі операторларының бірі болып табылады. L Шрёдингер операторы үшін көптеген іргелі нәтижелер алынды. Олардың ішінде, мысалы, шешілуі, бөлгіштігі (коэрцитивті бағалаулар), әртүрлі салмақтық бағалар, оператордың анықталу облысы бойынша аралық туындылардың бағалауы, меншікті мәндер және сингулярлық сандар (s-сандар) туралы мәселелер бар. Қазіргі кезде эллиптикалық операторлар үшін жоғарыда келтірілген нәтижелердің әртүрлі қорытындылары бар. Жалпы дифференциалды операторлар үшін бұл мәселенің шешілуі толық аяқталмаған. Атап айтқанда, біздің біліуміз бойынша, резольвентасының бар болуы мен коэрцитивтігі, сондай-ақ шексіз облыста өспелі және тербелісті коэффициенттерімен гиперболалық типті оператордың спектрінің дискретілігін көрсететін нәтиже болмады. Әрине, $L_2(R^{n+1})$ кеңістігінде анықталған гиперболалық типті дифференциалдық операторлардың кейбір кластарының зерттеуін Фурье әдісі арқылы теріс параметрлі Шрёдингер операторын зерттеуге алып келуге болады: $L_t = -\Delta + \left(-t^2 + itb(x) + q(x)\right)$, мұндағы t — $(-\infty < t < \infty)$, $i^2 = -1$ параметрі. Демек, L_t операторында $|t| \to \infty$ болса, онда $-t^2 \to -\infty$ екенін көру қиын емес. Демек, мұнда, $L = -\Delta + q(x)$ Шрёдингер операторына қарағанда, мүлдем басқа жағдай пайда болады, соның ішінде L Шрёдингер операторы үшін әзірленген әдістер L_t теріс параметрлі Шрёдингер операторына жарамсыз болып қалады. Барлық бұл мәселелер осы жұмыстың өзектілігі мен жаңалығын көрсетті. Осы мақалада теріс параметрлі Шрёдингер операторының резольвентасының бар болуы және коэрцитивтілігі жан-жақты зерттелді.

Кілт сөздер: Шрёдингер операторы, сингулярлы диффренциалдық оператор, гиперболалық оператор, теріс параметр, коэрцитивті бағалаулар, резольвента.

М.Б. Муратбеков, М.М. Муратбеков

Об ограниченной обратимости оператора Шрёдингера с отрицательным параметром в пространстве $L_2(R^n)$

Оператор Шрёдингера $L = -\Delta + q(x),\ x \in R^n$, является одним из основных операторов современной квантовой механики и теоретической физики. Известно, что для оператора Шрёдингера L получено немало фундаментальных результатов. Среди них, например, вопросы о существовании резольвенты, разделимости (коэрцитивная оценка), различные весовые оценки, оценки промежуточных производных функций из области определения оператора, оценки собственных и сингулярных чисел (s-чисел). В настоящее время имеются различные обобщения указанных выше результатов для эллиптических операторов. Для общих дифференциальных операторов решение такой задачи в целом далеко от завершения. В частности, насколько нам известно, до сих пор не было результата, показывающего существование резольвенты и коэрцитивности, а также дискретности спектра оператора гиперболического типа в бесконечной области с растущими и колеблющимися коэффициентами. Нетрудно заметить, что изучение некоторых классов дифференциальных операторов гиперболического типа, определенных в пространстве $L_2(R^{n+1})$, можно свести с помощью метода Фурье к изучению оператора Шрёдингера с отрицательным параметром: $L_t = -\Delta + \left(-t^2 + itb(x) + q(x)\right)$, где t – параметр $(-\infty < t < \infty)$, $i^2 = -1$. Отсюда ясно, что в операторе L_t при $|t| \to \infty$ — $-t^2 \to -\infty$. Следовательно, здесь возникает совершенно иная ситуация по сравнению с оператором Шрёдингера $L = -\Delta + q(x)$, и, в частности, методы, отработанные для оператора Шрёдингера L, оказываются мало приспособленными при изучении оператора Шрёдингера L_t с отрицательным параметром. Все эти вопросы свидетельствуют об актуальности и новизне данной работы. В настоящей статье изучены вопросы существования резольвенты и коэрцитивности оператора Шрёдингера с отрицательным параметром.

Ключевые слова: оператор Шрёдингера, сингулярный дифференциальный оператор, гиперболический оператор, отрицательный параметр, коэрцитивные оценки, резольвента.

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Ö. Özer[1], D. Bellaouar[2]

[1]*Kırklareli University, Turkey;*
[2]*University 08 Mai 1945 Guelma, Guelma, Algeria*
(E-mail: ozenozer39@gmail.com)

Some results on a special type of real quadratic fields

In this paper, we determine the real quadratic fields $Q(\sqrt{d})$ coincide with positive square-free integers d including the continued fraction expansion form of $w_d = \left[a_0; \underbrace{\overline{7, 7, \ldots, 7}}_{\ell-1}, \overline{a_\ell} \right]$. Furthermore, we deal with determining fundamental units and Yokoi's d-invariants n_d and m_d in the relation to continued fraction expansion of w_d where $\ell(d)$ is a period length of w_d for the such type of real quadratic number fields $Q(\sqrt{d})$. The present paper improve the theory of fundamental unit which generates the unit group of real quadratic fields and also determine the special form of continued fraction expansion of integral basis element in real quadratic fields.

Keywords: continued fraction, real quadratic fields, fundamental unit, Yokoi's invariants, integer sequences, integer basic element.

1 Introduction

Some relations among i-th approachment of quadratic irrationals were proved by ElezoviA'c in [1] . Jeongho obtained lower bound for regulator of real quadratic fields by considering quadratic integers with fixed norm in [2]. In [3], Benamar et al. described polynomials and also gave lower bound of the number of some types of polynomials. Badziahin and Shallit confirmed the conjecture of Hanna and Wilson by considering specific type of continued fraction of real numbers and got some results on transcendental numbers in [4]. Zhang and Yue [5] described some congruences relations between the coffecients of fundamental unit of real quadratic fields and odd class number. Also, Tomita [6] gave some results on fundamental unit by use of the continued fraction expansion of integral basis element where period length is equal to 3. Clemens and his co-authors [7] explored some relationship between continued fraction expansion and infinite series representation for real numbers. Louboutin [8] obtained significant results on principal or non principal real quadratic fields as well as significant conditions for principality of continued fraction expansion. Tomita and Kawamoto [9] showed a relation between real quadratic fields of class number one and minimal type of the simple continued fraction expansion of certain quadratic irrationals. Both Sasaki [10] and Mollin [11] achieved many useful results on lower bound of fundamental unit for real quadratic number fields. Williams and Buck [12] got comparision between period length of $\sqrt{d}$ and $\frac{1+\sqrt{d}}{2}$. Besides, first author in this paper obtained some special results for different forms of continued fraction expansion of w_d in [13] and [14] where $d \equiv 2, 3(mod4)$ is square-free positive integer. Also, in [15] she got significant results for varied types of continued fraction expansion of w_d where $d \equiv 1(mod4)$ or $d \equiv 2, 3(mod4)$ Yokoi defined n_d and m_d invariants important for class number problem [16–19]. Readers unfamiliar fundamental unit and continued fraction expansions are referred to books [11, 20–23].

Throughout this paper, $I(d)$ is the set of all quadratic irrational numbers in $Q(\sqrt{d})$, we say that α in $I(d)$ is reduced if $\alpha > 1$ and $-1 < \alpha' < 1$ where α' is the conjugate of α and denoted by $R(d)$ is the set includes of all reduced quadratic irrational numbers in $I(d)$. Then, it is well known that any number α in $R(d)$ is purely periodic in the continued fraction expansion and the denominator of its modular automorphism is equal to fundamental unit ϵ_d of $Q(\sqrt{d})$. Yokoi's invariants are expressed by $m_d = \left[\left[\frac{u_d^2}{t_d} \right]\right]$ and $n_d = \left[\left[\frac{t_d}{u_d^2} \right]\right]$. it is also well known $[[\ x\ ]]$ represents the floor of x for any number x.

In this paper, we deal with the problem for demonstrating the continued fraction expansions which have got partial constant elements equal each others and written as 7s (except the last digit of the period) according to period length for d square-free integer (for $d \equiv 1(mod4)$ or $d \equiv 2, 3(mod4)$).

Moreover, we demonstrate the general parametrization of d square-free positive integer and fundamental unit ϵ_d as well as t_d , u_d the coefficents of fundamental unit. Additionally, we get fix on Yokoi's invariants for such types of real quadratic fields. Then,we give some results on fundamental units, continued fraction expansion and Yokoi's invariants with numerical Tables.

2 Preliminaries and Basic Results

In this section, readers can find some basic and important definitions and theorems for using our new results.

Definition 2.1. $\{\tau_i\}$ is a sequence defined by the following recurrence relation

$$\tau_i = 7\tau_{i-1} + \tau_{i-2}$$

for $i \geq 2$ where $\tau_0 = 0$ and $\tau_1 = 1$.

Lemma 2.2. [6]. For a square-free positive integer d congruent to 1 modulo 4, let $\omega_d = \dfrac{1+\sqrt{d}}{2}$, $a_0 = [[omega_d]]$, $\omega_R = a_0 - 1 + \omega_d$. Then $\omega_d \notin R(d)$, but $\omega_R \in R(d)$ holds. Moreover for the period $l = \ell(d)$ of ω_R, we get $\omega_R = [\overline{2a_0 - 1, a_1, ..., a_{l-1}}]$ and $\omega_d = [a_0, \overline{a_1, ..., a_{l-1}, 2a_0 - 1}]$. Furthermore, let $\omega_R = \dfrac{(P_l\omega_R + P_{l-1})}{(Q_l\omega_R + Q_{l-1})} = [2a_0 - 1, a_1, ..., a_{l-1}, \omega_R]$ be a modular automorphism of ω_R, then the fundamental unit ϵ_d of $Q\left(\sqrt{d}\right)$ is given by the following formula

$$\epsilon_d = \frac{t_d + u_d\sqrt{d}}{2}$$

and

$$t_d = (2a_0 - 1).Q_{\ell(d)} + 2Q_{\ell(d)-1}, \quad u_d = Q_{\ell(d)},$$

where Q_i is determined by $Q_0 = 0$, $Q_1 = 1$ and $Q_{i+1} = a_iQ_i + Q_{i-1}$ $(i \geq 1)$.

Lemma 2.3. For a square-free positive integer d congruent to 2, 3 modulo 4, let $\omega_d = \sqrt{d}$, $a_0 = [[\ \omega_d\ ]]$, $\omega_R = a_0 + \omega_d$. Then $\omega_d \notin R(d)$, but $\omega_R \in R(d)$ holds. Moreover for the period $l = l(d)$ of ω_R, we get $\omega_R = [\overline{2a_0, a_1, ..., a_{l-1}}]$ and $\omega_d = [a_0, \overline{a_1, ..., a_{l-1}, 2a_0}]$. Furthermore, let $\omega_R = \dfrac{(P_l\omega_R + P_{l-1})}{(Q_l\omega_R + Q_{l-1})} =$ $= [2a_0, a_1, ..., a_{l-1}, \omega_R]$ be a modular automorphism of ω_R, then the fundamental unit ϵ_d of $Q\left(\sqrt{d}\right)$ is given by the following formula:

$$\epsilon_d = \frac{t_d + u_d\sqrt{d}}{2} = (a_0 + \sqrt{d})Q_{\ell(d)} + Q_{\ell(d)-1}\rangle 1$$

and

$$t_d = 2a_0.Q_{\ell(d)} + 2Q_{\ell(d)-1}, \quad u_d = 2Q_{\ell(d)},$$

where Q_i is determined by $Q_0 = 0$, $Q_1 = 1$ and $Q_{i+1} = a_iQ_i + Q_{i-1}$, $(i \geq 1)$.

Remark 2.4. Let $\{\tau_n\}$ be a sequence defined as in Definition 2.1. Then, we state that:

$$\tau_n \equiv \begin{cases} 0\,(mod4), & n \equiv 0\,(mod6); \\ 1\,(mod4), & n \equiv 1, 4, 5\,(mod6); \\ 3\,(mod4), & n \equiv 2\,(mod6); \\ 2\,(mod4), & n \equiv 3\,(mod6), \end{cases}$$

where $n \geq 0$.

3 Main Theorems and Results

In this section, we present our results as follows:

Theorem 3.1. Let d be the square-free positive integer and ℓ be a positive integer holding that ℓ is different from $0(mod3)$ and $\ell > 1$. We assume that parametrization of d is

$$d = \frac{(7 + (2t+1)\tau_\ell)^2}{4} + ((2t+1)\tau_{\ell-1}) + 1$$

for $t \geq 0$ positive integer. In this case, we get following:

(1) If $\ell \equiv 1(mod6)$ and $t \equiv 1(mod2)$ positive integer then $d \equiv 2(mod4)$ holds.
(2) If $\ell \equiv 2(mod6)$ and $t \equiv 0(mod2)$ positive integer then $d \equiv 3(mod4)$ holds.
(3) If $\ell \equiv 4(mod6)$ and $t \equiv 0(mod2)$ positive integer then $d \equiv 3(mod4)$ holds.
(4) If $\ell \equiv 5(mod6)$ and $t \equiv 0(mod2)$ positive integer then $d \equiv 2(mod4)$ holds.

In this case, we obtain

$$w_d = \left[\frac{(2t+1)\tau_\ell + 7}{2}; \overline{\underbrace{7, 7, \ldots, 7}_{\ell-1}, (2t+1)\tau_\ell + 7}\right]$$

and $\ell = \ell(d)$. Moreover, we have following equations:

$$\epsilon_d = \left(\frac{(2t+1)\tau_\ell^2}{2} + \frac{7\tau_\ell}{2} + \tau_{\ell-1}\right) + \tau_\ell\sqrt{d};$$

$$t_d = (2t+1)\tau_\ell^2 + 7\tau_\ell + 2\tau_{\ell-1} \quad \text{and} \quad u_d = 2\tau_\ell$$

for ϵ_d, t_d and u_d.

Remark 3.2. Note that d is not integer for $\ell \equiv 0(mod3)$. That's why we assume that ℓ different from $0(mod3)$.

Proof. We assume that $\ell \equiv 1(mod6)$ positive odd integer, $\ell > 1$ and $t \equiv 1(mod2)$ positive integer. So, we can get $d \equiv 2(mod4)$ by substituting the equivalents into the parametrization of d. We can easily obtain the other cases in a similar way. By using Lemma 2.3 , we put

$$w_R = \frac{(2t+1)\tau_\ell + 7}{2} + \left[\frac{(2t+1)\tau_\ell + 7}{2}; \overline{\underbrace{7, 7, \ldots, 7}_{\ell-1}, (2t+1)\tau_\ell + 7}\right],$$

so we get

$$w_R = ((2t+1)\tau_\ell + 7) + \cfrac{1}{7 + \cfrac{1}{7 + \cfrac{1}{\ddots}}} = ((2t+1)\tau_\ell + 7) + \frac{1}{7} + \cdots + \frac{1}{w_R} + \ddots + \frac{1}{7 + \frac{1}{w_R}}.$$

By induction, we get

$$w_R = ((2t+1)\tau_\ell + 7) + \frac{\tau_{\ell-1}w_R + \tau_{\ell-2}}{\tau_\ell w_R + \tau_{\ell-1}}.$$

If we rearrange and use the Definition 2.1 into the above equality, we have

$$w_R^2 - ((2t+1)\tau_\ell + 7)\, w_R - (1 + (2t+1)\tau_{\ell-1}) = 0.$$

This requires that $w_R = \frac{(2t+1)\tau_\ell + 7}{2} + \sqrt{d}$ since $w_R > 0$. If we consider Lemma 2.3, we get

$$w_d = \sqrt{d} = \left[\frac{(2t+1)\tau_\ell + 7}{2}; \overline{\underbrace{7, 7, \ldots, 7}_{\ell-1}, (2t+1)\tau_\ell + 7}\right]$$

and $\ell = \ell(d)$. This shows that the first part of proof is completed.

Now, we should determine ϵ_d, t_d and u_d using Lemma 2.3, we get

$$Q_1 = 1 = \tau_1, \quad Q_2 = a_1.Q_1 + Q_0 \Rightarrow Q_2 = 7 = \tau_2;$$

$$Q_3 = a_2Q_2 + Q_1 = 7\tau_2 + \tau_1 = 7^2 + 1 = 50 = \tau_3,\ Q_4 = \tau_4, \ldots$$

So, this implies that $Q_i = \tau_i$ by using mathematical induction for every $i \geq 0$. If we substitute these values of sequence into the $\epsilon_d = \frac{t_d + u_d\sqrt{d}}{2} = (a_0 + \sqrt{d})Q_{l(d)} + Q_{l(d)-1}\rangle 1$ and rearrange, we have

$$\epsilon_d = \left(\frac{(2t+1)\tau_\ell^2}{2} + \frac{7\tau_\ell}{2} + \tau_{\ell-1}\right) + \tau_\ell\sqrt{d};$$

$$t_d = (2t+1)\tau_\ell^2 + 7\tau_\ell + 2\tau_{\ell-1} \quad \text{and} \quad u_d = 2\tau_\ell$$

for ϵ_d, t_d and u_d.So, we complete the proof of Theorem 3.1.

Corollary 3.3. Let d be a square-free positive integer and ℓ be a positive integer holding that $\ell > 1$ different from $0(mod3)$. We assume that parametrization of d is

$$d = \frac{(7+\tau_\ell)^2}{4} + \tau_{\ell-1} + 1$$

then we obtain $d \equiv 2, 3(mod4)$ and

$$w_d = \left[\frac{\tau_\ell + 7}{2}; \overline{\underbrace{7, 7, \ldots, 7}_{\ell-1}, \tau_\ell + 7}\right]$$

and $\ell = \ell(d)$. Besides, we get following equalities

$$\epsilon_d = \left(\frac{\tau_\ell^2}{2} + \frac{7\tau_\ell}{2} + \tau_{\ell-1}\right) + \tau_\ell\sqrt{d};$$

$$t_d = \tau_\ell^2 + 7\tau_\ell + 2\tau_{\ell-1}; \quad \text{and} \quad u_d = 2\tau_\ell;$$

$$m_d = \begin{cases} 1, & \text{if } \ell = 2; \\ 3, & \text{if } \ell > 4 \end{cases}$$

for ϵ_d, t_d, u_d and Yokoi's invariant m_d.

Furthermore, we prepare Table 1 where fundamental unit is ϵ_d, integral basis element is w_d and Yokoi's invariant is m_d for $2 < \ell(d) \leq 11$. (In this Table, we will rule out $\ell(d) = 4, 8, 10$ since d is not a square-free positive integer with these periods. Besides, d is not congruent to 2 or 3 (modulo 4) for $\ell(d) = 7$).

Table 1

Square-free positive integers *d* with $2 \leq l(d) \leq 11$

d	$\ell(d)$	m_d	w_d	ϵ_d
51	2	1	$[7; \overline{7, 14}]$	$50+7\sqrt{51}$
1633642	5	3	$[1278; \overline{7, 7, 7, 7, 2556}]$	$3257979+2549\sqrt{1633642}$
28516639237941410	11	3	$[168868704; \overline{7, 7, 7, 7, 7, 7, 7, 7, 7, 7, 337737408}]$	$57033277246500097+$ $+337737401\sqrt{28516639237941410}$

Proof. This corollary is gotten if we substitute $t = 0$ in Theorem 3.1. Now, we have to prove that

$$m_d = \begin{cases} 1, & \text{if } \ell = 2; \\ 3, & \text{if } \ell > 4. \end{cases}$$

If we put t_d and u_d into the m_d and rearrange, then we obtain

$$m_d = \left[\left[\frac{u_d^2}{t_d}\right]\right] = \left[\left[\frac{4\tau_\ell^2}{\tau_\ell^2 + 7\tau_\ell + 2\tau_{\ell-1}}\right]\right].$$

By using the above equality, we have $m_d = 1$ for $\ell = 2$. From the assumption since τ_ℓ is increasing we get,

$$4 > 4.\left(1 + \frac{7}{\tau_\ell} + \frac{2\tau_{\ell-1}}{\tau_\ell^2}\right)^{-1} > 3,988$$

for $\ell > 4$. Therefore, we obtain $m_d = 3$ for $\ell > 4$ and we have $m_d = \begin{cases} 1, & \text{if } \ell = 2; \\ 3, & \text{if } \ell > 4. \end{cases}$ which completes the proof of Corollary 3.3.

Furthermore, Table 1 is gotten as a numerical results of the corollary.

Corollary 3.4. Let d be the square-free positive integer and ℓ be a positive integer such that ℓ is not congruent to 0 as for $(mod 3)$, $\ell > 1$. We suppose that parametrization of d is

$$d = \frac{(7+3\tau_\ell)^2}{4} + 3\tau_{\ell-1} + 1$$

then, we obtain $d \equiv 2 (mod 4)$ and

$$w_d = \left[\frac{3\tau_\ell+7}{2}; \overline{\underbrace{7,7,\ldots,7}_{\ell-1}, 3\tau_\ell+7}\right]$$

and $\ell = \ell(d)$. Moreover, we have following equalities:

$$\epsilon_d = \left(\frac{3\tau_\ell^2}{2} + \frac{7\tau_\ell}{2} + \tau_{\ell-1}\right) + \tau_\ell\sqrt{d};$$

$$t_d = 3\tau_\ell^2 + 7\tau_\ell + 2\tau_{\ell-1} \quad \text{and} \quad u_d = 2\tau_\ell$$

$m_d = 1$ for $\ell \geq 7$ for ϵ_d, t_d, u_d and Yokoi's invariant m_d. Besides, we state the following Table 2, where fundamental unit is ϵ_d, integral basis element is w_d and Yokoi's invariant is m_d for $2 < \ell(d) \leq 13$.

Table 2

Square-free positive integers d with $l(d) = 7$ or $l(d) = 13$

d	$\ell(d)$	m_d	w_d	ϵ_d
37996589930	7	1	$[194927; \overline{7,7,7,7,7,7,389854}]$	$25330586923+129949\sqrt{37996589930}$
667032581096785339826	13	1	$[25826973905; \overline{7,7,...,7,51653947810}]$	$444688387335182490505+$ $+17217982601\sqrt{667032581096785339826}$

Proof. Corollary is obtained if we substitute $t = 1$ in Theorem 3.1. We should prove that $m_d = 1$ for $\ell \geq 7$. If we put t_d and u_d into the m_d and rearrange, then we obtain

$$2 > 4.\left(3 + \frac{7}{\tau_\ell} + \frac{2\tau_{\ell-1}}{\tau_\ell^2}\right)^{-1} > 1,333$$

for $\ell \geq 7$ since τ_ℓ is increasing sequence.By using the above equality, we have $m_d = 1$ for $\ell \geq 7$. Also, Table 2 is given as an illustration of this corollary.

Corollary 3.5. We assume that d and ℓ are defined as in Theorem 3.1. If we choose the parametrization of d as

$$d = \frac{(7+5\tau_\ell)^2}{4} + 5\tau_{\ell-1} + 1$$

then $d \equiv 2, 3(mod 4)$ and

$$w_d = \left[\frac{5\tau_\ell+7}{2}; \overline{\underbrace{7,7,\ldots,7}_{\ell-1}, 5\tau_\ell+7}\right]$$

with $\ell = \ell(d)$. Also, we have the following equalities:

$$\epsilon_d = \left(\frac{5\tau_\ell^2}{2} + \frac{7\tau_\ell}{2} + \tau_{\ell-1}\right) + \tau_\ell\sqrt{d};$$

$$t_d = 5\tau_\ell^2 + 7\tau_\ell + 2\tau_{\ell-1} \quad \text{and} \quad u_d = 2\tau_\ell,$$

$$n_d = 1$$

for $\ell \geq 2$.

Additionally, we prepare the Table 3 where fundamental unit is ϵ_d, integral basis element is w_d and and Yokoi's invariant is n_d for $2 \leq \ell(d) \leq 11$. (In this table, we rule out both $\ell(d) = 10$ since d is not a square-free positive integer in this period and also d is not congruent to 2 or 3 (modulo4) for $\ell(d) = 7$).

Table 3

Square-free positive integers d with $2 \leq l(d) \leq 11$

d	$\ell(d)$	n_d	w_d	ϵ_d
447	2	1	$[21; \overline{7, 42}]$	$148+7\sqrt{447}$
803067	4	1	$[896; \overline{7, 7, 7, 1792}]$	$319922+357\sqrt{803067}$
40655162	5	1	$[6376; \overline{7, 7, 7, 7, 12752}]$	$16252781+2549\sqrt{40655162}$
5380595841067	8	1	$[2319611; \overline{7, 7, 7, 7, 7, 7, 7, 4639222}]$	$2152234959022+927843\sqrt{5380595841067}$
712915956360881002	11	1	$[844343506; \overline{7, 7, 7, 7, 7, 7, 7, 7, 7, 7, 1688687012}]$	$285166381314969699+ +337737401\sqrt{712915956360881002}$

Proof. It is gotten if we substitute $t = 2$ in Theorem 3.1. Let's prove that Yokoi's d- invariant is $n_d = 1$ for $\ell \geq 2$.

We know from H. Yokoi's references [16–19] that $n_d = \left[\left[\dfrac{t_d}{u_d^2} \right]\right]$. If we substitute t_d and u_d into n_d, then we get

$$n_d = \left[\left[\frac{t_d}{u_d^2} \right]\right] = \left[\left[\frac{5\tau_\ell^2 + 7\tau_\ell + 2\tau_{\ell-1}}{4\tau_\ell^2}\right]\right] = 1,$$

since τ_ℓ is increasing and $1 < \frac{5}{4} + \frac{7}{4\tau_\ell} + \frac{\tau_{\ell-1}}{2\tau_\ell^2} < 1{,}510$ for $\ell \geq 2$. Therefore, we obtain $n_d = 1$ for $\ell \geq 2$. As illustration, we give Table 3.

Theorem 3.6. Let d be a square-free positive integer and $\ell > 1$ be a positive integer.

(i) We suppose

$$d = (2t\tau_\ell + 7)^2 + 8t\tau_{\ell-1} + 4$$

for $t > 0$ positive integer. In this case, we obtain that $d \equiv 1(mod4)$ and

$$w_d = \left[t\tau_\ell + 4; \overline{\underbrace{7, 7, \ldots, 7}_{\ell-1}, 2t\tau_\ell + 7} \right]$$

and $\ell = \ell(d)$. Moreover, in this case it holds

$$t_d = 2t\tau_\ell^2 + 7\tau_\ell + 2\tau_{\ell-1} \quad \text{and} \quad u_d = \tau_\ell$$

for $\epsilon_d = \dfrac{t_d + u_d\sqrt{d}}{2}$.

(ii) If $\ell \equiv 0(mod3)$ and

$$d = (t\tau_\ell + 7)^2 + 4t\tau_{\ell-1} + 4$$

for $t > 0$ positive odd integer then $d \equiv 1(mod4)$ and

$$w_d = \left[\frac{t}{2}\tau_\ell + 4; \overline{\underbrace{7, 7, \ldots, 7}_{\ell-1}, t\tau_\ell + 7} \right]$$

and $\ell = \ell(d)$. Furthermore, in this case

$$t_d = t\tau_\ell^2 + 7\tau_\ell + 2\tau_{\ell-1} \quad \text{and} \quad u_d = \tau_\ell$$

hold for $\epsilon_d = \dfrac{t_d + u_d\sqrt{d}}{2}$.

Remark 3.7. For the case (ii) in Theorem 3.6, it is clear that τ_ℓ is odd number if ℓ is not divided by 3. For the case (ii), assume that ℓ is not divided by 3. Then we get d is not integer if we put t positive odd integer into the parametrization of d. Besides, if we consider t is positive even integer, then the parametrization of d will be the case (i). So, we approve that $\ell \equiv 0(mod3)$ and t is positive odd integer in the case (ii).

Proof. (1) It is clear that $d \equiv 1(mod4)$ holds since $(2t\tau_\ell + 7)^2$ is odd integer for any $t > 0$ and $\ell > 1$ positive integers. We prove the theorem in a similar to Theorem 3.1. From Lemma 2.2, we know that $\omega_d = \frac{1+\sqrt{d}}{2}$, $a_0 = [[\ \omega_d\ ]]$, $\omega_R = a_0 - 1 + \omega_d$.

Considering above equations, we have

$$w_R = t\tau_\ell + 3 + \left[t\tau_\ell + 4; \overline{\underbrace{7, 7, \ldots, 7}_{\ell-1}, 2t\tau_\ell + 7}\right]$$

so we get

$$w_R = (2t\tau_\ell + 7) + \cfrac{1}{7 + \cfrac{1}{7 + \cfrac{1}{\ddots + \cfrac{1}{7 + \frac{1}{w_R}}}}} = (2t\tau_\ell + 7) + \frac{1}{7} + \cdots + \frac{1}{w_R}.$$

Rearranging and using Lemma 2.2 with Definition 2.1 into the above equality, we obtain

$$w_R^2 - (2t\tau_\ell + 7)\, w_R - (1 + 2t\tau_{\ell-1}) = 0.$$

This requires that $w_R = (t\tau_\ell + 4) - 1 + \frac{1+\sqrt{d}}{2}$ since $w_R > 0$. If we consider Lemma 2.1, we get

$$w_d = \left[t\tau_\ell + 4; \overline{\underbrace{7, 7, \ldots, 7}_{\ell-1}, 2t\tau_\ell + 7}\right]$$

and $\ell = \ell(d)$.

We obtained that $Q_i = \tau_i$ by using mathematical induction for all $i \geq 0$ in Theorem 3.1. Now, we get t_d and u_d using Lemma 2.2 as follows

$$t_d = 2t\tau_\ell^2 + 7\tau_\ell + 2\tau_{\ell-1} \quad \text{and} \quad u_d = \tau_\ell$$

for $\epsilon_d = \frac{t_d + u_d\sqrt{d}}{2}$. This shows that the first part of proof is completed.

(2) If we assume that $\ell \equiv 0(mod3)$ and the parametrization of

$$d = (t\tau_\ell + 7)^2 + 4t\tau_{\ell-1} + 4$$

for $t > 0$, then we have $d \equiv 1(mod4)$ since τ_ℓ is even integer. By taking $\frac{t}{2}$ instead of t into the case (1), we get

$$w_d = \left[\frac{t}{2}\tau_\ell + 4; \overline{\underbrace{7, 7, \ldots, 7}_{\ell-1}, t\tau_\ell + 7}\right]$$

and $\ell = \ell(d)$ for $\ell \equiv 0(mod3)$. Furthermore,

$$t_d = t\tau_\ell^2 + 7\tau_\ell + 2\tau_{\ell-1} \quad \text{and} \quad u_d = \tau_\ell$$

hold for $\epsilon_d = \frac{t_d + u_d\sqrt{d}}{2}$ which completes the proof.

Corollary 3.8. Let d be the square-free positive integer and $\ell > 1$ is a positive integer. We assume that parametrization of d is

$$d = (2\tau_\ell + 7)^2 + 8\tau_{\ell-1} + 4$$

then we obtain $d \equiv 1(mod4)$ and

$$w_d = \left[\tau_\ell + 4; \overline{\underbrace{7, 7, \ldots, 7}_{\ell-1}, 2\tau_\ell + 7}\right]$$

and $\ell = \ell(d)$. Moreover, we have

$$t_d = 2\tau_\ell^2 + 7\tau_\ell + 2\tau_{\ell-1} \quad \text{and} \quad u_d = \tau_\ell$$

for $\epsilon_d = \dfrac{t_d + u_d\sqrt{d}}{2}$ and, Yokoi's invariant n_d is determined as follows:

$$n_d = \begin{cases} 3, & \text{if } \ell = 2; \\ 2, & \text{if } \ell > 2. \end{cases}$$

Also, we give Table 4 where fundamental unit is ϵ_d, integral basis element is w_d and Yokoi's invariant is n_d for $2 \leq \ell(d) \leq 11$. (In this table, we rule out $\ell(d) = 4, 5, 10$ since d is not a square-free positive integer with these periods).

Table 4

Square-free positive integers *d* with $2 \leq l(d) \leq 11$

d	$\ell(d)$	n_d	w_d	ϵ_d
453	2	3	$[11; \overline{7, 21}]$	$(149+7\sqrt{453})/2$
11509	3	2	$[54; \overline{7, 7, 107}]$	$(5364+50\sqrt{11509})/2$
1325490045	6	2	$[18204; \overline{7, 7, ...7, 36407}]$	$(662612498+18200\ \sqrt{1325490045})/2$
67550754629	7	2	$[129953; \overline{7, 7, ..., 7, 259905}]$	$(33774431245+129949\sqrt{67550754629})/2$
3443597549845	8	2	$[927847; \overline{7, 7, ..., 7, 1855693}]$	$(1721792020097+927843\sqrt{3443597549845})/2$
175554743008597	9	2	$[6624854; \overline{7, 7, ..., 7, 13249707}]$	$(87777323274636+6624850\sqrt{175554743008597})/2$
456266217972000829	11	2	$[337737405; \overline{7, 7, ..., 7, 675474809}]$	$(228133106527234995+ 337737401\sqrt{456266217972000829})/2$

Proof. The corollary is had if we substitute $t = 1$ into the case (1) in Theorem 3.6. Let's show that

$$n_d = \begin{cases} 3, & \text{if } \ell = 2; \\ 2, & \text{if } \ell > 2. \end{cases}$$

If we put t_d and u_d into the n_d and rearrange, then we obtain

$$n_d = \left[\left[\frac{t_d}{u_d^2}\right]\right] = \left[\left[\frac{2\tau_\ell^2 + 7\tau_\ell + 2\tau_{\ell-1}}{\tau_\ell^2}\right]\right].$$

By using the above equality, we have $n_d = 3$ for $\ell = 2$. From the assumption since τ_ℓ is increasing sequence, we get,

$$2,1456 \geq \left(2 + \frac{7}{\tau_\ell} + \frac{2\tau_{\ell-1}}{\tau_\ell^2}\right) > 2$$

for $\ell > 2$. Therefore, we obtain $n_d = \begin{cases} 3, & \text{if } \ell = 2; \\ 2, & \text{if } \ell > 2. \end{cases}$ Then Corollary 3.8 is proved. To give numerical examples for Corollary 3.8, we prepare Table 4.

Corollary 3.9. Let d be the square-free positive integer and $\ell > 1$ is a positive integer holding that $\ell \equiv 0(mod3)$. We assume that parametrization of d is

$$d = (\tau_\ell + 7)^2 + 4\tau_{\ell-1} + 4$$

then we have $d \equiv 1(mod4)$ and

$$w_d = \left[\frac{\tau_\ell}{2} + 4; \overline{\underbrace{7, 7, \ldots, 7}_{\ell-1}, \tau_\ell + 7}\right]$$

and $\ell = \ell(d)$. Moreover, we have

$$t_d = \tau_\ell^2 + 7\tau_\ell + 2\tau_{\ell-1} \quad \text{and} \quad u_d = \tau_\ell$$

for $\epsilon_d = \dfrac{t_d + u_d\sqrt{d}}{2}$ and, Yokoi's invariant $n_d = 1$.

Besides, we prepare Table 5 where fundamental unit is ϵ_d, integral basis element is w_d and Yokoi's invariant is n_d for $3 \le \ell(d) \le 12$ (in this Table, we rule out $\ell(d) = 9$ since d is not a square-free positive integer with these periods).

Table 5

Square-free positive integers *d* with $3 \le l(d) \le 12$

d	$\ell(d)$	n_d	w_d	ϵ_d
3281	3	1	$[29; \overline{7, 7, 57}]$	$(2864+50\sqrt{3281})/2$
331505049	6	1	$[9104; \overline{7, 7, ..., 7, 18207}]$	$(331372498+18200\sqrt{331505049})/2$
58151567292136400057	12	1	$[1205731804;]$ $\overline{7, 7, ...7, 2411463607}]$	$(5815156711680680002+$ $+2411463600\ \sqrt{58151567292136400057})/2$

Proof. If we substitute $t = 1$ into the case (2) in Theorem 3.6, we have the Corollary 3.9. Let's prove that $n_d = 1$. If we put t_d and u_d into the n_d and rearrange, then we obtain

$$n_d = \left[\left[\ \frac{t_d}{u_d^2}\ \right]\right] = \left[\left[\ \frac{\tau_\ell^2 + 7\tau_\ell + 2\tau_{\ell-1}}{\tau_\ell^2}\ \right]\right].$$

From the assumption since τ_ℓ is increasing sequence, we get

$$1,1416 \ge \left(1 + \frac{7}{\tau_\ell} + \frac{2\tau_{\ell-1}}{\tau_\ell^2}\right) > 1$$

for $\ell \ge 3$ which completes the proof of the Corollary 3.9. We prepare Table 5 as an numerical illustrations of Corollary 3.9.

Corollary 3.10. Let d be a square-free positive integer congruent to modulo 4. If we suppose that d is holding the conditions of Theorem 3.6, then always satisfy that Yokoi's invariant n_d is different from zero. It means that $m_d = 0$.

Proof. It can be proven as similar of Corollary 3.2 in [15].

Remark 3.11. We should say that the present paper has got the most general theorems for given type real quadratic fields. Also, we can obtain infinitely many values of d which corresponds to $Q(\sqrt{d})$ and determine the structures of such fields by using our results.

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Ö. Өзер, Д. Беллауар

Нақты квадраттық өрістің арнаулы түрі бойынша кейбір нәтижелер

Мақалада нақты квадраттық өрістер бүтін еркін оң $w_d = \left[a_0; \overline{\underbrace{7, 7, \ldots, 7}_{\ell-1}, a_\ell}\right]$ тізбекті бөлшектің жіктелу түрімен қоса, d квадраттарымен сәйкес келетіні анықталды. Сондай-ақ негізгі бірлік және Йокой бірліктері анықталып, n_d және m_d d-инварианттары w_d бөлшегін үздіксіз жіктеуге қолданылды, мұндағы $\ell(d)$ w_d — кезең ұзындығы $Q(\sqrt{d})$ нақты квадраттық сандар өрісі түрі үшін. Авторлар нақты квадраттық өрістің бірлік группасы туындайтын іргелі бірлік теориясын жаңартып, нақты квадраттық өрісте бүтінсанды базистік элементтің үздіксіз бөлшекті жіктелуінің ерекше түрін анықтаған.

Кілт сөздер: тізбекті бөлшек, квадраттық өріс, негізгі бірлік, Йокой инварианттары, бүтінсанды базистік элемент.

Ö. Öзер, Д. Беллауар

Некоторые результаты по специальному типу вещественных квадратичных полей

В статье определено, что действительные квадратичные поля совпадают с целыми свободными положительными квадратами d, включая форму разложения цепной дроби $w_d = \left[a_0; \overline{\underbrace{7, 7, \ldots, 7}_{\ell-1}, a_\ell} \right]$.
Кроме того, определены основные единицы и единицы Йокой, d-инварианты n_d и m_d применительно к непрерывному разложению дроби w_d, где $\ell(d)$ — длина периода w_d для такого типа поля действительных квадратичных чисел $Q(\sqrt{d})$. Авторами статьи улучшена теория фундаментальной единицы, которая порождает единичную группу вещественных квадратичных полей, а также определена особая форма непрерывного дробного разложения целочисленного базисного элемента в вещественных квадратичных полях.

Ключевые слова: цепная дробь, квадратичное поле, основная единица, инварианты Йокой, целочисленный базисный элемент.

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N.T. Orumbayeva, A.B. Keldibekova

Ye.A. Buketov Karaganda State University, Kazakhstan
(E-mail: OrumbayevaN@mail.ru)

On the solvability of the duo-periodic problem for the hyperbolic equation system with a mixed derivative

One of the main and most studied problems in the theory of second order hyperbolic equations is a periodic boundary value problem. To solve such problems apply Fourier method, method of successive approximations, methods of functional analysis, variational method, etc. The development of information technologies imposes new requirements on the developed methods, paying special attention to their constructibility. One of such constructive methods is the method of a parametrization proposed in the works of D. S.Dzhumabaev for solving two-point boundary-value problems for ordinary differential equations. In this paper, we consider a boundary value problem for both variables for a system of hyperbolic equations with a mixed derivative. To solve this problem, the notation is introduced and the periodic boundary value problem is reduced to an equivalent problem consisting of a family of periodic boundary value problems for an ordinary differential equation and an integral relation. To solve the problem obtained, the method of a parametrization is used. The application of this method allowed us to construct an algorithm for finding an approximate solution of the periodic boundary value problem for a system of hyperbolic equations with a mixed derivative. In addition, the coefficient conditions of convergence and feasibility of the proposed algorithm are obtained.

Keywords: the duo-periodic boundary value problem, the method of a parametrization, the hyperbolic equation system, mixed derivatives.

Introduction

One of the main and most studied problems of the theory of the hyperbolic equations of the second order is a periodic boundary value problem. A systematic study of periodic boundary value problems for hyperbolic equations with mixed partial derivatives started in 60s with the work of L. Cesari [1]. J.K. Hale [2], G. Hecquet [3], A.K. Aziz [4], V. Lakshmikantham [5], S.V. Zhestkov, A.M. Samoylenko, T.I. Kiguradze, B.I. Ptashnik, Yu.A. Mitropolskiy, G.P. Homa, M.I. Gromyak and others dealt with further investigations of the solvability of periodic boundary value problems. To solve periodic boundary value problems of second order hyperbolic equations, were applied the Fourier method, the method of successive approximations, the methods of functional analysis, the variational method, etc. Despite the presence of numerous methods for study of periodic boundary value problems, interest in them continues to this day. The application of different approaches, ideas and methods leads to results formulated in different terms. The development of information technologies and its comprehensive application in applied problems imposes new requirements of the developed methods. Particular attention got to be paid to the methods that are different from others in their constructiveness at the stage of approximate construction of solutions and in the study of such qualitative issues as the establishment of the existence of a solution, the rationale for the convergence of approximate solutions to the exact one, an estimate of the inaccuracy of the approximate solution.

One of such constructive methods is the method of a parametrization [6, 7], proposed for solving two-point boundary value problems of ordinary differential equations. The point of using this method is to enter additional parameters and bring the original problem to multipoint boundary value problem with a parameter. It allows in terms of initial data to set conditions for the solvability of the boundary value problem for ordinary differential equations and to propose a family of algorithms for finding its approximate solution.

A modification of the method of a parametrization is the method of introducing functional parameters, devised in the works [8–16], which finds its application in the study of nonlocal boundary value problems with data on characteristics for a system of hyperbolic equations with a mixed derivative with two independent variables. There were constructed two-parameter families of algorithms for finding solutions to nonlocal boundary-value problems, at each step of which the Goursat problems are solved. On the basis of this algorithm was established the solvability of the boundary value problem with data on the characteristics for the system of hyperbolic equations with mixed derivative. However, the problems related to solvability of duo-periodic boundary value problems for the system of hyperbolic equations with mixed derivative remain relevant.

Problem statement. On $\Omega = [0,\omega] \times [0,T]$ we consider a duo-periodic problem for a system of hyperbolic equations of the form

$$\frac{\partial^2 z}{\partial x \partial t} = A(x,t)\frac{\partial z}{\partial x} + z + f(x,t), \quad (x,t) \in \Omega; \tag{1}$$

$$z(0,t) = z(\omega,t), \quad t \in [0,T]; \tag{2}$$

$$z(x,0) = z(x,T), \quad x \in [0,\omega], \tag{3}$$

where $(n \times n)$ — matrix $A(x,t)$; n — vector-function $f(x,t)$ are continuous by Ω, $\|z(x,t)\| = \max\limits_{i=\overline{1,n}} |z_i(x,t)|$; $\|A(x,t)\| = \max\limits_{i=\overline{1,n}} \sum_{j=1}^{n} |a_{ij}(x,t)|$. Let $C(\Omega, R^n)$ — be the space of functions $z : \Omega \to R^n$ continuous on Ω, with norm $\|z\|_0 = \max\limits_{(x,t)\in\Omega} \|z(x,t)\|$. Function $z(x,t) \in C(\Omega, R^n)$ having partial derivatives $\frac{\partial z(x,t)}{\partial x} \in C(\Omega, R^n)$, $\frac{\partial^2 z(x,t)}{\partial x \partial t} \in C(\Omega, R^n)$ is called the solution of the problem (1)-(3), if it satisfies the system (1) for all $(x,t) \in \Omega$ and periodic conditions (2), (3).

We consider a periodic boundary value problem with one independent variable to find solution of this problem

$$\frac{\partial^2 u}{\partial x \partial t} = A(x,t)\frac{\partial u}{\partial x} + u + f(x,t), \quad (x,t) \in \Omega; \tag{4}$$

$$u(0,t) = 0, \quad t \in [0,T]; \tag{5}$$

$$u(x,0) = u(x,T), \quad x \in [0,\omega]. \tag{6}$$

We enter new unknown function $v(x,t) = \dfrac{\partial u(x,t)}{\partial x}$ and reduce the periodic boundary value problem for a system of hyperbolic equations to a family of periodic boundary value problems for ordinary differential equations and a functional relation. Next, we apply the method of a parametrization [7]. By step $h > 0 : Nh = T$ we produce a partition of $[0,T) = \bigcup\limits_{r=1}^{N} [(r-1)h, rh)$, $N = 1,2,\ldots$. The region Ω is divided into N parts. By using $v_r(x,t)$, $u_r(x,t)$ denote respectively the restriction of the function $v(x,t), u(x,t)$ on $\Omega_r = [0,\omega] \times [(r-1)h, rh)$, $r = \overline{1,N}$.

Then problem (4)–(6) be equivalent to the boundary value problem

$$\begin{gathered}
\frac{\partial v_r}{\partial t} = A(x,t)v_r + u_r(x,t) + f(x,t), \quad (x,t) \in \Omega_r; \\
v_1(x,0) - \lim_{t \to T-0} v_N(x,t) = 0, \quad x \in [0,\omega]; \\
\lim_{t \to sh-0} v_s(x,t) = v_{s+1}(x,sh), \quad s = \overline{1,N-1}; \\
u_r(x,t) = \int_0^x v_r(\xi,t)\, d\xi, \quad (x,t) \in \Omega_r, \quad r = \overline{1,N},
\end{gathered} \tag{7}$$

where (7) — is the condition of gluing the functions $v(x,t)$ in the inner split lines. In $\lambda_r(x)$ we denote the value of $v_r(x,t)$ when $t = (r-1)h$, i.e $\lambda_r(x) = v_r(x,(r-1)h)$ and will replace

$$\tilde{v}_r(x,t) = v_r(x,t) - \lambda_r(x), \quad r = \overline{1,N}.$$

We obtain an equivalent boundary value problem with unknown functions $\lambda_r(x)$:

$$\frac{\partial \tilde{v}_r}{\partial t} = A(x,t)\tilde{v}_r + A(x,t)\lambda_r(x) + u_r(x,t) + f(x,t), \quad (x,t) \in \Omega_r; \tag{8}$$

$$\tilde{v}_r(x,(r-1)h) = 0, \quad x \in [0,\omega], \quad r = \overline{1,N}; \tag{9}$$

$$\lambda_1(x) - \lambda_N(x) - \lim_{t \to T-0} \tilde{v}_N(x,t) = 0, \quad x \in [0,\omega]; \tag{10}$$

$$\lambda_s(x) + \lim_{t \to sh-0} \tilde{v}_s(x,t) - \lambda_{s+1}(x) = 0, x \in [0,\omega], s = \overline{1, N-1}; \tag{11}$$

$$u_r(x,t) = \int_0^x \tilde{v}_r(\xi,t)\, d\xi + \int_0^x \lambda_r(\xi)\, d\xi, (x,t) \in \Omega_r, r = \overline{1,N}. \tag{12}$$

The problem (8), (9) for fixed $\lambda_r(x)$, $u_r(x,t)$ is a single-parameter family of Cauchy problems for systems of ordinary differential equations, where $x \in [0,\omega]$ and is equal to the integral equation

$$\tilde{v}_r(x,t) = \int_{(r-1)h}^t A(x,\tau)\, \tilde{v}_r(x,\tau)\, d\tau +$$

$$+ \int_{(r-1)h}^t A(x,\tau)\, d\tau \cdot \lambda_r(x) + \int_{(r-1)h}^t [u_r(x,\tau) + f(x,\tau)]\, d\tau. \tag{13}$$

Instead of $\tilde{v}_r(x,t)$ substitute the appropriate right part (13) and by repeating this process ν ($\nu = 1, 2, ...$) times we will get

$$\tilde{v}_r(x,t) = D_{vr}(x,t)\, \lambda_r(x) + F_{vr}(x,t,u_r) + G_{vr}(x,t,\tilde{v}_r), r = \overline{1,N}, \tag{14}$$

where

$$D_{\nu r}(x,t) = \sum_{j=0}^{\nu-1} \int_{(r-1)h}^t A(x,\tau_1) \dots \int_{(r-1)h}^{\tau_j} A(x,\tau_{j+1})\, d\tau_{j+1} \dots d\tau_1;$$

$$F_{vr}(x,t,u_r) = \int_{(r-1)h}^t [u_r(x,\tau_1) + f(x,\tau_1)]\, d\tau_1 + \sum_{j=1}^{v-1} \int_{(r-1)h}^t A(x,\tau_1) ...$$

$$... \int_{(r-1)h}^{\tau_{j-1}} A(x,\tau_j) \int_{(r-1)h}^{\tau_j} [u_r(x,\tau_{j+1}) + f(x,\tau_{j+1})]\, d\tau_{j+1} d\tau_j ... d\tau_1;$$

$$G_{vr}(x,t,\tilde{v}_r) = \int_{(r-1)h}^t A(x,\tau_1) ... \int_{(r-1)h}^{\tau_{v-2}} A(x,\tau_{v-1}) \int_{(r-1)h}^{\tau_{v-1}} A(x,\tau_v)\, \tilde{v}_r(x,\tau_v)\, d\tau_v d\tau_{v-1} ... d\tau_1,$$

$\tau_0 = t, r = \overline{1,N}$. Moving to the limit at $t \to rh - 0$, in (14) we find $\lim\limits_{t \to rh-0} = \tilde{v}_r(x,t)$, $r = \overline{1,N}$, for unknown functions $\lambda_r(x), r = \overline{1,N}$, we obtain a system of functional equations:

$$Q_\nu(x,h)\lambda(x) = -F_\nu(x,h,u) - G_\nu(x,h,\tilde{v}), \tag{15}$$

where

$$Q_\nu(x,h) = \begin{bmatrix} I & \dots & 0 & -[I + D_{\nu N}(x,Nh)] \\ I + D_{\nu 1}(x,h) & \dots & 0 & 0 \\ 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots \\ 0 & \dots & I + D_{\nu,N-1}(x,(N-1)h) & -I \end{bmatrix},$$

$$F_v(x,h,u) = (-F_{vN}(x,Nh,u_N), F_{v1}(x,h,u_1), ..., F_{v,N-1}(x,(N-1)h,u_{N-1})),$$

$$G_v(x,h,\tilde{v}) = (-G_{vN}(x,Nh,\tilde{v}_N), G_{v1}(x,h,\tilde{v}_1), ..., G_{v,N-1}(x,(N-1)h,\tilde{v}_{N-1})),$$

where I — is a unit matrix of dimension n. To find system of three functions

$$\{\lambda_r(x), \tilde{v}_r(x,t), u_r(x,t)\}, r = \overline{1,N}$$

we have a closed system consisting of equations (15), (14) and (12). Assuming the reversibility of the matrix $Q_v(x,h)$, for all $x \in [0,\omega]$, from equation (15), where $\tilde{v}_r(x,t) = 0$, $u_r(x,t) = 0$, we find $\lambda^{(0)}(x) : \lambda^{(0)}(x) = -[Q_v(x,h)]^{-1} \{F_v(x,h,0) + G_v(x,h,0)\}$. Using equation (14), for $\lambda_r(x) = \lambda_r^{(0)}(x)$ we find the functions $\tilde{v}_r^{(0)}(x,t)$, $r = \overline{1,N}$, i.e $\tilde{v}_r^{(0)}(x,t) = D_{vr}(x,t)\, \lambda_r^{(0)}(x) + F_{vr}(x,t,\psi) + G_{vr}(x,t,0)$.

The functions $u_r^{(0)}(x,t)\,, r=\overline{1,N}$, are determined from the relations

$$u_r^{(0)}(x,t)=\int_0^x \tilde{v}_r^{(0)}(\xi,t)\,d\xi+\int_0^x \lambda_r^{(0)}(\xi)\,d\xi, \quad (x,t)\in\Omega_r.$$

For the initial approximation of the problem (8)-(12) we take $\lambda_r^{(0)}(x)\,,\ \tilde{v}_r^{(0)}(x,t)\,,\ u_r^{(0)}(x,t)\,,\ r=\overline{1,N}$, and construct successive approximations by the following algorithm:

Step 1. A) Assuming that $u_r(x,t)=u_r^{(0)}(x,t)$, $r=\overline{1,N}$, are the first approximations in $\lambda_r(x),\tilde{v}_r(x,t)$ we found by solving the problem (8)-(11).

Taking $\lambda^{(1,0)}(x)=\lambda^{(0)}(x),\tilde{v}_r^{(1,0)}(x,t)=\tilde{v}_r^{(0)}(x,t)$ system of couple $\{\lambda_r^{(1)}(x),\ \tilde{v}_r^{(1)}(x,t)\}$, $r=\overline{1,N}$, we find as the limit of the sequence $\lambda_r^{(1,m)}(x)$, $\tilde{v}_r^{(1,m)}(x,t)$, determined by the following way:

Step 1.1. Assuming the reversibility of the matrix $Q_v(x,h)\,,$ for all $x\in[0,\omega]\,,$ from equation (15), where $\tilde{v}_r(x,t)=\tilde{v}_r^{(1,0)}(x,t)\,,$ we find $\lambda^{(1,1)}(x)$

$$\lambda^{(1,1)}(x)=-\left[Q_v(x,h)\right]^{-1}\left\{F_v\left(x,h,u^{(0)}\right)+G_v\left(x,h,\tilde{v}^{(1,0)}\right)\right\}.$$

Substituting the found $\lambda_r^{(1,1)}(x)\,,r=\overline{1,N}$ at (14) we find

$$\tilde{v}_r^{(1,1)}(x,t)=D_{vr}(x,t)\,\lambda_r^{(1,1)}(x)+F_{vr}\left(x,t,u^{(0)}\right)+G_{vr}\left(x,t,\tilde{v}^{(1,0)}\right).$$

Step 1.2. From equation (15), where $\tilde{v}_r(x,t)=\tilde{v}_r^{(1,1)}(x,t)\,,$ we define

$$\lambda^{(1,2)}(x)=-\left[Q_v(x,h)\right]^{-1}\left\{F_v\left(x,h,u^{(0)}\right)+G_v\left(x,h,\tilde{v}^{(1,1)}\right)\right\}.$$

Using the expression (14) again, we find the functions $\tilde{v}_r^{(1,2)}(x,t)\,,\ r=\overline{1,N}$:

$$\tilde{v}_r^{(1,2)}(x,t)=D_{vr}(x,t)\,\lambda_r^{(1,2)}(x)+F_{vr}\left(x,t,u^{(0)}\right)+G_{vr}\left(x,t,\tilde{v}^{(1,1)}\right).$$

On the $(1,m)$ step, we obtain a system of $\{\lambda_r^{(1,m)}(x)\,,\tilde{v}_r^{(1,m)}(x,t)\}$, $r=\overline{1,N}$.

Assuming that the solution of the problem (8)–(11) is the sequence of a system of couples $\{\lambda_r^{(1,m)}(x)\,,$ $\tilde{v}_r^{(1,m)}(x,t)\}$ is defined, and when $m\to\infty$ converges to the continuous, respectively, on $x\in[0,\omega]\,,(x,t)\in\Omega_r$ functions $\lambda_r^{(1)}(x)\,,\tilde{v}_r^{(1)}(x,t)\,,r=\overline{1,N}$.

B) The functions $u_r^{(1)}(x,t)\,,r=\overline{1,N}$, are determined from the ratio

$$u_r^{(1)}(x,t)=\int_0^x \tilde{v}_r^{(1)}(\xi,t)\,d\xi+\int_0^x \lambda_r^{(1)}(\xi)\,d\xi,(x,t)\in\Omega_r.$$

Step 2. A) Assuming that $u_r(x,t)=u_r^{(1)}(x,t)$, $r=\overline{1,N}$, are the first approximations in $\lambda_r(x),\tilde{v}_r(x,t)$ we find by solving the problem (8)–(11).

Taking $\lambda^{(2,0)}(x)=\lambda^{(1)}(x),\tilde{v}_r^{(2,0)}(x,t)=\tilde{v}_r^{(1)}(x,t)$ system of couple $\{\lambda_r^{(2)}(x),\ \tilde{v}_r^{(2)}(x,t)\}$, $r=\overline{1,N}$, we find as the limit of the sequence $\lambda_r^{(2,m)}(x)$, $\tilde{v}_r^{(2,m)}(x,t)$, defined by the following way:

Step 2.1. Assuming the reversibility of the matrix $Q_v(x,h)\,,$ for all $x\in[0,\omega]\,,$ from equation (15), where $\tilde{v}_r(x,t)=\tilde{v}_r^{(2,0)}(x,t)\,,$ we find $\lambda^{(2,1)}(x)$

$$\lambda^{(2,1)}(x)=-\left[Q_v(x,h)\right]^{-1}\left\{F_v\left(x,h,u^{(1)}\right)+G_v\left(x,h,\tilde{v}^{(2,0)}\right)\right\}.$$

Substituting the found $\lambda_r^{(2,1)}(x)\,,r=\overline{1,N}$, at (14) we find

$$\tilde{v}_r^{(2,1)}(x,t)=D_{vr}(x,t)\,\lambda_r^{(2,1)}(x)+F_{vr}\left(x,t,u^{(1)}\right)+G_{vr}\left(x,t,\tilde{v}^{(2,0)}\right).$$

Step 2.2. From equation (15), where $\tilde{v}_r(x,t)=\tilde{v}_r^{(2,1)}(x,t)\,,$ we define

$$\lambda^{(2,2)}(x)=-\left[Q_v(x,h)\right]^{-1}\left\{F_v\left(x,h,u^{(1)}\right)+G_v\left(x,h,\tilde{v}^{(2,1)}\right)\right\}.$$

Using expression (14) again, we find functions $\tilde{v}_r^{(2,2)}(x,t)$, $r=\overline{1,N}$,

$$\tilde{v}_r^{(2,2)}(x,t) = D_{vr}(x,t)\,\lambda_r^{(2,2)}(x) + F_{vr}\left(x,t,u^{(1)}\right) + G_{vr}\left(x,t,\tilde{v}^{(2,1)}\right).$$

On the $(2,m)$ step, we obtain a system of couples $\{\lambda_r^{(2,m)}(x), \tilde{v}_r^{(2,m)}(x,t)\}$, $r=\overline{1,N}$.

Assuming that the solution of the problem (8)–(11) is the sequence of a system of couples

$$\{\lambda_r^{(2,m)}(x), \tilde{v}_r^{(2,m)}(x,t)\}$$

is defined, and when $m\to\infty$ converges to the continuous, respectively, on $x\in[0,\omega]$, $(x,t)\in\Omega_r$ the functions $\lambda_r^{(2)}(x), \tilde{v}_r^{(2)}(x,t)$, $r=\overline{1,N}$.

B) The functions $u_r^{(2)}(x,t)$, $r=\overline{1,N}$, are determined from the ratio

$$u_r^{(2)}(x,t) = \int_0^x \tilde{v}_r^{(2)}(\xi,t)\,d\xi + \int_0^x \lambda_r^{(2)}(\xi)\,d\xi,\ (x,t)\in\Omega_r.$$

And so on. Conditions of the following statement provide the feasibility and convergence of the proposed algorithm, as well as the single-valued solvability of the problem (8)–(12).

Theorem 1. Let for some $h>0$: $Nh=T, N=1,2,...,$ and $\nu,\ \nu=1,2,...,$ $(nN\times nN)$ — the matrix $Q_\nu(x,h)$ is reversible for all $x\in[0,\omega]$ and inequalities are satisfied:

1) $\left\|[Q_\nu(x,h)]^{-1}\right\| \le \gamma_\nu(x,h)$;

2) $q_\nu(x,h) = \frac{(\alpha(x)h)^\nu}{\nu!}\left[1+\gamma_\nu(x,h)\sum_{j=1}^{\nu}\frac{(\alpha(x)h)^j}{j!}\right] \le \beta < 1$.

Then there is only one solution of problem (8)–(12) and valid assessment:

a) $\left\|\lambda^*(x)-\lambda^{(0)}(x)\right\| + \sup\limits_{t\in[0,T]}\left\|\tilde{v}^*(x,t)-\tilde{v}^{(0)}(x,t)\right\| \le \rho(x,\nu,h)\sup\limits_{t\in[0,T]}\|f(x,t)\|$;

b) $\sup\limits_{t\in[0,T]}\left\|u^*(x,t)-u^{(0)}(x,t)\right\| \le \int_0^x\left(\left\|\lambda^*(\xi)-\lambda^{(0)}(\xi)\right\| + \sup\limits_{t\in[0,T]}\left\|\tilde{v}^*(\xi,t)-\tilde{v}^{(0)}(\xi,t)\right\|\right)d\xi$,

where $\alpha(x) = \max\limits_{t\in[0,T]}\|A(x,t)\|$, $\beta = const$, $b_1(x,\nu,h) = \gamma_v(x,h)\,h\sum\limits_{j=0}^{v-1}\dfrac{(\alpha(x)h)^j}{j!}$;

$$b_2(x,\nu,h) = \left[1+\gamma_v(x,h)\sum_{j=1}^{v}\frac{(\alpha(x)h)^j}{j!}\right] h\sum_{j=0}^{v-1}\frac{(\alpha(x)h)^j}{j!};$$

$$b_3(x,\nu,h) = \gamma_v(x,h)\frac{(\alpha(x)h)^v}{v!},\quad d_1(x,\nu,h) = \frac{1+b_3(x,\nu,h)}{1-q_v(x,h)}b_2(x,\nu,h) + b_1(x,\nu,h);$$

$$d_2(x,\nu,h) = \frac{1+b_3(x,\nu,h)}{1-q_v(x,h)}q_v(x,h) + b_3(x,\nu,h);$$

$$g(x,\nu,h) = \left[d_1(x,\nu,h)\int_0^x\left[b_1(\xi,\nu,h)+b_2(\xi,\nu,h)\right]d\xi + d_2(x,\nu,h)b_2(x,\nu,h)\right];$$

$$\rho(x,\nu,h) = d_1(x,\nu,h)\sum_{j=1}^{\infty}\frac{1}{j!}\left(\int_0^x d_1(\xi,\nu,h)\,d\xi\right)^j\int_0^x g(\xi,\nu,h)\,d\xi\ + g(x,\nu,h).$$

Proof. When assumptions about the data of the problem we have the inequality

$$\|F_\nu(x,h,u)\| \le h\sum_{j=0}^{\nu-1}\frac{(\alpha(x)h)^j}{j!}\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\|u_r(x,t)\| + \|f(x,t)\|;$$

$$\|G_\nu(x,h,\widetilde{v})\| \le \frac{(\alpha(x)h)^\nu}{\nu!}\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\|\widetilde{v}_r(x,t)\|;$$

$$\max_{r=\overline{1,N}} \sup_{t\in[(r-1)h,rh)} \|D_{\nu r}(x,t)\| \le \sum_{j=1}^{\nu} \frac{(\alpha(x)h)^j}{j!}.$$

The following estimates follow from the zero step of the algorithm:

$$\max_{r=\overline{1,N}} \|\lambda_r{}^{(0)}(x)\| \le b_1(x,\nu,h) \max_{t\in[0,T]} \|f(x,t)\|;$$

$$\max_{r=\overline{1,N}} \sup_{t\in[(r-1)h,rh)} \|\widetilde{v}_r^{(0)}(x,t)\| \le b_2(x,\nu,h) \max_{t\in[0,T]} \|f(x,t)\|;$$

$$\max_{r=\overline{1,N}} \sup_{t\in[(r-1)h,rh)} \|u_r^{(0)}(x,t)\| \le \int_0^x \Big[b_1(\xi,\nu,h)+b_2(\xi,\nu,h)\Big]d\xi \max_{(x,t)\in\overline{\Omega}} \|f(\xi,t)\|.$$

The following estimates are valid:

$$\max_{r=\overline{1,N}} \|\lambda_r{}^{(1,1)}(x)-\lambda_r{}^{(1,0)}(x)\| \le$$

$$\le b_1(x,\nu,h) \max_{r=\overline{1,N}} \sup_{t\in[(r-1)h,rh)} \|u_r^{(0)}(x,t)\| + b_3(x,\nu,h) \max_{r=\overline{1,N}} \sup_{t\in[(r-1)h,rh)} \|\widetilde{v}_r^{(0)}(x,t)\|;$$

$$\max_{r=\overline{1,N}} \sup_{t\in[(r-1)h,rh)} \|\widetilde{v}_r^{(1,1)}(x,t)-\widetilde{v}_r^{(1,0)}(x,t)\| \le$$

$$\le b_2(x,\nu,h) \max_{r=\overline{1,N}} \sup_{t\in[(r-1)h,rh)} \|u_r^{(0)}(x,t)\| + q_\nu(x,h) \max_{r=\overline{1,N}} \sup_{t\in[(r-1)h,rh)} \|\widetilde{v}_r^{(0)}(x,t)\|.$$

Select the inequality

$$\Delta^{(1,1)}(x) = \max_{r=\overline{1,N}} \sup_{t\in[(r-1)h,rh)} \|\widetilde{v}_r^{(1,1)}(x,t)-\widetilde{v}_r^{(1,0)}(x,t)\| + \max_{r=\overline{1,N}} \|\lambda_r{}^{(1,1)}(x)-\lambda_r{}^{(1,0)}(x)\| \le$$

$$\le \Big[b_1(x,\nu,h)+b_2(x,\nu,h)\Big] \int_0^x \Big[b_1(\xi,\nu,h)+b_2(\xi,\nu,h)\Big]d\xi \max_{(x,t)\in\overline{\Omega}} \|f(x,t)\| +$$

$$+\Big[b_3(x,\nu,h)+q_\nu(x,h)\Big]b_2(x,\nu,h) \max_{(x,t)\in\overline{\Omega}} \|f(x,t)\|.$$

Thus,

$$\max_{r=\overline{1,N}} \|\lambda_r{}^{(1,m+1)}(x)-\lambda_r{}^{(1,m)}(x)\| \le$$

$$\le b_3(x,\nu,h) \max_{r=\overline{1,N}} \sup_{t\in[(r-1)h,rh)} \|\widetilde{v}_r^{(1,m)}(x,t)-\widetilde{v}_r^{(1,m-1)}(x,t)\|;$$

$$\max_{r=\overline{1,N}} \sup_{t\in[(r-1)h,rh)} \|\widetilde{v}_r^{(1,m+1)}(x,t)-\widetilde{v}_r^{(1,m)}(x,t)\| \le$$

$$\le q_\nu(x,h) \max_{r=\overline{1,N}} \sup_{t\in[(r-1)h,rh)} \|\widetilde{v}_r^{(1,m)}(x,t)-\widetilde{v}_r^{(1,m-1)}(x,t)\|.$$

Due to the inequality $q_\nu(x,h)<1$ follows the uniform convergence $v_r^{(1,m+1)}(x,t)$, at $(x,t)\in\Omega_r$, to $v_r^{(1)}(x,t)$ and the convergence of a sequence of systems of functions $\lambda_r^{(1,m+1)}(x)$ to continuous $x\in[0,\omega]$ functions $\lambda_r^{(1)}(x)$ for all $r=\overline{1,N}$:

$$\max_{r=\overline{1,N}} \sup_{t\in[(r-1)h,rh)} \|\widetilde{v}_r^{(1,m+1)}(x,t)-\widetilde{v}_r^{(1,0)}(x,t)\| \le$$

$$\le \sum_{j=0}^{m}[q_\nu(x,h)]^j \max_{r=\overline{1,N}} \sup_{t\in[(r-1)h,rh)} \|\widetilde{v}_r^{(1,1)}(x,t)-\widetilde{v}_r^{(1,0)}(x,t)\|;$$

$$\max_{r=\overline{1,N}} \|\lambda_r{}^{(1,m+1)}(x)-\lambda_r{}^{(1,0)}(x)\| \le$$

$$\leq b_3(x,\nu,h)\sum_{j=0}^{m}[q_\nu(x,h)]^j \max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\|\widetilde{v}_r^{(1,1)}(x,t)-\widetilde{v}_r^{(1,0)}(x,t)\|+$$

$$+\max_{r=\overline{1,N}}\|\lambda_r{}^{(1,1)}(x)-\lambda_r{}^{(1,0)}(x)\|;$$

$$\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\|\widetilde{v}_r^{(1,m+1)}(x,t)-\widetilde{v}_r^{(1,0)}(x,t)\|+\max_{r=\overline{1,N}}\|\lambda_r{}^{(1,m+1)}(x)-\lambda_r{}^{(1,0)}(x)\|\leq$$

$$\leq\Big[1+b_3(x,\nu,h)\Big]\sum_{j=0}^{m}[q_\nu(x,h)]^j \max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\|\widetilde{v}_r^{(1,1)}(x,t)-\widetilde{v}_r^{(1,0)}(x,t)\|+$$

$$+\max_{r=\overline{1,N}}\|\lambda_r{}^{(1,1)}(x)-\lambda_r{}^{(1,0)}(x)\|.$$

Moving to the limit at $m\to\infty$, we obtain estimates:

$$\Delta^{(1)}(x)=\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\|\widetilde{v}_r^{(1)}(x,t)-\widetilde{v}_r^{(0)}(x,t)\|+$$

$$+\max_{r=\overline{1,N}}\|\lambda_r{}^{(1)}(x)-\lambda_r{}^{(0)}(x)\|\leq g(x,\nu,h)\max_{t\in[0,T]}\|f(x,t)\|;$$

$$\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\|u_r^{(1)}(x,t)-u_r^{(0)}(x,t)\|\leq\int_0^x\Delta^{(1)}(\xi)d\xi.$$

For difference systems $\lambda_r{}^{(k+1)}(x)-\lambda_r{}^{(k)}(x)$, $\widetilde{v}_r^{(k+1)}(x,t)-\widetilde{v}_r^{(k)}(x,t)$, $u_r^{(k+1)}(x,t)-u_r^{(k)}(x,t)$, $r=\overline{1,N}$, $k=1,2,...$, valid estimates:

$$\max_{r=\overline{1,N}}\|\lambda_r{}^{(k+1,1)}(x)-\lambda_r{}^{(k+1,0)}(x)\|\leq$$

$$\leq b_1(x,\nu,h)\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\|u_r^{(k)}(x,t)-u_r^{(k-1)}(x,t)\|;$$

$$\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\|\widetilde{v}_r^{(k+1,1)}(x,t)-\widetilde{v}_r^{(k+1,0)}(x,t)\|\leq$$

$$\leq b_2(x,\nu,h)\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\|u_r^{(k)}(x,t)-u_r^{(k-1)}(x,t)\|;$$

$$\max_{r=\overline{1,N}}\|\lambda_r{}^{(k+1,m+1)}(x)-\lambda_r{}^{(k+1,m)}(x)\|\leq$$

$$\leq b_3(x,\nu,h)\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\|\widetilde{v}_r^{(k+1,m)}(x,t)-\widetilde{v}_r^{(k+1,m-1)}(x,t)\|;$$

$$\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\|\widetilde{v}_r^{(k+1,m+1)}(x,t)-\widetilde{v}_r^{(k+1,m)}(x,t)\|\leq$$

$$\leq q_\nu(x,h)\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\|\widetilde{v}_r^{(k+1,m)}(x,t)-\widetilde{v}_r^{(k+1,m-1)}(x,t)\|;$$

$$\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\|\widetilde{v}_r^{(k+1,m+1)}(x,t)-\widetilde{v}_r^{(k+1,0)}(x,t)\|\leq$$

$$\leq\sum_{j=0}^{m}[q_\nu(x,h)]^j\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\|\widetilde{v}_r^{(k+1,1)}(x,t)-\widetilde{v}_r^{(k+1,0)}(x,t)\|;$$

$$\max_{r=\overline{1,N}}\|\lambda_r{}^{(k+1,m+1)}(x)-\lambda_r{}^{(k+1,0)}(x)\|\leq$$

$$\leq b_3(x,\nu,h)\sum_{j=0}^{m-1}[q_\nu(x,h)]^j\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\|\widetilde{v}_r^{(k+1,1)}(x,t)-\widetilde{v}_r^{(k+1,0)}(x,t)\|+$$

$$+\max_{r=\overline{1,N}}\|{\lambda_r}^{(k+1,1)}(x)-{\lambda_r}^{(k+1,0)}(x)\|.$$

Moving to the limit at $m\to\infty$, we obtain estimates:

$$\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\|\widetilde{v}_r^{(k+1)}(x,t)-\widetilde{v}_r^{(k)}(x,t)\|\leq$$

$$\leq\frac{b_2(x,\nu,h)}{1-q_\nu(x,h)}\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\|u_r^{(k)}(x,t)-u_r^{(k-1)}(x,t)\|; \tag{16}$$

$$\max_{r=\overline{1,N}}\|{\lambda_r}^{(k+1)}(x)-{\lambda_r}^{(k)}(x)\|\leq$$

$$\leq\Big[\frac{b_3(x,\nu,h)}{1-q_\nu(x,h)}b_2(x,\nu,h)+b_1(x,\nu,h)\Big]\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\|u_r^{(k)}(x,t)-u_r^{(k-1)}(x,t)\|; \tag{17}$$

$$\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\|u_r^{(k+1)}(x,t)-u_r^{(k)}(x,t)\|\leq$$

$$\leq\int\limits_0^x\Big[\max_{r=\overline{1,N}}\|{\lambda_r}^{(k+1)}(\xi)-{\lambda_r}^{(k)}(\xi)\|+\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\|\widetilde{v}_r^{(k+1)}(\xi,t)-\widetilde{v}_r^{(k)}(\xi,t)\|\Big]d\xi.$$

Summing, respectively, the left and right parts of inequalities (16), (17) we have

$$\Delta^{(k+1)}(x)=\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\|\widetilde{v}_r^{(k+1)}(x,t)-\widetilde{v}_r^{(k)}(x,t)\|+\max_{r=\overline{1,N}}\|{\lambda_r}^{(k+1)}(x)-{\lambda_r}^{(k)}(x)\|\leq$$

$$\leq d_1(x,\nu,h)\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\|u_r^{(k)}(x,t)-u_r^{(k-1)}(x,t)\|; \tag{18}$$

$$\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\|u_r^{(k+1)}(x,t)-u_r^{(k)}(x,t)\|\leq\int\limits_0^x\Delta^{(k+1)}(\xi)d\xi.$$

For function $\Delta^{(k+1)}(x)$ based on (18) we obtain inequalities

$$\Delta^{(k+1)}(x)\leq d_1(x,\nu,h)\int\limits_0^x\Delta^{(k)}(\xi)d\xi;$$

$$\Delta^{(k)}(x)\leq\frac{d_1(x,\nu,h)}{(k-1)!}\bigg(\int\limits_0^x z(\xi)L(\xi)d\xi\bigg)^{k-1}\int\limits_0^x\Delta^{(1)}(\xi)d\xi.$$

Set the inequalities

$$\max_{r=\overline{1,N}}\|\lambda_r^{(k+p)}(x)-\lambda_r^{(0)}(x)\|+\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\|\widetilde{v}_r^{(k+p)}(x,t)-\widetilde{v}_r^{(0)}(x,t)\|\leq$$

$$\leq\Delta^{(k+p)}(x)+\Delta^{(k+p-1)}(x)+...+\Delta^{(1)}(x)\leq$$

$$\leq d_1(x,\nu,h)\sum_{j=1}^p\frac{1}{j!}\bigg(\int\limits_0^x d_1(\xi,\nu,h)d\xi\bigg)^j\int\limits_0^x\Delta^{(1)}(\xi)d\xi+\Delta^{(1)}(x)\leq$$

$$\leq\bigg[d_1(x,\nu,h)\sum_{j=1}^p\frac{1}{j!}\bigg(\int\limits_0^x d_1(\xi,\nu,h)d\xi\bigg)^j\int\limits_0^x g(\xi,\nu,h)d\xi+g(x,\nu,h)\bigg]\max_{(x,t)\in\overline{\Omega}}\|f(x,t)\|;$$

$$\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\|u_r^{(k+p)}(x,t)-u_r^{(0)}(x,t)\|\leq$$

$$\leq \int_0^x \left[\max_{r=\overline{1,N}} \|\lambda_r^{(k+p)}(\xi) - \lambda_r^{(0)}(\xi)\| + \max_{r=\overline{1,N}} \sup_{t\in[(r-1)h,rh)} \|\widetilde{v}_r^{(k+1)}(\xi,t) - \widetilde{v}_r^{(0)}(\xi,t)\| \right] d\xi.$$

At $p \to \infty$, we obtain the estimates of Theorem 1. The uniqueness of the solution is proved analogously to the uniqueness of Theorem 1 [9]. Theorem 1 is proved.

Theorem 2. Let the conditions of Theorem 1 be fulfilled. Then the problem (4)–(6) has a unique solution $u^*(x,t)$.

Main result

Now let us return to this duo-periodic problem Denote by $\mu(t)$ the value of the unknown function $z(x,t)$ for $x = 0$ and perform the replacement $u(x,t) = z(x,t) - \mu(t)$. Then the problem (1)–(3) is reduced to the following equivalent problem with the functional parameter

$$\frac{\partial^2 u}{\partial x \partial t} = A(x,t)\frac{\partial u}{\partial x} + u + \mu(t) + f(x,t), \quad (x,t) \in \Omega; \tag{19}$$

$$u(0,t) = 0, \quad t \in [0,T]; \tag{20}$$

$$u(x,0) = u(x,T), \quad x \in [0,\omega]; \tag{21}$$

$$u(\omega,t) = 0, \quad t \in [0,T]; \tag{22}$$

$$\mu(0) = \mu(T). \tag{23}$$

By virtue of (23) the equality $u(x,0) + \mu(0) = u(x,T) + \mu(T)$ which follows from (3) is written in the form (21). With the found $\mu(t)$ the function $u(x,t)$ is a solution of the periodic boundary value problem (19)–(21). To solve the problem (19)–(21) we use the method of a parametrization. Since conditions (20), (22) imply the equality $\frac{\partial u(0,t)}{\partial t} = 0$, $\frac{\partial u(\omega,t)}{\partial t} = 0$, for all $t \in [0,T]$, then integrating both parts (19) by $x \in [0,\omega]$ we obtain a system of differential equations, not resolved with respect to the derivative, to determine the unknown function $\mu(t)$:

$$\mu(t) = -\frac{1}{\omega}\int_0^\omega A(\xi,t)\frac{\partial u(\xi,t)}{\partial x}d\xi - \frac{1}{\omega}\int_0^\omega u(\xi,t)d\xi - \frac{1}{\omega}\int_0^\omega f(\xi,t)d\xi. \tag{24}$$

Thus, to determine the unknown functions $v(x,t)$, $u(x,t)$, $\mu(t)$ we have a closed system of equations (19)–(21) and (24).

Assuming that $u(x,t) = 0$, from equation (19) we find $\mu^{(0)}(t)$. Suppose that the problem (19)–(21) for $\mu(t) = \mu^{(0)}(t)$ has a solution $u^{(0)}(x,t) \in C\left(\overline{\Omega}, R^n\right)$.

For the initial approximation of problem (19)–(21) we take a pair $\{\mu^{(0)}(t),\ u^{(0)}(x,t)\}$ and construct successive approximations using the following algorithm:

Step 1. Assuming that $u(x,t) = u^{(0)}(x,t)$ from equation (24) we find $\mu^{(1)}(t)$:

$$\mu^{(1)}(t) = -\frac{1}{\omega}\int_0^\omega A(\xi,t)\frac{\partial u^{(0)}(\xi,t)}{\partial x}d\xi - \frac{1}{\omega}\int_0^\omega u^{(0)}(\xi,t)d\xi - \frac{1}{\omega}\int_0^\omega f(\xi,t)d\xi.$$

The function $u^{(1)}(x,t)$ is defined as the solution of a periodic boundary value problem

$$\frac{\partial^2 u}{\partial x \partial t} = A(x,t)\frac{\partial u}{\partial x} + u + \mu^{(1)}(t) + f(x,t), \quad (x,t) \in \Omega;$$

$$u(0,t) = 0, \quad t \in [0,T];$$

$$u(x,0) = u(x,T), \quad x \in [0,\omega].$$

To solve a periodic boundary value problem, we use the method of a parametrization.

Step 2. Assuming that $u(x,t) = u^{(1)}(x,t)$ from equation (24) we find $\mu^{(2)}(t)$

$$\mu^{(2)}(t) = -\frac{1}{\omega}\int_0^\omega A(\xi,t)\frac{\partial u^{(1)}(\xi,t)}{\partial x}d\xi - \frac{1}{\omega}\int_0^\omega u^{(1)}(\xi,t)d\xi - \frac{1}{\omega}\int_0^\omega f(\xi,t)d\xi.$$

The function $u^{(2)}(x,t)$ is defined as the solution of a periodic boundary value problem

$$\frac{\partial^2 u}{\partial x \partial t} = A(x,t)\frac{\partial u}{\partial x} + u + \mu^{(2)}(t) + f(x,t), \quad (x,t) \in \Omega;$$

$$u(0,t) = 0, \quad y \in [0,T];$$

$$u(x,0) = u(x,T), \quad x \in [0,\omega].$$

To solve a periodic boundary value problem, we use the method of a parametrization. And so on.

Continuing the process, at the k step we obtain the system $\left(\mu^{(k)}(t), u^{(k)}(x,t)\right)$.

Sufficient conditions for feasibility, convergence of the proposed algorithm establishes

Theorem 3. Let for some $h > 0 : Nh = T$, $N = 1,2,...$ and ν, $\nu = 1,2,...$, $(nN \times nN)$ — matrix $Q_\nu(x,h)$ is reversible for all $x \in [0,\omega]$ and the inequalities hold:

1) $\left\| [Q_\nu(x,h)]^{-1} \right\| \le \gamma_\nu(x,h)$;

2) $q_\nu(x,h) = \frac{(\alpha(x)h)^\nu}{\nu!}\left[1 + \gamma_\nu(x,h)\sum_{j=1}^{\nu}\frac{(\alpha(x)h)^j}{j!}\right] \le \beta < 1$;

3) $\theta(x,\nu,h) = \alpha(x)\rho(x,\nu,h) + \int_0^x \alpha(\xi)\rho(\xi,\nu,h)d\xi < \sigma < 1$.

Then the duo-periodic boundary value problem (1)–(3) has a unique solution.

Proof. According to Theorem 1, there are estimates

$$\left\|\lambda^{(1,*)}(x) - \lambda^{(1,0)}(x)\right\| + \sup_{t\in[0,T]}\left\|\tilde{v}^{(1,*)}(x,t) - v^{(1,0)}(x,t)\right\| \le$$

$$\le \rho(x,\nu,h)\sup_{t\in[0,T]}\left[\left\|\mu^{(1.0)}(t)\right\| + \|f(x,t)\|\right]; \tag{25}$$

$$\sup_{t\in[0,T]}\left\|u^{(1.*)}(x,t) - u^{(1.0)}(x,t)\right\| \le \int_0^x \left\|\lambda^{(1.*)}(\xi) - \lambda^{(1,0)}(\xi)\right\| d\xi +$$

$$+ \int_0^x \sup_{t\in[0,T]}\left\|\tilde{v}^{(1.*)}(\xi,t) - v^{(1.0)}(\xi,t)\right\| d\xi, \tag{26}$$

here $\lambda^{(1,0)}(x) = \lambda^{(0)}(x)$, $\tilde{v}^{(1.0)}(x,t) = \tilde{v}^{(0)}(x,t)$, $u^{(1.0)}(x,t) = u^{(0)}(x,t)$, $\mu^{(1.0)}(t) = \mu^{(0)}(t)$, $\lambda^{(1.*)}(x) = \lambda^{(1)}(x)$, $\tilde{v}^{(1.*)}(x,t) = \tilde{v}^{(1)}(x,t)$, $u^{(1.*)}(x,t) = u^{(1)}(x,t)$, then inequalities (25), (26) are rewritten as

$$\left\|\lambda^{(1)}(x) - \lambda^{(0)}(x)\right\| + \sup_{t\in[0,T]}\left\|\tilde{v}^{(1)}(x,t) - v^{(0)}(x,t)\right\| \le$$

$$\le \rho(x,\nu,h)\sup_{t\in[0,T]}\left\|\mu^{(0)}(t)\right\| + \rho(x,\nu,h)\sup_{t\in[0,T]}\|f(x,t)\|;$$

$$\sup_{t\in[0,T]}\left\|u^{(1)}(x,t) - u^{(0)}(x,t)\right\| \le$$

$$\le \int_0^x \rho(\xi,\nu,h)d\xi \cdot \sup_{t\in[0,T]}\left\|\mu^{(0)}(t)\right\| + \int_0^x \rho(\xi,\nu,h)\cdot \sup_{t\in[0,T]}\|f(\xi,t)\| d\xi.$$

Valid assessment

$$\left\|\mu^{(1)}(t) - \mu^{(0)}(t)\right\| \le \alpha(x)\rho(x,\nu,h)\sup_{t\in[0,T]}\left\|\mu^{(0)}(t)\right\| + \alpha(x)\rho(x,\nu,h)\sup_{t\in[0,T]}\|f(x,t)\| +$$

$$+ \int_0^x \rho(\xi,\nu,h)d\xi \cdot \sup_{t\in[0,T]}\left\|\mu^{(0)}(t)\right\| + \int_0^x \rho(\xi,\nu,h)\cdot \sup_{t\in[0,T]}\|f(\xi,t)\| d\xi.$$

At the k step we obtain a system

$$\left\|\lambda^{(k)}(x) - \lambda^{(k-1)}(x)\right\| + \sup_{t\in[0,T]}\left\|\tilde{v}^{(k)}(x,t) - v^{(k-1)}(x,t)\right\| \le$$

$$\leq \rho(x,\nu,h) \sup_{t\in[0,T]} \left\| \mu^{(k-1)}(t) - \mu^{(k-2)}(t) \right\|;$$

$$\sup_{t\in[0,T]} \left\| u^{(k)}(x,t) - u^{(k-1)}(x,t) \right\| \leq$$

$$\leq \int_0^x \left(\left\| \lambda^{(k)}(\xi) - \lambda^{(k-1)}(\xi) \right\| + \sup_{t\in[0,T]} \left\| \tilde{v}^{(k)}(\xi,t) - v^{(k-1)}(\xi,t) \right\| \right) d\xi;$$

$$\left\| \mu^{(k)}(t) - \mu^{(k-1)}(t) \right\| \leq [\theta(x,\nu,h)]^k \sup_{t\in[0,T]} \left\| \mu^{(1)}(t) - \mu^{(0)}(t) \right\|,$$

$k = 2, 3, ...$ Due to the inequality $\theta(x,\nu,h) < 1$ follows the uniform convergence of $u^{(k)}(x,t)$, $\tilde{v}^{(k)}(x,t)$, $(x,t) \in \overline{\Omega}$ to $u^*(x,t), v^*(x,t)$ and the convergence of a sequence of systems of functions $\lambda^{(k)}(x), \mu^{(k)}(t)$ to the continuous, respectively, on $x \in [0,\omega]$, $t \in [0,T]$ to the functions $\lambda^*(x), \mu^*(t)$.

The uniqueness of the solution is proved analogously to the uniqueness of Theorem 1 [9]. Theorem 3 is proved.

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Н.Т. Орумбаева, А.Б. Кельдибекова

Аралас туындылы гиперболалық теңдеулер жүйесі үшін қос периодты есептің шешімділігі жайында

Екінші ретті гиперболалық теңдеулер теориясының негізгі және неғұрлым зерттелген есептерінің бірі периодты шеттік есеп болып табылады. Мұндай есептерді шешу үшін Фурье әдісі, біртіндеп жуықтау әдісі, функционалдық талдау әдістері, вариациялық әдіс және т.б. қолданылады. Осындай конструктивтік әдістердің бірі қарапайым дифференциалдық теңдеулердің екі нүктелі шеттік есептерін шешу үшін Д.С. Джумабаевтың жұмыстарында ұсынылған параметрлеу әдісі болып табылады. Ұсынылып отырған жұмыста аралас туындылы гиперболалық теңдеулер жүйесі үшін екі айнымалы бойынша периодты шеттік есеп қарастырылған. Осы есепті шешу үшін жаңа функциялар енгізілді және периодты шеттік есеп қарапайым дифференциалдық теңдеулер үйірі мен интегралдық теңдеуден тұратын эквивалентті есепке келтірілді. Алынған есепті шешу үшін параметрлеу әдісі қолданылады. Бұл әдісті пайдалану аралас туындылы гиперболалық теңдеулер жүйесі үшін периодты шеттік есептің шешімін табу алгоритмін құруға мүмкіндік берді. Сонымен қатар ұсынылған алгоритмнің жинақтылығы мен орындалуының коэффициентті шарттары алынды.

Кілт сөздер: қос периодты есеп, параметірлік әдіс, гиперболалық теңдеулер жүйесі, аралас туынды.

Н.Т. Орумбаева, А.Б. Кельдибекова

О разрешимости двоякопериодической задачи для системы гиперболических уравнений со смешанной производной

Одной из основных и наиболее изученных задач теории гиперболических уравнений второго порядка является периодическая краевая задача. Для решения таких задач применяются метод Фурье, метод последовательных приближений, методы функционального анализа, вариационный метод и др. Развитие информационных технологий предъявляет новые требования на разрабатываемые методы, уделяя особое внимание их конструктивности. Одним из таких конструктивных методов является метод параметризации, предложенный в работах Д.С. Джумабаева, для решения двухточечных краевых задач обыкновенных дифференциальных уравнений. В данной работе рассмотрена периодическая по обеим переменным краевая задача для системы гиперболических уравнений со смешанной производной. Для решения данной задачи вводятся обозначения и периодическая краевая задача сводится к эквивалентной задаче, состоящей из семейства периодических краевых задач для обыкновенного дифференциального уравнения и интегрального соотношения. Для решения полученной задачи применяется метод параметризации. Применение данного метода позволило построить алгоритм нахождения приближенного решения периодической краевой задачи для системы гиперболических уравнений со смешанной производной. Кроме того, получены коэффициентные условия сходимости и осуществимости предложенного алгоритма.

Ключевые слова: двоякопериодическая краевая задача, метод параметризации, система гиперболических уравнений, смешанные производные.

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M.T. Kosmakova, D.M. Akhmanova, S.A. Iskakov, Zh.M. Tuleutaeva, L.Zh. Kasymova

Ye.A. Buketov Karaganda State University, Kazakhstan
(E-mail: Svetik_mir69@mail.ru)

Solving one pseudo-Volterra integral equation

In this paper, we study the solvability of a second-kind pseudo-Volterra integral equation. By replacing the right-hand side and the unknown function, the integral equation is reduced to an integral equation, the kernel of which is not «compressible». Using the Laplace transform, the obtained equation is reduced to an ordinary first-order differential equation (linear). Its solution is found. The solution of the homogeneous integral equation corresponding to the original nonhomogeneous integral equation found in explicit form. Special cases of a homogeneous integral equation and its solutions are written for different values of the parameter k. Classes are indicated in which the integral equation has a solution. Singular integral equations were considered in works [1–3]. Their kernels were also «incompressible», but kernels had an another form. In this connection, the weight classes of the solution existence differ from the class of the solution existence for the equation considered in this work.

Keywords: kernel, integral operator, class of essentially bounded functions, Laplace transformation.

Introduction

This paper is devoted to the research of questions of solvability of the following pseudo-Volterra integral equation of the second kind

$$\nu(t) - \frac{a}{2\sqrt{\pi}}\int_0^t \frac{1}{\sqrt{t}\,\sqrt{\tau}\sqrt{t-\tau}}\, e^{-\frac{t-\tau}{4a^2}} \cdot \nu(\tau)d\tau - \frac{1}{k\,a\sqrt{\pi}}\int_0^t \sqrt{\frac{\tau}{t}}\frac{1}{\sqrt{t-\tau}}\, e^{-\frac{t-\tau}{4a^2}} \cdot \nu(\tau)d\tau = f(t), \tag{1}$$

where a, k — are positive constants, $f(t)$ — is the given function.

A similar kind of integral equation arises in solving the boundary value problems of heat conduction with heat generation, which describe the development of the one-dimensional unsteady heat processes with axial symmetry.

1 Reducing the equation (1) to a differential equation in images

We rewrite the equation (1) in the form

$$\frac{\nu(t)}{\sqrt{t}}e^{\frac{t}{4a^2}} - \frac{a}{2\sqrt{\pi}}\int_0^t \frac{1}{t\sqrt{t-\tau}} \cdot \frac{1}{\sqrt{\tau}}e^{\frac{\tau}{4a^2}} \cdot \nu(\tau)d\tau -$$
$$- \frac{1}{k\,a\sqrt{\pi}}\int_0^t \frac{\tau}{t\sqrt{t-\tau}}\left\{\frac{1}{\sqrt{\tau}}e^{\frac{\tau}{4a^2}} \cdot \nu(\tau)\right\}d\tau = \frac{f(t)}{\sqrt{t}}e^{\frac{t}{4a^2}}. \tag{2}$$

After replacements:

$$\frac{1}{\sqrt{t}}e^{\frac{t}{4a^2}}\nu(t) = \nu_1(t), \quad \frac{1}{\sqrt{t}}e^{\frac{t}{4a^2}}f(t) = f_1(t) \tag{3}$$

equation (2) takes the form

$$\nu_1(t) - \frac{a}{2\sqrt{\pi}}\int_0^t \frac{1}{t\sqrt{t-\tau}} \cdot \nu_1(\tau)d\tau - \frac{1}{k\,a\sqrt{\pi}}\int_0^t \frac{\tau}{t\sqrt{t-\tau}}\,\nu_1(\tau)\,d\tau = t\,f_1(t);$$

or

$$t \cdot \nu_1(t) - \frac{a}{2\sqrt{\pi}}\int_0^t \frac{1}{\sqrt{t-\tau}} \cdot \nu_1(\tau)d\tau - \frac{1}{k\,a\sqrt{\pi}}\int_0^t \frac{1}{\sqrt{t-\tau}} \cdot \tau\,\nu_1(\tau)\,d\tau = f_2(t), \tag{4}$$

where $f_2(t) = t\,f_1(t)$.

We note that the integral operator acting in the class of continuous functions $\nu_1(t) \in C(0;\ +\infty)$, of an equation with a kernel:

$$\mathrm{K}(t,\tau) = \frac{a}{2\sqrt{\pi}} \frac{1}{t\sqrt{t-\tau}} + \frac{1}{k\,a\sqrt{\pi}} \frac{\tau}{t\sqrt{t-\tau}}$$

is not bounded.

To equation (4) we apply the Laplace transform, introducing the notation

$$\hat{\nu}_1(p) = \int_0^\infty \nu_1(t)\ \cdot e^{-pt}dt \ \Leftrightarrow\ \nu_1(t) \div \hat{\nu}_1(p);$$

$$\hat{f}_2(p) = \int_0^\infty f_2(t)\ \cdot e^{-pt}dt \ \Leftrightarrow\ f_2(t) \div \hat{f}_2(p),$$

that is

$$\hat{\nu}_1(p) = L\left[\nu_1(t)\right];$$

$$\hat{f}_2(p) = L\left[f_2(t)\right].$$

Since

$$\int_0^\infty \frac{1}{\sqrt{t}} e^{-pt}dt = \left\| \begin{array}{l} pt = z^2, \\ dt = \frac{2z}{p}\,dz \end{array} \right\| = \frac{2}{\sqrt{p}} \int_0^\infty e^{-z^2}dz = \frac{\sqrt{\pi}}{\sqrt{p}};$$

$$t \cdot \nu_1(t) \div -\hat{\nu}_1'(p),$$

integral equation (4) becomes a differential equation in the image space

$$-\hat{\nu}_1'(p) - \frac{a}{2\sqrt{\pi}} \frac{\sqrt{\pi}}{\sqrt{p}}\ \hat{\nu}_1(p) - \frac{1}{k\,a\sqrt{\pi}} \cdot \frac{\sqrt{\pi}}{\sqrt{p}} \cdot (-\hat{\nu}_1'(p)) = \hat{f}_2(p),$$

which can be rewritten as

$$\left[\frac{1}{k\,a\sqrt{p}} - 1\right] \hat{\nu}_1'(p) - \frac{a}{2\sqrt{p}}\ \hat{\nu}_1(p) = \hat{f}_2(p). \tag{5}$$

2 Solving a homogeneous linear differential equation

We solve a homogeneous equation that corresponds to a linear equation (5)

$$\left[\frac{1}{k\,a\sqrt{p}} - 1\right] \hat{\nu}_1'(p) - \frac{a}{2\sqrt{p}}\ \hat{\nu}_1(p) = 0. \tag{6}$$

The solution to differential equation (6) has the form:

$$\hat{\nu}_1(p) = \frac{Ce^{1/k}}{(ka)^{1/k}} \cdot \frac{e^{-a\sqrt{p}}}{\left(\sqrt{p} - \frac{1}{ka}\right)^{1/k}}, \tag{7}$$

where $\mathcal{C} - const$.

Since from formula No. 149 [4; 272] and from formula No. 9 [4; 259] we have

$$\frac{e^{-as}}{\left(s - \frac{1}{ka}\right)^{1/k}} = L\left[\frac{\tau^{\frac{1}{k}-1}}{\Gamma\left(\frac{1}{k}\right)} e^{\frac{\tau}{ka}}\right],$$

then, taking into account formula No. 29 from [4; 261] and applying the inverse Laplace transformation to (7), we obtain

$$\nu_1(t) = \frac{C}{\Gamma\left(\frac{1}{k}\right)\ (ka)^{1/k}} \int_a^\infty \frac{\tau(\tau-a)^{\frac{1}{k}-1}}{2\sqrt{\pi}\,t^{3/2}} e^{-\frac{\tau^2}{4t}} \cdot e^{\frac{\tau-a}{ka}}d\tau =$$

$$= \frac{Ce^{-\frac{1}{k}}}{\Gamma\left(\frac{1}{k}\right)\ (ka)^{1/k}} \frac{1}{2\sqrt{\pi}\,t^{3/2}} \int_a^\infty \tau(\tau-a)^{\frac{1}{k}-1} e^{-\frac{\tau^2}{4t}+\frac{\tau}{ka}}d\tau.$$

Introducing the notation

$$I(t,k)=\int_a^{\infty}\tau(\tau-a)^{\frac{1}{k}-1}e^{-\frac{\tau^2}{4t}+\frac{\tau}{ka}}d\tau,$$

we get

$$\nu_1(t)=\frac{Ce^{-\frac{1}{k}}}{\Gamma\left(\frac{1}{k}\right)\,(ka)^{1/k}}\frac{1}{2\sqrt{\pi}\,t^{3/2}}I(t,k). \tag{8}$$

We calculate the integral

$$I(t,k)=\int_a^{\infty}\tau(\tau-a)^{\frac{1}{k}-1}e^{-\frac{\tau^2}{4t}+\frac{\tau}{ka}}d\tau=$$

$$=\int_a^{\infty}(\tau-a)^{\frac{1}{k}}\,e^{-\frac{\tau^2}{4t}+\frac{\tau}{ka}}d\tau+a\int_a^{\infty}(\tau-a)^{\frac{1}{k}-1}\,e^{-\frac{\tau^2}{4t}+\frac{\tau}{ka}}d\tau.$$

Taking into account the formula 2.3.15 (1) from [5], we obtain

$$I(t,k)=\Gamma\left(\frac{1}{k}+1\right)\left(\frac{1}{2t}\right)^{-\frac{1}{2k}-\frac{1}{2}}\exp\left\{\frac{t}{2k^2a^2}+\frac{1}{2k}-\frac{a^2}{8t}\right\}\,D_{-\left(\frac{1}{k}+1\right)}\left(\frac{ka^2-2t}{ak\sqrt{2t}}\right)+$$

$$+a\,\Gamma\left(\frac{1}{k}\right)\left(\frac{1}{2t}\right)^{-\frac{1}{2k}}\exp\left\{\frac{t}{2k^2a^2}+\frac{1}{2k}-\frac{a^2}{8t}\right\}\,D_{-\frac{1}{k}}\left(\frac{ka^2-2t}{ak\sqrt{2t}}\right)=$$

$$=\Gamma\left(\frac{1}{k}\right)\exp\left\{\frac{1}{2k}\right\}\,(2t)^{\frac{1}{2k}}\exp\left\{\frac{t}{2k^2a^2}-\frac{a^2}{8t}\right\}\times$$

$$\times\left[\frac{1}{k}\sqrt{2t}\,D_{-\left(\frac{1}{k}+1\right)}\left(\frac{ka^2-2t}{ak\sqrt{2t}}\right)+a\,D_{-\frac{1}{k}}\left(\frac{ka^2-2t}{ak\sqrt{2t}}\right)\right].$$

Substituting $I(t,k)$ into expression (8), we obtain the general solution of the homogeneous equation that corresponds to the integral equation (4)

$$\nu_1(t)=\frac{Ce^{\frac{1}{k}}}{(ka)^{1/k}}\frac{(2t)^{\frac{1}{2k}}}{2\sqrt{\pi}\,t^{3/2}}\exp\left\{\frac{t}{2k^2a^2}-\frac{a^2}{8t}\right\}\times$$

$$\times\left[\frac{1}{k}\sqrt{2t}\,D_{-\left(\frac{1}{k}+1\right)}\left(\frac{ka^2-2t}{ak\sqrt{2t}}\right)+a\,D_{-\frac{1}{k}}\left(\frac{ka^2-2t}{ak\sqrt{2t}}\right)\right],$$

where [6] (see formula 9.241(2))

$$D_{-p}(z)=\frac{e^{-\frac{z^2}{4}}}{\Gamma\left(p\right)}\int_0^{+\infty}e^{-zx-\frac{x^2}{2}}x^{p-1}dx,\quad Re p>0 \tag{9}$$

are Parabolic cylinder functions (Weber functions).

Using the replacement that is inverse to (3), we get

$$\nu(t)=\frac{Ce^{\frac{1}{k}}}{(ka)^{\frac{1}{k}}}\frac{(2t)^{\frac{1}{2k}}}{2\sqrt{\pi}t}\exp\left\{\frac{t}{2k^2a^2}-\frac{t}{4a^2}-\frac{a^2}{8t}\right\}\times$$

$$\times\left[\frac{1}{k}\sqrt{2t}\,D_{-\left(\frac{1}{k}+1\right)}\left(\frac{ka^2-2t}{ak\sqrt{2t}}\right)+a\,D_{-\frac{1}{k}}\left(\frac{ka^2-2t}{ak\sqrt{2t}}\right)\right]. \tag{10}$$

(10) is the general solution of the homogeneous integral equation that corresponds to the initial equation (1).

3 Case $k = 2$

From a practical point of view, the case $k = 2$ is interesting

$$\nu(t) = \frac{C\sqrt{e}}{\sqrt{2a\pi}(2t)^{\frac{3}{4}}} \exp\left\{-\frac{t}{8a^2} - \frac{a^2}{8t}\right\} \times$$

$$\times \left[\frac{\sqrt{2t}}{2} D_{-\frac{3}{2}}\left(\frac{a}{\sqrt{2t}} - \frac{\sqrt{t}}{a\sqrt{2}}\right) + a\, D_{-\frac{1}{2}}\left(\frac{a}{\sqrt{2t}} - \frac{\sqrt{t}}{a\sqrt{2}}\right)\right], \tag{11}$$

where [7] considering formula [9]

$$D_{-\frac{1}{2}}(z) = \sqrt{\frac{\pi\, z}{2}} K_{\frac{1}{4}}\left(\frac{z^2}{4}\right)$$

and $K_\nu(x)$ is the modified Bessel function of the second kind or the Macdonald function.

Since from formula 9.247 (2) [6] we have

$$D_{-\frac{3}{2}}(z) = z\, D_{-\frac{1}{2}}(z) + 2\frac{d}{dz} D_{-\frac{1}{2}}(z) = \sqrt{\frac{\pi\, z^3}{2}} K_{\frac{1}{4}}\left(\frac{z^2}{4}\right) + \frac{d}{dz}\left(\sqrt{2\pi\, z} K_{\frac{1}{4}}\left(\frac{z^2}{4}\right)\right),$$

then, taking into account the formula

$$K'_\nu(x) = -\frac{1}{2}\left(K_{\nu-1}(x) + K_{\nu-1}(x)\right),$$

we conclude that the expression in square brackets in (11) will be a linear combination of functions $K_\nu\left(\frac{z^2}{4}\right)$, where

$$\nu = \left\{\frac{1}{4}; \frac{3}{4}; \frac{5}{4}\right\}, \quad z = \frac{a}{\sqrt{2t}} - \frac{\sqrt{t}}{a\sqrt{2}}.$$

From asymptotic behavior

$$K_\nu(x) \approx \sqrt{\frac{\pi}{2}} \frac{e^{-x}}{\sqrt{x}} \left(1 + O\left(\frac{1}{x}\right)\right), \quad x \to +\infty$$

and from limit relation

$$\lim_{t\to 0;\, t\to+\infty} z^2 = \lim_{t\to 0;\, t\to+\infty} \left(\frac{a}{\sqrt{2t}} - \frac{\sqrt{t}}{a\sqrt{2}}\right)^2 = +\infty,$$

it follows that function (11) will be bounded when $t \in (0,\ +\infty)$.

Thus, the following theorem is proved.

Theorem 1. The integral equation

$$\nu(t) - \frac{a}{2\sqrt{\pi}} \int_0^t \frac{1}{\sqrt{t}\sqrt{\tau}\sqrt{t-\tau}} e^{-\frac{t-\tau}{4a^2}} \cdot \nu(\tau) d\tau - \frac{1}{2\, a\sqrt{\pi}} \int_0^t \sqrt{\frac{\tau}{t}} \frac{1}{\sqrt{t-\tau}} e^{-\frac{t-\tau}{4a^2}} \cdot \nu(\tau) d\tau = 0$$

in the class of functions $\nu(t) \in L_\infty(0,\ +\infty)$ has a solution defined by the formula (11).

4 Case $k = 1$

When $k = 1$ from representation (10) we get

$$\nu(t) = \frac{Ce}{a\sqrt{2\pi\, t}} \exp\left\{\frac{t}{4a^2} - \frac{a^2}{8t}\right\} \left[\sqrt{2t}\, D_{-2}\left(\frac{a}{\sqrt{2t}} - \frac{\sqrt{2t}}{a}\right) + a\, D_{-1}\left(\frac{a}{\sqrt{2t}} - \frac{\sqrt{2t}}{a}\right)\right]. \tag{12}$$

From formulas 9.254 (1) and 9.254 (2) [6] we have for (12):

$$\nu(t) = \frac{Ce}{a\sqrt{2\pi\, t}} \exp\left\{\frac{t}{4a^2} - \frac{a^2}{8t}\right\} \left[\sqrt{\frac{\pi}{2}}\sqrt{2t} \exp\left\{\frac{a^2}{8t} + \frac{t}{2a^2} - \frac{1}{2}\right\} \times\right.$$

$$\times\left\{\sqrt{\frac{2}{\pi}}\,\exp\left\{1-\frac{a^2}{4t}-\frac{t}{a^2}\right\}-\left(\frac{a}{\sqrt{2t}}-\frac{\sqrt{2t}}{a}\right)erfc\left(\frac{a}{2\sqrt{t}}-\frac{\sqrt{t}}{a}\right)\right\}+$$

$$+\,a\sqrt{\frac{\pi}{2}}\exp\left\{\frac{a^2}{8t}+\frac{t}{2a^2}-\frac{1}{2}\right\}erfc\left(\frac{a}{2\sqrt{t}}-\frac{\sqrt{t}}{a}\right)\Bigg]=$$

$$=\frac{Ce^{\frac{3}{2}}}{a\sqrt{\pi}}\left[\frac{1}{\sqrt{t}}\exp\left\{-\frac{t}{4a^2}-\frac{a^2}{4t}\right\}+\frac{\sqrt{\pi}}{ae}\exp\left\{\frac{3t}{4a^2}\right\}erfc\left(\frac{a}{2\sqrt{t}}-\frac{\sqrt{t}}{a}\right)\right].$$

So, when $k=1$, representation (10) has the form:

$$\nu(t)=\frac{Ce^{\frac{3}{2}}}{a\sqrt{\pi}}\left[\frac{1}{\sqrt{t}}\exp\left\{-\frac{t}{4a^2}-\frac{a^2}{4t}\right\}+\frac{\sqrt{\pi}}{ae}\exp\left\{\frac{3t}{4a^2}\right\}erfc\left(\frac{a}{2\sqrt{t}}-\frac{\sqrt{t}}{a}\right)\right]. \qquad (13)$$

Thus, the following theorem is valid.

Theorem 2. The integral equation

$$\nu(t)-\frac{a}{2\sqrt{\pi}}\int_0^t\frac{1}{\sqrt{t}\,\sqrt{\tau}\sqrt{t-\tau}}\,e^{-\frac{t-\tau}{4a^2}}\cdot\nu(\tau)d\tau-\frac{1}{a\sqrt{\pi}}\int_0^t\sqrt{\frac{\tau}{t}}\frac{1}{\sqrt{t-\tau}}\,e^{-\frac{t-\tau}{4a^2}}\cdot\nu(\tau)d\tau=0$$

in the class of functions $\exp\left\{-\frac{t}{a^2}\right\}\nu(t)\in L_\infty(0,\,+\infty)$ has a solution defined by the formula (13).

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М.Т. Космакова, Д.М. Ахманова, С.А. Искаков, Ж.М. Тулеутаева, Л.Ж. Касымова

Псевдо-Вольтерраның интегралдық теңдеуінің шешілуі

Мақалада псевдо-Вольтерраның екінші текті интегралдық теңдеуінің шешілу сұрақтары зерттелді. Интегралдық теңдеу оң жақтағы және ізделінді функцияны ауыстыру арқылы ядросы «сығылмалы» болмайтын интегралдық теңдеуге келтірілді. Алынған теңдеу Лаплас түрлендіруі арқылы қарапайым бірінші ретті (сызықтық) дифференциалдық теңдеуге келтірілді. Бұл теңдеудің шешуі табылды. Бастапқы біртекті емес интегралдық теңдеуге сәйкес келетін біртекті интегралдық теңдеудің шешуі айқын түрде табылды. Біртекті интегралдық теңдеудің дербес жағдайлары және оның k параметрінің әртүрлі мәндеріндегі шешулері жазылды. Шешулері болатын интегралдық теңдеулердің кластары көрсетілген. Сингулярлық интегралдық теңдеулер [1–3] жұмыстарда қарастырылған. Сонымен қатар олардын ядролары «сығылмайтын» болды, бірақ түрі өзгеше. Осыған байланысты шешудің бар болуының салмақтық кластарының аталған жұмыстағы зерттеліп отырған теңдеулердің шешулері бар болуының кластарынан айырмашылығы бар.

Кілт сөздер: ядро, интегралдық оператор, шектелген функциялар кластары, Лаплас түрлендіруі.

М.Т. Космакова, Д.М. Ахманова, С.А. Искаков, Ж.М. Тулеутаева, Л.Ж. Касымова

Решение одного псевдо-Вольтеррового интегрального уравнения

В статье исследованы вопросы разрешимости псевдо-Вольтеррового интегрального уравнения второго рода. С помощью замен правой части и искомой функции интегральное уравнение сведено к интегральному уравнению, ядро которого не является «сжимаемым». С помощью преобразования Лапласа полученное уравнение сведено к обыкновенному дифференциальному уравнению первого порядка (линейному). Найдено его решение. Решение однородного интегрального уравнения, соответствующего исходному неоднородному интегральному уравнению, найдено в явном виде. Выписаны частные случаи однородного интегрального уравнения и его решения при различных значениях параметра k. Указаны классы, в которых интегральное уравнение имеет решение. Сингулярные интегральные уравнения были рассмотрены в работах [1–3]. Их ядра также были «несжимаемы», но имели другой вид. В связи с этим весовые классы существования решения отличаются от класса существования решения уравнения, исследуемого в данной работе.

Ключевые слова: ядро, интегральный оператор, класс существенно ограниченных функций, преобразование Лапласа.

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M.I. Tleubergenov[1], D.T. Azhymbaev[2]

[1] *Institute of Mathematics and Mathematical Modeling, Almaty, Kazakhstan;*
[2] *K. Zhubanov Aktobe Regional State University, Kazakhstan*
(E-mail: marat207@mail.ru)

Stochastical problem of Helmholtz for Birkhoff systems

The Helmholtz problem is considered in a probabilistic formulation. By a given stochastic Langevin-Itô equation in an indirect representation, as the equation of the Hamiltonian structure and the equation of the Birkhoffian structure are constructed. The functional that takes a stationary value on solutions of a given stochastic Birkhoff equation, is defined by the method of moment functions. The obtained results are illustrated by two examples: 1) the plane motion of a symmetric satellite in a circular orbit under the action of gravity and aerodynamic forces, and 2) the fluctuation motion of a gyroscope in a gimbal caused by the stochastic fluctuating moment of forces along the suspension axis of the inner ring.

Keywords: stochastic Langevin-Itô equation, inverse problem, equation of the Hamiltonian (or Birkhoffian) structure.

Introduction

The theory of inverse problems of differential systems is sufficiently fully developed in [1-6, etc.] for deterministic systems, which are described by ordinary differential equations (ODE). Thus, the work of N.P. Erugin [1], in which a set of ODE is constructed according to a given integral curve, subsequently turned out to be fundamental in the formation and development of the theory of inverse problems of the dynamics of systems described by the ODE. In [2–6], the formulation, classification of inverse problems of differential systems and general methods for their solution in the class of ODE are presented. Also, in the ODE class, inverse problems of the automatic control systems' dynamics are considered [7–9]. It should be noted that one of the general methods for solving inverse problems of dynamics in the class of ODE is the quasi-inversion method proposed in [4, 5] and which makes it possible to obtain necessary and sufficient conditions for solvability.

A new stage in the research of inverse problems of differential systems is the increased interest in recent years in the study of the Helmholtz problem (see, for example, the monograph [10]). In the monograph of A.S. Galiullin [10], along with a review of works, the Hamilton systems' generalization in the sense of the reducibility of the non-conservative mechanical systems' motion equations to classical equations of dynamics is considered, and, in particular, the problem of the equations' Hamiltonization of program motion's systems is solved.

The classical Helmholtz problem [11] is the problem of construction the equivalent differential equations in the form of Lagrange on given second-order ordinary differential equations. Moreover, the equations for which such transition is possible are called Helmholtz systems. In the works of A. Mayer [12] and G.K. Suslov [13] independently it is shown that the classical Helmholtz conditions are not only necessary, but also sufficient conditions for the transition from Newtonian equations to Lagrangian ones.

The solving of the Helmholtz problem [11] in this or that class of differential equations allows us to extend to this class of equations well-developed mathematical methods of classical mechanics. It should be noted that the two-volume monograph by R. M. Santilli [14, 15], devoted to the problem of representation of ordinary differential equations of the second order in the form of Lagrange, Hamilton and Birkhoff, occupies a special place in the completeness of the material and the variety of Helmholtz problem's study aspects.

The development of methods for solving inverse problems in the class of partial differential equations is discussed in [16–18].

In [19–21], inverse problems of dynamics are considered in a probabilistic formulation under the additional assumption of the random perturbations' presence, and, in particular, the follow problems: 1) *the basic inverse problem of dynamics,* in which it is required to construct a set of second-order stochastic differential equations of Itô type having a given integral manifold, 2) *the problem of reconstructing the equations of motion,* in which it is required to construct a set of control parameters that enter into a given system of second-order stochastic differential equations of Itô type from a given integral manifold, and 3) *the problem of closing the equations of motion,* in which it is required to construct a set of closed stochastic second-order differential equations of Itô type with respect to a given system of equations and a given integral manifold, are solved by the quasi-inversion method.

In this paper, we consider the Helmholtz problem in the presence of random perturbations of white noise type for Hamilton systems and Birkhoff systems.

1 Formulation of the problem and its solving

It is required to construct an equivalent equation of the Hamiltonian (or Birkhoffian) structure by the equation given in the Langevin-Itô form

$$\ddot{x}_\nu = F_\nu(x, \dot{x}, t) + \sigma_{\nu j}(x, \dot{x}, t)\dot{\xi}^j. \tag{1}$$

Here $\xi^j = \xi_0^j + \int c^j(y) P^0(t, dy)$, where, following [22], ξ_0^j is the Wiener process, P^0 is the Poisson process, $P^0(t, dy)$ is the number of process jumps P^0 in the interval $[0, t]$, falling on the set dy, where $y = \left(x^T, \dot{x}^T\right)^T$.

We say that a function $g(y, t)$ from the class K, $g \in K$, if g is continuous on t and is Lipschitz on y in the whole space $R^{2n} \ni y$ and satisfies the linear growth condition with respect to y: $\|g(y, t)\| \leq M(1 + \|y\|)$ with some constant M.

Suppose that a given vector-valued function F and a matrix σ belong to the class K. And since the vector-valued function F and the $(n \times k)$ matrix σ are assumed from the class K, this ensures [22] the existence and uniqueness up to the stochastic equivalence of the solution $\left(x^T(t), \dot{x}^T(t)\right)^T$ of equation (1) with the initial condition $\left(x(t_0)^T, \dot{x}(t_0)^T\right)^T = \left(x_0{}^T, \dot{x}_0^T\right)^T$ being a strictly Markov process with probability 1.

This formulation of the problem in the absence of random perturbations ($\sigma_{\nu j} \equiv 0$) was considered in the works of R.M. Santilli [14, 15], and in a probabilistic formulation the Helmholtz problem it was previously studied in [23–25], where equations of the Lagrangian structure are constructed from the given equation (1), and, further, from the stochastic Lagrange equation, a stochastic analogue of the Hamilton variational principle is determined.

To solve the problem, we will introduce previously a new variable and we will rewrite the given equation (1) in a form

$$\begin{cases} \dot{x}_k = y_k; \\ \dot{y}_k = F_k(x, y, t) + \sigma_{kj}(x, y, t)\dot{\xi}^j. \end{cases} \tag{2}$$

And then, with the help of replacements

$$a_k = \begin{cases} x_k \\ y_k \end{cases}; \quad \eta_j = \begin{cases} 0, & j = 1, 2, ..., n; \\ \xi^{j-n}, & j = n+1, n+2, ..., n+m; \end{cases}$$

$$G_k = \begin{cases} x_k \\ F_k \end{cases}; \quad \Lambda = (\Lambda_{kj}) = \begin{pmatrix} 0_{n\times n} & 0_{n\times m} \\ 0_{n\times n} & \sigma_{n\times m} \end{pmatrix}; \qquad \sigma = (\sigma_{\mu j}),$$

we rewrite the equation (2) in a form

$$\dot{a}_k = G_k(a, t) + \Lambda_{kj}(a, t)\dot{\eta}_j. \tag{3}$$

Further, we rewrite the stochastic equation of the Hamiltonian structure

$$\begin{cases} \dot{q}_k = \dfrac{\partial H}{\partial p_k}; \\ \dot{p}_k = -\dfrac{\partial H}{\partial q_k} + \sigma'_{kj}(q, p, t)\dot{\xi}^j, \quad (k = \overline{1, n}). \end{cases} \tag{4}$$

in the form

$$\dot{z}_\mu - \alpha_{\mu\nu}\frac{\partial H}{\partial z_\nu} = \theta_{\mu\nu}\dot{\eta}_\nu, \tag{5}$$

where the following notations are

$$z_k = \begin{cases} q_k, & k = 1,2,...,n, \\ p_{k-n}, & k = n+1, n+2, ..., 2n, \end{cases}$$

$$\alpha = (\alpha_{\mu\nu}) = \begin{pmatrix} 0_{n\times n} & I_{n\times n} \\ -I_{n\times n} & 0_{n\times n} \end{pmatrix}, \qquad \Theta = (\theta_{\mu\nu}) = \begin{pmatrix} 0_{n\times n} & 0_{n\times m} \\ 0_{n\times n} & \sigma'_{n\times m} \end{pmatrix};$$

$$\begin{pmatrix} \dfrac{\partial H}{\partial q_k} \\ -\dfrac{\partial H}{\partial p_k} \end{pmatrix} = \left(\alpha_{\mu\nu}\frac{\partial H}{\partial z_\nu}\right).$$

Or, if we introduce a matrix $(\omega_{\mu\nu})$ which is the inverse to a matrix $(\alpha_{\mu\nu})$

$$(\omega_{\mu\nu}) = (\alpha_{\mu\nu})^{-1} = \begin{pmatrix} 0_{n\times n} & -I_{n\times n} \\ I_{n\times n} & 0_{n\times n} \end{pmatrix}$$

and $2n$-dimensional vector

$$a_\mu \equiv \omega_{\mu\nu} z_\nu = \begin{pmatrix} -p_\mu, & \mu = 1,2,...,n \\ q_{\mu-n}, & \mu = n+1, n+2, ..., 2n \end{pmatrix},$$

then the equation (5) will be transformed to the equivalent equation

$$\omega_{\mu\nu}\dot{a}_\nu - \frac{\partial H}{\partial a_\nu} = \omega_{\mu k}\theta_{k\nu}\dot{\eta}_\nu. \tag{6}$$

Construction of the Hamiltonian in the indirect representation. We consider the problem of the indirect representation of equation (3) in the form of an equation of the Hamiltonian structure (6), that is, with the aid of a certain matrix $\Gamma = (\gamma_\nu^k)$, we consider the relation

$$\gamma_\nu^k(\dot{a}_k - G_k - \Lambda_{kj}\dot{\eta}_j) \equiv \omega_{\nu\mu}\dot{a}_\mu - \frac{\partial H}{\partial a_\nu} - \omega_{\nu k}\theta^{kj}\dot{\eta}_j, \tag{7}$$

or

$$C_{\nu k}\dot{a}_k - D_\nu(a,t) - \gamma_\nu^k\Lambda_{kj}\dot{\eta}_j) \equiv \omega_{\nu\mu}\dot{a}_\mu - \frac{\partial H}{\partial a_\nu} - \omega_{\nu k}\theta^{kj}\dot{\eta}_j, \tag{7'}$$

where $C_{\nu k} = \gamma_\nu^k; \quad D_\nu(a,t) = \gamma_\nu^k G_k$.

To satisfy the identity (7), it is required the fulfillment of conditions

$$C_{\nu k} = \omega_{\nu k}, D_\nu(a,t) = -\frac{\partial H}{\partial a^\nu}, \tag{8}$$

$$\gamma_\nu^k\Lambda_{kj} = \omega_{\nu k}\theta^{kj}, (\nu, k = \overline{1,2n}, j = \overline{1,n+m}); \tag{9}$$

$$\gamma_\nu^k = \omega_{\nu k}. \tag{10}$$

From (9) and (10), it follows that the equality

$$\sigma_{kj} = \sigma'_{kj}, \ (k = \overline{1,n}, j = \overline{1,m}) \tag{11}$$

takes place.

Hence, we have

Theorem 1. The indirect representation of the stochastic equation (3) in the form of the stochastic Hamilton equation (6) is possible if and only if conditions (8), (10), (11) are satisfied.

Remark. To construct the Hamilton function, which determines the form of equation (6), it is necessary to check the Helmholtz conditions for the given equation, which, following R.M. Santilli [14], represent the next relations:

$$C_{\mu\nu} + C_{\nu\mu} = 0; \tag{12}$$

$$\frac{\partial C_{\mu\nu}}{\partial a_\tau} + \frac{\partial C_{\nu\tau}}{\partial a_\mu} + \frac{\partial C_{\tau\mu}}{\partial a_\nu} = 0; \tag{13}$$

$$\frac{\partial C_{\mu\nu}}{\partial t} = \frac{\partial D_\mu}{\partial a_\nu} - \frac{\partial D_\nu}{\partial a_\mu}. \tag{14}$$

Construction of the Birkhoffian in an indirect representation. We consider the stochastic Helmholtz problem in the following formulation: it is required to construct a stochastic equation of the Birkhoffian structure of the form

$$\left[\frac{\partial R_\nu(a,t)}{\partial a_\mu}-\frac{\partial R_\mu(a,t)}{\partial a_\nu}\right]\dot{a}_\nu-\left[\frac{\partial B(a,t)}{\partial a_\mu}+\frac{\partial R_\mu(a,t)}{\partial t}\right]=T_{\mu j}\dot{\eta}_j, \tag{15}$$

by the given equation

$$C_{\mu\nu}\dot{a}_\nu-D_\nu(a,t)=\Lambda_{\mu j}\dot{\eta}_j,\quad (\mu,\nu=\overline{1,2n}), \tag{16}$$

where $B=B(a,t)$ is called the Birkhoff function, and $W=(W_{\mu\nu})$ is the Birkhoff tensor [15] with components $W_{\mu\nu}=\dfrac{\partial R_\nu(a,t)}{\partial a_\mu}-\dfrac{\partial R_\mu(a,t)}{\partial a_\nu}$.

The Birkhoff system (15) is a direct generalization of Hamilton systems (6). Indeed, when $a_\nu=q_\nu$ $(\nu=\overline{1,n})$, $a_\nu=p_{\nu-n}$ $(\nu=\overline{n+1,2n})$; $R_\nu=p_\nu$ $(\nu=\overline{1,n})$, $R_\nu=0$ $(\nu=\overline{n+1,2n})$; $B(a,t)=H(q,p,t)$ equation (15) takes the form of the canonical equations (6).

To solve this problem, we consider the relation

$$C_{\mu\nu}\dot{a}_\nu-D_\nu(a,t)-\Lambda_{\mu j}\dot{\eta}_j\equiv\left(\frac{\partial R_\nu(a,t)}{\partial a_\mu}-\frac{\partial R_\mu(a,t)}{\partial a_\nu}\right)\dot{a}_\nu-\left(\frac{\partial B(a,t)}{\partial a_\mu}+\frac{\partial R_\mu(a,t)}{\partial t}\right)-T_{\mu j}\dot{\eta}_j,$$

which is fulfilled identically under the following conditions:

$$C_{\mu\nu}(a,t)=\frac{\partial R_\nu(a,t)}{\partial a_\mu}-\frac{\partial R_\mu(a,t)}{\partial a_\nu}; \tag{17}$$

$$D_\nu(a,t)=\frac{\partial B(a,t)}{\partial a_\mu}+\frac{\partial R_\mu(a,t)}{\partial t}; \tag{18}$$

$$\Lambda_{\mu j}=T_{\mu j}. \tag{19}$$

Consequently, we have

Theorem 2. The direct representation of the stochastic equation (16) in the form of the stochastic Birkhoff equation (15) is possible if and only if conditions (17)–(19) are satisfied.

Indirect representation of the stochastic Hamilton equation in the form of a stochastic Birkhoff equation. We consider the problem of indirect construction of the Birkhoff equation (15) from a given Hamiltonian equation (6) in the presence of random perturbations.

In other words, we will define R_ν and B on given functions H and h_μ^α so that the relation

$$h_\mu^\alpha\left[\omega_{\alpha\beta}\dot{a}_\beta-\frac{\partial H}{\partial a_\alpha}-T^{\alpha j}\dot{\eta}_j\right]=\left(\frac{\partial R_\nu(a,t)}{\partial a_\mu}-\frac{\partial R_\mu(a,t)}{\partial a_\nu}\right)\dot{a}_\nu-\left(\frac{\partial B(a,t)}{\partial a_\mu}+\frac{\partial R_\mu(a,t)}{\partial t}\right)-T'_{\mu j}\dot{\eta}_j \tag{20}$$

be satisfied. The relation (20) turns into an identity when the relations

$$h_\mu^\alpha\omega_{\alpha\nu}=\frac{\partial R_\nu(a,t)}{\partial a_\mu}-\frac{\partial R_\mu(a,t)}{\partial a_\nu}; \tag{21}$$

$$h_\mu^\alpha\frac{\partial H}{\partial a_\alpha}=\frac{\partial B(a,t)}{\partial a_\mu}+\frac{\partial R_\mu(a,t)}{\partial t}; \tag{22}$$

$$h_\mu^\alpha T^{\alpha j}=T'_{\mu j}. \tag{23}$$

are performed.

Consequently, we have

Theorem 3. The indirect representation of the stochastic Hamilton equation (6) in the form of a stochastic Birkhoff equation (15)is possible if and only if there exist $4n^2$ functions h_μ^α such that conditions (21)–(23) are satisfied for given functions H, R_μ, B, T, T'.

Birkhoffian action in the stochastic Helmholtz problem. The Helmholtz problem in the class of Langevin-Ito stochastic differential equations is divided into two interrelated problems. At the first stage, a stochastic analog of the Lagrange, Hamilton or Birkhoff equations is constructed from the given equation. And further, in the second stage, the required functional (Hamiltonian or Birkhoffian action) must be constructed from the constructed L, H or B with R.

In this section, one of the options for constructing a stochastic analog of the Birkhoff action is considered.

Let us consider the stochastic equation of the Lagrangian structure

$$\frac{d}{dt}(\frac{\partial L}{\partial \dot{q}_\nu}) - \frac{\partial L}{\partial q_\nu} = \sigma_{\nu j}(q,\dot{q},t)\dot{\xi}^j, \quad (\nu = \overline{1,n}), \tag{24}$$

which is assumed, following the works [23–25], constructed in direct or indirect representation by the given equation (1).

Then the averaged Lagrangian ML will satisfy [23] the following equation

$$\frac{d}{dt}(\frac{\partial ML}{\partial \dot{q}_\nu}) - \frac{\partial ML}{\partial q_\nu} = 0.$$

From the function ML by the Legendre transform, we define the averaged Hamiltonian $\tilde{H} = p_i q_i - ML$, which generates the following canonical equation

$$\begin{cases} \dfrac{dq_i}{dt} = \dfrac{\partial \tilde{H}}{\partial p_i}; \\ \dfrac{dp_i}{dt} = -\dfrac{\partial \tilde{H}}{\partial q_i}, \quad (i = \overline{1,n}), \end{cases}$$

or in variables $a = (a_1, a_2, ..., a_{2n})$ the following canonical equation of the type of equation (6)

$$\omega_{\mu\nu}\dot{a}_\nu - \frac{\partial \tilde{H}}{\partial a_\nu} = 0. \tag{24$'$}$$

Further, on the basis of Theorem 3 by the equation (24'), we construct a set $(\tilde{R}, \tilde{B})$ generating the Birkhoff equation (25)

$$\left[\frac{\partial \tilde{R}_\nu(a,t)}{\partial a_\mu} - \frac{\partial \tilde{R}_\mu(a,t)}{\partial a_\nu}\right] a_\nu - \left[\frac{\partial \tilde{B}(a,t)}{\partial a_\mu} + \frac{\partial \tilde{R}_\mu(a,t)}{\partial t}\right] = 0, \tag{25}$$

which is equivalent to the indirect Hamilton equation (26)

$$h^\alpha_\mu \left(\omega_{\alpha\beta}\dot{a}_\beta - \frac{\partial \tilde{H}}{\partial a_\alpha}\right) = 0 \tag{26}$$

under the conditions (27), (28)

$$h^\alpha_\mu \omega_{\alpha\nu} = \frac{\partial \tilde{R}_\nu(a,t)}{\partial a_\mu} - \frac{\partial \tilde{R}_\mu(a,t)}{\partial a_\nu}; \tag{27}$$

$$h^\alpha_\mu \frac{\partial \tilde{H}}{\partial a_\alpha} = \frac{\partial \tilde{B}(a,t)}{\partial a_\mu} + \frac{\partial \tilde{R}_\mu(a,t)}{\partial t}. \tag{28}$$

Then the functional taking the stationary value on the solutions of equation (1) is constructed in the form of an average Birkhoffian action in the form

$$\tilde{S} = \int_{t_1}^{t_2} \left[\tilde{R}_\nu(a,t)\dot{a}_\nu - \tilde{B}(a,t)\right] dt.$$

Examples. We will consider the problem of constructing the Hamiltonian and Birkhoff functions for specific stochastic equations using the statements proved above.

Example 1. Let us consider the plane motion of a symmetric satellite along a circular orbit under the assumption of a pitch change under the influence of gravitational forces and aerodynamic forces [26, 27]

$$\ddot{\theta} = f(\theta, \dot{\theta}) + \sigma(\theta, \dot{\theta})\dot{\xi}, \tag{29}$$

where θ is the pitch angle and the functions f and σ have the form

$$f = Ml\sin 2\theta - M[g(\theta) + \eta\dot{\theta}], \quad \sigma = M\delta[g(\theta) + \eta\dot{\theta}].$$

In work [23], the problem of indirect construction of the Lagrangian on the given equation (29) was considered

$$h[\ddot{\theta} - f(\theta,\dot{\theta}) - \sigma(\theta,\dot{\theta})\dot{\xi}] = 0 \tag{29'}$$

at $h = e^{-Q\eta t}$. The required Lagrangian for (29$'$) was constructed in the form

$$L = e^{-Q\eta t}[\frac{1}{2}\dot{\theta}^2 - Q(\frac{1}{2}l\cos 2\theta + G)], \quad \text{where} \quad G = \int g(\theta)d\theta, \tag{30}$$

which provides a representation (29$'$) in the form of an equation of the Lagrangian structure

$$\frac{d}{dt}\frac{\partial L}{\partial\dot{\theta}} - \frac{\partial L}{\partial\theta} = e^{-Q\eta t}\sigma(\theta,\dot{\theta})\dot{\xi}. \tag{31}$$

We will consider the problem of indirect construction of the Hamiltonian according to given equation (29). Namely, using the Lagrange function (30) and the Legendre transformation, we will define the Hamilton function in the form

$$H = \chi\dot{\theta} - L(\theta,\dot{\theta},t)\Big|_{\dot{\theta}=\dot{\theta}(\theta,\chi,t)}.$$

And since $\chi = \dfrac{\partial L}{\partial\dot{\theta}}$, then $\chi = e^{-Q\eta t}\dot{\theta}$, and, consequently, $\dot{\theta} = e^{Q\eta t}\chi$. Then the canonical equation corresponding to equation (31) will take the form

$$\begin{cases} \dot{\theta} = \dfrac{\partial H}{\partial\chi}; \\ \dot{\chi} = -\dfrac{\partial H}{\partial\theta} + \tilde{\sigma}(\theta,\chi,t)\dot{\xi}, \end{cases} \tag{32}$$

where $\tilde{\sigma} = \sigma'(\theta,\dot{\theta},t)\Big|_{\dot{\theta}=\dot{\theta}(\theta,\chi,t)}$, and the Hamilton function is defined in the form

$$H = \frac{1}{2}e^{Q\eta t}\chi^2 - e^{-Q\eta t}\beta(\theta). \tag{33}$$

To solve the problem of the indirect representation of the Birkhoffian for a given equation (29), we will use Theorem 3. By the equation (32) constructed above and the Hamilton function (33) from relation (20) with $(h_{\mu\nu}) = \begin{pmatrix} h & 0 \\ 0 & h \end{pmatrix}$, functions $R_\mu(\mu = 1,2)$ and B are defined in the follow form $R_\mu = \{\chi, (1+h)\theta\}$, $B = \dfrac{1}{2}he^{Q\eta t}\chi^2 - he^{-Q\eta t}\beta(\theta)$, where h is an arbitrary constant.

Example 2. Let us consider a second-order nonlinear differential equation describing the motion of the inner ring of a gyroscope in a gimbal [28]

$$\ddot{\beta} + 2\nu\dot{\beta} + f(\beta) = \dot{\xi}, \tag{34}$$

where β is the angle of rotation of the inner ring. Here there is the coefficient at white noise $\sigma = 1$.

In [23], the problem of indirect construction of the Lagrangian on the given equation (34)

$$h[\ddot{\beta} + 2\nu\dot{\beta} + f(\beta)] = \dot{\xi} \tag{34'}$$

was considered at $h = e^{2\nu t}$. And the required Lagrangian for (34$'$) was constructed in the form

$$L = e^{2\nu t}[\frac{1}{2}\dot{\beta}^2 - \gamma(\beta)], \quad \text{where} \quad \frac{d}{dt}\gamma(\beta) = f(\beta),$$

providing the representation (34$'$) in the form of an equation of Lagrangian structure

$$\frac{d}{dt}(\frac{\partial L}{\partial\dot{\beta}}) - \frac{\partial L}{\partial\beta} = e^{2\nu t}\dot{\xi}. \tag{35}$$

Similarly to Example 1, we consider the problem of the indirect representation of the Hamiltonian and the Birkhoffian. We will define the Hamilton function by the Legendre transformation of the Lagrange function

$$H = \chi\dot{\beta} - L(\beta,\dot{\beta},t)\Big|_{\dot{\beta}=\dot{\beta}(\beta,\chi,t)} = \frac{1}{2}e^{-2\nu t}\chi^2 + e^{2\nu t}\gamma(\beta),$$

which generates the stochastic Hamilton equation of the form

$$\begin{cases} \dot{\theta} = \dfrac{\partial H}{\partial \chi}; \\ \dot{\chi} = -\dfrac{\partial H}{\partial \theta} + e^{2\nu t}\dot{\xi}, \end{cases} \tag{36}$$

that is equivalent to the Lagrange equation (35). Further, according to the equation (36) and the relation (20) with $(h_{\mu\nu}) = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}$, functions $R_\mu (\mu = 1,2)$ and B are defined in the follow form

$$R_\mu = \{\chi, (1+\alpha)\beta\}, \quad B = \frac{1}{2}\alpha e^{-2\nu t}\chi^2 + \alpha e^{2\nu t}\gamma(\beta),$$

where α is an arbitrary constant.

In particular, the unknown functions take the form $R_\mu = \{\chi, 2\beta\}$, $B \equiv H$ at $\alpha = 1$.

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М.Ы. Тілеубергенов, Д.Т. Әжімбаев

Биркгоф жүйелер үшін стохастикалық Гельмгольц есебі

Ықтималдық қойылымда Гельмгольц есебі қарастырылды. Тура емес көрсетілуде берілген Ланжевен-Ито стохастикалық теңдеуі бойынша Гамильтон және Биркгоф құрылымды теңдеулер тұрғызылды. Моменттік функциялар әдісі арқылы берілген стохастикалық Биркгоф теңдеуінің шешімдерінде стационарлық мән қабылдайтын функционал анықталды. Алынған нәтижелер екі мысалда суреттеледі: 1) тартылыс күші мен аэродинамикалық күш әсерімен айналмалы орбитада симметриялық спутниктің жазық қозғалысы және 2) ішкі сақинаның ілу осі бойынша күштердің стохастикалық флуктуалатын сәтінен туындаған кардан аспадағы гироскоптың флуктуациялық қозғалысы.

Кілт сөздер: стохастикалық Ланжевен-Итоның теңдеуі, кері есебі, Гамильтон (немесе Биркгоф) құрылымды теңдеуі.

М.И. Тлеубергенов, Д.Т. Ажымбаев

Стохастическая задача Гельмгольца для систем Биркгофа

Рассмотрена задача Гельмгольца в вероятностной постановке. По заданному стохастическому уравнению Ланжевена-Ито в непрямом представлении строится как уравнение гамильтоновой структуры, так и уравнение биркгофиановой структуры. Методом моментных функций определяется функционал, принимающий стационарное значение на решениях заданного стохастического уравнения Биркгофа. Полученные результаты иллюстрируются на двух примерах: 1) плоское движение симметричного спутника по круговой орбите под действием сил тяготения и аэродинамических сил и 2) флуктуационное движение гироскопа в кардановом подвесе, вызванное стохастическим флуктуирующим моментом сил по оси подвеса внутреннего кольца.

Ключевые слова: стохастическое уравнение Ланжевена-Ито, обратная задача, уравнение Гамильтоновой (или Биркгофиановой) структуры.

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A.R. Yeshkeyev, A.K. Issayeva

Ye.A. Buketov Karaganda State University, Kazakhstan
(E-mail: modth1705@mail.ru)

∇-cl-atomic and prime sets

In this article the model-theoretic properties of special formula subsets of the semantic model of some fixed Jonsson theory are considered. The main purpose of this paper is the study of concepts of models' primeness and atomness in the study of inductive theories which admit the property of joint embedding and amalgama property. For this purpose is determined special sets, each element of which realise some type which is which is the main type in the sense of an existential formulas. Definable closures of such sets form an existential closed model. The main result obtained in this paper describes the properties of atomic and prime sets regarding strongly convex Jonsson theories.

Keywords: strongly convex theory, center of Jonsson theory, semantic model, atomic set, algebraically prime set, core set.

In the well-known paper [1], R. Vaught have proved the fundamental theorem-criterion on the behavior of countable prime and atomic models for complete theories in countable language. The essence of this criterion is that in a complete theory any countable-prime model is at the same time an atomic model of this theory. After some time A. Robinson in [2] have defined the concept of an algebraically prime model. This concept is a generalization of the concept of a prime model. Further, in the well-known work [3], D. Baldwin and D. Kueker considered the concept of new types of atomicity of a countable model. Naturally, appear the question about an analogue of Vaught's theorem for an algebraically prime model. We denote this problem by AAP (atomicity & algebraically primeness).

After some time, A. Robinson in [2] defined the concept of an algebraically prime model, and this concept is a generalization of the concept of a prime model. Further, in the well-known paper [3], J.T. Baldwin and D.W. Kueker considered the concept of new types of atomic of a countable model. Obviously, the question arose about the analogue of the theorem of Vaught for an algebraically prime model. We denote this problem symbolically by AAP (atomicness & algebraically primeness). Unfortunately, in [3], the authors were unable to obtain a criterion for an algebraically prime model in the language of new types of atomicity; moreover, a sufficient number of examples given in this paper suggests that this issue is unlikely to be resolved positively, i.e. a criterion or some conditions connecting the concepts of algebraic simplicity and the corresponding form of atomicity from [3] are obtained.

In this paper, we transfer the main ideas from [3] to countable models of some fixed Jonsson theory. Interest in the study of Jonsson theories is due to the following factors. Firstly, the class of Jonsson theories contains a sufficient number of well-known classical examples of algebras that are widely used in various sections of mathematics. For example, to Jonsson theories we can relate the theory of groups, Abelian groups, a large number of different types of rings, in particular, fields of fixed characteristic, also linear orders and Boolean algebras and such universal object as polygons over a monoid or S-actions, where S is a monoid. Secondly, arbitrary Jonsson theory is, generally speaking, not complete, and since the technical apparatus of the modern Model theory is adapted for the study of complete theories, the conditions that determine the jonssonness, naturally, distinguish among all, generally speaking, incomplete theories, which more or less adapted to the model-theoretic study of the class of theories. Nevertheless, some completeness of the considered Jonsson theory is necessary and, as a rule, it does not exceed $\forall$, $\exists$ or $\forall\exists$ completeness. Thirdly, when studying Jonsson theories, an important role is played the types of morphisms, with the help of which the classes of models of these theories are studied. If in the case of a complete theory, we are dealing with elementary monomorphisms (embeddings or extensions), then in the case of a Jonsson theory we will deal with an isomorphic and homomorphic morphisms (embeddings or extensions). When studying the Jonsson theories, we distinguish some special subclasses in which the behavior of countable models is more predictable with respect to the AAP problem. These are the following classes of theory: the class of convex theories defined by A. Robinson [2] and the class of existentially prime theories [4].

Studing the latest results of the modern model theory, it became clear, that a model-theoretic approach to the study of formula-definable subsets of some considered model is great importance. For complete theories, this model is associated with the monster model; in the Jonsson's case, analog of theory is the semantic model of considered theory. In this article, we will consider special formula subsets that, on the one hand, define atomicity in the sense of [3], but, on the other hand, firstly, give some geometrical interpretation in the sense of pregeometry given on the Boolean of semantic model, secondly, gives a new tool for study of the corresponding type of atomicity. So in this paper, we continue to investigate the AAP problem within the above paper and restrictions. We give the necessary definitions and related ones for further paper in this article.

We give the definitions [5] and related results necessary for further work in this article. Recall that

Definition 1. A theory T is Jonsson if:

1) theory T has infinite models;

2) theory T is inductive;

3) theory T has the joint embedding property (JEP);

4) theory T has the property of amalgam (AP).

Examples of Jonsson theories are:

1) group Theory;

2) theory of Abelian groups;

3) theory of fields of fixed characteristics;

4) theory of Boolean algebras;

5) theory of polygons over a fixed monoid;

6) theory of modules over a fixed ring;

7) theory of linear order.

When studying the model-theoretic properties of Jonsson theory, the semantic method plays an important role. It consists in the following: the elementary properties of the center of Jonsson theory are in a certain sense associated with the corresponding first-order properties of Jonsson theory itself. The center of Jonsson theory is a syntactic invariant and its properties are well defined in the case when Jonsson theory is perfect. The following concepts define the essence of the semantic model and the center of Jonsson theory [6].

Definition 2. Let $\kappa \geq \omega$. Model M of theory T is called κ-universal for T, if each model T with the power strictly less κ isomorphically imbedded in M; κ- homogeneous for T, if for any two models A and A_1 of theory T, which are submodels of M with the power strictly less then κ and for isomorphism $f : A \to A_1$ for each extension B of model A, wich is a submodel of M and is model of T with the power strictly less then κ there is exist the extension B_1 of model A_1, which is a submodel of M and an isomorphism $g : B \to B_1$ which extends f.

Definition 3. Model C of Jonsson theory T is called semantic model, if it is ω^+-homogeneous-universal.

Definition 4. The center of Jonsson theory T is called an elementary theory of the its semantic model. And denoted through T^*, i.e. $T^* = Th(C)$.

The following two facts speak about the «good» exclusivity of the semantic model.

Fact 1 [6; 160]. Each Jonsson theory T has k^+-homogeneous-universal model of power 2^k. Conversely, if a theory T is inductive and has infinite model and ω^+-homogeneous-universal model then the theory T is a Jonsson theory.

Fact 2 [6; 160]. Let T is a Jonsson theory. Two k-homogeneous-universal models M and M_1 of T are elementary equivalent.

Definition 5. Jonsson theory T is called a perfect theory, if each a semantic model of theory T is saturated model of T^*.

The following theorem is a criterion of perfectness of Jonsson theory.

Theorem 1 [6; 158]. Let T is a Jonsson theory. Then the following conditions are equivalent:

1) Theory T is perfect;

2) Theory T^* is a model companion of theory T.

Theorem 2 [6; 162]. If T is a perfect Jonsson theory then $E_T = ModT^*$.

We will select some special subsets of the semantic model.

Definition 6. Let $X \subseteq C$. We will say that a set X is $\nabla - cl$-Jonsson subset of C, if X satisfies the following conditions:

1) X is ∇-definable set (this means that there is a formula from ∇, the solution of which in the C is the set X, where $\nabla \subseteq L$, that is ∇ is a view of formula, for example $\exists, \forall, \forall\exists$ and so on.);

2) $cl(X) = M$, $M \in E_T$, where cl is some closure operator defining a pregeometry over C (for example $cl = acl$ or $cl = dcl$).

When studying the model-theoretic properties of an inductive theory, so called existentially closed models play an important role. Recall their definitions.

Definition 7. Model A of a theory T is called existentially closed if for any model B and any existential formula $\varphi(\overline{x})$ with constants of A we have $A \models \exists\overline{x}\varphi(\overline{x})$ provided that A is a submodel of B and $B \models \exists\overline{x}\varphi(\overline{x})$.

Through E_T we denote the class of all existentially closed models of the theory T.

In connection with this definition in the frame of the study of inductive theories, the following two remarks are true:

Remark 1: For any inductive theory E_T is not empty.

Remark 2: Any countable model of the inductive theory is isomorphically embedded in some countable existentially closed model of this theory.

An analogue of a prime model (in the sense of a complete theory) for an inductive model, generally speaking, incomplete theory, is the concept of an algebraically prime model, which introduced A. Robinson [2].

Definition 8. A is an algebraically prime model of theory T, if A is a model of T and A may be isomorphically embedded in each model of the theory T.

Note that since the class of Jonsson theories of a fixed signature is a subclass of inductive theories of this signature, then the above remarks 1,2 are true for Jonssons theories and, by criterion of Jonsson theory's perfectness, class of existentially closed models of considered Jonsson theory coincides with the class of center's model of this theory.

In connection with the interest to the AAP problem in the frame of the study of Jonsson theory in [7] a new class of theories was defined, in which there is an algebraically prime model which is existentially closed.

Recall the definition of this class.

Definition 9. The inductive theory T is called the existentially prime if: 1) it has a algebraically prime model, the class of its AP (algebraically prime models) denote by AP_T; 2) class E_T non trivial intersects with class AP_T, i.e. $AP_T \bigcap E_T \neq 0$.

The following definition of a theory's convexity belongs to A. Robinson [2].

Definition 10. The theory is called convex if for any its model A and for any family $\{B_i \mid i \in I\}$ of substructures of A, which are models of the theory T, the intersection $\bigcap_{i\in I} B_i$ is a model of T, provided it is non-empty. If in addition such an itnersection is never empty, then T is called strongly convex.

The concept of a core model which introduced by A. Robinson is also an example of a particular case of an algebraically prime model.

Definition 11. A signature model of a given theory (hereinafter structure) is called core if it is isomorphic to the unique substructure of each model of the given theory. The core structure that is the model of the theory of a given signature will be called the core model of the theory.

The following result from Kueker's paper [8] gives a criterion of the existence of a core structure.

Theorem 3. For any T the following conditions are equivalent:

(1) C is a core structure for T;

(2) C is a model of every universal sentence consistent with T, and there are existential formulas $\varphi_i(x)$ and $k_i \in \omega$, for $i \in I$, such that

$$C, T \models \exists^{=k_i} x\varphi_i \quad \text{for all} \quad i \in I,$$

and

$$C \models \forall x \bigvee_{i\in I} \varphi_i.$$

The following definitions are taken from J. Baldwin and D. Kueker's work [3]. These definitions distinguish a whole class of new types of atomic models, and this new type of atomic models differs significantly from the concept of the atomic model from [1].

Definition 12. A formula $\varphi(\overline{x})$ is a Δ-formula, if exist existential formulas (from Σ) $\psi_1(\overline{x})$ and $\psi_2(\overline{x})$ such that

$$T \models (\varphi \leftrightarrow \psi_1) \quad \text{и} \quad T \models (\neg\varphi \leftrightarrow \psi_2).$$

Definition 13.

(i) $(A, a_0, ..., a_{n-1}) \Rightarrow_\Gamma (B, b_0, ..., b_{n-1})$ means that for every formula $\varphi(x_1, ..., x_{n-1})$ of Γ, if $A \models \varphi(\overline{a})$, then $B \models \varphi(\overline{b})$.

(ii) $(A, \overline{a}) \equiv_\Gamma (B, \overline{b})$ means that $(A, \overline{a}) \Rightarrow_\Gamma (B, \overline{b})$ and $(B, \overline{b}) \Rightarrow_\Gamma (A, \overline{a})$.

As classes Γ we consider $\triangle$ or Σ.

The following definition of an atomic model refers to [1].

Consider a complete theory T in L. A formula $\varphi(x_1...x_n)$ is said to be complete (in T) iff for every formula $\psi(x_1...x_n)$ exactly one of

$$T \models \varphi \to \psi, \quad T \models \varphi \to \neg\psi$$

holds. A formula $\theta(x_1...x_n)$ is said to be completable (in T) iff there is a complete formula $\varphi(x_1...x_n)$ with $T \models \varphi \to \theta$. If $\theta(x_1...x_n)$ is not completable it is said to be incompleatable.

A theory T is said to be atomic iff every formula of L which is consistent with T is completable in T. A model A is said ti be an atomic model iff every n- tuple $a_1...a_n \in A$ satisfies a complete formula in $Th(A)$

Definition 14. A model is called atomic if every tuple of its elements satisfies some complete formula. In connection with the new conceptof atomicity from [3], the following concept will be analogous to the definition of a complete formula

Definition 15. A formula $\varphi(x_1, ..., x_n)$ is complete for Γ-formulas if φ is consistent with T and for every formula $\psi(x_1, ..., x_n)$ in Γ, having no more free variables than φ, or

$$T \models \forall\overline{x}(\varphi \to \psi).$$

Equivalently, a consistent $\varphi(\overline{x})$ is complete for Γ — formulas provided whenever as $\psi(\overline{x})$ is a Γ — formula and $(\varphi \wedge \psi)$ is consistent with T, then $T \models (\varphi \to \psi)$.

And the concept of the atomic model from [1] is transformed into the following concept from [3].

Definition 16. B is a $(\Gamma_1, \Gamma_2,)$ — atomic model of T, if B is a model of T and for every n every n-tuple of elements of A satisfies some formula from B in Γ_1, which is complete for Γ_2-formulas.

The following notion of a weakly atomic model from [3] is a generalization of above definition.

Definition 17. B is a weak (Γ_1, Γ_2) — atomic model of T, if B is a model of T and for every n every n-tuple $\overline{a}$a of elements of A satisfies in B some formula $\varphi(\overline{x})$ of Γ_1 such that $T \models (\varphi \to \psi)$ as soon as $\psi(\overline{x})$ of Γ_2 and $B \models \psi(\overline{a})$.

In this paper we will not give examples of the (Γ_1, Γ_2) — atomic model and the weak (Γ_1, Γ_2) atomic model, leaving the reader to do this on their own, referring to a sufficient the number of examples of these concepts given in [3].

Before discussing the results obtained, concerning to $\nabla - cl$ atomic models, we note that we fix some Jonsson theory T and its semantic model C in the countable language L and $\nabla \subseteq L$:∇ is consistent with T, that is, any finite subset of formulas from ∇ is consistent with T. Let $A \subseteq C$.

Let cl be, as in Definition 6, and it is true that $cl = acl$ and at the same time $cl = dcl$. It is clear that such the operator is a special case of the closure operator and its example is the a closure operator defined on any linear space as a linear shell.

We also assume that the pregeometry given by the cl operator is modular [9].

Definition 18. The set A will be called $(\nabla_1, \nabla_2) - cl$ atomic in the theory T, if

1) $\forall a \in A, \exists\varphi \in \nabla_1$ such that for any formula $\psi \in \nabla_2$ follows that φ is complete formula for ψ and $C \models \varphi(a)$;

2) $cl(A) = M, M \in E_T$.

Definition 19. A set A will be called weakly $(\nabla_1, \nabla_2) - cl$ is atomic in T, if

1) $\forall a \in A, \exists\varphi \in \nabla_1$ such that in $C \models \varphi(a)$ for any formula $\psi \in \nabla_2$ follow that $T \models (\varphi \to \psi)$ whenever $\psi(x)$ of ∇_2 and $C \models \psi(a)$;

2) $cl(A) = M, M \in E_T$.

It is easy to understand that definitions 18 and 19 are naturally generalized the notion of atomicity and weak atomicity to be ∇_1 atomic and weak ∇_1 atomic for any tuple of finite length from set A.

Thus, we have generalized the concepts (Γ_1, Γ_2) of the atomic model and weakly (Γ_1, Γ_2) of the atomic model dividing in to $(\nabla_1, \nabla_2) - cl$ atomic and weakly $(\nabla_1, \nabla_2) - cl$ atomic set. Also note that the concept $(\nabla_1, \nabla_2) - cl$ atomic and weakly $(\nabla_1, \nabla_2) - cl$-atomic sets are some special modifications of definition 6.

Let $i \in \{1, 2\}$, $M_i = cl(A_i)$, where $A_i = (\nabla_1, \nabla_2)$ is a $cl-$ atomic set . $a_0, ..., a_{n-1} \in A_1$, $b_0, ..., b_{n-1} \in A_2$.

Definition 20.

(i) $(M_1, a_0, ..., a_{n-1}) \Rightarrow_\nabla (M_2, b_0, ..., b_{n-1})$ means that for every formula $\varphi(x_1, ..., x_{n-1})$ of ∇, if $M_1 \models \varphi(\overline{a})$, then $M_2 \models \varphi(\overline{b})$.

(ii) $(M_1, \overline{a}) \equiv_\nabla (M_2, \overline{b})$ means that $(M_1, \overline{a}) \Rightarrow_\nabla (M_2, \overline{b})$ and $(M_1, \overline{b}) \Rightarrow_\nabla (M_1, \overline{a})$.

Definition 21. A set A is said to be $(\nabla_1, \nabla_2) - cl$-algebraically prime in the theory T, if

1) If A is $(\nabla_1, \nabla_2) - cl$-atomic set in T;

2) $cl(A) = M, M \in AP_T$.

From the definition of an algebraically prime set in the theory T follows that the Jonsson theory T which has an algebraically prime set is automatically existentially prime. It is easy to understand that an example of such a theory is the theory of linear spaces.

Definition 22. The set A is said to be $(\nabla_1, \nabla_2) - cl$-core in the theory T, if

1) If A is (∇_1, ∇_2) a cl - atomic set in the theory T;

2) $cl(A) = M$, M is the core model of the T theory.

We formulate some obtained results regarding these new concepts.

Lemma 1. Let T be complete for existential sentences perfect Jonsson theory. 1) If A is weakly $(\nabla, \Delta) - cl$-atomic set in the theory T, then A is $(\nabla, \Delta) - cl$-atomic set, 2) If A is weak $(\nabla, \Delta) - cl$-atomic set in the theory T, then A is $(\nabla, \Delta) - cl$-atomic set.

Proof. Note, that due to the perfectness of the theory T we use theorems 1,2 and definition 19. Since $dcl(A) = M \in E_T$, then $M \in ModT^*$, where T^* is a center of T. Since the theory T is perfect, then T^* is model companion of T, and accordingly is a model complete theory. So any formula of T^* is equivalent to some Σ-formula.

It follows that any $(\nabla_1, \nabla_2) - cl$ set A is $(\Delta, \Delta) - cl$ set A. It follows that both points of Lemma 1 are satisfied.

Let $i \in \{1, 2\}$, $M_i = cl(A_i)$, where $A_i = (\Sigma, \Sigma) - cl$-is a atomic set. $a_0, ..., a_{n-1} \in A_1$, $b_0, ..., b_{n-1} \in A_2$.

Theorem 4. Let T - be complete for $\exists$-sentences a strongly convex Jonsson perfect theory and let A is $(\nabla_1, \nabla_2) - cl$-atomic set in T.

Then $(i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (iv) \wedge (vi), (i) \Rightarrow (i)^* \Rightarrow (v) \Rightarrow (vi), (ii) \Rightarrow (ii)^* \Rightarrow (vi)$, $(i)^* \Rightarrow (ii)^*$ and $(iv)^* \Rightarrow (iv)$, where:

(i) A is $(\Delta, \Sigma) - cl$-atomic set in theory T,

$(i)^*$ A is weakly $(\Delta, \Pi) - cl$-atomic set in theory T,

(ii) A is $(\Sigma, \Sigma) - cl$-atomic set in theory T,

$(ii)^*$ A is weakly $(\Sigma, \Pi) - cl$-atomic set in theory T,

(iii) A is weakly $(\Sigma, \Sigma) - cl$-atomic set in theory T,

(iv) $cl(A) \in AP_T$,

$(iv)^*$ A is core in theory T,

(v) A is weakly $(\Delta, \Delta) - cl$-atomic set in theory T,

(vi) A is weakly $(\Sigma, \Delta) - cl$-atomic set in theory T,

Lemma 2. Let A_1 will be weak $(\Sigma, \Sigma) - cl$-atomic set of T. Assume that

$$(M_1, a_0, ..., a_{n-1}) \Rightarrow_\exists (M_2, b_0, ..., b_{n-1}).$$

Then for any $a_n \in M_1$ there is some $b_n \in M_2$ such that

$$(M_1, a_0, ..., a_n) \Rightarrow_\exists (M_2, b_0, ..., b_n).$$

Proof. Let $\varphi(x_0, ..., x_{n-1})$ be existential, satisfied by $a_0, ..., a_{n-1}$ in M_1, and which imply every existential formula satisfied by M_1 $a_0, ..., a_{n-1}$. It follows from the definition 19. Let $\psi(x_0, ..., x_n)$ be satisfy for the some $a_0, ..., a_n$. Then $T \models (\varphi \rightarrow \exists x_n \psi)$ and $M_2 \models \varphi(b_0, ..., b_{n-1})$, is follows, that exists some b_n, such that $M_2 \models \psi(b_0, ..., b_n)$, and this b_n, will be what we need.

We can show, that (iii)⇒(iv). Let M_1 be countable and weak (Σ, Σ)-atomic, and let M_2 be any model of T. Then $M_1 \Rightarrow_\exists M_2$ since T is a complete theory for existential sentences, and Lemma 2 can be applied repeatedly where $A = \{a_i : i \in \omega\}$ to build step by step an embedding of A_1 into M_2.

Remark 3: By the perfectness of T, we can apply Lemma 1 and then, by Lemma 1, we can replace ∇_i on Δ, where $i \in \{1, 2\}$. Due to the strongly convexity of the theory, the theory T has a unique core model. This follows from the fact that if the theory satisfies the property of joint embedding and is additionally strongly convex, then its core model in the theory T is unique up to isomorphism [8]. Based on this fact, we can conclude that under the conditions of this theorem we have a unique core model, since its existence follows from strongly convexity, and its uniqueness follows from the combination with Jonssonness.

Proof. The only implication that is not follows directly from the definitions is $(iii) \Rightarrow (iv)$, which is a consequence of the previous Lemma 2, and $(iv) \Rightarrow (iv)^*$ follows from the remark 3.

All concepts that are not defined here can be extracted from [6].

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А.Р. Ешкеев, А.Қ. Исаева

∇-cl-атомдық және жай жиындар

Мақалада белгілі бекітілген йонсондық теориясының семантикалық моделінің арнайы формулалар ішкі жиындарының модельді-теориялық қасиеттері қарастырылған. Бұл жұмыстың негізгі мақсаты жай және атомдық модельдердің индуктивті теориялар аясында үйлесімді енгізу және амальгама қасиеттерінің түсініктері болып табылады. Бұл үшін арнайы жиындар анықталды, олардың әр элементі экзистенционалдық формулалар аясында кейбір басты типті жүзеге асырады. Осындай жиындарының анықталған тұйықталуы экзистенционалды тұйық модельді қалыптастырады. Осы мақалада алынған негізгі нәтиже салыстырмалы түрде дөңес йонсондық теориясының атомдық және жай жиынтығының қасиеттерін сипаттайды.

Кілт сөздер: қатты дөнес теория, йонсон теориясының орталығы, семантикалық модель, атомдық жиын, алгебралық жай жиын, ядролық жиын.

А.Р. Ешкеев, А.К. Исаева

∇-cl-атомные и простые множества

В работе рассмотрены теоретико-модельные свойства специальных формульных подмножеств семантической модели некоторой фиксированной йонсоновской теории. Основной целью данной работы является изучение понятий простоты и атомности моделей в раках изучения индуктивных теорий, допускающих свойства совместного вложения и свойства амальгамы. Для этой цели определяются специальные множества, каждый элемент которых реализует некоторый тип, являющийся главным в смысле экзистенциальных формул. Определимые замыкания таких множеств образуют экзистенциальную замкнутую модель. Основной результат, полученный в этой работе, описывает свойства атомных и простых множеств относительно сильно выпуклых йонсоновских теорий.

Ключевые слова: сильно выпуклая теория, центр йонсоновской теории, семантическая модель, атомное множество, алгебраически простое множество, ядерное множество.

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A.R. Yeshkeyev, M.T. Omarova

Ye.A. Buketov Karaganda State University, Kazakhstan
(E-mail: modth1705@mail.ru)

Central types of convex fragments of the perfect Jonsson theory

In this paper, the central types of convex fragments of the perfect Jonsson theory are considered. The main goal of this paper is to redefine the A.D. Taimanov's question for complete theories in the sphere of Jonsson fragments of formula-definable subsets of a Jonsson fixed theory's semantic model. Also the relationships between the center and Jonsson theory in the permissible enrichment of signature are considered. Herewith, the considered theories are hereditary. It is also assumed that an algebraic closure coincides with a definable closure. Within the framework of the above restrictions for the considered Jonsson theory, results on the existence of core models for such theories were obtained.

Keywords: Jonsson theory, semantic model, existentially prime theory, pregeometry, model companion, core model.

In the investigation of inductive theories, the study of Jonsson theories and their classes of models is one of the actual problems of the Model theory and universal algebra concurrently. This is due to the fact, that Jonsson theories are a subclass of inductive theories and the Jonsson conditions are satisfied for all known and important examples from algebra. For example, these are the theories of group, abelian groups, fields of fixed characteristic, of the modules, linear orders and Boolean algebras. The basic concepts and methods of Jonsson theories' studying were defined as the subject of research after the publication of an original B. Jonsson's works [1,2], in which he defined some conditions for algebraic systems. Jonsson conditions are natural algebraic requirements that arise when we studying a wide class of algebras. All the above examples are important both in algebra and in various areas of mathematics. As can be seen, from the listed list, the scope of application of the technique developed for the study of Jonsson theories can be quite wide.

We will adhere to the standard designations adopted in Model theory [3]. The class of inductive theories is wider than the class of Jonsson theories. As Jonsson theories are, generally speaking, uncomplete theories, this causes additional difficulties and accordingly requires the creation of their own methods of study from researchers. To date, the main of existing method's of study is a semantic method. Its essence is a checking properties of first-order sentences which belong to the Jonsson theory's center with respect to the pre-image, that is, with respect to the Jonsson theory itself. The first-order formulas subsets in the existentially closed submodels of the considered Jonsson theory's semantic model are automatically compared. Moreover, in the case of perfect theories, sufficiently a complete descriptions of such theories and their classes of models were obtained. All this is reflected in the book [2].

We have already noticed, that the class of Jonsson theories, as a proper subclass of inductive theories, is quite wide. Also in algebra there are many natural examples of algebras whose axioms satisfy the axioms of Jonsson theories. But still, it's still a fairly wide class if we take into account its uncompleteness, so if we add the convexity condition, then although the class is narrowed, we still have such classical examples as group theory, theory of Abelian groups, theory of fields of fixed characteristic and many other classical algebraic objects that are Jonsson and convex concurrently.

As a rule, we can consider two different ways to define an arbitrary Jonsson theory. In the first case, we must consider the K class of models for some signature. Next, select all Γ-sentences such that they are true in every model of this class, where Γ-is the kind of sentences in the language of this signature. For example, $\Gamma = \forall$, $\Gamma = \exists$ or their various combinations that are negotiated. The second way is a choose sets of sentences as initial or axioms. The set of sentences that is equivalent to $Th(K)$, is called the set of axioms for K. Examples are the axioms of group theory, abelian groups, full groups and torsion-free groups, etc. In both cases, the considered class must satisfy the Jonsson conditions. This can be achieved by various means. In particular, an important subclass of the considered class of models is the class of existentially closed models. The following fact is true, that the set of all $\forall\exists$-sentences which true in existentially closed models form a Jonsson theory. Therefore, both of the above methods in the obvious way by closing the subclass of existentially closed models with respect to amalgam and the property of joint embedding will determine the Jonsson theory. Also We can move from an arbitrary complete theory to the corresponding Jonsson theory using the following method: the addition of some additional symbols to the signature. As a rule, three types of symbols are used: predicates, function symbols and constants. But at the same time, the enrichment of signature must preserve the important certain theoretic model qualities of the old signature. Without less generality depending on the specific problem, such enrichments of a signature will be called admissible. For example, the requirements of preserving the stability of a theory or a type definability. We say that the conservative extension of the theory T in the language L is the theory $T' \supseteq T$ in the extended language $L' \supseteq L$ such that any model A of theory T can be enriched to model A' of theory T'. It is known that any theory T has a conservative extension T', which is inductive and admits elimination of quantifiers. It is also known that every theory T has a conservative extension that is universal and model-complete.

Moreover, it can be noted that if the complete theory is considered with infinite models, then both of these extensions for this theory are Johnsonian ones.

Thus, all of the above suggests that the study of model-theoretic properties of Jonsson theories is an important task.

We give the necessary definitions related to the Jonsson theories [4].

Definition 1. A theory T has the property of joint embedding if for any models A, B of the theory T there is a model M of the theory T and isomorphic embeddings $f : A \to M, g : B \to M$.

Definition 2. A theory T has the amalgam property if for any models A, B, C theories T and isomorphic embeddings $f_1 : A \to B, f_2 : A \to C$ there are such $M \models T$ and isomorphic embeddings $g_1 : B \to M$, $g_2 : B \to M$, such that $g_1 \circ f_1 = g_2 \circ f_2$.

A theory is inductive if it is stable with respect to the union of chains. The following theorem is known:

Theorem 1. (Cheng-Los-Sushko). A theory is stable with respect to the union of chains if and only if it is $\forall\exists$ - axiomatizable, i.e. is equivalent to the set of $\forall\exists$- sentences.

Definition 3. A theory T is Jonsson if:

1) Theory T has infinite models;
2) Theory T is inductive;
3) Theory T has the joint embedding property (JEP);
4) Theory T has the property of amalgam (AP).

As noted above, the following theories are examples of Jonsson theories:

1) groups;
2) Abelian groups;
3) Boolean algebras;
4) linear orders;
5) characteristic fields (- a prime number or zero);
6) ordered fields.

Definition 4. Jonsson theory T is called a perfect theory, if each a semantic model of theory T is saturated model of T^*.

Definition 5. Let T be a Jonsson theory. Then a companion Jonsson theory T is a theory T^* of the same signature if:

1) $(T^*)_\forall = T_\forall$;
2) for any Jonsson theory T', if $T_\forall = T'_\forall$, then $T^* = (T')^*$.
3) $T_{\forall\exists} \subseteq T^*$.

The natural interpretations of the companion $T^\#$ are T^*, T^0, T^f, T^M, T^e, where the T^0 is the Kaiser shell, the T^* is the center, the T^M is the model companion, the T^f is a finite forcing companion in the Robinson's sence, the T^e is an elementary theory of all existentially closed models' of class of T.

Definition 6. A theory is called convex if for any its model A and for any family $\{B_i \mid i \in I\}$ of substructures of A, which are models of the theory T, the intersection $\bigcap_{i\in I} B_i$ is a model of T.

Definition 7. If a theory is strongly convex, then the intersection of all its models is contained in some of its models.

This model is called a core model of this theory.

Definition 8. The inductive theory T is called an existentially-prime if:

1) It has an algebraically prime model. The class of its AP denote by AP_T;
2) Class E_T is not trivial intersects with class AP_T, i.e. $AP_T \cap E_T \neq 0$.

Since Jonsson theories are inductive, we can consider Jonsson theories which an existentially prime and then among them we can consider the convex theories. The most striking example, which is showing, that such examples a lot of is an example of group theory. This example is characterized in that it is an example of an imperfect Jonsson theory. In the case of theory of Abelian groups, we have a perfect example of a convex Jonsson theory.

Definition 9. A signature model of a given theory (structure) is called a core if it is isomorphic to an unique substructure of each model of the given theory.

The core structure that is the model of the theory of a given signature will be called the core model of theory.

Definition 10. Let $X \subseteq C$. We will say that a set X is $\nabla - cl$-Jonsson subset of C, if X satisfies the following conditions:

1) X is ∇-definable set (this means that there is a formula from ∇, the solution of which in the C is the set X, where $\nabla \subseteq L$, that is ∇ is a view of formula, for example $\exists, \forall, \forall\exists$ and so on.);

2) $cl(X) = M$, $M \in E_T$, where cl is some closure operator defining a pregeometry [5; 289] over C (for example $cl = acl$ or $cl = dcl$).

Definition 11. If (X, cl) is a Jonsson pregeometry, we say that A is Jonsson independent if $a \notin cl(A \setminus \{a\})$ for all $a \in A$ and that B is a J-basis for Y if is J-independent and $Y \subseteq acl(B)$.

Definition 12. We say that a J-pregeometry (X, cl) is J-geometry if $cl(\emptyset) = \emptyset$ and $cl(\{x\}) = \{x\}$ for any $x \in X$.

If (X, cl) is a J-pregeometry, then we can naturally define a J-geometry. Let $X_0 = X \setminus cl(\emptyset)$. Consider the relation $\sim$ on X_0 given by $a \sim b$ if and only if $cl(\{a\}) = cl(\{b\})$. By exchange, $\sim$ is an equivalence relation. Let $\widehat{X}$ be $X_0/\sim$. Define $\widehat{cl}$ on $\widehat{X}$ by $\widehat{cl}(A/\sim) = \{b/\sim : b \in cl(A)\}$.

Definition 13. Let (X, cl) be J-pregeometry. We say that (X, cl) is trivial if $cl(A) = Y_{a\in A} cl\{a\}$ for any $A \subseteq X$. We say that (X, cl) is modular if for any finite-dimensional closed $Jdim(A \cup B) = Jdim(A) + Jdim(B) - Jdim(A \cap B)$.

We say that (X, cl) is locally modular if (X, cl_a) is modular for some $a \in X$.

Definition 14. We say that (X, cl) is modular if for any finite-dimensional closed $A, B \subseteq X$

$$dim(A \cup B) = dimA + dimB - dim(A \cap B)$$

Let T — be an arbitrary Jonsson theory in the language of signature σ. Let C be the semantic model of T. $A \subseteq C$. Let $\sigma_\Gamma(A) = \sigma \cup \{c_a | a \in A\} \cup \Gamma$, where $\Gamma = \{g\} \cup \{c\} \cup \{P\}$. Consider the following theory, where $T_\Gamma(A) = Th_\nabla(C, a)_{a\in A} \cup \{g(a) = a | a \in A\} \cup g(c) \cup T_g \cup \{P(c)\} \cup \{''P \subseteq''\}$, where T_g — expresses the fact, that for any model $(M, g^M) \models T_g$ takes place:

1) g^M — automorphism M;

2) $\{m \in M \mid g^M(m) = m\}$ is a universe of some existentially closed submodel M of the semantic model C of theory T of signature σ.

There is question that is to find the condition for preserving the definability of a type with the appropriate stability obtained as a result of enriching the language.

In connection with the above question, well known are the results of T.G. Mustafin [6] and E.A. Palyutin [7] about new types of stability for complete theories.

Also in the work [7] had noted that an each of above enrichment (unary predicate, automorphism, constant) is satisfied to admissibility property that is saved the definibality of type in the frame of new kind of stability.

For the predicate P we write the expression $\{''P \subseteq''\}$, which is essentially an infinite set of sentences, this mean that an interpretation of the symbol P is an existentially closed submodel in signature σ. Due to incompleteness, we do not write down the exact connection between the elements $\Gamma = \{g\} \cup \{c\} \cup \{P\}$, but they are supposed to be consistent within the framework of the theory $T_\Gamma(A)$.

The theory $T_\Gamma(A)$ is not necessarily complete. Suppose that it is Jonsson, i.e. it has a center $T^*_\Gamma(A)$. We consider all replenishment of the center T^* of the theory T in the new signature σ_Γ, where $\Gamma = \{g\} \cup \{c\} \cup \{P\}$.

By virtue of the Jonssoness theory $T_\Gamma(A)$, T^* it will also be a Jonsson theory, generally speaking, not a complete theory, then its center, respectively, exists and we denote it by T^c. With the restriction of T^c to the signature $T_\Gamma(A)\backslash\{c\}$, the theory T^c becomes a complete type. This type is called the central type of the theory T relative to the given enrichment $T_\Gamma(A)\backslash\{c\}$.

Unfortunately, even while preserving the definability of the type, as mentioned above, not all Jonsson theories preserve their Jonssoness with allowable enrichment of the signature, for example, field theory does not necessarily admit amalgam when enriching with a single predicate, those there is a counterexample. Therefore, we will assume that the closure operator cl will define a modular geometry. We note that, in particular, the algebraic closure operator in the sense of fields does not generate a modular pregeometry.

Definition 15. If $X = C$ and (X, cl) is a modular, then the Jonsson theory T is called modular.

The following assertions will be used to prove the main assertions in the framework of the perfect Jonsson theory T regarding the connection of lattices of formulas $E_n(T)$ and the Boolean algebras $F_n(T)$.

Fact 1 [4]. For any Jonsson theory T the following conditions are equivalent:

1) T is perfect;

2) T^* is model complete.

Fact 2 [4]. For any complete for $\exists$-sentences Jonsson theory T, the following conditions are equivalent:

1) T^* is model complete;

2) for each $n < \omega$, $E_n(T)$ is the Boolean algebra, where $E_n(T)$ is a lattice of $\exists$-formulas with n free variables.

At the moment, we do not know the amalgama criterion when enriching the signature, so we will work under some assumptions.

Next we give the following definitions:

It is well known that the perfect Jonsson theory can be studied using of the first-order properties of the center of this theory and its semantic model, since the center of the Jonsson theory is the model companion of it. Imperfect Jonsson theories at the moment have not been studied. For example, a bright example of this fact is the theory of all groups. We know that this theory is Jonsson, but does not have a model companion, and the structure of its semantic model is unknown to us. In this work, in order to avoid the above mentioned situation with non amalgamation under enrichment, we will consider a special subclass of Jonsson theories, namely, hereditary Jonsson theories.

We introduce the following definitions necessary for the above purposes.

Definition 16. The Jonsson theory is said to be hereditary, if in any of its admissible enrichment any expansion of it in this enrichment will be Jonsson theory.

All considered Jonsson theories in the future under this article are assumed to be hereditary.

Let T be a perfect, complete for existential sentences, Jonsson theory of signature $\sigma_\Gamma(A)$.

Let A is ∇-cl-Jonsson subset of C, C is semantic model of the theory T, $\nabla = \forall\exists$, $cl = acl$, with $acl = dcl$ and a pregeometry generated by cl on the set of all subsets of C is modular. $cl(X) = M$, $M \in E_T$. We consider the theory $Th_{\forall\exists}(M)$, which we call the fragment of the set A and denote by $Fr(A)$.

We formulate the question of A.D. Taimanov, given that this problem was defined for complete theories.

Namely, in studying the properties of models of the first order complete theories information about the Boolean algebras (Lindenbaum-Tarsky algebras) $F_n(T)$ is usually, $n \in \omega$ [8]. In connection with these Boolean algebras $F_n(T)$, $n \in \omega$, the A.D. Taimanov's question is well known (can be found in the works [9]):

(*) What properties should have to have the Boolean algebras B_n, $n \in \omega$, to exist a complete theory T, such that B_n is isomorphic to $F_n(T)$, $n \in \omega$?

We will say that the question (*) is solved positively for the complete theory T, if there exists a sequence of Boolean algebras B_n, $n \in \omega$, such that B_n is isomorphic to $F_n(T)$, $n \in \omega$.

It is well known, that in some cases working with Jonsson theories we have the opportunity to restrict ourselves to existential formulas and existentially closed models of the considered Jonsson theory. In this case, instead of the Lindenbaum-Tarskian algebras $F_n(T)$, $n \in \omega$, one should consider lattices of existential formulas $E_n(T)$, $n \in \omega$. Thus, the above of A.D.Taimanov's question can be formulated as follows:

(**) What properties must have the lattices E_n, $n \in \omega$, that there was Jonsson theory T, such that E_n is isomorphic to $E_n(T)$, $n \in \omega$?

Similarly, we will say, that question (**) is solved positively for the Jonsson theory T, if there exists a sequence of lattices E_n, $n \in \omega$, that E_n is isomorphic to $E_n(T)$, $n \in \omega$.

In connection with these questions (*), (**) the following results were obtained:

Theorem 2. Let $Fr(A)$ be a perfect complete for existential sentences Jonsson theory of signature $\sigma_\Gamma(A)$.

Then the following conditions are equivalent:

1) a positive solution to question (**) with respect to the theory $Fr(A)^*$;

2) a positive solution to the question (*) with respect to the theory of T^c, where T^c is the center of the theory of T^*.

Proof. We prove from 1)⇒2). Suppose that question (**) has a positive solution with respect to the theory $Fr(A)^*$. Within theory $T_\Gamma(A) = Th_\nabla(C, a)_{a\in A} \cup \{g(a) = a | a \in A\} \cup g(c) \cup T_g \cup \{P(c)\} \cup \{''P \subseteq''\}$ we can see, that $Fr(A)^*$ is generally speaking uncomplete theory, but is Jonsson due to consistency with $T_\Gamma(A)$. Then it has a center, let it be equal to $(Fr(A)^*)^*$. Then by virtue of the perfect theory $Fr(A)$, its center $Fr(A)^*$ will also be perfect in the new signature $\sigma_\Gamma(A)$. Then by virtue of the perfect any formula with respect to $(Fr(A)^*)^*$ will be equivalent to some $\exists$-formula by virtue of the model completeness of the theory $(Fr(A)^*)^*$. But this theory is complete. Therefore, since the question (**) has a positive solution, the question (*) will have a positive solution, since the lattice $E_n(Fr(A)^*)$ is a Boolean algebra by virtue of facts 1,2. By the construction of $Fr(A)^*$, T^* in the new signature $\sigma_\Gamma(A)$ is also consistent with $T_\Gamma(A)$ as $Fr(A)^*$. By virtue of existentially closed of M, where M is equal to $dcl(A)$, this means, that question (**) has a positive solution concerning the theory $Fr(A)^*$ if and only if, it has a positive solution concerning the theory of T^* in the new signature $\sigma_\Gamma(A)$. It remains to apply Fact 1 and Fact 2 again regarding to T^* and T^c, i.e. by the model completeness of T^c we can conclude, that $E_n(T^*)$ is a Boolean algebra and, accordingly, the positive question (*) is solved for the theory T^c.

Now we prove from 2)$\Rightarrow$1). Suppose the opposite, i.e. question (**) has not positive solution for the theory $Fr(A)^*$. This means, that for any sequence of lattices $E_n(Fr(A)^*)$ of Jonsson theory $Fr(A)^*$, it is true, that E_n is not isomorphic to $E_n(Fr(A)^*)$, $n \in \omega$. But this is incorrect because the question (*) is satisfied for T^c and, accordingly, for $(Fr(A)^*)^*$ because of their consistency with $T_\Gamma(A)$. And since $Fr(A)^* \subseteq (Fr(A)^*)^*$, we have a contradiction with our assumption, since we can take as a sequence of lattices a sequence of Boolean algebras from a positive solution of the question (*) for $(Fr(A)^*)^*$.

Theorem 3. Let a theory T be a perfect Jonsson strongly convex theory and let it be existentially prime. A is as in the previous Theorem 1 and $Fr(A)$ is existentially prime perfect Jonsson strongly convex fragment.

Then the following conditions are equivalent:

1) $Fr(A)^*$ has a core model;

2) $Fr(A)^c$ has a core model;

3) theory T has a core model.

Proof. We prove from 1)$\Rightarrow$2). Let $Fr(A)^*$ has a core model. Since $Fr(A)^* \subseteq (Fr(A)^*)^*$, the class of models $Mod((Fr(A)^*)^*)$ of the theory $(Fr(A)^*)^*$ is contained in the class of models $Mod(Fr(A)^*)$ of the theory $(Fr(A)^*$. Well known [10; 166], that if theory T satisfies joint embedding property in addition to being strongly convex, then the core model of T is unique up to isomorphism. It follows, that $(Fr(A)^*)^*$ has a core model, since $Fr(A)^*$ и $(Fr(A)^*)^*$ are model-consistent, but $(Fr(A)^*)^* = Fr(A)^c$.

We prove from 2)$\Rightarrow$3). Let $Fr(A)^c$ has a core model K. Since $Fr(A)^*$ has the property of joint embedding due to the strong convexity of the theory T, the model K is unique up to isomorphism and by virtue of the fact, that $(Fr(A)^*)^* \subseteq T^*$ in the language of the new signature $K \in ModT$.

We prove from 3)$\Rightarrow$1). Let a theory T have a core model K. Since the theory T is strongly convex and admits the property of joint embedding, this model K is unique up to isomorphism. Since C is a semantic model of the theory T, then K is isomorphically embedded into C. Let a model $M \in E(T)$, be such that $A = dcl(M)$. We know, that the theory $Th_{\forall\exists}(M) = Fr(A)^* \subseteq Th_{\forall\exists}(C)$. Suppose, that $Fr(A)^*$ have not a core model, but due to the existential primeness of the theory T there is an algebraic prime model P and $P \in E(T)$. In the model C there is a substructure K', which is an isomorphic image of model K and which is a core substructure of the theory T. Since the theory T is perfect, $E_n(T)$ is a Boolean algebra and, $T^* = Th(C)$ is a model-complete theory, i.e. any formula in the language of the theory T^* is equivalent to some $\forall$ formula. It follows, that the core substructure T^* is the core model of T^*. But $Mod(T^*) \subseteq ModT$, i.e. $K' \in Mod(Fr(A)^*)$. Since $Fr(A)^*$ is a Jonsson theory, consistent with $T_\Gamma(A)$, follow that K' is the core substructure of the semantic model C' of the theory $Fr(A)^*$. By virtue of the perfectness and the existential primeness of $Fr(A)$ it follows, that K' is the core model of $Fr(A)$. But since C' a semantic model $Fr(A)^*$, then K' is a core model of $Fr(A)^*$.

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А.Р. Ешкеев, М.Т. Омарова

Йонсон теориясының дөңес фрагменттерінің централдық типтері

Мақалада мінсіз йонсон теориясының дөңес фрагменттерінің орталық типтері қарастырылған. Осы жұмыстың негізгі мақсаты — А.Д. Таймановтың мәселені қайта анықтау толық теориялар үшін тұрақты йонссон теориясының семантикалық моделі йонссон фрагменттері саласына формалды анықталған ішкі жиын екенін қайта анықтау. Сондай-ақ сигнатураны жеткілікті түрде байытудағы йонсондық теория мен центрдің арасындағы байланыстар қарастырылды. Бұл жағдайда зерттелген теориялар мұрагерлік болып табылады. Алгебралық тұйықталу анықталған тұйықталумен сәйкес келеді деп болжанды. Қарастырылатын йонсондық теорияларға көрсетілген шектеулер шеңберінде осындай теориялар үшін ядролық модельдердің бар болуы туралы нәтижелер алынды.

Кілт сөздер: йонсондық теория, семантикалық модель, экзистенциалды жай теория, предгеометрия, модельді компаньон, ядролық модель.

А.Р. Ешкеев, М.Т. Омарова

Центральные типы выпуклых фрагментов совершенной йонсоновской теории

В работе рассмотрены центральные типы выпуклых фрагментов совершенной йонсоновской теории. Основная цель данной работы — переопределение вопроса А.Д. Тайманова для полных теорий в сферу йонсоновских фрагментов формульно определимых подмножеств семантической модели фиксированной йонсоновской теории. Также рассмотрены связи между центром и йонсоновской теорией в допустимом обогащении сигнатуры. При этом рассматриваемые теории являются наследственными. А также предполагается, что алгебраическое замыкание совпадает с определимым замыканием. В рамках указанных выше ограничений на рассматриваемую йонсоновскую теорию получены результаты о существовании ядерных моделей для таких теорий.

Ключевые слова: йонсоновская теория, семантическая модель, экзистенциально простая теория, предгеометрия, модельный компаньон, ядерная модель.

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G.A. Yessenbayeva[1], A.N. Yesbayev[2], H. Poppell[3]

[1] Ye.A. Buketov Karaganda State University, Kazakhstan;
[2] Nazarbayev Intellectual School, Nur-Sultan, Kazakhstan;
[3] Concordia International School Shanghai, Shanghai, China
(E-mail: esenbaevagulsima@mail.ru)

On the inequality of different metrics for trigonometric polynomials

The article is devoted to the research question of inequalities for different metrics with trigonometric polynomials. The structure of this exploring, its main components and types, as well as its classical approaches are presented in this article. Nikolsky's inequalities in different metrics are well known for trigonometric polynomials. In this paper, inequalities of different metrics are proved in the Lorentz and Lebesgue spaces for trigonometric polynomials of one variable. A similar result is obtained for trigonometric polynomials of many variables. The article is focused mainly on mathematicians.

Keywords: Lebesgue spaces, Lorentz spaces, trigonometric polynomials, inequalities of different metrics.

Inequalities of different metrics play an important role among the essential attributes of mathematician's tools, exploring various mathematical structures. They are successfully used in many areas of modern theoretical and applied mathematics, so inequalities of different metrics have become an essential element of serious mathematical research, in particular research in functional analysis.

Denote by $L_p[0;2\pi)$ the space of functions $f(x)$, where the functions $f(x)$ are scalar-valued, measurable in the sense of Lebesgue on an interval $[0;2\pi)$ and integrable on $[0;2\pi)$ to the p-th degree

$$\|f\|_p = \left(\int_0^{2\pi} |f(x)|^p\, dx\right)^{\frac{1}{p}},$$

for which the quantity C is finite provided that $1 \leq p < \infty$.

As usual, it is meant that in the limiting case $p = \infty$ the functions $f \in L_\infty[0;2\pi)$ are measurable and essentially limited with a finite essential maximum [1]

$$\|f\|_\infty = \sup_{x\in[0;2\pi)} vrai\, |f(x)| < \infty.$$

A function of type

$$T_m(z) = \sum_{k=-m}^{m} c_k e^{ikz}$$

is called a trigonometric polynomial of order m, where c_k $(k = -m, ..., m)$ are complex numbers and C is a variable.

The function

$$D_m(x) = \frac{1}{2} + \sum_{k=1}^{m} \cos kx = \frac{\sin\left(m+\frac{1}{2}\right)x}{2\sin\frac{x}{2}}$$

is called the Dirichlet kernel.

Lorentz spaces $L_{p,q}$ are a more subtle scale of spaces than the scale of Lebesgue spaces L_p, and have a great use in the theory of Fourier series, in differential equations, in the theory of functional spaces.

Consider a space with a positive measure (U,μ). For a scalar-valued μ-measurable function f, which takes almost everywhere finite values, we introduce the distribution function $m(\sigma, f)$ by the formula

$$m(\sigma, f) = \mu\left\{x : |f(x)| > \sigma\right\}.$$

For every measurable function f, we denote its non-increasing permutation by f^*, if f^* is defined by the following relation

$$f^*(t) = \inf \{\sigma : m(\sigma, f) \leq t\}.$$

Lorentz spaces $L_{p,q}$ are defined as follows: a function f belongs to the space $L_{p,q}$, $1 \leq p \leq \infty$, if and only if

$$\|f\|_{L_{p,q}} = \left(\int_0^\infty \left(t^{\frac{1}{p}} \cdot f^*(t) \right)^q \frac{dt}{t} \right)^{\frac{1}{q}} < \infty,$$

when $1 \leq q < \infty$,

$$\|f\|_{L_{p,\infty}} = \sup_t t^{\frac{1}{p}} \cdot f^*(t) < \infty,$$

when $q = \infty$ [2].

In the case $p = q$, the Lorentz spaces $L_{p,q}$ coincide with Lebesgue spaces L_p

$$L_{p,p} = L_p, \quad 1 \leq p \leq \infty.$$

Let E be a normalized functional space whose elements are defined up to equivalence with respect to Lebesgue measure. In other words, the elements of E are classes of equivalent functions, that is, almost everywhere coinciding functions. In the record $f \in E$, under the f designation we shall mean either a class of equivalent functions, or some function (a representative) of this class.

For the function $f \in E$, defined on the set $G \subset E$, the restriction f to $G^* \subset G$ is the function $f^* = f\,|_{G^*}$, defined on G^* by the equality

$$f^*(x) = f(x) \quad \forall x \in G^*.$$

Let E and F be two functional spaces. We say that E is embedded in F and write $E \subset_{\rightarrow} F$ if, firstly, all the elements of E (or of their restrictions to the domain of the elements of F) are contained in F and, secondly, there is a constant C independent of f such that the following inequality holds

$$\|f\|_F \leq C \|f\|_E \quad \forall f \in E.$$

Theorem. Let $m \in \mathrm{N}$, $1 \leq p < q \leq \infty$, T_m be a trigonometric polynomial of order m, then the following inequality of different metrics holds

$$\|T_m\|_{L_{q,1}} \leq C \cdot m^{\frac{1}{p} - \frac{1}{q}} \cdot \|T_m\|_{L_{p,\infty}}, \tag{1}$$

where the parameter C is independent of m and T_m.

Proof. Applying the Jung – O.Neil inequality [3]

$$\|f * g\|_{L_{q,s}} \leq C \cdot \|f\|_{L_{p,t_1}} \cdot \|g\|_{L_{r,t_2}},$$

where

$$\frac{1}{s} = \frac{1}{t_1} + \frac{1}{t_2}, \qquad \frac{1}{p} + \frac{1}{r} = \frac{1}{q} + 1, \tag{2}$$

and putting in it

$$s = 1, \quad t_1 = \infty, \quad t_2 = 1,$$

we receive the inequality

$$\|f * g\|_{L_{q,1}} \leq C \cdot \|f\|_{L_{p,\infty}} \cdot \|g\|_{L_{r,1}}. \tag{3}$$

Since

$$S_M(f) = f * D_M; \tag{4}$$

$$(D_M)^*(t) \leq C \cdot \min \left(\frac{1}{t}, M \right), \tag{5}$$

where S_M is the partial sum of the Fourier series, D_M is the Dirichlet kernel, and $(D_M)^*(t)$ is a non-increasing permutation [4], then denoting

$$g = D_M$$

and using estimate (5), we obtain the following relation

$$\|g\|_{L_{r,1}} = \int\limits_0^\infty t^{\frac{1}{r}} \cdot g^*(t)\frac{dt}{t} \le \int\limits_0^\infty t^{\frac{1}{r}} \cdot \min\left(\frac{1}{t},\, M\right)\frac{dt}{t} =$$

$$= \int\limits_0^{\frac{1}{M}} t^{\frac{1}{r}} \cdot \min\left(\frac{1}{t},\, M\right)\frac{dt}{t} + \int\limits_{\frac{1}{M}}^\infty t^{\frac{1}{r}} \cdot \min\left(\frac{1}{t},\, M\right)\frac{dt}{t}. \tag{6}$$

Taking into account that

$$\min_{0\le t\le \frac{1}{M}}\left(\frac{1}{t},\, M\right) = M, \quad \min_{\frac{1}{M}\le t\le\infty}\left(\frac{1}{t},\, M\right) = \frac{1}{t},$$

we transform the relation (6) to the form

$$\|g\|_{L_{r,1}} \le M \cdot \int\limits_0^{\frac{1}{M}} t^{\frac{1}{r}-1}dt + \int\limits_{\frac{1}{M}}^\infty t^{\frac{1}{r}-2}dt. \tag{7}$$

Since we have a condition $p < q$, from the relations (2) it follows that

$$\frac{1}{r} - 1 = \frac{1}{q} - \frac{1}{p} < 0,$$

and then the ratio (7) is converted to the following form

$$\|g\|_{L_{r,1}} \le M \cdot \int\limits_0^{\frac{1}{M}} t^{\frac{1}{r}-1}dt + \int\limits_{\frac{1}{M}}^\infty t^{\frac{1}{r}-2}dt = M \cdot \left.\frac{t^{\frac{1}{r}}}{\frac{1}{r}}\right|_0^{\frac{1}{M}} + \left.\frac{t^{\frac{1}{r}-1}}{\frac{1}{r}-1}\right|_{\frac{1}{M}}^\infty =$$

$$= rM \cdot \left(\frac{1}{M}\right)^{\frac{1}{r}} - \frac{r}{1-r}\left(\frac{1}{M}\right)^{\frac{1}{r}-1} =$$

$$= rM^{1-\frac{1}{r}} - \frac{r}{1-r}M^{1-\frac{1}{r}} = \frac{r^2}{r-1} \cdot M^{1-\frac{1}{r}} = C \cdot M^{1-\frac{1}{r}}$$

or

$$\|g\|_{L_{r,1}} \le C \cdot M^{1-\frac{1}{r}} = C \cdot M^{\frac{1}{p}-\frac{1}{q}}. \tag{8}$$

Using (4) and (8), from (3) we have an inequality of the form

$$\|f * g\|_{L_{q,1}} = \|S_M(f)\|_{L_{q,1}} \le C \cdot M^{\frac{1}{p}-\frac{1}{q}} \cdot \|f\|_{L_{p,\infty}}.$$

The last inequality holds for any functions$f \in L_{p,\infty}$.

Therefore, if we take f as a trigonometric polynomial of order m, that is, $f = T_m$, and, taking into account that $S_m(T_m) = T_m$, we obtain the sought-for inequality

$$\|T_m\|_{L_{q,1}} \le C \cdot m^{\frac{1}{p}-\frac{1}{q}} \cdot \|T_m\|_{L_{p,\infty}}.$$

The theorem is proved.

Remark. We note that the inequality of different metrics from the theorem is more accurate than the classical Nikolsky's inequality of different metrics for trigonometric polynomials of order m [1]

$$\|T_m\|_{L_q} \le C \cdot m^{\frac{1}{p}-\frac{1}{q}} \cdot \|T_m\|_{L_p}, \tag{9}$$

because

$$L_{q,1} \subset\!\!\to L_{q,q} = L_q, \quad L_p = L_{p,p} \subset\!\!\to L_{p,\infty},$$

that is,

$$\|T_n\|_{L_q} \le C \cdot \|T_n\|_{L_{q,1}}, \qquad \|T_m\|_{L_{p,\infty}} \le C \cdot \|T_m\|_{L_p}.$$

Corollary. Let $m \in \mathrm{N}$, $1 \le p < q \le \infty$, $1 \le r \le \infty$, then

$$\|T_m\|_{L_{q,r}} \le C \cdot m^{\frac{1}{p}-\frac{1}{q}} \cdot \|T_m\|_{L_p}, \tag{10}$$

where the parameter C is independent of m and T_m.

Proof. Since, under condition $r < r_1$, the relation is satisfied for Lorentz spaces

$$\|f\|_{L_{p,r}} \le C \cdot \|f\|_{L_{p,r_1}},$$

then from (1) we obtain (10)

$$\|T_m\|_{L_{q,r}} \le \|T_m\|_{L_{q,1}} \le C \cdot m^{\frac{1}{p}-\frac{1}{q}} \cdot \|T_m\|_{L_{p,\infty}} \le$$

$$\le C \cdot m^{\frac{1}{p}-\frac{1}{q}} \cdot \|T_m\|_{L_{p,p}} \le C \cdot m^{\frac{1}{p}-\frac{1}{q}} \cdot \|T_m\|_{L_p}.$$

The required inequality is proved.

S.M. Nikolsky has obtained for any trigonometric polynomial from R^n the inequality, similar to the relation (9),

$$\|T_{m_1 \dots m_n}\|_q \le 2^n \cdot \left(\prod_{k=1}^{n} m_k\right)^{\frac{1}{p}-\frac{1}{q}} \cdot \|T_{m_1 \dots m_n}\|_p, \quad 1 \le p < q \le \infty,$$

where an arbitrary trigonometric polynomial of order $m_1, ..., m_n$ with variables $x_1, ..., x_n$ can be written in the form

$$T_{m_1 \dots m_n}(x_1, ..., x_n) = \sum_{k_1=-m_1}^{m_1} ... \sum_{k_m=-m_n}^{m_n} c_{k_1 \dots k_n} e^{i \sum\limits_{s=1}^{n} k_s x_s},$$

and L_p^* is the space of functions $f(x_1, ..., x_n)$, which are measurable in R^n, periodic with period 2π with respect to each of the variables x_i and integrable to the p-th power ($1 \le p < \infty$) on the period. Thus, for each function $f \in L_p^*$ the following relation takes place [1]

$$\|f\|_p^* = \left(\int_0^{2\pi} ... \int_0^{2\pi} |f(x_1, ..., x_n)|^p \, dx_1 ... dx_n\right)^{\frac{1}{p}} < \infty,$$

in the case of $p = \infty$, we have

$$\|f\|_\infty^* = \sup_{x_i} vrai \, |f(x_1, ..., x_n)|.$$

After conducting a similar proof, for any trigonometric polynomial in R^n, we obtain inequalities similar to relations (1) and (10),

$$\|T_{m_1 \dots m_n}\|_{L_{q,1}^*} \le 2^n \cdot \left(\prod_{k=1}^{n} m_k\right)^{\frac{1}{p}-\frac{1}{q}} \cdot \|T_{m_1 .. m_n}\|_{L_{p,\infty}^*}, \quad 1 \le p < q \le \infty;$$

$$\|T_{m_1 \dots m_n}\|_{L_{q,r}^*} \le C \cdot \left(\prod_{k=1}^{n} m_k\right)^{\frac{1}{p}-\frac{1}{q}} \cdot \|T_{m_1 .. m_n}\|_{L_p^*}, \quad 1 \le p < q \le \infty, \quad 1 \le r \le \infty,$$

where the parameter C does not depend on m and T_m.

Here, the space $L_{p,q}^*[0, 2\pi]^n$ is defined as the set of functions for which the inequality holds

$$\|f\|_{L_{p,q}^*} = \left(\int_0^{2\pi} \left(t^{\frac{1}{p}} \cdot f^*(t)\right)^q \frac{dt}{t}\right)^{\frac{1}{q}} < \infty$$

and $f^*(t) = f^{*_1, \dots, *_n}(t_1, ..., t_n)$ denotes a function obtained by applying a non-increasing permutation sequentially in variables $x_1, ..., x_n$ with fixed other variables and this function is called a non-increasing permutationof a measurable function $f(x_1, ...x_n)$ in $[0, 2\pi]^n$ [5].

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Г.А. Есенбаева, А.Н. Есбаев, Х. Поппелл

Тригонометриялық көпмүшеліктер үшін түрлі метрикадағы теңсіздіктер жайлы

Мақала тригонометриялық көпмүшеліктер үшін түрлі метриканың теңсіздіктерін зерттеуге арналған. Авторлар осы зерттеудің құрылымын берген, оның негізгі компонеттері мен түрлері, сонымен қатар оның классикалық тәсілдерін көрсеткен. Тригонометриялық көпмүшеліктер үшін Никольскийдің әртүрлі метрикадағы теңсіздіктері жақсы белгілі. Осы жұмыста Лоренц және Лебег кеңістіктеріндегі біртүрлі айнымалыдан тұратын тригонометриялық көпмүшеліктер үшін түрлі метрикалардың теңсіздіктері дәлелденген. Аналогиялық нәтиже ретінде бірнеше айнымалылардан тұратын тригонометриялық көпмүшеліктер үшін алынған. Мақала негізінен математиктерге арналған.

Кілт сөздер: Лебег кеңістігі, Лоренц кеңістігі, тригонометриялық көпмүшеліктер, түрлі метрикадағы теңсіздіктер.

Г.А. Есенбаева, А.Н. Есбаев, Х. Поппелл

О неравенстве разных метрик для тригонометрических полиномов

Статья посвящена вопросу исследования неравенств разных метрик для тригонометрических полиномов. Авторами представлены структура данного исследования, его основные компоненты и виды, а также его классические подходы. Для тригонометрических полиномов хорошо известны неравенства Никольского в разных метриках. В данной работе доказаны неравенства разных метрик в пространствах Лоренца и Лебега для тригонометрических полиномов одного переменного. Аналогичный результат получен для тригонометрических полиномов многих переменных. Статья ориентирована, главным образом, на математиков.

Ключевые слова: пространства Лебега, пространства Лоренца, тригонометрические полиномы, неравенства разных метрик.

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R.B. Baimakhan[1], A.R. Baimakhan[2], Z.M. Abdiakhmetova[2], A.A. Seynasinova[3], G.M. Baimakhanova[4]

[1]*Al-Farabi Kazakh State Women's Pedagogical University, Almaty, Kazakhstan;*
[2]*Kazakh National University, Almaty, Kazakhstan;*
[3]*Academy of Civil Aviation, Almaty, Kazakhstan;*
[4]*South Kazakhstan State Pedagogical University, Shymkent, Kazakhstan*
(E-mail: baimakhan.aigerim@gmail.com)

Generalization of Tsytovich strength conditions for soils of anisotropic structure

The problem of developing a new generalized strength (plasticity)condition is applied to soils of anisotropic, in particular a transversal-isotropic (trans-structural) structure. This condition is derived by generalizing the known Coulomb-Moore plasticity condition σ_c for two directions: along the layers parallel and crosswise to layers, that is, for the perpendicular directions to the layers, relative to the isotropy plane for the transtropic soil, systematized for the first time by A.K. Bugrov and A.I. Golubev. It is proposed to analyze the possibility of generalizing the conditions of plasticity (strength) in the principal stresses σ_1 and σ_2, as well as on the values of the critical principal stress σ_{1c} and σ_{2c} , proposed and developed in due time for soils of the isotropic structure N.A. Tsytovich and N.S. Bulychev. Following the approach of W. Witke, who proposed to apply such a criterion to rocks of an orthotropic structure, the formulated criterion is proposed, which allows to determine the moment of the onset of plastic destruction of anisotropic soil and the direction of its further spread from the initial point. The table contains critical values of anisotropic soil in two orthogonal directions of the coordinate axes, calculated with the help of the proposed new criterion, which allows us to solve a new class of problems of fracture mechanics.

Keywords: soil, loam, isotropy, anisotropy, plasticity, isotropy plane.

Introduction

Most soils in nature have an anisotropic structure. For example, cover soils of mountain slopes, which are accompanied by landslide processes or construction sites, leading to a heel of the foundations of a building, and so on. Existing methods and approaches in soil mechanics are based on the assumption that soils are inherently isotropic in structure. Mechanical structures of such soils are determined by two parameters: Young's modulus E and Poisson coefficient ν. Whereas the properties of soils of an anisotropic structure are determined by five parameters: two Young's moduli E_1, E_2, two Poisson's coefficients ν_1, ν_2 and a shear modulus G_2. This limits the ability to solve soil stability problems by analytical methods. Moreover, there are no criteria to determine the strength of the soil. Therefore, there the question of developing a criterion of strength for soils of an anisotropic structure arises.

Task 1. Below in Figure, on the left, the subsurface sandy sand (upper layer) and loamy structure (lower layer) is shown, and on the right, the denser one — solid pebble soil and anisotropic. Such structures have soil construction sites. And in the mountain slopes, where landslide processes are often observed, the cover soils have not only sandy-loamy and pebbly structures, but also have more complex, sloping structures. It is not difficult to see that if the soils on the left figure have horizontally layered structures, then on the right figure are visible the destroyed soils, under the road, which have sloping structures.

Figure. Natural soils layered anisotropic structure

Now for the development of the criterion of strength we start from the well-known classical Coulomb-Moore strength (plasticity)condition developed for soils and rocks of an isotropic structure [1, 2], which looks like:

$$\tau_c = C + \sigma_n tg\varphi, \tag{1}$$

where τ_c is the tangential component of the tear-off voltage, C is the adhesion force, σ_n is the normal stress at the slip site; ϕ is an angle of internal friction. Unaxial compression strength expressed by next formula:

$$\sigma_c = \frac{2C\cos\varphi}{1-\sin\varphi}. \tag{2}$$

The condition of plasticity (strength) in the principal stresses (σ_3-does not affect the strength) expressed by next formula:

$$\sigma_{1c} = \sigma_{maxc} = \sigma_c + \beta\sigma_{2c}, \tag{3}$$

where β is the bulk strength parameter [2]:

$$\beta = \frac{1+\sin\varphi}{1-\sin\varphi}. \tag{4}$$

The condition of limiting equilibrium for disconnected loose soils is written according to N.A. Tsytovich [1]:

$$\frac{\sigma_{c1}-\sigma_{c2}}{\sigma_{c1}+\sigma_{c2}} = \sin\varphi, \tag{5}$$

where σ_{c1} and σ_{c2} are the limiting principal stresses. Hence, to obtain the expressions for σ_{c2}, we transform the expression (5)

$$\sigma_{2c} = \xi\sigma_{1c}, \tag{6}$$

where

$$\xi = \frac{1}{\beta} = \frac{1-\sin\varphi}{1+\sin\varphi}. \tag{7}$$

Also, from (6) using (4) we get

$$\sigma_{1c} = \beta\sigma_{2c}. \tag{8}$$

Now expression (6) with regard for (7) can be represented as

$$\frac{\sigma_2}{\sigma_1} = tg^2(45^\circ \pm \frac{\varphi}{2}). \tag{9}$$

This expression is widely used in the theory of pressure of soils on fences. The minus sign in parentheses corresponds to the active pressure, and the plus sign indicates the passive resistance of loose soils. Now we write down the condition of maximum equilibrium for cohesive soils

$$\frac{\sigma_1 - \sigma_2}{\sigma_1 + \sigma_2 + cctg\varphi} = \sin\varphi, \tag{10}$$

whence

$$\sigma_1 - \sigma_2 = 2\sin\varphi(\frac{\sigma_1 + \sigma_2}{2} + c\,\mathrm{ctg}\,\varphi, \tag{11}$$

or

$$\sigma_1 = \sigma_2 + 2\sin\varphi(\frac{\sigma_1 + \sigma_2}{2}) + c\,\mathrm{ctg}\,\varphi. \tag{12}$$

This well-known N.A. Tsytovich criteria should be generalized taking into account the anisotropy of the soil shown in Figure. It should be noted that in recent years the issues of anisotropy of soils have been actively pursued by Kazakhstan scientists. For example, the authors of [5–10], in addition to researching the stressful-deformed state of soils and rocks of anisotropic structure, surface, underground and other engineering structures, develop and special criteria for destruction.

Results

Following V. Vitke's [10] approach, proposed in due time for rocks of an orthotropic structure, the criterion of strength (1)–(12) is extended for soils of a transtropically anisotropic structure. Here the plasticity in soils can develop along the isotropy plane (parallel) or (and) in directions crosswise to it (perpendicular). They will differ significantly from each other. Therefore, condition (1) is written separately for these two directions

$$\tau_{||c} = C_{||} + \sigma_{||n} tg\varphi_{||}; \tag{13}$$

$$\tau_{\perp c} = C_{\perp} + \sigma_{\perp n} tg\varphi_{\perp}, \tag{14}$$

where $\tau_{||c}$, $\tau_{\perp c}$ are the tangential stresses on slip sites; $C_{||}$, $C_{\perp}$ are cohesion forces, normal stresses on slip planes in parallel $\sigma_{||n}$, and perpendicular $\sigma_{\perp n}$ directions to the isotropy plane are determined from the experiment or removed from Mohr's circles, $\varphi_{||}$, $\varphi_{\perp}$ are internal friction angles. For oblique-laminated anisotropic materials, we denote the slope angle of the isotropy planes by $\bar{\bar{\varphi}}$. It should be recalled that the angles $\bar{\bar{\varphi}}$ and φ have completely different meanings that are not related to each other. The tensile strength for uniaxial compression is written in the form

$$\sigma_{||c} = \frac{2C_{||}\cos\varphi_{||}}{1 - \sin\varphi_{||}}; \tag{15}$$

$$\sigma_{\perp c} = \frac{2C_{\perp}\cos\varphi_{\perp}}{1 - \sin\varphi_{\perp}}. \tag{16}$$

Also, the plasticity(strength) condition in the principal stresses will have the form

$$\sigma_{||c} = \sigma_{max||c} = \sigma_{||c} + \beta_{||}\sigma_{2||c}; \tag{17}$$

$$\sigma_{\perp c} = \sigma_{max\perp c} = \sigma_{\perp c} + \beta_{\perp}\sigma_{2\perp c}, \tag{18}$$

where $\beta_{||}$, $\beta_{\perp}$ — are parameters of bulk strength:

$$\beta_{||} = \frac{1 + \sin\varphi_{||}}{1 - \sin\varphi_{||}}; \tag{19}$$

$$\beta_{\perp} = \frac{1 + \sin\varphi_{\perp}}{1 - \sin\varphi_{\perp}}. \tag{20}$$

The condition of limiting equilibrium for non-cohesive bulk solids written by next expressions:

$$\frac{\sigma_{c1||} - \sigma_{c2||}}{\sigma_{c1||} + \sigma_{c2||}} = \sin\varphi_{||}; \tag{21}$$

$$\frac{\sigma_{c1\perp} - \sigma_{c2\perp}}{\sigma_{c1\perp} + \sigma_{c2\perp}} = \sin\varphi_{\perp}, \tag{22}$$

where $\sigma_{c1\perp}$ and $\sigma_{c1||}$ are the limiting principal stresses. Similarly, we write expression (6) in the following form

$$\sigma_{2c||} = \xi_{||}\sigma_{1c||}; \tag{23}$$

$$\sigma_{2c\perp} = \xi_{\perp}\sigma_{1c\perp}, \tag{24}$$

where

$$\xi_{||} = \frac{1}{\beta_{||}} = \frac{1-\sin\varphi_{||}}{1+\sin\varphi_{||}}; \tag{25}$$

$$\xi_{\perp} = \frac{1}{\beta_{\perp}} = \frac{1-\sin\varphi_{\perp}}{1+\sin\varphi_{\perp}}, \tag{26}$$

or from the expressions (11), (12) using (7), (8) we obtain

$$\sigma_{1c||} = \beta_{||}\sigma_{2c||}; \tag{26}$$

$$\sigma_{1c\perp} = \beta_{\perp}\sigma_{2c\perp}. \tag{27}$$

Expressions (11) and (12) with allowance for (13) and (14) with respect to the isotropy plane can be represented in the form

$$\frac{\sigma_{2c||}}{\sigma_{1c||}} = tg^2(45^\circ \pm \frac{\varphi_{||}}{2}); \tag{28}$$

$$\frac{\sigma_{2c\perp}}{\sigma_{1c\perp}} = tg^2(45^\circ \pm \frac{\varphi_{\perp}}{2}). \tag{29}$$

It is known that, in the isotropic version, these expressions in the form (9) are used in the theory of pressure of soils on fences. And here, in expressions (28) and (29) the minus sign in parentheses corresponds to the active pressures, and the plus sign to the passive resistances of free-flowing soils.If the barrier is a retaining wall, then the pressure on this wall according to expression (29) acts either perpendicularly or at an angle, depending on the slope of the layers of the isotropy planes of soils with an inclined anisotropic structure.Since the pressure in expression (28) acts across the layers of isotropy, then relative to the wall, they act parallel to the wall at the points of adhesion, that is, on the boundary layer. Now we write the condition of limiting equilibrium for connected soils by analogy with N.A. Tsytovich

$$\frac{\sigma_{1c||} - \sigma_{2c||}}{\sigma_{1c||} + \sigma_{2c||} + C_{||}ctg\varphi_{||}} = \sin\varphi_{||}; \tag{30}$$

$$\frac{\sigma_{1c\perp} - \sigma_{2c\perp}}{\sigma_{1c\perp} + \sigma_{2c\perp} + C_{\perp}ctg\varphi_{\perp}} = \sin\varphi_{\perp}. \tag{31}$$

We transform these expressions to the next form:

$$\sigma_{1||} - \sigma_{2||} = 2\sin\varphi_{||}(\frac{\sigma_{1||} + \sigma_{2||}}{2} + C_{||}ctg\varphi_{||}); \tag{32}$$

$$\sigma_{1\perp} - \sigma_{2\perp} = 2\sin\varphi_{\perp}(\frac{\sigma_{1\perp} + \sigma_{2\perp}}{2} + C_{\perp}ctg\varphi_{\perp}), \tag{33}$$

or to calculate the largest principal stresses, we represent them in the form

$$\sigma_{1||} = \sigma_{2||} + 2\sin\varphi_{||}(\frac{\sigma_{1||} + \sigma_{2||}}{2} + C_{||}ctg\varphi_{||}); \tag{34}$$

$$\sigma_{1\perp} = \sigma_{2\perp} + 2\sin\varphi_{\perp}(\frac{\sigma_{1\perp} + \sigma_{2\perp}}{2} + C_{\perp}ctg\varphi_{\perp}). \tag{35}$$

The need to develop such generalized criteria, allowing to determine not only the state of pre-failure, but also the direction of damage propagation, confirms Figure. There are their experimentally determined values. Such single data is available, for example, in [3]. The following table shows the critical values β for some of the main types of surface soils. For comparison, limit values β for limestone and concrete are also given. These data also confirm the need to develop a criterion for destruction for soils of an anisotropic structure.

Table

Critical parameters of anisotropic plasticity for soils of an anisotropic structure

No	Soils	Volume weight	Strength parameter	Young's Modules		Shear modulus	Poisson's coefficients		Clutch Strength		Internal friction		Main stresses	
		$\gamma, kH/m^3$	ξ_0	$E_{\parallel}$, Мпа	$E_{\perp}$, Мпа	$G_{\parallel\perp}$, Мпа	$\nu_{\parallel}$	$\nu_{\perp}$	$C_{\parallel}$, Мпа	$C_{\perp}$, Мпа	$\varphi^0_{\parallel}$	$\varphi^0_{\perp}$	$\sigma_{c\parallel}$	$\sigma_{c\perp}$
1	Loam	17.0	0.60	13.4	26.4	7.6	0.16	0.24	0.025	0.050	26	26	0.80	0.160
2	Sand	17.0	0.43	23.0	16.0	7.0	0.30	0.30	0.005	0.005	27	33	0.16	0.180
3	Loam saturated	20.0	0.48	30.0	15.0	7.6	0.36	0.24	0.030	0.060	19	23	0.084	0.197
4	Priming	19.0	1.00	10.0	20.0	7.4	0.30	0.40	0.080	0.120	20	24	0.230	0.370
5	Loam	9.4	0.65	12.0	8.0	3.4	0.39	0.35	0.010	0.014	20	24	0.029	0.043
6	Loam	8.0	0.65	6.0	4.0	1.7	0.39	0.35	0.050	0.007	15	18	0.130	0.019
7	Loam	9.2	0.65	9.0	6.0	2.5	0.39	0.35	0.006	0.008	18	22	0.017	0.024
8	Sandy loam	19.8	0.53	19.6	18.4	7.1	0.31	0.30	0.003	0.003	18	21	0.008	0.009
9	Loam is hard	19.9	0.58	39.8	27.0	10.0	0.36	0.35	0.02	0.02	13	17	0.005	0.054
10	Sand fine	21.1	0.25	81.3	85.0	32.7	0.28	0.30	0.002	0.002	35	37	0.008	0.008
11	Rock Limestone	2.5	0.33	3200	1600	1185	0.38	0.32	47	0.25	31	29	116.1	0.849
12	Concrete BIT-PE polyester	1.65		4941	4941	1930	0.28	0.28					201.4	0.849

Conclusion

A justified criterion of strength is proposed, which makes it possible to determine the direction of propagation of a fracture of earth fractures relative to the isotropy plane of inhomogeneous layers. They are directions parallel to the layers and perpendicular.The developed parameters of strength (plasticity) for soils of natural anisotropic structure allows to solve the class of geomechanics tasks associated with the definition of stress-deformed condition of the covering soils of mountain slopes necessary for the prediction of landslide processes and the state of stability of under foundation soils of construction sites.

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Р.Б. Баймахан, А.Р. Баймахан, З.М. Абдиахметова,
А.А. Сейнасинова, Г.М. Баймаханова

Анизотропты құрылымды топырақтарға арналған Цытовичтың қатаңдық шарттарын жалпылауға арналған трифотикалық күшейтулердің синтезі

Тасымалдаудың трансверсалды-изотоптық (транстроптық) құрылыстарына қатысты жаңа анизотроптық грунтты қолдануға арналған жаңа беріктік (пластикалық) жалпыланған шартын құру мәселесі талқыланды. Мұндай шарт белгілі Кулон-Мор σ_c беріктік шартын екі бағытта жалпылаудан шығады: қабаттарға параллель және қабаттарға қарсы, яғни, А.К. Бугров және А.И. Голубев алғаш рет жүйелеген транстропты құрылымы бар топырақтардың изотропия жазықтығына қатысты қабаттарға перпендикуляр бағыттар үшін. Негізгі σ_1 және σ_2, сонымен қатар кезінде Цытович және Н.С. Булычевпен ұсынылып дамытылған σ_{1c} және σ_{2c} негізгі критикалық кернеулер мәні бойынша иілгіштіктің (беріктіктің) шарттарын жалпылау мүмкіндігіне талдау ұсынылды. В.Витке ұсынысы бойынша, осы критерийді ортотропты құрылымды тау жыныстарына қолдануды ескере отырып, анизотропты құрылымы бар топырақтардың пластикалық қирау моментін және бастапқы нүктеден ары қарай таралу бағытын анықтауға мүмкіндік беретін критерий құру ұсынылды. Қирау механикасы есептерінің жаңа класын шешуге мүмкіндік туғызатын жаңа критериймен есептелген екі ортогоналды осьтер бағытында анизотропты құрылымы бар топырақтардың критикалық мәндерін қамтитын кесте берілген.

Кілт сөздер: топырақ, изотроптық, анизотропиялық, иілгіштік, изотропты жазықтығы.

Р.Б. Баймахан, А.Р. Баймахан, З.М. Абдиахметова,
А.А. Сейнасинова, Г.М. Баймаханова

Обобщение условий прочности Цытовича для грунтов анизотропного строения

Рассмотрен вопрос разработки нового обобщенного условия прочности (пластичности) применительно к грунтам анизотропного, в частности, трансверсально-изотропного (транстропного) строения. Такое условие выводится обобщением известного условия пластичности Кулона-Мора σ_c для двух направлений: вдоль слоев параллельно и вкрест слоям, т.е. для направлений, перпендикулярных к слоям, относительно плоскости изотропии для грунтов транстропного строения, систематизированного впервые А.К. Бугровым и А.И. Голубевым. Сделан анализ возможности обобщения условий пластичности (прочности) по главным напряжениям σ_1 и σ_2, а также по значениям критических главных напряжений σ_{1c} и σ_{2c}, предложенный и развитый в свое время для грунтов изотропного строения Н.А. Цытовичем и Н.С. Булычевым. Следуя подходу В.Витке, который предложил применить такой критерий к горным породам ортотропного строения, предложен сформулированный критерий, который позволяет определить момент наступления пластического разрушения грунтов анизотропного строения и направление его дальнейшего распространения от начальной точки. Приведена таблица, содержащая критические значения грунтов анизотропного строения в двух ортогональных направлениях координатных осей, вычисленные с помощью предложенного нового критерия, который позволяет решить новый класс задач механики разрушения.

Ключевые слова: грунт, изотропия, анизотропия, пластичность, плоскость изотропии.

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K.B. Begaliyeva[1,2], M.T. Arshidinova[1,2], E. Arynov[3],
A.K. Kudaykulov[1], A.A. Tashev[1], T.B. Duishenaliev[4]

[1] *Institute of Information and Computing Technologies, Almaty, Kazakhstan;*
[2] *Al-Farabi Kazakh National University, Almaty, Kazakhstan;*
[3] *O.A. Baikonurov Zhezkazgan University, Kazakhstan;*
[4] *I. Razzakov Kyrgyz State University, Bishkek, Kyrgyzstan*
(E-mail: kalamkas_b@mail.ru)

The energy method for solving a nonlinear problem of thermoelasticity for arod of variable cross section

A horizontal rod of limited length is considered. Radius of the rod varies linearly along its length. The cross-sectional area of the left end is larger than the cross-sectional area of the right end. The lateral surface of the test rod is completely insulated. The heat flow is fed to the cross-sectional area of the left end. Through the cross-sectional area of the right end of the rod, heat exchange takes place with the surrounding medium. The field of distribution of temperature, displacement, three components of deformation and stresses are determined in the work, provided that both ends of the rod are rigidly fixed. And also, the magnitude of the elongation of the rod is determined when one end of the rod is fixed and when the other is free.In the case of fixing the two ends of the rod, the magnitude of the resulting axial compressive force is also calculated. When studying the rod, the fundamental laws of conservation of energy were used.

Keywords: elongation, axial force, cross-section, temperature, displacement, deformation, stress.

Introduction

Many load-bearing elements of gas-generator, nuclear and thermal power stations, jet engines and the processing industry are rods of variable cross-section. To ensure reliable operation of these equipments, it is necessary to provide the thermal strength of load-bearing elements in the form of variable-section rods that operate with the simultaneous action of dissimilar kinds of heat sources. Because of the variability of the cross section, nonlinear thermomechanical processes appear in such rods.

To study the nature of such processes, consider a horizontal rod of limited length, of variable cross-section. In this case, the radius of the section varies linearly along the length of the investigated rod, i.e. $r = ax + b$, $0 \leq x \leq l$), where is the l-length of the rod, $a, b - const$. The cross-sectional area of the rod varies nonlinearly along the length of the rod in the following manner $F(x) = \pi(ax + b)^2[m^2]$. The lateral surface of the test rod along the entire length is heat-insulated. On the cross-sectional area of the left end of the rod $F(x = 0) = \pi b^2$, a heat flux with a constant intensity $q \left[\frac{watt}{cm^2}\right]$. Through the cross-sectional area of the right end of the rod $F(x = l) = \pi(al + b)^2$, heat exchange takes place with the surrounding medium. At the same time, the heat transfer coefficient $h \left[\frac{watt}{cm^2\ {}^0C}\right]$, ambient temperature $T_{oc}[K]$, the physical and mechanical properties of the core material is characterized by the coefficient of thermal expansion $\alpha \left[\frac{1}{K}\right]$, thermal conductivity $K_{xx} \left[\frac{watt}{cm \cdot K}\right]$ and modulus of elasticity $E \left[\frac{kg}{cm^2}\right]$. The scheme of the investigated rod is shown in Figure 1.

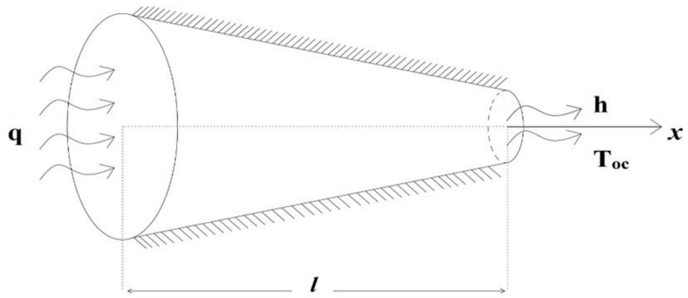

Figure 1. Scheme of the investigated rod

Overviev

In the presence of heat flow, heat insulation and heat transfer, the functional of the total thermal energy for the investigated rod has the form [1]:

$$J = \int_{F(x=0)} qTds + \int_V \frac{K_{xx}}{2}(\frac{\partial T}{\partial x})^2 dv + \int_{F(x=l)} \frac{h}{2}(T - T_{oc})^2 ds, \qquad (1)$$

where $T = T(x)$ the field of distribution of temperatures along the length of the rod, which is approximated by a complete polynomial of the fourth order

$$T(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 = \varphi_i(x)T_i + \varphi_j(x)T_j + \varphi_k(x)T_k + \varphi_m(x)T_m + \varphi_n(x)T_n, \qquad (2)$$

where $\varphi(x)$ — are spline functions:

$$\varphi_i(x) = \frac{(3l^4 - 25l^3x + 70l^2x^2 - 80lx^3 + 32x^4)}{3l^4};$$
$$\varphi_j(x) = \frac{(48l^3x - 208l^2x^2 + 288lx^3 - 128x^4)}{3l^4};$$
$$\varphi_k(x) = \frac{(-36l^3x + 228l^2x^2 - 384lx^3 + 192x^4)}{3l^4}; \qquad (3)$$
$$\varphi_m(x) = \frac{(16l^3x - 112l^2x^2 + 224lx^3 - 128x^4)}{3l^4};$$
$$\varphi_n(x) = \frac{-3l^3x + 22l^2x^2 - 48lx^3 + 32x^4}{3l^4},$$

$0 \leq x \leq l$, where the nodal temperature values are determined by the formulas

$$T_j = T(x = 0); \quad T_i = T\left(x = \frac{l}{4}\right); \quad T_k = T\left(x = \frac{l}{2}\right); \quad T_m = T\left(x = \frac{3l}{4}\right); \quad T_n = T(x = l). \qquad (4)$$

Taking into account (2)–(4), minimizing (1) with T_j, T_i, T_k, T_m and T_n we obtain a resolving system of algebraic equations taking into account existing natural boundary conditions.Solving the system we determine the nodal values of temperature (4), and by (2) we construct the field of temperature distribution along the length of the rod. If one end of the rod is fixed and the other end is free, then the length of the rod Δl_T[cm] is determined according to the general law of thermophysics [1]

$$\Delta l_T = \int_0^l \alpha T(x)dx.$$

If both ends of the rod are rigidly fixed, then an axial compressive force R[kG] arises in the rod, which is determined from the compatibility condition of the deformation [1]

$$R = \frac{\Delta l_T \cdot E \int_0^l F(x)dx}{l^2}.$$

In this case, a distribution field of the thermo-elastic component of the voltage $\sigma(x)\left[\dfrac{kg}{cm^2}\right]$ arises in the rod:

$$\sigma(x)=\frac{R}{F(x)},0\leq x\leq l.$$

Then, according to Hooke's law, we can determine the distribution field of the thermoelastic deformation component $\varepsilon(x)$ [dimensionless]:

$$\varepsilon(x)=\frac{\sigma(x)}{E}.$$

The temperature component of deformations $\varepsilon_{T(x)}$ [dimensionless] is determined according to the general law of thermophysics [1]:

$$\varepsilon_{T(x)}=-\alpha T(x).$$

Then, according to Hooke's law, the field of distribution of the temperature component of the stress $\sigma_T(x)\left[\dfrac{kg}{cm^2}\right]$:

$$\sigma_T(x)=E\cdot\varepsilon_T(x)=-\alpha E\cdot T(x).$$

According to the theory of thermoelasticity, the laws of distribution of elastic components of deformations $\varepsilon_{x(x)}$ [dimensionless] and stresses $\sigma_T(x)\left[\dfrac{kg}{cm^2}\right]$:

$$\varepsilon_x(x)=\varepsilon(x)-\varepsilon_T(x);$$

$$\sigma_x(x)=E\cdot\varepsilon_x(x)=\sigma(x)-\sigma_T(x).$$

The potential energy of elastic deformations is used to determine the displacement field [2]:

$$\Pi=\int_V\frac{\sigma_x(x)}{2}\varepsilon_x(x)dv-\int_V\alpha E\cdot T(x)\cdot\varepsilon_x(x)dv.$$

According to the Cauchy relation [2], we have:

$$\varepsilon_x(x)=\frac{\partial U}{\partial x};$$

$$U=U(x)=\varphi_i(x)U_i+\varphi_j(x)U_j+\varphi_k(x)U_k+\varphi_m(x)U_m+\varphi_n(x)U_n,$$

where$U$ is the displacement field. Minimizing Π from the nodal values of the displacement, a system of linear algebraic equations is constructed. To solve this system, it is necessary to specify the conditions for securing the two ends of the rod, i.e. $U_i=U(x=0)=0$ and $U_n=U(x=l)=0$. Further, defining U_i,U_j,U_k,U_m,U_n, a displacement field is constructed. For practical application of the above method and algorithm, we take the following initial data $l=20$ см, $a=\frac{1}{10}$, $b=4$ см, $\alpha=0,0000125\frac{1}{K}$, $E=2\cdot10^6\frac{kG}{cm^2}$, $K_{xx}=100\frac{watt}{cm\cdot K}$, $h=10\frac{watt}{cm^2\cdot K}$, $T_{oc}=40^0K$, $q=-500\frac{watt}{cm^2}$.

Figure 2 shows that the temperature is higher near the left end of the rod, where the heat flow is supplied. Due to the thermal insulation of the lateral surface, heat is lost minimally, so that the temperature at the right end of the rod is maintained at 2400 K.

With these initial data, the obtained solutions are shown in Figures 2–5.

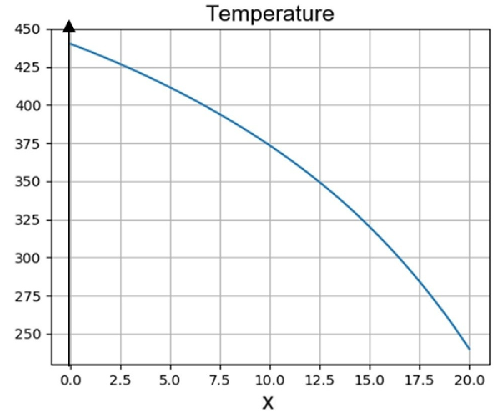

Figure 2. Dependences of the temperature T along the length of the rod

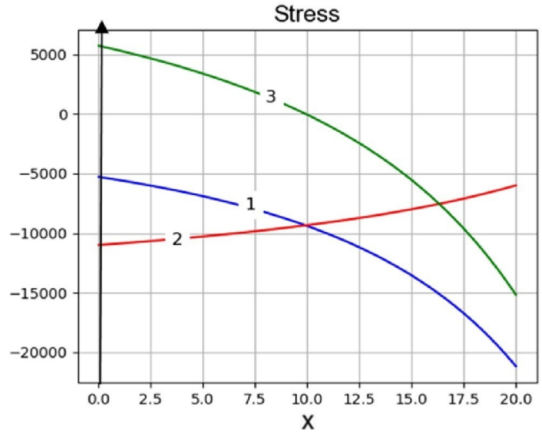

Figure 3. Stress Dependencies along the length of the rod

Decision

The stresses along the length of the rod are shown in Figure 3 ($1-\sigma(x)$ is the thermoelastic, $2-\sigma_{(x)}$ is the temperature, $3-\sigma_E(x)$ is the elastic component of the stress). It can be seen from the figure that the thermoelastic $-\sigma(x)$ and temperature $-\sigma_{(x)}$ are the components of the stress along the entire length of the rod are of a compressive nature. While the elastic $-\sigma_E(x)$ component of the stress in the area $0 \leq E \leq \frac{l}{2}$ has a tensile character, and in the area $\frac{l}{2} \leq E \leq l$ it is compressive.

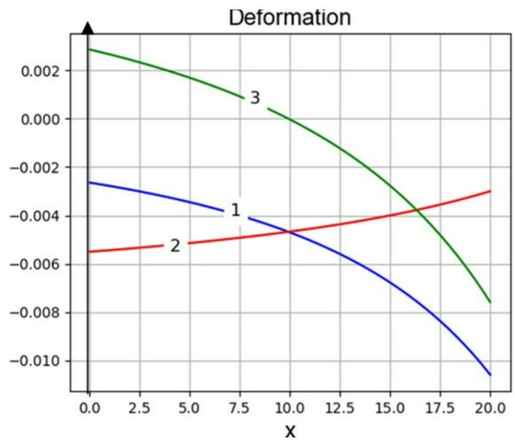

Figure 4. Dependence of the deformation along the length of the rod

Dependences of deformations along the length of the rod are shown in Figure 4 ($1 - \sigma(x)-$thermoelastic, $2 - \sigma_{(x)}-$ temperature, $3 - \sigma_E(x)-$ elastic component deformation). The distribution field of the deformation components is proportional to the corresponding stresses.

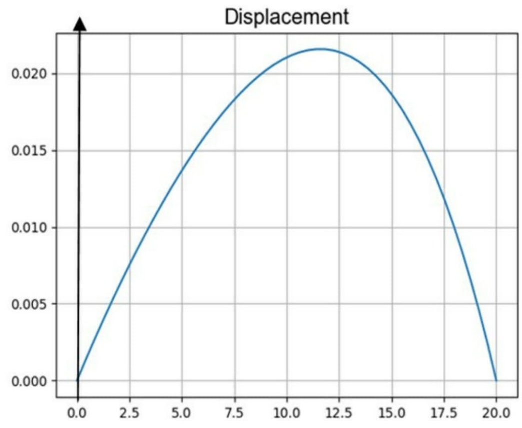

Figure 5. Dependences of displacement along the length of the rod

Figure 5 shows the field of distribution of displacements of a rod fixed at two ends. From this it can be seen that all sections (except for exceptions) move in the direction of the x axis. The greatest amplitude of displacement corresponds to the coordinate of$x \approx \frac{3l}{5}$.

Conclution

A numerical model of nonlinear thermomechanical processes in a rod of variable cross-section is developed, based on the fundamental law of conservation of energy. This allows to obtain reliable numerical results taking into account all natural boundary conditions. The results obtained are consistent with the corresponding laws of physics. This method can be used for the numerical solution of a class of problems determined by the steady-state thermomechanical state of load-bearing structural elements operating under the influence of dissimilar kinds of heat sources.

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К.Б. Бегалиева, М.Т. Аршидинова, Е. Аринов,
А.К. Кудайкулов, А.А. Ташев, Т.Б. Дуйшеналиев

Айнымалы көлденең қимасы бар сырықтың сызықты емес термоэластикалық есебін шешуде энергетикалық әдісті қолдану

Газ генераторларының, ядролық және жылу электр станцияларының, реактивті қозғалтқыштарының және өңдеу өнеркәсібінің көптеген элементтері айнымалы көлденең қимасы бар сырық болып табылады. Осы жабдықтардың сенімді жұмыс істеуін қамтамасыз ету үшін, жылу көздерінің әртүрлі түрлерінің бір уақытта әсер етуімен жұмыс істейтін айнымалы көлденең қимасы бар сырықты қарастырып, мойынтіректер элементтерінің жылу берілуін қамтамасыз ету қажет. Мақалада айнымалы көлденең қимадағы шектеулі ұзындықтағы сырық қарастырылды. Көлденең қимасы дөңгелек, оның радиусы ұзындығы бойымен сызықты түрде өзгереді. Сол жақтың көлденең қимасы оң жақтан үлкен. Зерттелген сырықтың бүйір беті толығымен термалды түрде оқшауланған. Жылу ағыны сол жақтың көлденең қимасына қолданылады. Сырықтың оң жақ шетінің көлденең қимасы арқылы қоршаған ортаға жылу алмасуы жүргізіледі. Жұмыста температура, ығысу, сырықтың екі жағы қатаң бекітілген жағдайдағы деформация және стрестің үш құрамдас бөліктері анықталған. Сондай-ақ бір шетіне бекітіп, екіншісі еркін болған кезде сырықтың ұзарту шамасы анықталды. Сырықтың екі ұшын бекіту нәтижесінде алынған осьтік қысымды күштің мәні есептелді. Сырықты зерттеу кезінде энергияны сақтаудың іргелі заңы пайдаланылды.

Кілт сөздер: ұзару, осьтік күш, қима, температура, жылжу, деформация, стресс, энергетикалық әдіс, сырық.

К.Б. Бегалиева, М.Т. Аршидинова, Е. Аринов,
А.К. Кудайкулов, А.А. Ташев, Т.Б. Дуйшеналиев

Энергетический метод для решения нелинейной задачи термоэластичности для стержня переменного поперечного сечения

Многие несущие элементы реактивных двигателей, оборудования а также газогенераторных, атомных и тепловых электростанций и перерабатывающей промышленности являются стержнями переменного сечения. Для обеспечения надежной работы этих оборудований необходимо обеспечить термопрочность несущих элементов в виде стержней переменного сечения, которые работают при одновременном воздействии разнородных видов источников тепла. В статье рассмотрен горизонтальный стержень ограниченной длины переменного поперечного сечения. Радиус стержня меняется линейно по его длине. Площадь поперечного сечения левого конца больше площади поперечного сечения правого конца. Боковая поверхность исследуемого стержня полностью теплоизолирована. На площадь поперечного сечения левого конца подводится тепловой поток. Через площадь поперечного сечения правого конца стержня происходит теплообмен с окружающей средой. В работе определены поле распределения температуры, перемещения, три составляющие деформации и напряжения при условии, что оба конца стержня жестко закреплены. А также определена величина удлинения стержня, когда один конец стержня закреплен, а другой — свободен. В случае закрепления двух концов стержня вычислена величина возникающего осевого сжимающего усилия. При исследовании стержня использовался фундаментальный закон сохранения энергии.

Ключевые слова: удлинение, осевая сила, сечение, температура, перемещение, деформация, напряжение.

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S.K. Matveev[1], N.Zh. Jaichibekov[2], B.S. Shalabayeva[2], D. Zhussupova[2]

[1]*Saint Petersburg State University, Russia;*
[2]*L.N. Gumilyov Eurasian National University, Nur-Sultan,Kazakhstan*
(E-mail: smat@rambler.ru)

Two-dimensional calculations of stratified turbulent flow in a pipe

In this paper, we consider the stratified turbulent flow of a two-phase medium in inclined pipes. Based on the new turbulence model [1], a program code for calculating two-dimensional flows for the study of two-phase stratified flows in pipes was developed, including taking into account the rough of the pipeline wall. The technique for calculating two-phase flows in extended pipelines is described. The problem of stationary stratified two-phase flow in a pipe of constant cross section in the case of turbulent regime is numerically solved. Calculations of the resistance of a rough pipe are carried out and the results on the influence of roughness on pipe resistance and velocity distribution are presented.

Keywords: stratified turbulent flow, resistance, two-dimensional calculations, rough surface.

Introduction

Calculations of two-phase flows in long pipelines remain relevant in our time [2]. In connection with the specific nature of the flow, the known turbulence models for such flows require an appropriate correction. Despite a satisfactory description with the proposed modification of the resistance of the pipeline [3], there remains some dissatisfaction with the description of velocity profiles. In the experiments, a small systematic deviation of the velocity profiles in the axial region of the pipe from the logarithmic law

$$\frac{u_m - u}{u_*} = f\left(\frac{r}{R}\right),$$

where u_* is the velocity in the laminar sublayer. This deviation obeys the so-called «speed defect law» in the form $\frac{u_m - u}{u_*} = f\left(\frac{r}{R}\right)$, where the function f reflects the speed excess over the logarithmic law calculated (u_m is the maximum speed on the pipe axis). Since the CAM-1 turbulence model described in [3] does not give such a deviation, a modification of the turbulence model, called CAM-2, was proposed in [1], where the logic of choosing the necessary dependencies is shown.

When calculating a two-phase stratified flow with a horizontal interfacial surface, we will use the simplest square grid in the cross section of the pipe, as was done in the case of laminar flow [4, 5]. Note that when approximating the equations of laminar flow on a square grid, the main error appeared near the walls of the pipe. In the case of turbulent flow, the situation becomes more complicated in connection with the special role of the wall laminar sublayer and the buffer region. Even in the one-dimensional problem, in connection with the singularity of the equations, it was necessary to use a grid of up to 5000 nodes with a condensation of the grid near the wall [1]. In the two-dimensional case, this would lead to an unjustified increase in the counting time and high demands on computational resources. A compromise solution in this situation is the use the «near – wall» functions, i.e. special approximations of the unknown functions in the near-wall region.

1 Distribution of speed

An example of such an approach is the method of determining frictional stress by measurements of velocity near the wall. Assuming that the velocity distribution obeys the logarithmic law:

$$\frac{u}{u_*} = C_1 \ln\left(\frac{yu_*}{\nu}\right) + C_2,$$

from this relationship, can be determined $u_* = \sqrt{\frac{\tau_w}{\rho}}$ by the measured values of y and u (τ_w is the frictional stress on the wall). To apply method the near-wall functions, it is necessary to have velocity approximations suitable at any point close to the wall, including in the laminar sublayer and in the buffer zone. As such, an approximation near a smooth wall, we can propose the following relationships:

$$\frac{u}{u_*} = C_1 \ln(y_*) + C_2 - \frac{C_3}{y_*} + \frac{C_4}{y_*^2}$$

and $\frac{u}{u_*} = y_*$ for $y_* < 5$. The values: C_3, C_4 are determined by the smooth conjugation of these formulas for $y_* = 5$. In the CAM-2 model [1] the following values of constants: $C_1 = 2.439$, $C_2 = 5.386$, $C_3 = 30.31$, $C_4 = 43.76$ are accepted.

Special attention was consider to velocity distributions near the rough wall and approximations were obtained

$$\frac{u}{u_*} = C_1 \ln\left(\frac{y_*}{k_c}\right) + B_2 - \frac{C_3}{y_*} + \frac{C_4}{y_*^2}, B_2 = 8.5 + 1.78b - 0.89b^2, \tag{1}$$

where $b = (\lg k_c - 2)^2$.

Thus, for a point, spaced by a distance y along the normal from the surface, the velocity of the fluid will be known. These formulas allow us to determine the frictional stress, which is considered at this point equal to friction on the wall. After this, the differentiation of the approximation formulas determines the viscosity, from which the boundary condition for the turbulent viscosity transport equation can be obtained. This solves the problem of the boundary condition for $y \geq h_s$ (h_s is the height of roughness), which was mentioned in the consideration of roughness [6].

2 Calculation of two-dimensional two-phase flows in a pipe

Consider in more detail the application of the CAM-2 turbulence model in the two-dimensional approximation, changing the notation somewhat. Now let x, y be the Cartesian coordinates in the cross section of the pipe, and the z coordinate is directed along the pipe axis. Accordingly, the longitudinal velocity is denoted by w, ν_t is the turbulent viscosity, the total kinematic viscosity $\nu_\Sigma = \nu_t + \nu$ (in the section of logarithmic velocity distribution $\tau_\Sigma \approx \tau_w = u_*^2 \rho$), on the interface ($h$ is the depth of the lower layer of the liquid), R is the radius of the pipe.

For stationary problems for a circular pipe with allowance for the axial symmetry, the balance equation for the turbulent viscosity is written in the form

$$\frac{k_y \bar{C}_b \nu_\Sigma}{\tau_m^2} \left(\left(\frac{\partial w}{\partial x} \right)^2 + \left(\frac{\partial w}{\partial y} \right)^2 \right)^3 + \frac{\partial}{\partial x} \left(\nu_t \frac{\partial \nu_t}{\partial x} \right) + \frac{\partial}{\partial y} \left(\nu_t \frac{\partial \nu_t}{\partial y} \right) =$$

$$= \bar{C}_d \left(\left(\frac{\partial \nu_t}{\partial x} \right)^2 + \left(\frac{\partial \nu_t}{\partial y} \right)^2 \right) + \bar{C}_w \left(\frac{\nu_t}{R - r} \right)^2. \tag{2}$$

For speed, we have

$$\frac{1}{\rho} \frac{dp}{dz} = \frac{\partial}{\partial x} \left(\nu_\Sigma \frac{\partial w}{\partial x} \right) + \frac{\partial}{\partial y} \left(\nu_\Sigma \frac{\partial w}{\partial y} \right).$$

To approximate these equations on a square grid, the simplest approximation of the second order of accuracy on a five-point template is used. It is important that equation (2) in the near-wall nodes of the grid is not approximated, and the turbulent viscosity in them is determined with help of the near-wall functions, as was described [1].

To organize iterations for solving difference equations with respect to the value of the desired function, in the central node of the grid template it is possible. In addition, for convergence it is necessary to apply lower relaxation [7]. The convergence turns out to be very slow, but due to the not too shallow grid, the counting time is quite acceptable.

The program code of this solution by on the modified CAM-2 turbulence model for the two-dimensional approximation made it possible to obtain the following results.

3 Results

Figure 1 shows the results of a two-dimensional calculation of drag and velocity profiles in a smooth pipe on a square grid with 50 cells per radius (the fragment of the grid is shown in the left part of the Fig.)

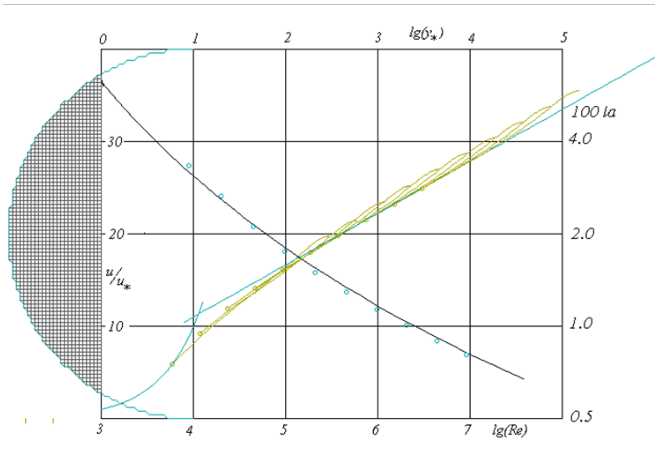
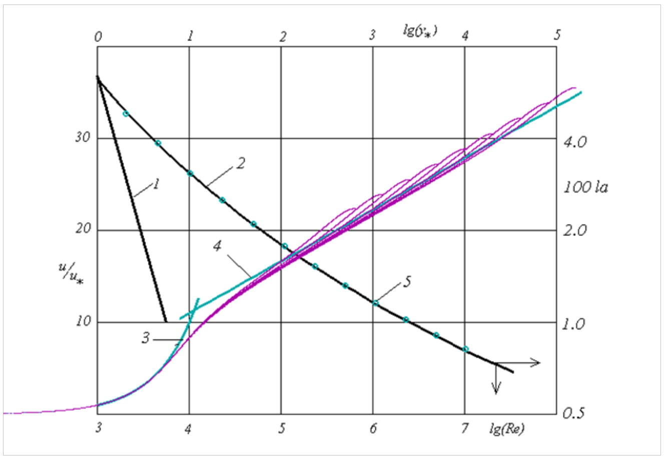


Figure 1. Results of a two-dimensional flow calculation
in a smooth pipe on the grid $h/R = 0.02$

A comparison with a more accurate one-dimensional calculation, the results of which are shown in [1]. For convenience of comparison, we represent this result in Figure 2.



1 — theoretical in the laminar regime; 2 — experimental Prandtl under turbulent regime; 3 — the velocity distribution in the laminar sublayer; 4 — in the turbulent core of the current; 5 — coefficients of resistance over the CAM-2

Figure 2. Coefficient of resistance and velocity distribution in a smooth
pipe (the CAM-2 turbulence model) for the one-dimensional case

As a result of the numerical solution to the modified turbulence model (CAM-2), a graph of the dependence of the drag coefficient on Re in logarithmic coordinates is constructed for the one-dimensional case and is shown in blue circles in Figure 2. The results shown in Figure 2 give the best agreement with the experimental data on frictional resistance given in [8].

In Figure 1, you can see that the use the near-wall functions provides acceptable accuracy. Let is attention that in yellow circles speed values are marked in the grid node closest to the surface.

To judge the influence of the grid step on the accuracy of the calculation, in Figure 3 the results obtained for 20 cells per radius are given. In this figure, you can see the principle of approximation of the boundary of the calculated area, shown by a blue broken line.

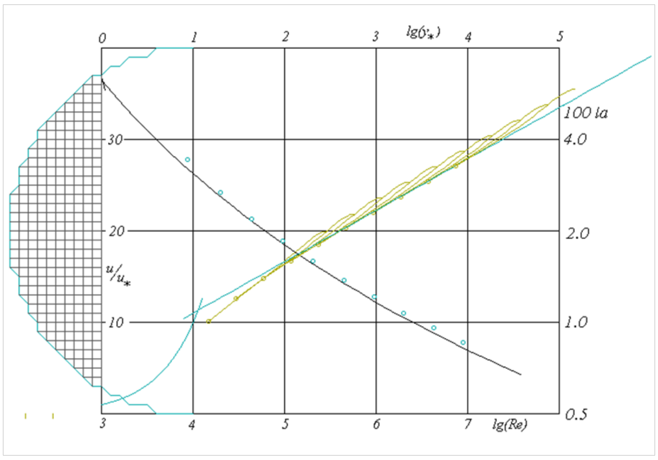


Figure 3. Results of two-dimensional flow computation
in a smooth pipe on a grid $h/R = 0.05$

As mentioned above, the near-wall functions can also be used for rough surfaces. The most famous results on the influence of rough on pipe resistance and velocity distribution were obtained by I. Nikuradze [9] and are given in many monographs and textbooks, for example [8, 10].

As a demonstration of this fact, and to check the accuracy of the approximations (1), we give in Figures 4–7 the results of two-dimensional calculations of flows in rough pipes. We note that, with a significant rough and a fine grid, several layers of grid cells can enter the zone of near – wall approximation, which complicates the calculations. For this reason, the results of calculations with a strong rough are not given here. The results shown in Figures 4–7 show that even calculations on a coarse grid have satisfactory accuracy. In these figures, the blue circles are the results of calculating the resistance at a fixed roughness, the horizontal line to the right is an experimentally determined resistance with full roughness, and the dark blue circle is the experimental value of the minimum coefficient of resistance.

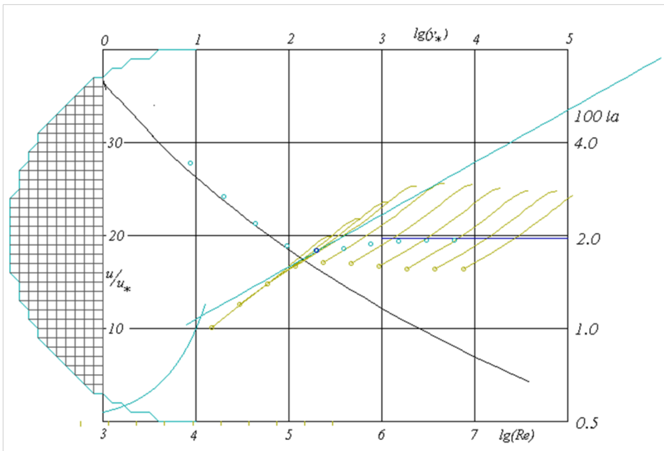


Figure 4. Results of two-dimensional calculation of flow in a pipe
with the roughness $R/h_s = 507$, on a grid $h/R = 0.05$

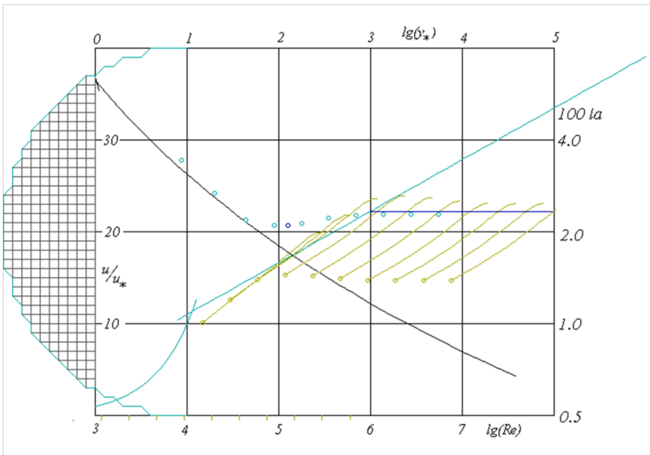


Figure 5. Results of two-dimensional calculation of flow in a pipe with the roughness $R/h_s = 252$, on a grid $h/R = 0.05$

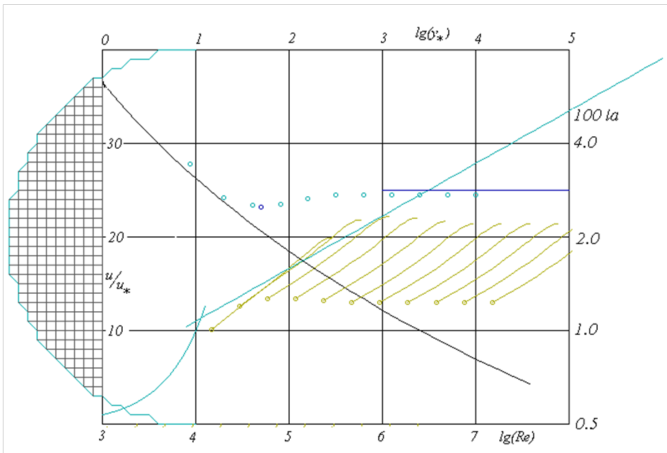


Figure 6. Results of two-dimensional calculation of flow in a pipe with the roughness $R/h_s = 126$, on a grid $h/R = 0.05$

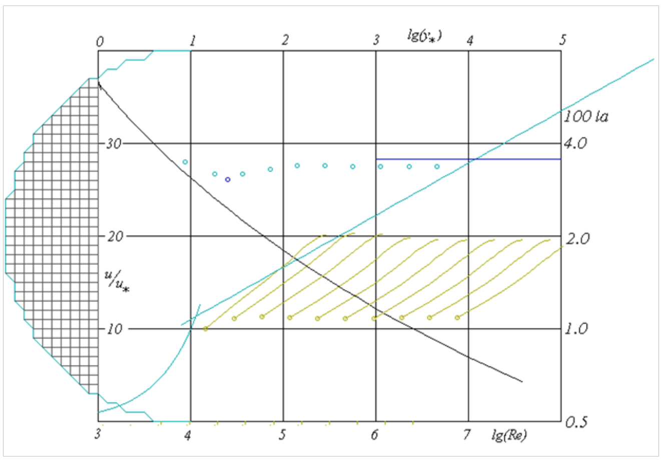


Figure 7. Results of two-dimensional calculation of flow in a pipe with the roughness $R/h_s = 60$, on a grid $h/R = 0.05$

4 Conclusion

In this paper, we present the results of a two-dimensional calculation of the flow in a pipe based on the CAM-2 turbulence model, where the turbulent viscosity balance equations are used. The technique for calculating two-phase flows in long pipelines is described, and with roughness. In the formulation of the corresponding conditions at the level of the maximum roughness height (or higher), this method is combined with the method of applying the near-wall functions describing the distribution of parameters near the wall and resting on experimental data. The results of calculations with this roughness description are given in the form of dependences of laminar and turbulent resistances on the Reynolds number, and in the form of a velocity distribution calculated from the equations of the model.

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С.К. Матвеев, Н.Ж. Жәйшібеков, Б.С. Шалабаева, Д. Жусупова

Құбырдағы стратифицирланған турбуленттік ағымның екіөлшемді есептеулері

Мақалада қосфазалы ортада көлбеу құбырлардағы стратифицирланған турбуленттік ағым қарастырылды. Жаңа турбуленттік модель [1] негізінде қосфазалы стратифицирланған, сонымен қатар құбырдың беті кедір-бұдыр жағдайдағы, турбуленттік ағымдарды зерттеу мақсатында, екіөлшемді ағымдар үшін программалық код жазылды. Созылған құбырдағы қосфазалы ағымдарды есептеудің әдістемесі ұсынылды. Турбулентті режимдегі тұрақты қима жағдайда стационарлы стратифицирланған қосфазалы ағымның есебі шешілді. Кедір-бұдырлы құбырлардың кедергісі және кедір-бұдырлықтың құбырдың кедергісіне әсері және жылдамдықтардың таратылуы есептелінді.

Кілт сөздер: стратифицирланған турбуленттік ағым, кедергі, екіөлшемді есептеулер, кедір-бұдыр бет.

С.К. Матвеев, Н.Ж. Жәйшібеков, Б.С. Шалабаева, Д. Жусупова

Двумерные расчеты стратифицированного турбулентного потока в трубе

В статье рассмотрено стратифицированное турбулентное течение двухфазной среды в наклонных трубах. На основе новой модели турбулентности [1] был разработан программный код для расчета двумерных течений для изучения двухфазных стратифицированных течений в трубах, в том числе с учетом шероховатости стенки трубопровода. Описана методика расчета двухфазных течений в протяженных трубопроводах. Численно решена проблема стационарного стратифицированного двухфазного потока в трубе постоянного сечения в случае турбулентного режима. Проведены расчеты сопротивления шероховатой трубы, и представлены результаты по влиянию шероховатости на сопротивление трубы и распределение скоростей.

Ключевые слова: стратифицированный турбулентный поток, сопротивление, двумерные расчеты, шероховатая поверхность.

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S. Pylypaka[1], M. Klendii[2], V. Trokhaniak[3]

[1]*National University of Life and Environmental Sciences of Ukraine, Kiev, Ukraine;*
[2]*Separated Subdivision of National University of Life and Environmental Sciences of Ukraine, Kiev, Ukraine;*
[3]*Berezhany Agrotechnical Institute, Ukraine*
(E-mail: trohaniak.v@gmail.com)

Particle motion over a plane, which rotates about a horizontal axis and makes a certain angle with it

Differential equations of relative material particle motion over a plane, which rotates about a horizontal axis, have been set up. Plane location relative to a rotation axis is set by a certain angle, which value can range from zero to ninety degrees. If angle value is equal to zero, a plane passes through a rotation axis; if angle value equals to 90 degrees, it is perpendicular to a rotation axis. The equations have been solved using numerical methods. In case of end positions of an angle, analytical solution has been found.

Keywords: soil-tilling disk, rotary motion, differential equations of motion.

Introduction

Particle motion over a horizontal plane in the form of a rough disk, which rotates about a vertical axis, is considered to be the most investigated one. Such disks with blades attached to them are used in scatters of centrifugal type. Operating elements with a horizontal axis of rotation in the form of a shaft with flat blades attached to it are used for spreading organic fertilizers. They can be also used to mix particles or scatter them in a centrifugal direction. It is interesting from theoretical perspective and it is to the point for the possibility of practical implementation to investigate particle motion over a plane, which rotates about a horizontal axis and makes a certain angle with it.

Compound particle motion over rough surfaces of operating elements of agricultural machinery has been considered in the major works [1–4]. They investigate particle motion over a horizontal disk, which rotates about a vertical axis, both without blades and with blades of the simplest designs. Paper [5] considers particle motion over a flat disk, which rotates about the axis that is inclined to the horizon. Patterns of particle motion over a disk both without blades and with straight blades arranged in radial direction from the axis of rotation have been investigated. The research presented in paper [6] is similar to ours but with the difference that the axis of rotation is not horizontal but a vertical one. This research considers relative particle motion in a wide range of angles of inclination of a plane to a rotation axis, beginning from a horizontal position and finishing with a vertical one. The development of a bladed operating element of a conveyer-mixer has been considered in paper [7]. There is a separate group of scientific papers, which investigate particle motion on a rough surface under the action of weight force [4, 8–15, 16, 17].

Material and method

Let us locate a plane in the form of a rectangle, which will be rotated, in three-dimensional coordinates $OXYZ$. Firstly, let it be in agreement with a horizontal plane OXY, here, we set our own coordinate axes: Ou axis lies in OX solid axis and Ov axis lies in OY axis (Fig. 1, a).

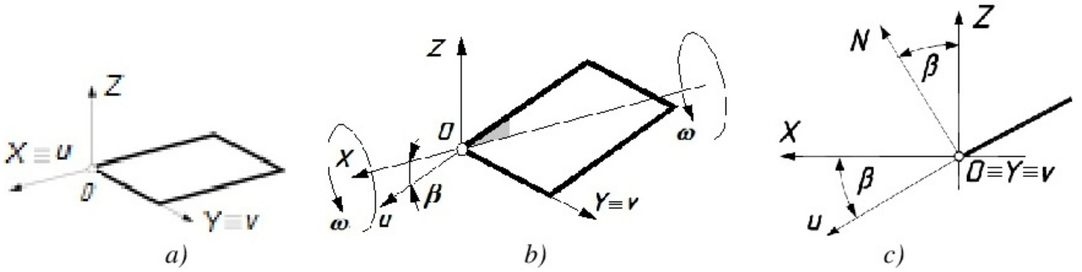


Figure 1. Location of a rectangular section of a plane in three-dimensional coordinates OXYZ

Let us rotate a plane about OY axis by a certain angle β (Fig. 1, b) and fix it in this position. Figure 1, c shows plane location in three-dimensional coordinates $OXYZ$, when OY axis of the latter is projected to a point and the plane itself is projected to a line.

Parametric equations of a plane in projections onto solid axes $OXYZ$ are written as:

$$X = u\cos\beta;\quad Y = v;\quad Z = u\sin\beta. \tag{1}$$

Let us rotate a plane about OZ axis with constant angular velocity ω. Then, during time t a moving plane rotates through an angle φ relative to a fixed system. The degree of rotation angle φ is determined by the familiar formula:

$$\phi = \omega t. \tag{2}$$

Parametric equations of a plane after its turn by the angle φ=ωt about OX axis are written as:

$$\begin{aligned} X &= u\cos\beta; \\ Y &= v\cos\omega t - u\sin\beta\sin\omega t; \\ Z &= v\sin\omega t + u\sin\beta\cos\omega t. \end{aligned} \tag{3}$$

Figure 2 shows a set of separate plane positions built at equal time intervals. It bends round a cone with axis.

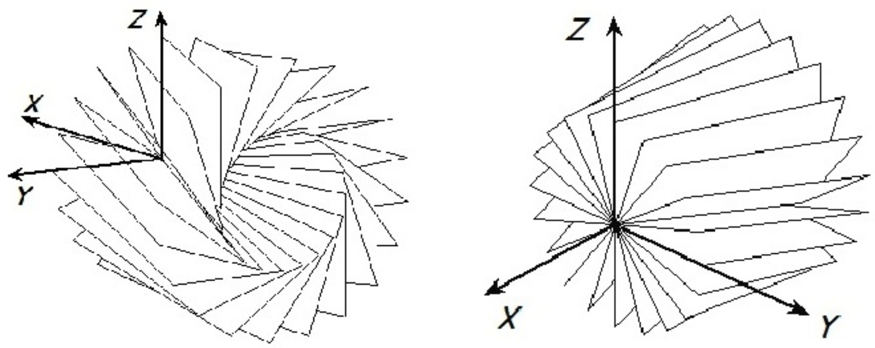


Figure 2. Set of separate plane positions built at its rotation about OX axis

During plane rotation a material particle slides over it having relative motion. Here, its coordinates u and v in a plane will vary with time, that is why, we consider them to be time-varying functions t: u=$u(t)$, v=$v(t)$. Such a relation between independent variables u and v through the third variable t describes a certain line in a plane – a trajectory of relative motion. Provided that $u = u(t)$ and $v = v(t)$, the equations (3) also describe a line in three-dimensional coordinate system – a trajectory of absolute motion of a particle.

Projections of absolute velocity and absolute acceleration of a particle on the axes of a fixed $OXYZ$ coordinate system are determined by successive differentiation of the equations (3), considering u and v to be unknown functions.

After differentiation of (3) and grouping of terms we get projections of absolute velocity:

$$\begin{aligned} x' &= u'\cos\beta; \\ y' &= (v' - u\omega\sin\beta)\cos\omega t - (u'\sin\beta + v\omega)\sin\omega t; \\ z' &= (v' - u\omega\sin\beta)\sin\omega t + (u'\sin\beta + v\omega)\cos\omega t. \end{aligned} \tag{4}$$

Here, let us denote coordinates x, y, z not by capital letters as it is in (3) but by lowercase letters, since we moved from the equations with two variables to the equations with one variable t. After differentiating the equations (4) and grouping of terms we get projections of absolute acceleration:

$$\begin{aligned} x'' &= u''\cos\beta; \\ y'' &= \left(v'' - v\omega^2 - 2u'\omega\sin\beta\right)\cos\omega t - \left(u''\sin\beta + 2v'\omega - u\omega^2\sin\beta\right)\sin\omega t; \\ z'' &= \left(v'' - v\omega^2 - 2u'\omega\sin\beta\right)\sin\omega t + \left(u''\sin\beta + 2v'\omega - u\omega^2\sin\beta\right)\cos\omega t. \end{aligned} \tag{5}$$

Let us establish a motion equation in the form of $m\overline{w} = \overline{F}$, where m — mass of a particle, $\overline{w}$ — vector of absolute acceleration, $\overline{F}$ — resultant vector of the forces exerted upon a particle. Such forces are weight force $mg(g = 9,81m/s2)$, reaction N of a plane and friction force fN at particle sliding over a plane f — friction coefficient). All the forces must be projected on the axes of a fixed coordinate system.

Weight force is downward-directed, thus, its projections can be written as: $\{0; 0; -mg\}$.

Plane reaction N is perpendicular to it (Fig. 1, c) and its projections are the following:

$$\{-N\sin\beta; 0; N\cos\beta\}. \tag{6}$$

Friction force is directed at a tangent to the trajectory of relative motion opposite to the direction of velocity. Let us find projections of velocity vector of relative motion by differentiating the expressions (1), assuming that *u=u(t)* and *v=v(t)*:

$$\begin{aligned} x' &= u'\cos\beta; \\ y' &= v'; \\ z' &= u'\sin\beta. \end{aligned} \tag{7}$$

Relative velocity value is determined by geometric sum of the components (7)

$$V = \sqrt{x'^2 + y'^2 + z'^2} = \sqrt{u'^2 + v'^2}. \tag{8}$$

A unit vector, which is at a tangent to the trajectory of relative motion, is determined from dividing the projections (7) by the value of velocity (8). Taking into account that friction force fN is directed opposite to the direction of relative particle motion, its projections can be written as:

$$\left\{-fN\frac{u'\cos\beta}{\sqrt{u'^2+v'^2}}; \quad -fN\frac{v'}{\sqrt{u'^2+v'^2}}; \quad -fN\frac{u'\sin\beta}{\sqrt{u'^2+v'^2}}\right\}. \tag{9}$$

Weight force (6) does not change its direction during plane rotation. Plane reaction force N (7) and friction force (9) depend on rotation angle φ (2) of a plane. Thus, they must be also turned through the angle $\varphi = \omega t$ about axis. Having taken that into consideration, projections of plane reaction force N take the following form:

$$\left\{\begin{array}{l} -N\sin\beta; \\ -N\cos\beta\sin\omega t; \\ N\cos\beta\cos\omega t \end{array}\right\}. \tag{10}$$

Projections of friction force after a turn through the angle $\varphi = \omega t$ can be written as:

$$\left\{\begin{array}{l} -fN\frac{u'\cos\beta}{\sqrt{u'^2+v'^2}}; \\ \\ -fN\frac{v'\cos\omega t - u'\sin\beta\sin\omega t}{\sqrt{u'^2+v'^2}}; \\ \\ -fN\frac{v'\sin\omega t + u'\sin\beta\cos\omega t}{\sqrt{u'^2+v'^2}}. \end{array}\right\}. \tag{11}$$

Let us set up a vector equation $m\overline{w} = \overline{F}$ in projections on the axes of a fixed three-dimensional coordinates, taking into account the applied forces (6), (10) and (11):

$$\begin{aligned} mx'' &= -N\left(\sin\beta + f\frac{u'\cos}{\sqrt{u'^2+v'^2}}\right); \\ my'' &= -N\left(\begin{array}{l}\cos\beta\sin\omega t + \\ +f\frac{v'\cos\omega t - u'\sin\beta\sin\omega t}{\sqrt{u'^2+v'^2}}\end{array}\right); \\ mz'' &= N\left(\begin{array}{l}\cos\beta\cos\omega t - \\ -f\frac{v'\sin\omega t + u'\sin\beta\cos\omega t}{\sqrt{u'^2+v'^2}}\end{array}\right) - mg. \end{aligned} \tag{12}$$

Let us substitute other derivatives (projections of absolute acceleration) from (5) into the equation (12). The obtained set of three equations contains three unknown functions: $N = N(t), u = u(t)$ and $v = v(t)$. Let us solve it for N, u'' and v'':

$$\begin{cases} u'' = -\sin\beta\left[g\cos\omega t + \omega\left(2v' - u\omega\sin\beta\right)\right] - u' f\cos\beta\frac{g\cos\omega t + \omega\left(2v' - u\omega\sin\beta\right)}{\sqrt{u'^2+v'^2}}; \\ v'' = -g\sin\omega t + \omega\left(v\omega + 2u'\sin\beta\right) - v' f\cos\beta\frac{g\cos\omega t + \omega\left(2v' - u\omega\sin\beta\right)}{\sqrt{u'^2+v'^2}}. \end{cases} \quad (13)$$

$$N = m\cos\beta\left[g\cos\omega t + \omega\left(2v' - u\omega\sin\beta\right)\right].$$

The first two equations make a system of nonlinear differential equations of the second order for the functions that describe the trajectory of relative particle motion over a plane.

Let us consider a partial case, when angle $\beta = 0$. In such a case a plane passes through axis which is its rotational axis (Fig. 1, a). Here, particle motion is possible in radial direction parallel to v axis at $u = const$. In this case the system (13) is simplified:

$$\begin{cases} u'' = -u' f\frac{g\cos\omega t + 2v'\omega}{\sqrt{u'^2+v'^2}}; \\ v'' = -g\sin\omega t + v\omega^2 - v' f\frac{g\cos\omega t + 2v'\omega}{\sqrt{u'^2+v'^2}}. \end{cases} \quad (14)$$

At u=*const* the first equation of the system (14) is changed into an identical equation and we obtain only one linear differential second-order equation:

$$v'' = -g\sin\omega t + v\omega^2 - f\left(g\cos\omega t + 2v'\omega\right). \quad (15)$$

The equation (15) has its analytical solution:

$$v = e^{-\left(f+\sqrt{1+f^2}\right)\omega t}\left(c_1 + c_2 e^{2\sqrt{1+f^2}\omega t}\right) + g\frac{2f\cos\omega t + \left(1-f^2\right)\sin\omega t}{2\left(1+f^2\right)\omega^2}. \quad (16)$$

For further determination of the values of integration constants $_1$ and $_2$ let us find the expression of relative velocity by differentiating the equation (16):

$$V = \frac{dv}{dt} = -e^{-\left(f+\sqrt{1+f^2}\right)\omega t}\omega\left[c_1\left(f + \sqrt{1+f^2}\right) + c_2\left(f - \sqrt{1+f^2}\right)e^{2\sqrt{1+f^2}\omega t}\right] + $$
$$+ g\frac{\left(1-f^2\right)\cos\omega t - 2f\sin\omega t}{2\left(1+f^2\right)\omega}. \quad (17)$$

Let a particle be separated from the axis of rotation by the distance v_0 at the initial time point, when a plane is in a horizontal position (that is to say, at $t = 0$), and have initial velocity V_0. Let us substitute $t = 0$ into (16) and (17) and equate these expressions to relative initial values of position and velocity and we obtain a set of two equations for unknown constants c_1 and c_2:

$$\begin{cases} v_0 = \frac{fg}{(1+f^2)\omega^2} + c_1 + c_2; \\ V_0 = \frac{g\left(1-f^2\right)}{2\omega(1+f^2)} - \omega\left[\sqrt{1+f^2}\left(c_1 - c_2\right) + f\left(c_1 + c_2\right)\right]. \end{cases} \quad (18)$$

Having solved (18) for c_1 and c_2, we obtain expressions for the determination of these constants:

$$c_1 = \frac{g\left(1 + f^2 - 2f\sqrt{1+f^2}\right) - 2\omega\left(1+f^2\right)\left[V_0 + v_0\omega\left(f - \sqrt{1+f^2}\right)\right]}{4\omega^2\left(1+f^2\right)^{3/2}};$$

$$c_2 = \frac{-g\left(1 + f^2 + 2f\sqrt{1+f^2}\right) + 2\omega\left(1+f^2\right)\left[V_0 + v_0\omega\left(f + \sqrt{1+f^2}\right)\right]}{4\omega^2\left(1+f^2\right)^{3/2}}.$$

Results

Figure 3 represents dependency graphs (16) and (17) under set initial conditions $v_0 = 0,2m$ and $V_0 = 0$ and friction coefficient $f = 0,3$. They show that a particle can move in radial direction either away from a rotation axis or towards it. It depends on the value of angular velocity of plane rotation. At $\omega = 5s^{-1}$- a particle moves towards a rotation axis, here, in the line of about $0,1s$ it practically does not move and only at its turn from a horizontal position to the angle φ, which is close to a friction angle, it begins to move.

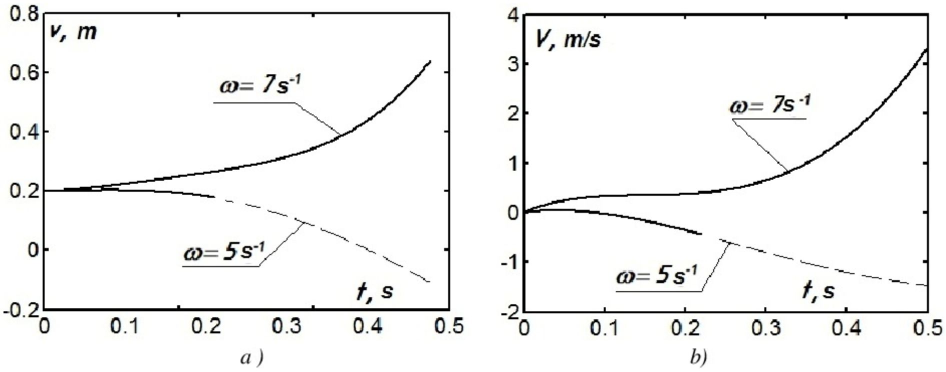
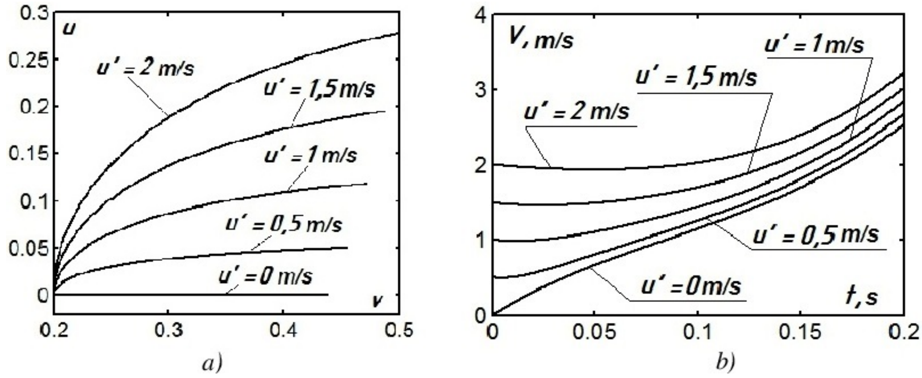


Figure 3. Graphs of distance $\nu = \nu(t)$ – a) and relative velocity $V = V(t)$ - b) at $\beta = 0$

However, without reaching a rotation axis, after 0,22 s plane reaction force is equal to zero, that is to say, a particle does not presses the surface (from this moment on the graphs of distance and velocity are represented using a dashed line). At $\omega = 7s^{-1}$ a particle moves away from a rotation axis and after $0,5s$ it is more than $0,6m$ away from it and it gains velocity of more than $3m/s$. Plane reaction force increases as well.

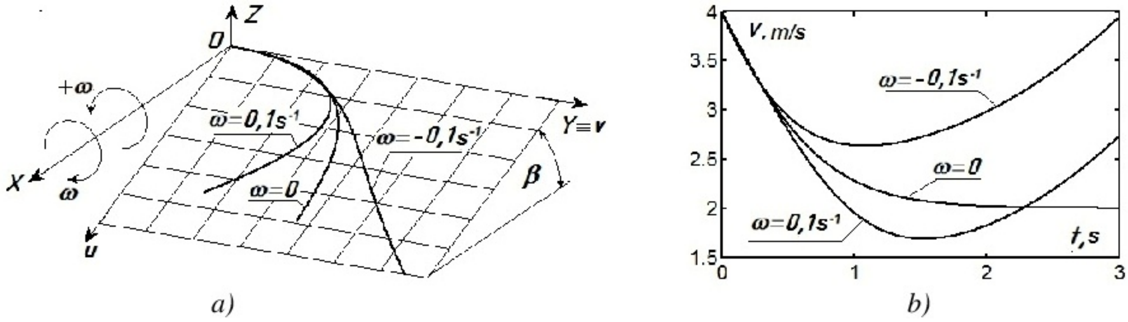
The equation (17) enables finding the parameters of particle motion in radial direction only. However, initial velocity of a particle entering a plane may have another direction, for example, the one that is parallel to a rotation axis as is the case in snow throwing machines for delivering snow on a rotor blade. In this case it is necessary to solve the system (14) using numerical methods at $u \neq const$ or the system (13) at $\beta = 0$. Figure 4 shows graphical representations of particle motion at $\beta = 0$, which enters a plane with various initial velocities along u axis and zero initial velocity in radial direction ($v' = 0$).



a) trajectories of particle sliding; b) graphs of velocities

Figure 4. Graphical representations of particle motion ($\beta = 0, \omega = 10s^{-1}, f = 0,3, \nu_0 = 0,2$ m)

In addition, the system of differential equations (13) makes it possible to find trajectories of particle sliding at $\omega = 0$, that is to say, when a plane is fixed. For example, let us take an inclination angle β of a plane which is equal to a friction angle. For f=0,3 this angle has the following value: $\beta = arctgf = 16,70$. Figure 5, a shows trajectories of particle motion at various angular velocities including $\omega = 0$ that have been built using numerical methods. At the beginning of its motion a particle was given initial velocity $V = 4m/s$ along $Y \equiv v$ axis. It traces out a curvilinear trajectory and after a certain time (3s) it enters a rectilinear trajectory, which coincides with the line of the greatest inclination of a plane. Here, its velocity is stabilized and it becomes twice less than the initial one (Fig. 5, b).



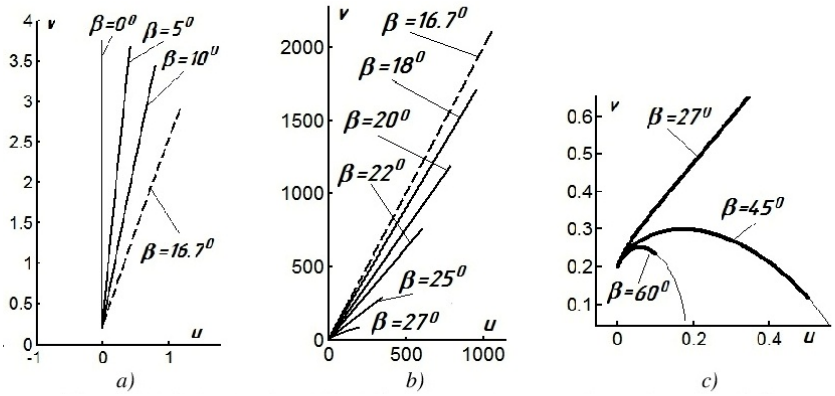
a) trajectories of sliding over a plane; b) graphs of change in velocity over time

Figure 5. Graphical representations of particle motion over a plane, which has an inclination angle β equal to a friction angle

Then a particle moves along the line of the greatest inclination with a stabilized velocity. Such a result is in a complete agreement with the result obtained in paper [18], which considers particle motion over an inclined plane at its side feed.

If a plane is given a rotating motion with low angular velocity under previous initial conditions, motion trajectories and graphs of velocity change (Fig. 5).

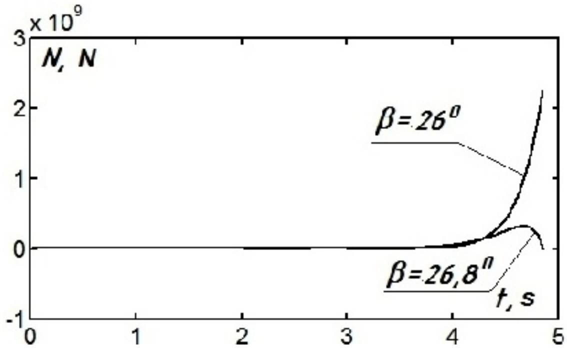
Let us consider a general case when angle $\beta 0$. At $\beta = 0$ and at $u = const$ (that is to say, without relative initial velocity along a rotation axis) a particle moves in radial direction (Fig. 4, a at $u' = 0$). Let us find out how a particle moves, if angle β is gradually increased. Figure 6 shows trajectories of particle sliding at various values of angle β. Figure 6, a represents trajectories that a particle traces in 0,5s. At $\beta = 0$ it moves in radial direction. If angle β increases, a particle trajectory begins to deviate from radial direction but it moves practically rectilinearly. The length of the distance covered gradually decreases. A dashed line is used to represent a trajectory in case when angle β is equal to a friction angle $\beta = arctgf$. Figure 6, b shows relative particle motion at further increase of angle β over the time of 1,5s. The figure shows that even an insignificant increase in angle β results in a sharp decrease in the distance covered. Here, the angle of deviation from radial direction increases and the trajectory remains rectilinear. At further increase of angle β, the trajectory of a particle becomes curvilinear and there is a moment when reaction of a plane becomes equal to zero, that is to say, there is detachment of a particle from a plane. Figure 6, c shows trajectories of relative particle motion during 0,5 s, here, a heavy line represents the trajectory where particles press a plane. At angle $\beta = 45^0$ the detachment of a particle from a plane happens in 0,4 s and at $\beta = 60^0$ – in 0,175 s. Thus, if angle β is close to 90^0, relative particle motion over a plane becomes impossible. At $\beta = 90^0$ it is impossible from a logical point of view, since a plane is vertical and there is no interaction with a particle.



$a) - \omega = 10s^{-1}$; $b) - f = 0,3$; $c) - \nu_0 = 0,2m$

Figure 6. Trajectories of particle sliding over a plane at various values of angle β

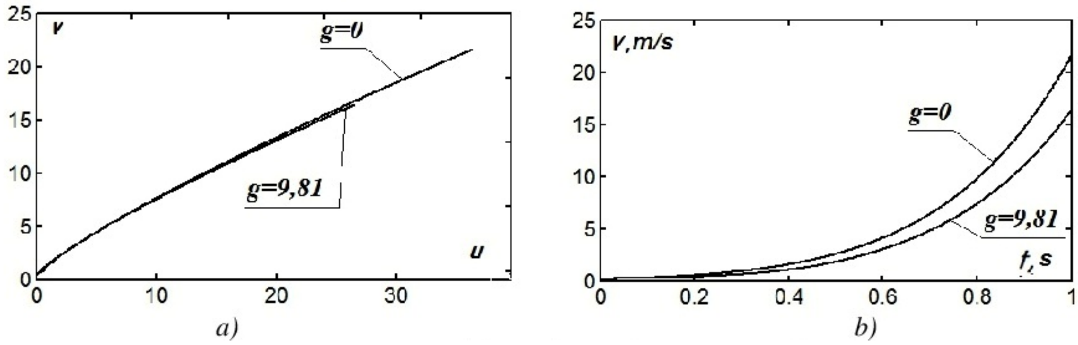
Thus, under our initial conditions and at angular velocity $\omega = 10^{s-1}$, transition from rectilinear to curvilinear trajectory of relative motion occurs, if angle β is close to 27^0. We have considered this transition in details during 5 s after the beginning of motion, when a particle covers a great distance. Figure 7, a represents trajectories of relative motion, which show that a change in angle β from 26*0* to 27*0*, that is to say, within the range of one degree, results in the change of a trajectory from rectilinear to curvilinear. If a trajectory is rectilinear, particle pressure on a plane increases and if a trajectory is curvilinear, it becomes equal to zero over a certain time. Figure 7, b shows graphs of pressure change for a particle weighting 0,01 kg. For a particle that moves along a curvilinear trajectory (at $\beta = 26,8^0$), pressure becomes equal to zero in 4,86 s. At great values of angle β the detachment of a particle happens almost immediately after its entering a plane as it has been previously shown.



a) motion trajectories; b) graph of pressure change

Figure 7. Graphical representations of particle motion during 5 s at $\omega = 10s^{-1}, f = 0,3, \nu_0 = 0,2m$

If there is curvilinear motion, a particle turns in the direction of u axis (Fig. 6, c). If a plane is positioned as it is shown in Figure 1, b, this direction coincides with the direction of the line of the greatest inclination of a plane. We can assume that such a change in the direction of a trajectory is the result of the action of weight force of a particle. However, calculations with dropping particle weight (at g=0) show that trajectories and graphs of velocity change differ with and without weight (Fig. 8). It can be explained by the fact that the main force, which causes particle deviation from a rectilinear trajectory, is the component of Coriolis force.



a) motion trajectories; b) a graph of relative velocity change

Figure 8. Graphical representations of particle motion during 1 s at $\beta = 27^0,\ \omega = 10s^{-1},\ f = 0,3, \nu_0 = 0,2s$:

If angle $\beta = 0$, Coriolis force is directed perpendicular to a plane and does not shift a particle from a rectilinear trajectory. A particle moves in radial direction under the action of centrifugal force. At $\beta \neq 0$ there is a component of Coriolis force that, at first, causes deviation of rectilinear motion from radial direction and then, if angle β increases, there is trajectory bending towards u axis.

If weight is ignored, a particle picks up speed faster and covers a greater distance. Obviously, it is due to the fact that friction force decreases in those moments of rotational motion of a plane when a particle is located under it.

Conclusions

When a particle enters an inclined plane, which rotates about a horizontal axis with constant angular velocity, a particle traces out a relative trajectory on a plane (a sliding trace), which form depends on the angle β between a plane and a rotation axis. If angle $\beta = 0^0$, a relative trajectory is a straight line along which a particle picks up speed in radial direction. At side material feed, that is to say, if a particle enters a plane having certain initial velocity in the direction that is parallel to a rotation axis, a particle traces a curvilinear trajectory on a plane, which approaches rectilinear radial direction over a certain time. If $\beta \neq 0$, the pattern of particle motion changes. At low angle values it moves in rectilinear direction, which makes a certain angle with radial direction. This angle increases, if angle β increases, here, the distance of particle sliding over a plane during the same time period decreases rapidly. If there is a further increase of angle β, there is a moment when the trajectory of particle motion becomes curvilinear; here, there is trajectory bending towards u axis. Here, a particle moves along a curvilinear trajectory only until a certain moment when plane reaction becomes equal to zero and there is particle detachment from a plane. If angle values are close to 90^0, particle motion over a plane becomes impossible, since it is detached from a plane almost immediately after entering. If $\beta = 90^0$, motion is impossible due to the fact that a plane is vertical and there is no interaction with a particle.

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С.Ф. Пилипака, Н.Б. Клендий, В.И. Троханяк

Көлденең осьті айналатын жазықтықта жататын және онымен белгілі бұрыш жасайтын бөлшектің қозғалысы

Көлденең осьті айналатын жазықтықтағы материалдық бөлшектердің салыстырмалы қозғалысының дифференциалдық теңдеулері құрастырылды. Айналу өсіне байланысты жазықтықтың орны мәні нөлден тоқсан градусқа дейінгі аралықтағы бұрышпен берілді. Бұрышы нөлге тең болған кезде жазықтың айналу осі арқылы өтеді, ал тоқсан градус болғанда ол айналу осіне перпендикуляр болады. Теңдеулер сандық әдістермен шешілді. Бұрыштық шеткі позициясы үшін аналитикалық шешім табылды.

Кілт сөздер: топырақ өңдейтін диск, айналмалы қозғалыс, қозғалыстың дифференциалдық теңдеулері.

С.Ф. Пилипака, Н.Б. Клендий, В.И. Троханяк

Движение частицы по плоскости, вращающейся вокруг горизонтальной оси и составляющей с ней определенный угол

Составлены дифференциальные уравнения относительного движения материальной частицы по плоскости, которая вращается вокруг горизонтальной оси. Положение плоскости по отношению к оси вращения задается углом, который может иметь значение из промежутка от нуля до девяноста градусов. При угле, равном нулю, плоскость проходит через ось вращения, при девяноста градусах она перпендикулярна оси вращения. Уравнения решены численными методами. Для крайних положений угла найдено аналитическое решение.

Ключевые слова: почвообрабатывающий диск, вращательное движение, дифференциальные уравнения движения.

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M.O. Kenzhebayeva

Al-Farabi Kazakh National University, Almaty, Kazakhstan;
U.A. Dzholdasbekov Institute of Mechanics and Mechanical Engineering, Almaty, Kazakhstan
(E-mail: merey-mex-2017@mail.ru)

Comparison the readings of gravity field and it‘s gradient potential

The problem of studying the deep structure of the earth's crust is one of the strategic directions of geophysical research, ensuring the development of Earth sciences. Therewith, gravimetry is one of the main methods for studying the structure of the earth's crust. This study is associated with the concepts of gravitational field‘s potential and gradient of the underground anomaly. Solving the inverse problem of restoring density of the underground anomaly, thorough analysis of the direct problem plays an important part. This study analyzes the characteristic features of the readings of anomaly gravitational field‘s potential and gradient. Based on the results obtained by the author, it has revealed and verified the necessity of choosing a gradient as the boundary conditions of the direct problem, which will significantly improve the inverse problem calculations results – finding the density of anomaly. This research demonstrates quantitatively that anomaly gravitational field‘s gradient more accurately describes the anomaly, compared to the gravitational field‘s potential.

Keywords: inverse problem, gravitational potential, gradient method.

Introduction

Inverse problems cover a wide range of applied problems. Among hyperbolic, elliptic and parabolic inverse problems, elliptic problems are very inaccurate. In this regard, the problem itself is very complicated. In the book [1] a wide range of tasks of all types is considered. The inverse problems of hyperbolic type [1] are represented especially widely. Previous results obtained by the authors [2, 3] were published in the studies. There is a general overview of the situation on the issues of gravimetry at the field [4, 5]. On the basis of gravimetric readings on the Earth surface in the studied area, we use the mathematical apparatus for solving inverse problems of elliptic type.

When constructing a mathematical model, we simplified the objects under study as much as possible. Consider a vertical section of the ground. For simplicity, we chose it in the shape of a rectangle. It is known that inside this area in a certain place there is an anomaly, but it is unknown what type of anomaly it is (what is its density). The area of anomaly is known, and we will designate it by Ω. On the earth surface we have gravimetric readings of the field potentials $\eta_1(x)$ and its gradient $\eta_2(x)$. We denote the lower and lateral subsurface boundaries by Γ as shown in Figure 1. We artificially expand the study area so that the value of the gradient potential of the anomaly field does not affect the external boundaries.

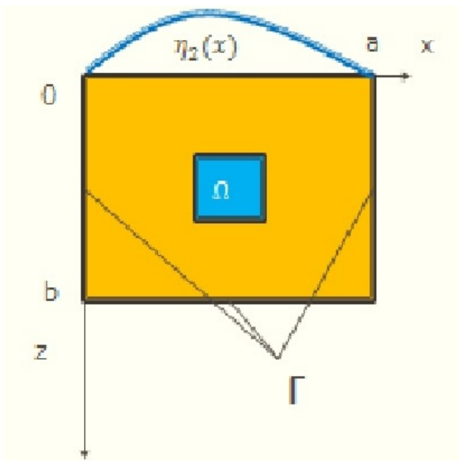

Figure 1. Interpretation of the simplified model of the problem

The equations of condition (*direct problem*) are described by the following formulas. The potential difference between perturbed and non-perturbed gravitational fields is described by the Poisson equation with boundary conditions (1)–(4). The expression (3)–(4) characterizes the results of measuring the potential and its derivative on the outer surface. Condition (5) characterizes the difference in the density of the soil with and without anomaly over the entire study area.

$$\Delta\ \eta(x,z) = -\ 4\pi G\psi(x,z); \tag{1}$$

$$\eta(x,z)|_{\Gamma} = 0; \tag{2}$$

$$\eta(x,0) = \eta_1(x); \tag{3}$$

$$\frac{\partial\eta(x,0)}{\partial z} = \eta_2(x); \tag{4}$$

$$\psi(x,z) = \begin{cases} 0, & \text{out of } \Omega; \\ \psi_0, & \text{in } \Omega, \end{cases} \tag{5}$$

where $\eta(x,z)$ is the potential of the gravitational field, G is the gravitational constant, $\psi(x,z)$ is the anomaly density, $\eta_1(x)$ is the measured readings of the gravitational field, $\eta_2(x)$ are the measured values of the gradient of the gravitational field, Γ is the boundary of the study area without the earth's surface, Ω is the anomaly region.

In the direct problem, we consider density as a known value. We calculate the value of the gravitational field potential and its gradient on the studied area using one of the conditions (3) or (4).

Inverse problem is a search for the anomaly density based on the results of measuring the potential and its derivative on the outer surface. The inverse problem is solved by optimization method, more precisely - a gradient method. It is necessary to introduce a functional using the standard deviation as a minimization parameter.

First statement of optimization problem looks as follows:

$$\Delta\ \eta(x,z) = -\ 4\pi G\psi(x,z);$$

$$\eta(x,z)|_{\Gamma} = 0;$$

$$\eta(x,0) = \eta_1(x);$$

$$\frac{\partial\eta(x,0)}{\partial z} = \eta_2(x);$$

$$\psi(x,z) = \begin{cases} 0, & \text{out of } \Omega; \\ \psi_0, & \text{in } \Omega; \end{cases}$$

$$\mathrm{I}\,(\psi_0) = \int_0^{\mathrm{L}} \left(\frac{\partial\eta(\mathrm{x},0)}{\partial \mathrm{z}} - \eta_2(\mathrm{x})\right)^2 \mathrm{dx} \to \min.$$

Second statement of optimization problem looks like this:

$$\Delta\ \eta(x,z) = -\ 4\pi G\psi(x,z);$$

$$\eta(x,z)|_{\Gamma} = 0;$$

$$\eta(x,0) = \eta_1(x);$$

$$\frac{\partial\eta(x,0)}{\partial z} = \eta_2(x);$$

$$\psi(x,z) = \begin{cases} 0, & \text{out of } \Omega; \\ \psi_0, & \text{in } \Omega; \end{cases}$$

$$\mathrm{I}\,(\psi_0) = \int_0^{\mathrm{L}} (\eta(x,0) - \eta_1(\mathrm{x}))^2 \mathrm{dx} \to \min.$$

We do not know yet, which of these forms is better. However, some information about the properties of these optimization problems can be obtained on the basis of a quantitative analysis of the direct problem. We want to find out what happens at the upper boundary (earth surface) at different locations of the anomaly inside the area.

Suppose we need to examine a region with a size of 100 horizontally and 50 vertically. We artificially expand the region up by 100 in order to analyze what will happen at the upper boundary z = 50. All calculations were performed on COMSOL Multiphysics 5.2 (Fig. 2). The anomaly has a dimension of 2 to 2. We will change the location of the position of the anomaly horizontally along j = 0, 10, 20, 30, 35, 40, 45. It will be sufficient to change the anomaly location to the middle of the study area, since earlier in the studies we found that the results are symmetrical. The vertical shifts along i = 5, 10, 15, 20, 30, 40. Figure 3 shows the surge of the gravitational potential on the surface z = 50, that is, theoretically, the readings of a gravimeter on the surface of the earth. Figure 4 shows a graph of the value of the anomaly gravitational field gradient, located as in Figure 2. Later in the tables, we analyzed the indication of the potential and its gradient for different variations of the anomaly location.

Figure 2. Anomaly location at the extended upper boundary.
The angle of the lower left edge of the anomaly is located at (20;10)

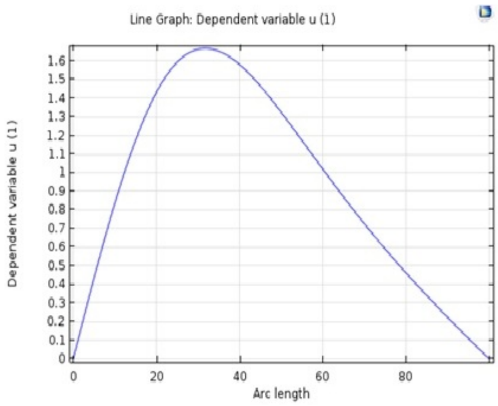

Figure 3. The value of the gravitational field potential on the surface z = 50
at the anomaly location in the coordinate (20;10)

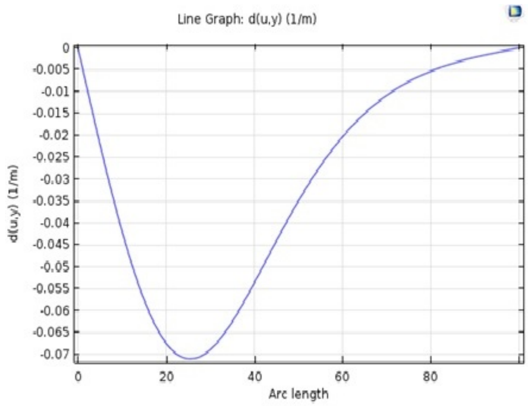

Figure 4. Gradient value of the gravitational field on the surface z = 50
at the anomaly location in the coordinate (20;10)

Below there are table of symbols describing the parameters of the gravitational field potential and gradient.

• i is the line – horizontal position, bottom up;
• j is the column – vertical position, left to right;
• a is the horizontal coordinate of the anomaly;
• ap is the coordinate of the potential peak on the outer surface;
• ag is the coordinate of the potential gradient peak on the outer surface;
• bp is the value of the potential peak on the outer surface;
• bg is the value of the gradient peak on the outer surface;
• cpl is the coordinate of the point to the left of the peak with a drop in the potential value by an order of magnitude compared with the peak value;
• cgl is the coordinate of the point to the left of the peak with the gradient value falling by an order of magnitude compared with the peak value;
• cpr is the coordinate of the point to the right of the peak with a drop in the potential value by an order of magnitude compared with the peak value;
• cgr is the coordinate of the point to the right of the peak with the gradient value falling by an order of magnitude compared with the peak value.

Consider the most extreme point of the anomaly both vertically and horizontally (Tables 1, 2).

Table 1

Horizontally anomaly location at x = 5 (i = 5 line). The anomaly «runs through» from left to right to the middle of the area (depending on the j column)

j	a	ap	ag	bp	bg	cpl	cgl	cpr	cgr
0	5	19	20	0.04	-0.01	18.54	13.91	38.82	28.26
10	5	29	21	0.51	-0.02	19.43	15.18	40.67	29.96
20	5	32	26	0.88	-0.04	22.26	18.22	44.27	34.82
30	5	37	32	1.14	-0.04	26.72	23.88	49.78	42.13
35	5	40	37	1.22	-0.04	29.15	27.53	53.13	46.67
40	5	43	41	1.28	-0.05	31.97	31.97	56.27	51.35
45	5	47	46	1.31	-0.05	35.21	36.42	59.51	56.06

Table 2

Vertical position of the anomaly at y = 0 (j = 0 column). The anomaly «runs» from bottom to top to the middle (from the depth to the surface) of the region z = 50 (depending on the i line)

i	a	ap	ag	bp	bg	cpl	cgl	cpr	cgr
5	5	19	21	0.04	-0.002	18.54	13.91	38.82	28.26
10	10	26	19	0.09	-0.004	17.85	13	37.99	26.69
15	15	25	18	0.13	-0.007	16.95	12.11	36.25	24.83
20	20	23	15	0.19	-0.011	15.11	10.67	33.49	22.43
30	30	17	10	0.37	-0.028	10.89	7.26	25.83	15.28
40	40	9	5	0.86	-0.128	5.53	3.65	13.9	7.31

Now consider the central location of the anomaly (Table 3).

Table 3

Vertical location of the anomaly at y = 45 (j = 45 column). The anomaly «runs» from bottom to top to the middle (from the depth to the surface) of the region z = 50 (depending on the i line)

i	a	ap	ag	bp	bg	cpl	cgl	cpr	cgr
5	5	47	46	1.31	-0.04	35.21	36.42	59.51	56.06
10	10	47	46	2.45	-0.08	35.17	36.72	59.35	55.64
15	15	46	46	3.68	-0.13	35.56	37.31	59.03	54.99
20	20	47	45	5.05	-0.18	36.01	37.96	58.2	54.12
30	30	46	46	8.59	-0.35	37.32	40.01	55.96	51.82
40	40	46	45	14.71	-0.84	40.34	43.05	52.17	49.01

Studying the obtained results, it was found that the value of the gravitational field gradient more accurately describes the anomaly location and provides the most accurate anomaly center readings and boundaries. All calculation tables (all possible combinations of location) were in favor of the gravitational field gradient. Thus, it is better to be guided by the readings of the gravitational field gradient in the search for the vertical anomaly location.

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М.О. Кенжебаева

Аномалиядағы гравитациялық өрістің потенциалын және градиенттін талдау

Жер қыртысының терең құрылымын зерттеу мәселесі Жер туралы ғылымды дамытуды қамтамасыз ететін геофизикалық зерттеулердің стратегиялық бағыттарының бірі болып табылады. Сонымен қатар гравитациялық барлау жер қыртысының құрылымын зерттеудің негізгі әдістерінің бірі болып есептеледі. Бұл жұмыс жер асты аномалиясының гравитациялық өрісінің потенциалы мен градиенті туралы түсініктермен байланысты. Жер асты аномалиясының тығыздығын қалпына келтірудің кері есебін шешу тікелей есепті мұқият талдауда маңызды рөл атқарады. Мақалада аномалияның гравитациялық өрісінің потенциалы мен градиентінің белгілеріне тән ерекшеліктері талданды. Авторлармен алынған нәтижелер негізінде тікелей есептің шекаралық шарттары ретінде градиентті таңдау қажеттілігі анықталды және дәлелденді. Бұл кері есепті шығару нәтижелерін – аномалияның тығыздығын іздеуді жақсартады. Аталған жұмыста аномалияның гравитациялық өрісінің градиенті, гравитациялық өрістің потенциалына салыстырмалы түрде қарағанда, аномалияны нақтырақ анықтайтындығы сандық көрсеткіштермен сипатталды.

Кілт сөздер: кері есеп, гравитациялық потенциал, градиент әдісі, тікелей есептің шекаралық шарттары.

М.О. Кенжебаева

Анализ градиента и потенциала гравитационного поля аномалии

Проблема изучения строения земной коры является одним из стратегических направлений геофизических исследований, обеспечивающих развитие науки о Земле. При этом гравиразведка является одним из основных методов изучения строения земной коры. Данная работа связана с понятиями потенциала и градиента гравитационного поля подземной аномалии. При решении обратной задачи восстановления плотности подземной аномалии важную роль играет тщательный анализ прямой задачи. В статье проанализированы характерные особенности показания потенциала и градиента гравитационного поля аномалии. На основе результатов, полученных автором, выявлена и обоснована необходимость выбора градиента в качестве граничных условий при решении прямой задачи. Кроме того, количественно показано, что градиент гравитационного поля аномалии точнее описывает аномалию, по сравнению с потенциалом гравитационного поля.

Ключевые слова: обратная задача, гравитационный потенциал, градиентный метод, решение прямой задачи.

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S.V. Voronenko

Kherson State Maritime Academy, Ukraine
(E-mail: vr.sergey@ukr.net)

Research of possibilities of characteristics and parameters increase of asynchronous machines

Improving the efficiency and reliability of ship electric power systems is a priority task for modern shipbuilding. One of the ways to solve this problem is to use the power of the main power plant for the production of electricity. In this case, it is advisable to give preference to asynchronous machines as the simplest in production and reliable in operation. It should also be noted that the asynchronous machine can be made with a massive ferromagnetic rotor or a sleeve with the use of nanotechnology, which will improve the energy parameters and performance of the electrical machine and the system as a whole. The article analyzes the characteristics of the device and the operation of an asynchronous machine (AM) with a short-circuited rotor with a ferromagnetic sleeve. Dependence of AM parameters and characteristics on the geometric dimensions of the sleeve is shown, and a technique for their determination using the system of Maxwell differential equations is proposed. An algorithm for calculating the characteristics of an AM machine and the equivalent parameters of a ferromagnetic sleeve is developed.

Keywords: asynchronous machine, rotor, ferromagnetic sleeve, slip, equivalent active and inductive resistances.

Introduction

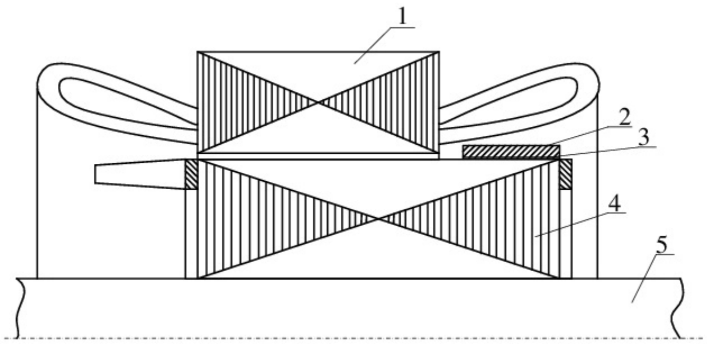
Target setting. The modern stage of development of shipbuilding is characterized by an increase in the efficiency and reliability of ship power systems. One of the main directions of an increase in the efficiency is the improvement of waste-heat recovery system, including heat recovery of exhaust gases of main and auxiliary diesel engines using turbochargers with built-in electric machines [1]. Taking into account high speeds of rotation of turbochargers as electric machines in these systems it is expedient to use machines with a massive ferromagnetic rotor or with a ferromagnetic sleeve. At the same time, the efficiency of exhaust gas heat recovery is improved, and the operation of the compressor in different operating modes of the diesel engine is improved.

Analysis of basic researches and publications. Applications of integral turbocharger systems using an electric machine with permanent magnets, as well as asynchronous machines with a short-circuited rotor are known [2, 3]. At the same time, the problematic issues are ensuring the mechanical strength of the rotor and problems of regulating the voltages and powers of electric machines [4]. The solution of the above problems is greatly simplified by using asynchronous machines with a massive ferromagnetic rotor using nanotechnologies in the design of ferromagnetic sleeves. Regulation of voltages, frequencies and powers in such systems is provided by means of semiconductor frequency converters.

Objective of the work is to analyze the characteristics of asynchronous machines with a ferromagnetic sleeve and to develop an algorithm for determining the parameters of the sleeve and calculating the operating characteristics of machines.

Materials, methods and results of research

Improvement of reliable and mass-dimension characteristics of asynchronous machines can be achieved by applying a special design rotor with frequency-dependent parameters, including using nanotechnology. One of the directions of the solution of such a problem can be the use of a rotor with a ferromagnetic sleeve [3, 5]. In such a design, the length of a conventional short-circuited rotor pack is 20–35 % greater than the active length of the stator pack. The elongated part of the rotor is pressed with a sleeve with soft magnetic steel (Fig. 1).



1 – stator, 2 – ferromagnetic sleeve, 3 – dielectric interval, 4 – rotor with short-circuit winding, 5 – shaft

Figure 1. The design of a rotor with a ferromagnetic sleeve

A distinctive feature of this design is that the ferromagnetic sleeve is placed under the frontal parts of the stator winding and it does not require an increase in the overall length of the machine. The processes of energy conversion in the AM of the proposed design proceeds in the same way as in the AM of the standard design but the difference consists in the following. The main magnetic flux of the machine induces an electromotive force $\dot{E}_2$ in the rotor winding, under the action of this EMF, a current $\dot{I}_2$ flows in the rotor winding, creating a magnetic flux going through the circuit: a massive sleeve, rotor clutch, a rotor back. The resulting magnetic flux passing through this circuit and formed by the combined action of the MMS of rotor winding and the eddy currents of the massive sleeve leads to an EMF$\dot{E}_5$ in the rotor winding. This EMF significantly limits the current flowing in the short-circuited rotor winding, and leads to a significant reduction in the starting current. This EMF can be represented as a sum of two components, which are voltage drops on the equivalent active r_5 and inductive x_5 resistances of a massive sleeve:

$$-\dot{\mathrm{E}}_5 = r_5\dot{I}_2 + jx_5\dot{I}_2.$$

Thus, additional active-inductive resistance is put into the circuit of rotor winding, where the voltage drop from the rotor current is equivalent to the mentioned EMF$\dot{\mathrm{E}}_5$. In this case, the value of the resistance significantly depends on the frequency of the rotor current [5]. In low slip modes of the machine, and therefore at a low rotor current frequency, a small resistance is introduced into the rotor circuit, so the rigidity of the mechanical characteristic, the power factor and the efficiency factor of the AM with the rotor of the proposed design is reduced insignificantly, compared to AM with a short-circuited female rotor. As the rotor speed of the AM female rotor decreases, and the sliding increases, a complex additional resistance with the predominance of the active component is automatically introduced into the rotor circuit, which leads to an increase in the electromagnetic moment of AM, while limiting the growth of the consumption current, and also to widening the control range. In this case, the necessary AM characteristics are obtained by choosing the appropriate geometric dimensions: the length and thickness of the sleeve.

At present, two methods for determining the parameters of the AM equivalent circuit with distributed secondary parameters is applied in practical calculations:

1. The calculation method, based on the classical works of Neumann on the theory of skin effect [5]. The advantage of these formulas is that they take into account both the impermanence of the magnetic permeability, and the loss of eddy currents and hysteresis. However, they are valid with sharp cases of the skin effect, when the cross dimensions of the magnetic circuit exceed twice depth at which the electromagnetic wave completely damps.

2. A calculation technique based on solving a system of differential equations of the electromagnetic field with finding the Pointing vector. The components of the complex Pointing vector is the flow of active and reactive power through a unit of surface. By the found active and reactive power, equivalent parameters of the equivalent circuit are determined.

In the study of electromagnetic processes in AM with a ferromagnetic sleeve on the rotor, the main attention is paid to the consideration of issues associated with the reflection of an electromagnetic wave from the outer surface of the sleeve, which has a thickness less than twice equivalent depth of penetration of the electromagnetic wave. The expediency of choosing such a sleeve thickness is explained by the fact that at this thickness it is possible to achieve high rigidity of the mechanical characteristic and the best energy parameters in the nominal mode [3].

The method based on solving a system of differential Maxwell equations makes it possible to calculate the electromagnetic field for any sleeve thickness, so we choose it for further calculations. In order to obtain a simple solution in the form of functional dependencies, the problem is solved in a rectangular coordinate system with the following assumptions:

– the magnetic permeability μ in the entire volume of the massive sleeve at the derivation of the main relations is assumed to be constant and only in the final formulas it is considered as a function of the magnetic field strength;

– the inner surface of the sleeve (Fig. 2) turns into a plane.

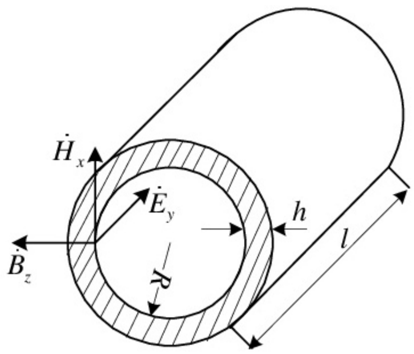


Figure 2. Surface of a ferromagnetic sleeve

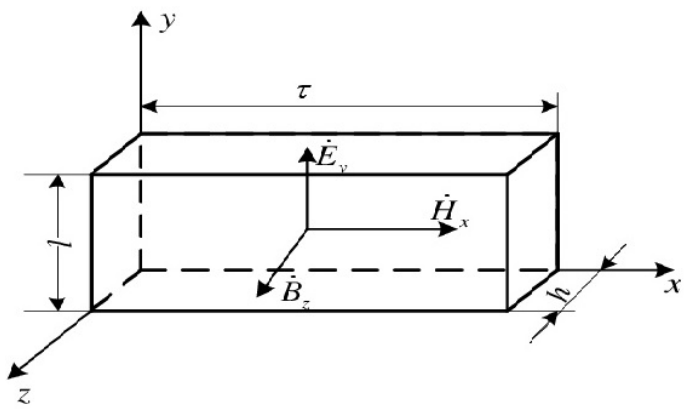


Figure 3. View of the surface of a ferromagnetic sleeve in a rectangular coordinate system

The base of the rectangular coordinate system is located on the inner surface of the massive sleeve: the x axis is directed along the circumference of the rotor, the y axis is parallel to the rotor axis, the z axis is along the radius (Fig. 3):

– the influence of higher harmonics is not taken into account;

– the short-circuited rotor winding is represented as an infinite current layer distributed on the surface of the steel of rotor body, the magnetic permeability of the rotor steel is assumed to be equal to infinite distance;

– in the coordinate system that is stationary relative to the AM rotor, the electric field strength is constant along the length of the sleeve, and along the coordinate x it varies according to the law:

$$E_y(x,z,t) = E_{ym}(z)\cos\left(\omega t - \frac{\pi x}{\tau}\right),$$

where ω — angular frequency of the rotor current; τ — pole pitch.

Substituting given equation into Maxwell system of differential equations and performing the transformations, we obtain following equation:

$$\dot{E}_{ym} = A \cdot sh(\lambda \cdot z) + B \cdot ch(\lambda \cdot z),$$

where A and B — constant integration,

$$\lambda = \alpha(1+j), \alpha = \sqrt{\frac{\mu\omega\gamma}{2}} = \frac{1}{\delta_M},$$

where δ_M — the equivalent depth of penetration of the electromagnetic wave into the sleeve in the direction of the z coordinate.

Constant integration is determined from the boundary conditions:

a) we assume that the tangential component of the magnetic field strength on the inner surface of the ferromagnetic sleeveH_s is:

$$A = \frac{j\omega\mu}{\alpha\,(1+j)} H_s;$$

b) the normal components of the induction of the magnetic field at the interface between the two environments (the outer surface of the ferromagnetic sleeve - air space) are equal to each other. Since the magnetic resistance of the air is much greater than the magnetic resistance of the sleeve, the normal component of the induction of the magnetic field on the outer surface of the massive sleeve must be zero, so:

$$B = -\frac{j\omega\mu}{\alpha\,(1+j)} th\left[\alpha\,(1+j)\,h\right] H_s.$$

The total power of the electromagnetic field penetrating into the ferromagnetic sleeve through a unit of surface is:

$$\dot{S}_z = S_p + jS_q = -\frac{1}{2}\left[\dot{E}_{ym}\cdot H^*_{xm}\right]_{z=0}.$$

And the active and reactive components are determined accordingly:

$$S_p = \frac{1}{2} n \sqrt{\frac{\mu\omega}{2\gamma}} H_s^2;$$

$$S_q = \frac{1}{2} m \sqrt{\frac{\mu\omega}{2\gamma}} H_s^2,$$

where n and m are coefficients that take into account the geometric dimensions of the ferromagnetic sleeve, γ is electrical conductivity of steel.

$$n = \frac{th(\alpha h) + tg^2(\alpha h)\cdot th(\alpha h) + th^2(\alpha h)\cdot tg(\alpha h) - tg(\alpha h)}{1 + th^2(\alpha h)\cdot tg^2(\alpha h)};$$

$$m = \frac{th(\alpha h) + tg^2(\alpha h)\cdot th(\alpha h) - th^2(\alpha h)\cdot tg(\alpha h) + tg(\alpha h)}{1 + th^2(\alpha h)\cdot tg^2(\alpha h)}.$$

Analytical expressions for the active and inductive resistances of the sleeve, which are equivalently introduced, are got in a result of solving the system of Maxwell equations under the condition of the constancy of the magnetic permeability

$\mu = const$:

$$r_5' = 4n\sqrt{\frac{\mu_0\omega_0}{2\gamma}}\cdot\sqrt{\mu_s s}\cdot\frac{m_1\,(k_{w1}w_1)^2}{\pi d}\cdot l;$$

$$x_5' = 4m\sqrt{\frac{\mu_0\omega_0}{2\gamma}}\cdot\sqrt{\mu_s s}\cdot\frac{m_1\,(k_{w1}w_1)^2}{\pi d}\cdot l,$$

where μ_0 — magnetic constant; ω_0 — angular frequency of the current of the AM male rotor; s — AM slip value, m_1; k_{w1}, w_1 — the number of phases, the winding factor, the number of turns of the winding of the AM male rotor; μ_s — relative magnetic permeability on the inner surface of the ferromagnetic sleeve, determined from the main magnetization curve from the value of the magnetic field strength on this surface H_s.

Final expressions for the determination of resistances, which are equivalently introduced, taking into account the effect of the impermanence of the magnetic permeability and hysteresis losses on the equivalent parameters of the sleeve, and also the effect of the edge effect are taken into account with the help of the corresponding coefficients:

$$r_5' = k\alpha_p n \cdot \sqrt{\mu_s s} \cdot l \cdot k_e;$$

$$x_5' = k\alpha_q m \cdot \sqrt{\mu_s s} \cdot l \cdot k_e,$$

where α_p α_q — coefficients, that take into account the change in magnetic permeability in the sleeve and losses due to hysteresis; $k_e = 1 + \frac{d}{pl}$ — coefficient of edge effect; p — number of pairs of AM poles.

$$k = 4\sqrt{\frac{\mu_0\omega_0}{2\gamma}} \cdot \frac{m_1\left(k_{w1}w_1\right)^2}{\pi d}.$$

So, the active and inductive resistances of the ferromagnetic sleeve, which are equivalently introduced, depend on the geometric dimensions of the sleeve, the relative magnetic permeability on its inner surface, the number of pole pairs and the AM slip.

In turn, the magnetic permeabilityμ_s depends on the strength of the magnetic field H_s, and hence on the value of the current of the female rotor I_2', which acts.

On the basis of the above calculation procedure for the parameters of the ferromagnetic sleeve, which are equivalently introduced, the sequence of calculating the static performance of an asynchronous machine of a special design can be described by the following algorithm (Fig. 4):

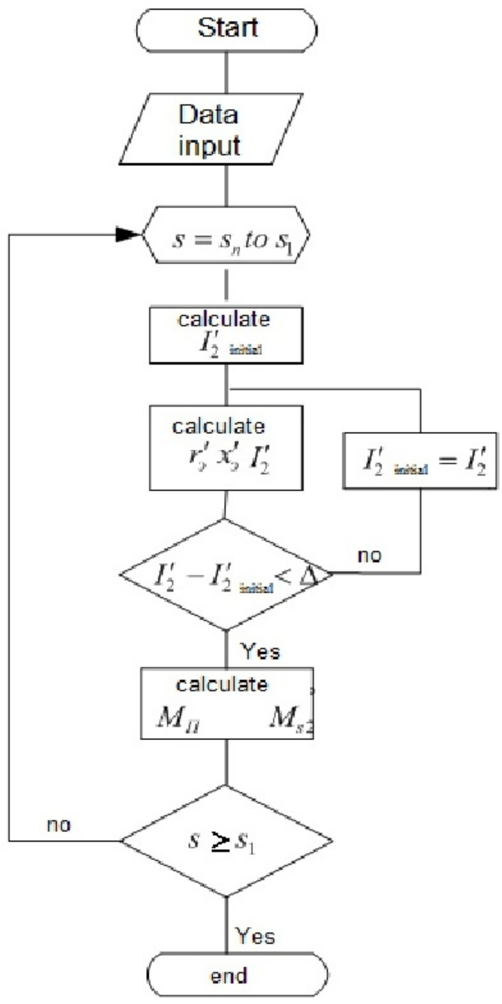


Figure 4. Algorithm for calculating the operating and starting characteristics of AM with a massive sleeve on the rotor

1. We set the initial values of the slip $s_{initial}$ of AM, the axial length l_{initial} and the radial thickness h_{initial} of the ferromagnetic sleeve.

2. From the above formulas, we find the initial approximation of the current of the female rotor I'_{20} without taking into account the parameters of the sleeve ($z'_{\text{equivalent}} = 0$).

3. Determine the magnetic field strength on the inner surface of a massive sleeve, taking into account the connection of the tangential component of the magnetic field strength on the inner surface of the sleeve and the value of the current of female rotor: $H_s = \sqrt{2} \cdot \dfrac{2m_1 k_{w1} w_1}{\pi d} I'_2$

4. With the magnetization curve of the sleeve material $\mu_s = f(H_s)$, we find the magnetic permeability μ_s. In this case, it is expedient to use the approximated expression of the magnetization curve.

5. The found value of the magnetic permeability, taking into account the parameters n and m, determined from the above equations, correspond to the equivalent parameters of a massive sleeve r'_e and x'_e.

6. Calculate the current of the female rotor I'_2, reduced to the winding of the male rotor, taking into account the parameters of the ferromagnetic sleeve found in paragraph 4.

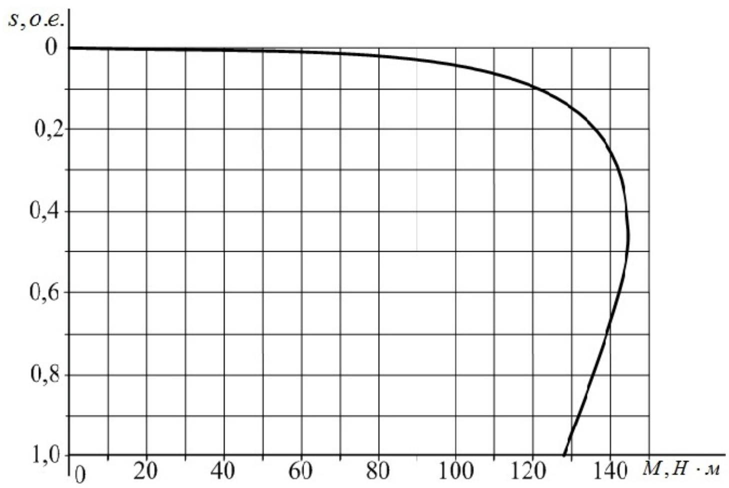
7. Specify the initial approximation of the current of the female rotor and repeat the calculation of the parameters until the initial approximation and the calculated value of the current I'_2 become equal to each other with the necessary accuracy.

8. Calculate the working and starting characteristics of AM.

9. Set the increment Δs and perform the calculation of the characteristics for a given slip range.

When implementing the algorithm, it is assumed that the known values are: the number of pairs of poles of the synchronous and asynchronous machines, the number of turns and the winding coefficient of the male rotor, the parameters of the AM equivalent circuit, the coefficients of the curve for approximating the magnetization curve of the ferromagnetic sleeve material (in our case, the material is steel 3).

The calculation is performed for some slip points s in the range from s_{initial} up to $s_{\max}$ with increments Δs each time for specific geometric dimensions of the sleeve. In accordance with the algorithm, the program is made and the calculation of characteristics is performed (Fig. 5).



h = 0.008 m, l = 0.094 m

Figure 5. Mechanical characteristics of AM at a constant speed for the length and thickness of the ferromagnetic sleeve

The initial section of the graph (Fig. 5) for small slip is stiff, there is no pronounced maximum in the regulation zone. Analysis of the curve (Fig. 5) confirms the expediency of choosing an AM with a ferromagnetic sleeve on the female rotor. AM of such a design has good regulatory properties.

Calculated by the above methodic equivalent parameters of the ferromagnetic sleeve are shown on Figure 6.

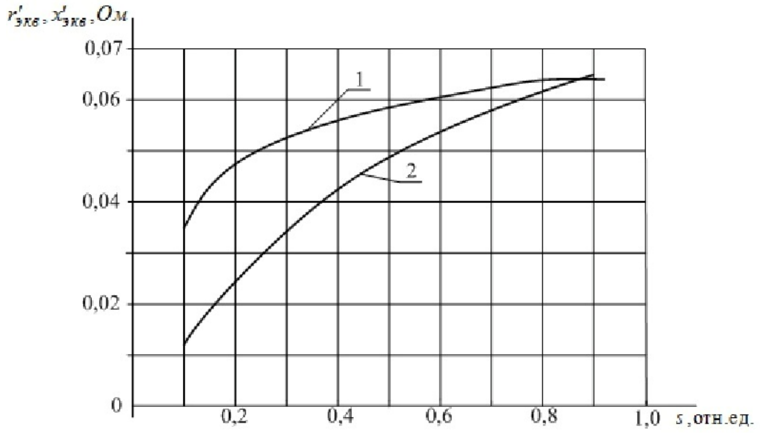


Figure 6. The results of calculating the equivalent active (2) and inductive (1) resistances of a ferromagnetic sleeve with a length $l = 0,11$ m and thickness $h = 0,008$ m

From the analysis of graphs (Fig. 6) it follows that the equivalent parameters of the ferromagnetic sleeve essentially depend on the AM slip value. For small AM slip, the parameters have small values and have an insignificant effect on the AM characteristics.

This allows to provide high rigidity of mechanical characteristics of AM. With an increase in the frequency of the rotor currents, a rapid rate of increase in the active component of the resistance is observed, which is equivalently introduced, in comparison with the inductive one, which provides an increase in the starting torque of AM.

Conclusion

The proposed method for calculating the parameters and characteristics of a ferromagnetic sleeve with the help of Maxwell differential equations allows simplify the development of asynchronous machines with a massive rotor. The developed algorithm for calculating the operating and starting characteristics of AM is expedient for use in the design of integral turbocharger systems of marine diesel engines.

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С.В. Вороненко

Асинхронды машиналардың көрсеткіштері мен параметрлерін арттыру мүмкіндіктерін зерттеу

Кеме электр энергетикалық жүйелерінің үнемділігі мен сенімділігін арттыру қазіргі заманғы кеме жасаудың басым міндеті болып табылады. Бұл проблеманы шешу бағыттарының бірі — электр энергиясын өндіру үшін басты энергетикалық қондырғының қуатын пайдалану. Осы ретте асинхронды машиналарға өндірісте ең қарапайым және жұмыста сенімді ретінде артықшылық беру орынды. Сондай-ақ асинхронды машина нанотехнологияларды қолдана отырып, массивті ферромагниттік ротормен немесе гильзамен іске қосылуы мүмкін екенін атап өту қажет, бұл электр машиналары мен жалпы жүйенің энергетикалық параметрлері мен көрсеткіштерін арттыруға мүмкіндік береді. Мақалада ферромагнитті гильзасы бар қысқа тұйықталған роторы бар асинхронды машинаның (АМ) құрылымы мен жұмысының ерекшеліктеріне талдау жүргізілді. Гильзаның геометриялық өлшемдеріне АМ параметрлері мен сипаттамаларының тәуелділігі көрсетілген, оларды Максвелл дифференциалдық теңдеулер жүйесін қолдану арқылы анықтау әдістемесі ұсынылған. Машина мен ферромагниттік гильзаның эквивалентті параметрлерін есептеу алгоритмі әзірленді.

Кілт сөздер: асинхронды машина, ротор, ферромагниттік гильза, сырғу, белсенді және индуктивті кедергіге балама.

С.В. Вороненко

Исследование возможностей повышения показателей и параметров асинхронных машин

Повышение экономичности и надежности судовых электроэнергетических систем является приоритетной задачей современного судостроения. Одним из направлений решения данной проблемы является использование мощности главной энергетической установки для производства электроэнергии. При этом предпочтение целесообразно отдавать асинхронным машинам как наиболее простым в производстве и надежным в работе. Необходимо отметить также то, что асинхронная машина может быть выполнена с массивным ферромагнитным ротором или гильзой с применением нанотехнологий, что позволит повысить энергетические параметры и показатели электрической машины и системы в целом. В статье проведен анализ особенностей устройства и работы асинхронной машины (АМ) с короткозамкнутым ротором с ферромагнитной гильзой. Показана зависимость параметров и характеристик АМ от геометрических размеров гильзы, предложена методика их определения с использованием системы дифференциальных уравнений Максвелла. Разработан алгоритм расчета характеристик АМ машины и эквивалентных параметров ферромагнитной гильзы.

Ключевые слова: асинхронная машина, ротор, ферромагнитная гильза, скольжение, эквивалентные активное и индуктивное сопротивления.

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G.A. Yessenbayeva[1], D.N. Yesbayeva[2], N.K. Syzdykova[1], I.A. Akpanov[1]

[1] *Ye.A. Buketov Karaganda State University, Kazakhstan;*
[2] *Shanghai Factory-Amigo EC Technology Co., Ltd, China*
(E-mail: esenbaevagulsima@mail.ru)

On the calculation of rectangular plates by the collocation method

The article is devoted to the application of the collocation method to solving differential equations, which are the basis for calculating many problems of mechanics. In this article the structure of this method is presented, its main components are highlighted; its types are characterized, as well as its classical approaches. The research of the problem of rectangular plates bending is carried out by the method of collocations in this article. The collocation method, like all numerical-analytical approximate methods, has a number of advantages and disadvantages, which are also noted in this article. The article is focused mainly on mechanics, engineers and technical specialists.

Keywords: collocation method, collocation points, bending of a rectangular plate, plate deflection function, plate equilibrium equation.

A collocation method is one of the classical methods, which has been repeatedly used to solve many problems of structural mechanics. A collocation method is a method for the numerical solution of ordinary differential equations, partial differential equations and integral equations. The idea is to choose a finite-dimensional space of candidate solutions and a number of points in the domain (these points are called collocation points), and to select that solution which satisfies the given equation at the collocation points.

The collocation method for solving differential equations

The collocation method is a numerical-analytical approximate method for solving a differential equation

$$Ly(x) = f(x), \tag{1}$$

where L is a differential operator; $y(x)$ is a function, which satisfies the given boundary conditions at the boundaries of the interval (a, b); (a, b) is the domain of definition of the function $y(x)$.

The solution is sought as a finite series

$$y(x) = \sum_{m=1}^{M} A_m \varphi_m(x). \tag{2}$$

Here $\varphi_m(x)$ is the coordinate functions satisfying the given boundary conditions; A_m are unknown coefficients; M is the number of members of the series.

To determine the coefficients A_m, the solution (2) is substituted into the differential equation (1), which is satisfied at the points x_i $(i = 1, ..., M)$, i.e. at the collocation points from the interval (a, b)

$$Ly(x_i) = \sum_{m=1}^{M} A_m L\varphi_m(x_i) = f(x_i). \tag{3}$$

As a result, we obtain the system (3) M of algebraic equations. Having solved this system, we determine the unknown coefficients A_m. After determining the coefficients, the function $y(x)$ and the necessary derivatives of this function are calculated at any point of the interval (a, b), as well as outside the interval. The accuracy of the solution depends on both the choice of functions $\varphi_m(x)$ and the choice of collocation points.

The collocation method refers to the simplest approximate methods for solving differential equations, requiring only differentiation, functions calculation, and solution of a system of equations. In contrast to the grid method, after determining unknown coefficients numerical analytical methods allow to use the methods of mathematical analysis, to differentiate, to integrate, to determine the maximum-minimum points, etc. [1]

Internal collocation method

As mentioned above, the coefficients A_m are chosen so that equation (1) is satisfied at the points of collocations within the domain of definition of this equation.

Boundary collocation method

For the boundaries of a complicated form, the representation of the solution in the form (2) may be useful, where the coordinate functions $\varphi_m(x)$ satisfy equation (1), but do not satisfy the boundary conditions. The equations for determining the coefficients A_m are obtained from satisfying the boundary conditions at the points of the boundary [2].

The calculation of rectangular plates in bending by the collocation method

Consider a rectangular plate. We take the plate deflection function in the form

$$w(x,y) = \sum_{m=1}^{M} \sum_{n=1}^{N} A_{mn} X_m(x) Y_n(y), \tag{4}$$

where $X_m(x)$, $Y_n(y)$ are functions satisfying the boundary conditions of the plate support at the boundaries $x = 0$, $x = a$ and $y = 0$, $y = b$ respectively; A_m are unknown coefficients.

We define the collocation points x_i, y_i, $i = 1, ..., K$, where $K = M \times N$ is the number of points of collocations. K is equal to the number of members of the series.

The plate equilibrium equation is satisfied at the collocation points

$$\nabla^4 w(x_i, y_i) = \frac{q(x_i, y_i)}{D}, \tag{5}$$

where $D = \dfrac{E\,h^3}{12}$ is the cylindrical rigidity of the plate, q is the intensity of the external distributed load. We substitute (4) into (5)

$$\sum_{m=1}^{M} \sum_{n=1}^{N} A_{mn} \left[\frac{\partial^4 w_{mn}(x_i, y_i)}{\partial x^4} + 2\frac{\partial^4 w_{mn}(x_i, y_i)}{\partial x^2 \partial y^2} + \frac{\partial^4 w_{mn}(x_i, y_i)}{\partial y^4} \right] =$$

$$= \sum_{m=1}^{M} \sum_{n=1}^{N} A_{mn} \left[X_m^{IV}(x_i) Y_n(y_i) + 2X_m''(x_i) Y_n''(y_i) + X_m(x_i) Y_n^{IV}(y_i) \right] = \frac{q(x_i, y_i)}{D}. \tag{6}$$

Consider a rectangular plate, hinged on the contour, with the following dimensions $0 \le x \le a$, $0 \le y \le b$.

The boundary conditions of the plate bearing are

$$\begin{gathered} w(0,y) = w(a,y) = 0; \quad M_x(0,y) = M_x(a,y) = 0; \\ w(x,0) = w(x,b) = 0; \quad M_y(x,0) = M_y(x,b) = 0. \end{gathered} \tag{7}$$

From the condition that the bending moments on the contour are zero, we have

$$\frac{\partial^2 w(0,y)}{\partial x^2} = \frac{\partial^2 w(a,y)}{\partial x^2} = 0;$$

$$\frac{\partial^2 w(x,0)}{\partial y^2} = \frac{\partial^2 w(x,b)}{\partial y^2} = 0.$$

Taking into account the boundary conditions, we accept

$$X_m = \sin m\pi \frac{x}{a}, \quad Y_n = \sin n\pi \frac{y}{b};$$

and obtain a solution in the form of a double series

$$w(x,y) = \sum_{m=1}^{M} \sum_{n=1}^{N} A_{mn} \sin m\pi \frac{x}{a} \sin n\pi \frac{y}{b}.$$

Obviously, the boundary conditions (7) are satisfied.

The system of equations for the collocation method (6) is obtained in the form [1]

$$\left(\frac{\pi}{a}\right)^4 \sum_{m=1}^{M} \sum_{n=1}^{N} A_{mn} \left[m^4 + 2(\lambda mn)^2 + (\lambda n)^4\right] \sin m\pi \frac{x_i}{a} \sin n\pi \frac{y_i}{b} = \frac{q(x_i, y_i)}{D};$$

$$i = 1, ..., K; \quad \lambda = \frac{a}{b}$$

or

$$\sum_{m=1}^{M} \sum_{n=1}^{N} A_{mn} C_{mn} \sin m\pi \frac{x_i}{a} \sin n\pi \frac{y_i}{b} = \frac{q(x_i, y_i)}{D} \frac{a^4}{\pi^4},$$

where

$$C_{mn} = m^4 + 2(\lambda mn)^2 + (\lambda n)^4.$$

Examples of the calculation of a rectangular plate with one member of the series (4) and with three members of the series (4) are given in [1]. As can be seen from these examples, the accuracy of the calculation depends on the number of members of the series and also on the ratio of the plate sides. For a square plate, the accuracy of the calculation by three series members as compared to the calculation with one series member increased about three times for both deflections and bending moments. For a rectangular plate with $\lambda = 1.5$ the accuracy increased significantly only for deflections, for bending moments the accuracy changed slightly.

It is also shown in [1] that the accuracy of the calculation results depends on the choice of collocation points. It can be seen from the results of calculation that for a square plate the change of collocation points led to a certain increase in the accuracy of deflections and a decrease in the accuracy of bending moments. At the same time, the values of deflection and bending moments were greater than the exact values, while in the previous calculation their values were less than the corresponding exact values. For a rectangular plate the change in collocation points led to a certain increase in the accuracy of the deflection and a more significant increase in the accuracy of the bending moments [1].

In the case of a bending problem for a rectangular plate, the desired deflection function $w(x,y)$ can be represented as a sum

$$w(x,y) = \sum_{m=1}^{M} A_m \varphi_m(x,y), \tag{8}$$

where A_m are the sought-for constant coefficients,

$$\varphi_m(x,y) = \xi_m(x)\eta_m(y)$$

are pre-selected functions that determine the possible deformation of the plate and satisfy all boundary conditions.

Substituting (8) into the plate equilibrium equation

$$D\,\nabla^2\nabla^2\,w = q\,(x,y)\,,$$

where $\nabla^2\nabla^2 w$ is the biharmonic operator, we get the expression

$$\sum_{m=1}^{M} A_m \left[\xi_m^{IV}(x)\eta_m(y) + 2\xi_m''(x)\eta_m''(y) + \xi_m(x)\eta_m^{IV}(y)\right] = \frac{q(x,y)}{D}, \tag{9}$$

which is generally not satisfied for any values of the constants A_m.

We require that the expression (9) be satisfied at M points $(x_1, y_1),\ (x_2, y_2), \ldots,\ (x_n, y_n)$ in the considered domain. Then from (9) we get the system of algebraic equations

$$\begin{cases} \sum\limits_{m=1}^{M} A_m \left[\xi_m^{IV}(x_1)\eta_m(y_1) + 2\xi_m''(x_1)\eta_m''(y_1) + \xi_m(x_1)\eta_m^{IV}(y_1)\right] = \dfrac{q(x_1,y_1)}{D}; \\ \sum\limits_{m=1}^{M} A_m \left[\xi_m^{IV}(x_2)\eta_m(y_2) + 2\xi_m''(x_2)\eta_m''(y_2) + \xi_m(x_2)\eta_m^{IV}(y_2)\right] = \dfrac{q(x_2,y_2)}{D}; \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ \sum\limits_{m=1}^{M} A_m \left[\xi_m^{IV}(x_M)\eta_m(y_M) + 2\xi_m''(x_M)\eta_m''(y_M) + \xi_m(x_M)\eta_m^{IV}(y_M)\right] = \dfrac{q(x_M,y_M)}{D}, \end{cases} \tag{10}$$

of which M constants A_m are defined.

In this case, if the selected functions $\varphi_m(x,y)$ do not satisfy all the boundary conditions, then in addition to the equations (10), it is necessary to write for several points of the plate contour such equations that satisfy the given boundary conditions.

For example, if the edge of the plate $x = a$ is free from fixings and loads, and the selected functions $\varphi_m(x,y)$ do not satisfy the conditions

$$M_x(a,y) = Q_x(a,y) = 0,$$

then for k points of this edge one should write down

$$\begin{cases} M_x(a,y_k) = -D\sum\limits_{m=1}^{M} A_m \left[\xi_m''(a)\eta_m(y_k) - \nu\xi_m(a)\eta_m''(y_k)\right] = 0; \\ Q_x(a,y_k) = -D\sum\limits_{m=1}^{M} A_m \left[\xi_m'''(a)\eta_m(y_k) + (2-\nu)\xi_m'(a)\eta_m''(y_k)\right] = 0. \end{cases} \tag{11}$$

Naturally, the total number of equations of type (10) and (11) should be equal M, i.e. should be equal the number of constants A_m to be determined. From this it follows that in case of an unsuccessful choice of functions $\varphi_m(x,y)$, the accuracy of solution of the main differential equation of the problem decreases due to the fact that in the domain occupied by the plate, it is necessary to reduce the number of collocation points.

As an example, we consider a square hinged plate loaded with a uniformly distributed load q. We confine ourselves to the first approximation, i.e. we keep in (8) only the first member of the series

$$w(x,y) = A_1\varphi_1(x,y) = A_1\xi_1(x)\eta_1(y).$$

For coordinate functions, we take the following expressions

$$\begin{cases} \xi_1(x) = x^4 - 2ax^3 + a^3x; \\ \eta_1(y) = y^4 - 2ay^3 + a^3y, \end{cases} \tag{12}$$

which are functions of deflections of a hinged beam of length a and satisfy the given boundary conditions on the plate contour. Derivatives of these functions (12), included in the equation (10) have the form

$$\xi_1''(x) = 12x^2 - 12ax; \quad \eta_1''(y) = 12y^2 - 12ay;$$
$$\xi_1^{IV}(x) = 24; \qquad \eta_1^{IV}(y) = 24.$$

We choose the collocation point in the center of the plate, i.e. we write (10)

$$A_1\left[\xi_1^{IV}(x_1)\eta_1(y_1) + 2\xi_1''(x_1)\eta_1''(y_1) + \xi_1(x_1)\eta_1^{IV}(y_1)\right] = \frac{q}{D},$$

for $x_1 = y_1 = a/2$

$$A_1\left(7,5a^4 + 18a^4 + 7,5a^4\right) = \frac{q}{D},$$

then we obtain

$$A_1 = 0,03\frac{q}{a^4D}.$$

Deflection in the center of the plate is

$$w\left(\frac{a}{2},\frac{a}{2}\right)=A_1\xi_1(\frac{a}{2})\eta_1(\frac{a}{2})=0,03\frac{q}{a^4D}\cdot\left(\frac{5}{16}a^4\right)^2=0,00293\frac{qa^4}{D}.$$

Comparing this result with the more accurate solution $0,00406\frac{qa^4}{D}$ given earlier in [3], one can see that the error is 28 %.

If we took a point with coordinates $x_1=y_1=a/4$ as a collocation point, we would get

$$A_1=0,048\frac{q}{a^4D};\quad w\left(\frac{a}{2},\frac{a}{2}\right)=0,00469\frac{qa^4}{D},$$

that is, a slightly more accurate solution with an error of + 15.5 %. It follows that the collocation point does not always need to be taken in the place of the least rigidity of the structure, as some researchers have recommended.

If the number of members in the row (8) increases, the accuracy of the solution naturally increases. Thus, for two terms in a series (8), for coordinate functions represented by power polynomials of type (12) and for two collocation points $x_1=y_1=a/2$, the deflection in the center of the plate is equal to $0,00394qa^4/D$, i.e. it differs from the exact solution by 3 %.

It can be seen that the use of the collocation method is connected, as in the variation methods, with the intuitive choice of functions. Compared to variation methods, the collocation method gives less accurate solutions with the same number of held constants. If in variation methods the error appears only when we choose approximating functions, then here, moreover, it arises when we choose collocation points.

Moreover, in the collocation method, the reciprocity of the coefficients of resolving algebraic equations is violated; as a result, these equations do not have symmetry. In addition, the collocation method is simpler than the variation methods. There is no need to integrate functions $\varphi_i(x,y)$ within the considered domain and, therefore, less time is required for the preparation of algebraic equations.

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Г.А. Есенбаева, Д.Н. Есбаева, Н.К. Сыздыкова, И.А. Ақпанов

Коллокация әдісімен тікбұрышты пластиналарды есептеу туралы

Мақала механиканың көптеген есептерін есептеудің негізі болып табылатын дифференциалдық теңдеулерді шешуге коллокация әдісін қолдану мәселесіне арналған. Мақалада осы әдістің құрылымы берілген, оның негізгі компоненттері көрсетілген, сондай-ақ оның түрлері мен классикалық тәсілдері сипатталған. Авторлар коллокация әдісімен тікбұрышты пластиналардың иілуі туралы есептерді зерттеді. Сондай-ақ барлық сандық-аналитикалық жуықталған әдістер сияқты коллокация әдісінің бірқатар артықшылықтары мен кемшіліктері бар екені көрсеткен. Мақала, негізінен, механиктерге, инженерлерге және техникалық мамандықтағы мамандарға бағытталған.

Кілт сөздер: коллокация әдісі, коллокация нүктелері, тікбұрышты пластинаның иілуі, пластинаның иілу функциясы, пластинаның тепе-теңдік теңдеуі.

Г.А. Есенбаева, Д.Н. Есбаева, Н.К. Сыздыкова, И.А. Акпанов

О расчете прямоугольных пластин методом коллокаций

Статья посвящена вопросу применения метода коллокаций к решению дифференциальных уравнений, являющихся основой расчета многих задач механики. В статье представлена структура данного метода, выделены его основные компоненты, охарактеризованы его виды, а также его классические подходы. Авторами проведено исследование задачи об изгибе прямоугольных пластин методом коллокаций. Метод коллокаций, как и все численно-аналитические приближенные методы, имеет ряд преимуществ и недостатков, которые также отмечены в данной работе. Статья ориентирована, главным образом, на механиков, инженеров и специалистов технических специальностей.

Ключевые слова: метод коллокаций, точки коллокаций, изгиб прямоугольной пластины, функция прогиба пластины, уравнение равновесия пластины.

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Доктору физико-математических наук, профессору, академику НАН РК С.Н. Харину — 80 лет

Станислав Николаевич Харин, доктор физико-математических наук, профессор, академик НАН РК, родился 4 декабря 1938 г. в селе Каскелен Алма-Атинской области. В 1956 г. с золотой медалью окончил среднюю школу в г. Воронеже. В этом же году поступил на механико-математический факультет Казахского государственного университета, который с отличием закончил в 1961 г. В 1961–1964 гг. он обучался аспирантуре кафедры уравнений математической физики КазГУ. Уже в эти годы С.Н. Харин проявил незаурядные способности ведения научно-исследовательских работ в области уравнений математической физики и их приложений.

Свою трудовую деятельность Станислав Николаевич начал в Секторе математики и механики АН КазССР, преобразованном в 1965 г. в Институт математики и механики (ИММ АН КазССР), где проходит вся его дальнейшая трудовая деятельность. В 1968 г. С.Н. Харин защитил кандидатскую диссертацию, которая была посвящена решению сингулярных интегральных уравнений, связанных с проблемами математического моделирования тепловых процессов в электрических контактах. С 1969 г. по 1980 г. С.Н. Харин занимал должность старшего научного сотрудника лаборатории уравнений математической физики (ЛУМФ), а в 1980–1994 гг. — заместителя директора по научной работе ИММ (позднее ИТПМ МН АН РК). В 1990 г. им была защищена докторская диссертация на тему «Математические модели теплофизических процессов в электрических контактах» в Институте теплофизики Сибирского отделения АН СССР (г. Новосибирск).

В 1994 г. С.Н. Харин был избран член-корреспондентом Национальной академии наук и в 1994–1997 гг. являлся академиком-секретарём Отделения физико-математических наук Республики Казахстан, одновременно занимая должность заведующего лабораторией УМФ ИТПМ МН АН РК. С.Н. Харин в 1994–1997 гг. являлся членом Президиума Национальной академии наук Республики Казахстан.

Многие исследования, проводимые в системе Академии наук РК, связаны с именем С.Н. Харина. Им разработаны и изучены математические модели, описывающие нестационарные и электромагнитные поля в электроконтактных системах; процессы в низкотемпературной плазме электрической дуги в коммутационных аппаратах и плазмотронах; явления мостиковой и дуговой эрозии и сваривания электродов. В исследованиях С.Н. Харина построены теория и эффективные методы решения краевых задач в областях с подвижными границами. В ходе исследования им разработан аппарат новых специальных функций типа Хартри, с помощью которого, в частности, получено аналитическое решение двухфазной задачи Стефана с граничным условием потока в форме ряда функций Хартри. На основе решения пространственной задачи стефановского типа в новой постановке разработана математическая модель, описывающая процессы плавления и сваривания электрических контактов при сквозных токах, а также дано теоретическое обоснование обнаруженным ранее экспериментально трём зонам сваривания. С.Н. Хариным также изучены новые типы краевых задач для параболических уравнений, описывающих процессы тепло- и массообмена в телах с переменным сечением. Для этих задач построены тепловые потенциалы, исследованы соответствующие сингулярные интегральные уравнения псевдовольтерровского типа и разработаны методы приближённого решения, в частности, метод мажорантных функций и новая модификация метода Кармана-Полгаузена. Полученные результаты нашли применение в теории низкотемпературной электродуговой плазмы и коммутационных процессов в электрических аппаратах.

С.Н. Хариным активно развивается математический аппарат, позволяющий производить оптимальный выбор композиции контактных пар с минимальной или самоограничивающейся эрозией. За разработку электрического соединителя SP-063 С.Н. Харин в 1981 г. удостоен Золотой медали ВДНХ СССР, а в 1982 г. награжден знаком «Изобретатель СССР» за внедрение методов расчёта биметаллических распределителей аппаратов защиты. Работы С.Н. Харина по тепловой теории контактной мостиковой эрозии известны во всём мире. В 2015 г. С.Н. Харин получил премию Ragnar Holm Award в США за исследования в области электрических контактов.

Анализ математического моделирования температурных и электромагнитных полей в контактных системах позволил выяснить роль туннельного эффекта, оценить влияние на него адгезионных и пассивирующих плёнок и разработать защищённые авторскими свидетельствами электроконтактные конструкции со специальными устройствами, нейтрализующими туннельный перегрев и обеспечивающими стабильное переходное сопротивление.

Станислав Николаевич Харин является автором более 300 публикаций, включая 4 монографии и 12 авторских свидетельств на изобретения.

Большое внимание ученый уделял и уделяет подготовке научных кадров. Под его научным руководством успешно защищено 10 кандидатских и 4 PhD диссертации. Он был председателем Комитета по научно-техническому сотрудничеству между Республикой Казахстан и Исламской Республикой Пакистан (1996–2001) и президентом Малой академии наук школьников. Сейчас он успешно продолжает свою исследовательскую деятельность, и доказательством этого является выход в свет новой монографии «Математические модели явлений в электрических контактах», опубликованной в прошлом году в Новосибирске.

Кроме того, С.Н. Харин является профессором КБТУ, членом Диссертационного совета КБТУ по защите диссертаций на присуждение ученой степени доктора философии (PhD), доктора по профилю в 2016 г. по направлению «Математическое и компьютерное моделирование».

Редколлегия научного журнала «Вестник Карагандинского университета. Серия Математика» сердечно поздравляет Станислава Николаевича с 80-летним юбилеем и желает ему крепкого здоровья и творческого долголетия.

МЕРЕЙТОЙ ИЕСІ
НАШИ ЮБИЛЯРЫ
OUR ANNIVERSARIES

Доктору физико-математических наук, профессору М.И. Рамазанову — 70 лет

Мурат Ибраевич Рамазанов, доктор физико-математических наук, профессор, родился 24 февраля 1949 г. в г. Булаево Северо-Казахстанской области. После окончания механико-математического факультета КазНУ им. аль-Фараби (КазГУ им. С.М. Кирова) в 1971 г. он начал свою трудовую деятельность в Карагандинском государственном университете и по сей день вносит большой вклад в развитие университета. В 1981 г. была защищена кандидатская диссертация, а в 2006 г. – докторская диссертация.

М.И. Рамазанов является высококвалифицированным специалистом в области нагруженных дифференциальных уравнений в частных производных, интегральных уравнений и их приложений к прикладным задачам. Он постоянно выступает с научными докладами на международных конгрессах, конференциях, симпозиумах, посвященных обсуждению современных проблем математики, которые проводятся в Республике Казахстан и за рубежом.

Научные достижения М.И. Рамазанова опубликованы в рейтинговых журналах из базы Web of Science «Boundary Value Problems», «Siberian Mathematical Journal», «Advances in Difference Equations» и других, в приоритетных изданиях по математике: «Дифференциальные уравнения» (Москва, РАН), «Сибирский математический журнал» (Новосибирск, СО РАН), Труды Института математики им. С.Л.Соболева (Новосибирск, СО РАН), Труды Института математики НАН Беларуси, в Трудах Международных математических конгрессов: г. Пекин, Хайдарабад (Индия) и других. За последние три года им опубликовано более 15 научных статей в высокорейтинговых журналах.

В терминах (комплексного) спектрального параметра, являющегося коэффициентом нагруженного слагаемого, профессором М.И. Рамазановым выполнено описание резольвентного множества и спектра для спектрально-нагруженного параболического оператора, дана характеристика кратности собственных функций в пространстве ограниченных и непрерывных функций в зависимости от значения спектрального параметра. По результатам этих исследований издана монография «Нагруженные уравнения как возмущения дифференциальных уравнений».

В последние годы М.И. Рамазанов вместе с сотрудниками ведет исследования по однородным краевым задачам теплопроводности в вырождающихся нецилиндрических областях. Установлено, что здесь, наряду с тривиальным решением, существуют и нетривиальные.

Неоценим вклад Мурата Ибраевича в подготовке научных кадров. Под его научным руководством защищены 3 кандидатские диссертации, более 10-ти магистерских диссертаций. На данный момент М.И. Рамазанов является научным руководителем 3 докторантов PhD по специальности 6D060100 — «Математика».

Также с именем М.И. Рамазанова связаны научные исследования, проводимые в рамках грантового финансирования МОН РК: он руководитель темы №1164/ГФ4 «Неклассические задачи математической физики и сингулярные интегральные уравнения Вольтерры» (2014–2017 гг.), исполнитель темы

№0823/ГФ4 «Спектрально-нагруженные операторы и их приложения» (2014–2017 гг); исполнитель темы №0052/ПЦФ (международный) «Операторные методы решения общих краевых задач для уравнений с частными производными и их приложения» (2013–2015 гг.), в данный момент он является руководителем темы «Псевдо-Вольтерровые интегральные уравнения и неклассические эволюционные граничные задачи» ИРН проекта: AP05132262 (2018–2020 гг.).

Долгие годы он является председателем Диссертационного совета по специальности 6D060100 — «Математика» при Карагандинском государственном университете имени академика Е.А. Букетова.

За большой вклад в науку М.И. Рамазанов был удостоен Государственной научной стипендии для ученых и специалистов, внесших выдающийся вклад в развитие науки и техники 2008–2010 гг., награжден нагрудным знаком «За заслуги в развитии науки Республики Казахстан», юбилейной медалью «40 лет КарГУ им. академика Е.А. Букетова», является обладателем гранта на звание «Лучший преподаватель вуза» (2009 г.). Лауреат Премии имени д-ра ф.-м.н., профессора Т.Г. Мустафина, заслуженный работник Карагандинского государственного университета им. Е.А. Букетова; имеет ряд почетных грамот Акима Карагандинской области «За активное участие в общественно-политической жизни области и личный трудовой вклад в дело построения нового казахстанского общества», Национальной палаты предпринимателей РК «За большие заслуги перед казахстанской наукой и неоценимый вклад в развитие высшей школы, подготовку высокопрофессиональных специалистов для Республики Казахстан». М.И. Рамазанов вошел в ТОР-50 Генерального рейтинга ППС вузов РК с баллом 1500 (Национальный рейтинг востребованности вузов РК–2018, Астана, 2018).

Редколлегия научного журнала «Вестник Карагандинского университета. Серия Математика» сердечно поздравляет Мурата Ибраевича с 70-летним юбилеем и желает ему крепкого здоровья и творческого долголетия.

АВТОРЛАР ТУРАЛЫ МӘЛІМЕТТЕР
СВЕДЕНИЯ ОБ АВТОРАХ
INFORMATION ABOUT AUTHORS

Abdiakhmetova, Z.M. — PhD, Acting assosiated professor, Al-Farabi Kazakh National University, Almaty, Kazakhstan.

Akhmanova, D.M. — Candidate of physical-mathematical sciences, Docent of Chair «Mathematics and Information sciences teaching methodics», Ye.A. Buketov Karaganda State University, Kazakhstan.

Akpanov, I.A. — Master student of specialty «Mechanics», Faculty of mathematics and information technologies, Ye.A. Buketov Karaganda State University, Kazakhstan.

Arynov, E. — Doctor of physical and mathematical sciences, O.A. Baikonurov Zhezkazgan University, Kazakhstan.

Arshidinova, M.T. — Phd doctoral student of the 2nd course, Institute of Information and Computing Technologies CS MES RK, Almaty, Kazakhstan; Al-Farabi Kazakh National University, Almaty, Kazakhstan.

Azhymbaev, D.T. — Candidate of physical and mathematical sciences, Associate professor of the Department of Mathematics, K. Zhubanov Aktobe Regional State University, Kazakhstan.

Baimakhan, A.R. — PhD, Al-Farabi Kazakh National University, Almaty, Kazakhstan.

Baimakhan, R.B. — Doctor of technical sciences, Professor, Kazakh State Women's Pedagogical University, Almaty, Kazakhstan.

Baltabayeva, A.M. — Master student, Faculty of mathematics and information technologies, Ye.A. Buketov Karaganda State University, Kazakhstan.

Begaliyeva, K.B. — PhD doctoral student of the 2nd course, Institute of Information and Computing Technologies,Almaty, Kazakhstan; Al-Farabi Kazakh National University, Almaty, Kazakhstan.

Bellaouar, D. — PhD, Assosiated professor, Laboratory of Pure and Applied Mathematics (LMPA), University 08 Mai 1945 Guelma, Guelma, Algeria.

Duishenaliev, T.B. — Doctor of physical and mathematical sciences, Professor of the Department of Mechanics, I.Razzakov Kyrgyz State University, Bishkek, Kyrgyzstan.

Iskakov, S.A. — PhD doctoral student, Ye.A. Buketov Karaganda State University, Kazakhstan.

Issayeva, A.K. — PhD doctoral student, Ye.A. Buketov Karaganda State University, Kazakhstan.

Jaichibekov, N.Zh. – Doctor of physical and mathematical sciences, Professor, L.N. Gumilyov Eurasian National University, Nur-Sultan, Kazakhstan.

Jenaliyev, M.T. — Doctor of physical and mathematical sciences, Professor, Chief Researcher of the Institute of Mathematics and Mathematical Modeling, Almaty, Kazakhstan.

Kalidolday, A.H. — Master student of the 2nd course, S.Toraigyrov Pavlodar State University, Kazakhstan.

Kasymova, L.Zh. — PhD doctoral student, Ye.A. Buketov Karaganda State University, Kazakhstan.

Keldibekova, A.B. — PhD doctoral student, Ye.A. Buketov Karaganda State University, Kazakhstan.

Kenzhebayeva, M.O. — PhD doctoral student, U.A. Dzholdasbekov Institute of Mechanics and Mechanical Engineering, al-Farabi Kazakh National University, Almaty, Kazakhstan.

Kervenev, K.E. — Master of natural sciences, Senior teacher, Ye.A. Buketov Karaganda State University, Kazakhstan.

Klendii, M.B. — Associate professor, Phd, Associate professor of Department of General Engineering Training of Separated Subdivision of National University of Life and Environmental Sciences of Ukraine Berezhany Agrotechnical Institute, Ukraine.

Kosmakova, M.T. — PhD, Docent of chair mathematical analysis and differential equations, Ye.A. Buketov Karaganda State University, Kazakhstan.

Kudaikulov, A.K. — Doctor of physical and mathematical sciences, Institute of Information and Computational Technologies, Almaty, Kazakhstan.

Matveev, S.K. – Doctor of physical and mathematical sciences, Professor, Saint Petersburg State University, Russia.

Muhtarov, M. — Candidate of physical and mathematical sciences, Professor, S. Toraigyrov Pavlodar State University, Kazakhstan.

Muratbekov, M.B. — Doctor of physics and mathematics, Professor, Head of the Department «Mathematics and Methods of Teaching Mathematics», Taraz State Pedagogical University, Kazakhstan.

Muratbekov, M.M. — PhD, Candidate of physical and mathematical sciences, Director of the Information Technology Center of the Kazakh University of Economics, Finance and International Trade, Nur-Sultan, Kazakhstan.

Omarova, M.T. — PhD doctoral student, Ye.A. Buketov Karaganda State University, Kazakhstan.

Orumbayeva, N.T. — Candidate of physical-mathematical sciences, Docent of Chair «Mathematical Analysis and Differential Equations», Ye.A. Buketov Karaganda State University, Kazakhstan.

Özer, Ö. — PhD, Assosiated professor, Department of Mathematics, Faculty of Engineering, Faculty of Science and Arts, Kirklareli University, Turkey.

Poppell, H. — Scientific consultant, Concordia International School Shanghai, China.

Pylypaka, S.F. — Professor, Doctor of technical sciences, Professor of Department of Descriptive Geometry, computer Graphics and Design of National University of Life and Environmental Science of Ukraine, Kiev, Ukraine.

Ramazanov, M.I. — Doctor of physical and mathematical science, Professor, Ye.A. Buketov Karaganda State University, Kazakhstan.

Rysbayeva, G.P. — Candidate of physical and mathematical sciences, Associate professor, Academy of Border Service of the National Security Committee, Almaty, Kazakhstan.

Seynasinova, A.A. — Candidate of Physical and mathematical sciences, Associate professor, Academy of Civil Aviation, Almaty, Kazakhstan.

Shalabayeva, B.S. – Candidate of physical and mathematical sciences, Associate professor, L.N. Gumilyov Eurasian National University, Nur-Sultan, Kazakhstan.

Syzdykova, N.K. — Candidate of physical and mathematical sciences, Head of Chair «Mathematics and Information sciences teaching methodics», Ye.A. Buketov Karaganda State University, Kazakhstan.

Tanin, A.O. — PhD doctoral student, Ye.A. Buketov Karaganda State University, Kazakhstan.

Tashev, A.A. — Doctor of technical science, Institute of Information and Computational Technologies, Almaty, Kazakhstan.

Tleubergenov, M.I. — Doctor of physical and mathematical sciences, Full Professor, Chief Researcher of the Department of Differential Equations, Institute of Mathematics and Mathematical Modeling, Almaty, Kazakhstan.

Trokhaniak, V.I. — Phd, Senior Lecturer of Heat and Power Engineering Department of National University of Life and Environmental Sciences of Ukraine, Kiev, Ukraine.

Tuleutaeva, Zh.M. — PhD doctoral student, Ye.A. Buketov Karaganda State University, Kazakhstan.

Voronenko, S.V. — Lecturer at the Department of Operation of Ship Electrical Equipment and Automation, Kherson State Maritime Academy, Ukraine.

Yesbayev, A.N. — Mathematics teacher, Nazarbayev Intellectual School, Nur-Sultan, Kazakhstan.

Yesbayeva, D.N. — Research assistant and translator, Shanghai Factory-Amigo EC Technology Co., Ltd, China.

Yeshkeyev, A.R. — Doctor of physical and mathematical science, Professor, Ye.A. Buketov Karaganda State University, Kazakhstan.

Yessenbayeva, G.A. — Candidate of physical and mathematical sciences, Docent of Prof. T.G. Mustafin algebra, mathematical logic and geometry chair, Ye.A. Buketov Karaganda State University, Kazakhstan.

Zhussupova, D. – Master degree, L.N. Gumilyov Eurasian National University, Nur-Sultan, Kazakhstan.