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Special issue

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Preface

About the conference ICAAM2018

This issue is a collection of 14 selected papers. These papers are presented at the Fourth International Conference on Analysis and Applied Mathematics (ICAAM 2018) organized by Near East University, Lefkosa (Nicosia), Mersin 10, Turkey. The meeting was held on September 6–9, 2018 in North Cyprus, Turkey.

The main organizer of the conference is Near East University, Nicosia (Lefkosa), Mersin 10, Turkey. The conference was also supported by Institute of Mathematics and Mathematical Modeling, Almaty, Kazakhstan.

The conference is organized biannually. Previous conferences were held in Gumushane, Turkey in 2012; in Shymkent, Kazakhstan in 2014; and in Almaty, Kazakhstan in 2016. The proceedings of ICAAM 2012, ICAAM 2014, and ICAAM 2016 were published in AIP Conference Proceedings (American Institute of Physics) and in some rating scientific journals.

Near East University was pleased to host the fourth conference which was focused on various topics of analysis and its applications, applied mathematics and modeling.

The main aim of the International Conferences on Analysis and Applied Mathematics (ICAAM) is to bring mathematicians working in the area of analysis and applied mathematics together to share new trends of applications of mathematics. In mathematics, the developments in the field of applied mathematics open new research areas in analysis and vice versa. That is why, we planed to found the conference series to provide a forum for researches and scientists to communicate their recent developments and to present their original results in various fields of analysis and applied mathematics.

This issue presents papers by authors from different countries: Albania, Bulgaria, France, Libya, Russia, Turkey, Turkmenistan, USA, Kazakhstan. Especially we are pleased with the fact that many articles are written by co-authors who work in different countries. We are confident that such international integration provides an opportunity for a significant increase in the quality and quantity of scientific publications.

Finally, but not least, we would like to thank the Editorial board of the "Bulletin of the Karaganda University. «Mathematics» series", who kindly provided an opportunity for the formation of this special issue.

July 2018

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On some approximate calculations for certain pseudo-differential equations

We consider discrete pseudo-differential operatots and equations as approximate operators and equations for their continuous analogues. For this purpose we study a solvability for such equations in appropriate discrete spaces and give some error estimates for discrete and continuous solutions. This approach is based on the discrete Fourier transform and factorization tecnique which is used for special canonical domains in Euclidean space.

Keywords: discrete pseudo-differential operator, periodic factorization, solvability, approximate solution, error estimates

0.1 Introduction

A theory of pseudo-differential operators and equations is a very developed part of mathematics now [1–3]. But almost there are results on an approximate solution for such equations and related boundary value problems. Therefore one suggests to start a studying pseudo-differential equations and boundary value problems on discrete structures for which is very convenient to construct computational algorithms. Here we will study model operators and equations in special canonical domains. We are interested in a solvability of discrete equations and a comparison of discrete and continue solutions.

Let $A(\xi)$ be a function defined in $\mathbb{R}^m$ and satisfying the condition

$$c_1(1+|\xi|)^\alpha \leq |A(\xi)| \leq c_2(1+|\xi|)^\alpha, \tag{1}$$

with positive constants c_1, c_2, and let $S(\mathbb{R}^m)$ be the Schwartz space of infinitely differentiable rapidly decreasing at infinity functions. Such a function $A(\xi)$ generates a pseudo-differential operator

$$(Au)(x) = \int\limits_{\mathbb{R}^m}\int\limits_{\mathbb{R}^m} A(\xi)e^{i(x-y)\cdot\xi}u(y)d\xi dy, \quad x \in \mathbb{R}^m, \tag{2}$$

which is defined firstly for $u \in S(\mathbb{R}^m)$, and then it will extend on more general spaces. This function $A(\xi)$ is called a symbol of pseudo-differential operator A.

Remark 1. Usually they consider more general pseudo-differential operators

$$(Au)(x) = \int\limits_{\mathbb{R}^m}\int\limits_{\mathbb{R}^m} A(x,\xi)e^{i(x-y)\cdot\xi}u(y)d\xi dy, \quad x \in \mathbb{R}^m,$$

generated by the symbol $A(x,\xi)$ defined in $\mathbb{R}^m \times \mathbb{R}^m$. But taking into account so-called «a local principle» our nearest problem is studying more simple operator (2) and its discrete analogue.

Let $A_d(\xi)$ be a periodic function in $\mathbb{R}^m$ so that

$$c_1(1+|\zeta_h^2|)^{\frac{\alpha}{2}} \leq |A_d(\xi)| \leq c_2(1+|\zeta_h^2|)^{\frac{\alpha}{2}}, \tag{3}$$

where $\zeta_h^2 = h^{-2}\sum\limits_{k=1}^{m}(e^{-ih\xi_k}-1)^2$, and positive constants c_1, c_2 do not depend on h.

Let $D \subset \mathbb{R}^m$ be a domain (finite or infinite). We will consider functions $u_d(\tilde{x})$ defined in $D_d \equiv D \cap h\mathbb{Z}^m, h > 0$, and introduce the following operator

$$(A_d u_d)(\tilde{x}) = \sum_{\tilde{y}\in h\mathbb{Z}^m}\int\limits_{\hbar\mathbb{T}^m} A_d(\xi)u_d(\tilde{y})e^{i(\tilde{x}-\tilde{y})\cdot\xi}h^m d\xi, \quad \tilde{x} \in D_d,$$

where $\hbar \equiv h^{-1}, \mathbb{T}^m \equiv [-\pi,\pi]^m$.

Definition 1. The operator A_d is called a discrete pseudo-differential operator or shortly h-operator. The periodic function $A_d(\xi)$ is called its $\hbar$-symbol.

Let us remind that a symbol (operator) is called elliptic if

$$ess \inf_{\xi\in\hbar\mathbb{R}^m} |A_d(\xi)| > 0,$$

and obviously all symbols under consideration are elliptic.

0.1.1 The discrete Fourier transform

If $u_d(\tilde{x}), \tilde{x} \in h\mathbb{Z}^m$ is a function of a discrete variable then we say «discrete function». For such discrete functions one can define the discrete Fourier transform

$$(F_d u_d)(\xi) \equiv \tilde{u}_d(\xi) = \sum_{\tilde{x}\in h\mathbb{Z}^m} e^{-i\tilde{x}\cdot\xi} u_d(\tilde{x}) h^m, \quad \xi \in \hbar\mathbb{T}^m,$$

if the latter series converges. The obtained function $\tilde{u}_d(\xi)$ is periodic in $\mathbb{R}^m$ with basic cube of periods $\hbar\mathbb{T}^m$. Such discrete Fourier transform preserves all key properties of the integral Fourier transform, particularly the inverse discrete Fourier transform is given by the formula

$$(F_d^{-1}\tilde{u}_d)(\tilde{x}) = \frac{1}{(2\pi)^m} \int_{\hbar\mathbb{T}^m} e^{i\tilde{x}\cdot\xi} \tilde{u}_d(\xi) d\xi, \quad \tilde{x} \in h\mathbb{Z}^m.$$

The discrete Fourier transform is an isomorphism between the spaces $L_2(h\mathbb{Z}^m)$ and $L_2(\hbar\mathbb{T}^m)$ with norms

$$||u_d||_2 = \left(\sum_{\tilde{x}\in h\mathbb{Z}^m} |u_d(\tilde{x})|^2 h^m \right)^{1/2} \quad \text{and} \quad ||\tilde{u}_d||_2 = \left(\int_{\xi\in\hbar\mathbb{T}^m} |\tilde{u}_d(\xi)|^2 d\xi \right)^{1/2}.$$

0.1.2 Discrete spaces

Since a definition of Sobolev–Slobodetskii spaces uses patrial derivatives we will use their discrete analogues, namely divided differences of first order

$$(\Delta_k^{(1)} u_d)(\tilde{x}) = h^{-1}(u_d(x_1, \cdots, x_k + h, \cdots, x_m) - u_d(x_1, \cdots, x_k, \cdots, x_m)),$$

for which their discrete Fourier transform looks as follows

$$(\widetilde{\Delta_k^{(1)} u_d})(\xi) = h^{-1}(e^{-ih\cdot\xi_k} - 1)\tilde{u}_d(\xi).$$

For a divided difference of a second order we have obviously

$$(\Delta_k^{(2)} u_d)(\tilde{x}) = h^{-2}(u_d(x_1, \cdots, x_k + 2h, \cdots, x_m);$$

$$-2u_d(x_1, \cdots, x_k + h, \cdots, x_m) + u_d(x_1, \cdots, x_k, \cdots, x_m)),$$

and its discrete Fourier transform is

$$(\widetilde{\Delta_k^{(2)} u_d})(\xi) = h^{-2}(e^{-ih\cdot\xi_k} - 1)^2\tilde{u}_d(\xi).$$

Then for the discrete Laplacian

$$(\Delta_d u_d)(\tilde{x}) = \sum_{k=1}^{m} (\Delta_k^{(2)} u_d)(\tilde{x}),$$

we have

$$(\widetilde{\Delta_d u_d})(\xi) = h^{-2} \sum_{k=1}^{m} (e^{-ih\cdot\xi_k} - 1)^2\tilde{u}_d(\xi).$$

Further we introduce the space $S(h\mathbb{Z}^m)$ consisting of functions with finite semi-norms

$$|u_d| = \sup_{\tilde{x}\in h\mathbb{Z}^m} (1+|\tilde{x}|)^l |\Delta^{(\mathbf{k})} u_d(\tilde{x})|$$

for all $l \in \mathbb{N}, \mathbf{k} = (k_1, \cdots, k_m), k_r \in \mathbb{N}, r = 1, \cdots, m$, where

$$\Delta^{(\mathbf{k})} u_d(\tilde{x}) = \Delta_1^{k_1} \ldots, \Delta_m^{k_m} u_d(\tilde{x}).$$

In other words the space $S(h\mathbb{Z}^m)$ is a discrete analogue of the Schwartz space $S(\mathbb{R}^m)$.

Definition 2. By definition the space $H^s(h\mathbb{Z}^m)$ is a closure of the space $S(h\mathbb{Z}^m)$ with respect to the norm

$$||u_d||_s = \left(\int\limits_{\hbar\mathbb{T}^m} (1+|\zeta_h^2|)^s |\tilde{u}_d(\xi)|^2 d\xi \right)^{1/2}. \tag{4}$$

Definition 3. The space $H^s(D_d)$ consists of discrete functions from $H^s(h\mathbb{Z}^m)$ with supports in $\overline{D_d}$. A norm in the space $H^s(D_d)$ is induced by a norm of the space $H^s(h\mathbb{Z}^m)$. The space $H_0^s(D_d)$ consists of discrete functions (distributions from $S'(\mathbb{R}^m)$) u_d with supports in D_d, additionally these discrete functions must admit a continuation ℓ onto $H^s(h\mathbb{Z}^m)$. A norm in the space $H_0^s(D_d)$ is given by the formula

$$||u_d||_s^+ = \inf ||\ell u_d||_s,$$

where infimum is taken over all continuations ℓ.

Of course all norms (4) are equivalent to L_2-norm, but all equivalence constants will depend on h. That is why we would like to note that all constants below do not depend on h.

0.2 Equations and approximations

We will consider the pseudo-differential equation

$$(Au)(x) = v(x), \quad x \in D, \tag{5}$$

and suggest for its solution some computational schemes.

Since we know solvability conditions for pseudo-differential equations in $\mathbb{R}^m$ and $\mathbb{R}_+^m$ [3] we will select such discrete pseudo-differential operators which reserve all needed properties of their continuous analogues.

Let P_h be a restriction operator on $h\mathbb{Z}^m$, i. e. for $u \in S(\mathbb{R}^m)$

$$(P_h u)(x) = \begin{cases} u(\tilde{x}), & x = \tilde{x} \in h\mathbb{Z}^m; \\ 0, & x \notin h\mathbb{Z}^m. \end{cases}$$

We tried this projector for simplest pseudo-differential operators, namely Calderon–Zygmund operators, these operators can be treated as pseudo-differential operators of order 0, and we obtained very acceptable results [4–7]. But now we will use another restriction operator.

A construction for the restriction operator Q_h for functions $u \in S(\mathbb{R}^m)$ is the following. We take the Fourier transform $\tilde{u}(\xi)$, then its restriction on $\hbar\mathbb{T}^m$ and periodically continue it onto a whole $\mathbb{R}^m$. Further we apply the inverse discrete Fourier transform F_d^{-1} and obtain a discrete function which is denoted by $(Q_h u)(\tilde{x}), \tilde{x} \in h\mathbb{Z}^m$. In our opinion the projector Q_h is more convenient than P_h although the projectors P_h and Q_h are almost the same according to the following result.

Lemma 1. For $u \in S(\mathbb{R}^m), \forall\beta > 0$, we have

$$|(P_h u)(\tilde{x}) - (Q_h u)(\tilde{x})| \leq Ch^\beta, \quad \forall\tilde{x} \in h\mathbb{Z}^m,$$

where the constant C depends on u only.

Proof. Indeed, we need to compare two Fourier transforms. By definition

$$(P_h u)(\tilde{x}) = \frac{1}{(2\pi)^m} \int\limits_{\mathbb{R}^m} e^{i\tilde{x}\cdot\xi} \tilde{u}(\xi) d\xi,$$

and respectively

$$(Q_h u)(\tilde{x}) = \frac{1}{(2\pi)^m} \int\limits_{\hbar\mathbb{T}^m} e^{i\tilde{x}\cdot\xi}\tilde{u}(\xi)d\xi,$$

thus this difference is given by the integral

$$(P_h u)(\tilde{x}) - (Q_h u)(\tilde{x}) = \frac{1}{(2\pi)^m} \int\limits_{\mathbb{R}^m\setminus\hbar\mathbb{T}^m} e^{i\tilde{x}\cdot\xi}\tilde{u}(\xi)d\xi.$$

A conclusion of the lemma 1 follows from an invariance of the Schwartz class $S(\mathbb{R}^m)$ with respect to the Fourier transform and the simple estimate

$$|\tilde{u}(\xi)| \leq C_u|\xi|^{-\gamma}$$

for $\forall\gamma > 0$.

Further, the symbol $A_d(\xi)$ will be defined by the following way. We take a restriction of $A(\xi)$ on the cube $\hbar\mathbb{T}^m$ and periodically extend it onto a whole $\mathbb{R}^m$. We consider such h-operator as an approximate operator for A. So, to find an approximate discrete solution for the equation

$$(A_d u_d)(\tilde{x}) = v_d(\tilde{x}), \quad \tilde{x} \in D_d, \tag{6}$$

for $D = \mathbb{R}^m$ we can use the following discrete equation

$$A_d u_d = Q_h v. \tag{7}$$

Its solution is given by the formula

$$u_d(\tilde{x}) = \frac{1}{(2\pi)^m} \int\limits_{\hbar\mathbb{T}^m} e^{i\tilde{x}\cdot\xi}A^{-1}(\xi)\tilde{v}(\xi)d\xi, \quad \tilde{x} \in h\mathbb{Z}^m,$$

so that we do not need to find an approximate solution for an infinite system of linear algebraic equations like [4, 5]. For our case we need to apply any kind of cubature formulas for calculating the latter integral and a cubature formula for calculating the Fourier transform $\tilde{v}(\xi)$.

According to the Lemma 1 one can compare discrete and continuous solutions for enough smooth right-hand sides and symbols.

Theorem 1. If the symbol $A(\xi)$ satisfies the condition (1) and is infinitely differentiable on $\mathbb{R}^m$, u is a solution of the equation (5), u_d is a solution of the equation (7) then for $v \in S(\mathbb{R}^m)$ we have the following error estimate

$$|u(\tilde{x}) - u_d(\tilde{x})| \leq Ch^\beta, \quad \forall\tilde{x} \in h\mathbb{Z}^m,$$

for arbitrary $\beta > 0$.

0.3 Equations in a half-space

This case is very different from $\mathbb{R}^m$, and an ellipticity condition is not sufficient for a solvability. A principal role for the solvability is plaid by an index of the periodic factorization which is defined for an elliptic symbol.

Let us denote $\Pi_\pm = \{(\xi', \xi_m \pm i\tau), \tau > 0\}, \xi = (\xi', \xi_m) \in \mathbb{T}^m$.

Definition 4. The periodic factorization for an elliptic symbol $A_d(\xi)$ is called its representation in the form

$$A_d(\xi) = A_{d,+}(\xi)A_{d,-}(\xi),$$

where the factors $A_{d,\pm}(\xi)$ admit an analytical continuation into half-strips $\hbar\Pi_\pm$ with respect to a last variable ξ_m for almost all fixed $\xi' \in \hbar\mathbb{T}^{m-1}$ and satisfy the estimates

$$|A_{d,+}^{\pm 1}(\xi)| \leq c_1(1 + |\hat{\zeta}^2|)^{\pm\frac{æ}{2}}, \quad |A_{d,-}^{\pm 1}(\xi)| \leq c_2(1 + |\hat{\zeta}^2|)^{\pm\frac{\alpha-æ}{2}},$$

with constants c_1, c_2 non-depending on h,

$$\hat{\zeta}^2 \equiv \hbar^2\left(\sum_{k=1}^{m-1}(e^{-ih\xi_k} - 1)^2 + (e^{-ih(\xi_m+i\tau)} - 1)^2\right), \quad \xi_m + i\tau \in \hbar\Pi_\pm.$$

The number æ $\in \mathbb{R}$ is called an index of the periodic factorization.

Theorem 2. If the elliptic symbol $\tilde{A}_d(\xi)$ admits the periodic factorization with the index æ so that $|æ-s|<1/2$ then the equation (6) has a unique solution in the space $H^s(D_d)$ for arbitrary right-hand side $v_d \in H_0^{s-\alpha}(D_d)$,

$$\tilde{u}_d(\xi) = \tilde{A}^{-1}_{d,+}(\xi) P^{per}_{\xi'}(\tilde{A}^{-1}_{d,-}(\xi)\widetilde{\ell v}_d(\xi)),$$

$$(P^{per}_{\xi'}\tilde{u}_d)(\xi) \equiv \frac{1}{2}\left(\tilde{u}_d(\xi) + \frac{h}{2\pi i} v.p. \int_{-\hbar\pi}^{\hbar\pi} \tilde{u}_d(\xi', \eta_m)\cot\frac{h(\xi_m-\eta_m)}{2}d\eta_m\right), \tag{8}$$

Remark 2. One can easily conclude that the solution does not depend on a continuation ℓv_d.

Theorem 3. Let æ $- s = n+\delta, n \in \mathbb{N}, |\delta| < 1/2$. Then a general solution of the equation (6) in Fourier images has the following form

$$\tilde{u}_d(\xi) = \tilde{A}^{-1}_{d,+}(\xi) X_n(\xi) P^{per}_{\xi'}(X_n^{-1}(\xi)\tilde{A}^{-1}_{d,-}(\xi)\widetilde{\ell v}_d(\xi)) + \tilde{A}^{-1}_{d,+}(\xi)\sum_{k=0}^{n-1} c_k(\xi')\hat{\zeta}^k_m,$$

where $X_n(\xi)$ is arbitrary polynomial of order n of variables $\hat{\zeta}_k = \hbar(e^{-ih\xi_k}-1)$, $k=1,\cdots,m$ satisfying the condition (3), $c_j(\xi'), j=0,1,\cdots,n-1$, are arbitrary functions from $H_{s_j}(h\mathbb{T}^{m-1}), s_j = s -$ æ $+ j - 1/2$.

For the case æ $- s = -n+\delta, n\in\mathbb{N}, |\delta|<1/2$, we consider the following general equation

$$(A_d u_d)(\tilde{x}) + \sum_{j=0}^{n} K_j\left(\tilde{b}_j(\tilde{x}')\otimes\delta(\tilde{x}_m)\right) = v_d(\tilde{x}), \quad \tilde{x}\in D_d, \tag{9}$$

with unknowns $u_d, \tilde{b}_j, j=0,1,\cdots,n$, and K_j are given pseudo-differential operators with symbols $K_j(\xi)$ satisfying the condition (3) with power α_j.

Remark 3. The operator K_j acts as follows. If we denote by $\hat{K}_j(\tilde{x})$ a kernel of the pseudo-differential operator K_j, we obtain

$$K_j\left(\tilde{b}_j(\tilde{x}')\otimes\delta(\tilde{x}_m)\right) = \sum_{\tilde{y}\in h\mathbb{Z}^{m-1}} \hat{K}_j(\tilde{x}'-\tilde{y}', \tilde{x}_m) b_j(\tilde{y}')h^{m-1}.$$

Continuing the right-hand side onto a whole $\mathbb{R}^m$ and applying the discrete Fourier transform, we obtain the system of linear algebraic equations

$$\sum_{j=0}^{n} t_{kj}(\xi')\tilde{b}_j(\xi') = f_k(\xi'), \quad k=0,1,\cdots,n,$$

where

$$t_{kj}(\xi') = \frac{1}{2\pi}\int_{-\hbar\pi}^{\hbar\pi}\left(\frac{e^{-ih\xi_m}-1}{h}\right)^k \frac{K_j(\xi',\xi_m)}{A_{d,-}(\xi',\xi_m)}d\xi_m;$$

$$f_k(\xi') = \frac{1}{2\pi}\int_{-\hbar\pi}^{\hbar\pi}\left(\frac{e^{-ih\xi_m}-1}{h}\right)^k A^{-1}_{d,-}(\xi',\xi_m)\widetilde{(\ell v_d)}(\xi',\xi_m)d\xi_m.$$

Theorem 4. Let æ $- s = -n+\delta, n\in\mathbb{N}, |\delta|<1/2$. Then the equation (9) has a unique solution $u_d \in H^s(D_d), c_j \in H^{s_j}(h\mathbb{Z}^{m-1}), s_j = s-\alpha+\alpha_j+1/2, j=0,1,\cdots,n$, iff

$$ess \inf_{\xi'\in h\mathbb{T}^{m-1}} |\det(t_{kj}(\xi'))^n_{k,j=0} > 0.$$

The following estimate

$$||u_d||_s \le a||v_d||^+_{s-\alpha}, \quad ||b_j||_{s_j} \le a_j||v_d||^+_{s-\alpha}, \quad j=0,1,\cdots,n,$$

holds with constants $a, a_1, \cdots, a_n$, non-depending on h.

0.3.1 A limit case

It is well-known [8] that

$$\cot x = \frac{1}{x} - \sum_{n=1}^{\infty} \frac{2^{2n}|B_{2n}|}{(2n)!} x^{2n-1}, \quad -\pi < x < \pi,$$

where B_{2n} are Bernoulli numbers.

In the formula (8) a kernel of the operator $P_{\xi'}^{per}$, i. e. $h \cot \frac{h\xi_m}{2}$ has the following representation

$$h \cot \frac{h\xi_m}{2} = \frac{2}{\xi_m} - h \sum_{n=1}^{\infty} \frac{2^{2n}|B_{2n}|}{(2n)!} \left(\frac{h\xi_m}{2}\right)^{2n-1},$$

so that we will obtain under $h \to 0$ a well-known kernel of the Hilbert transform $\frac{1}{\pi i}\frac{1}{\xi_m}$ with respect to a last variable. Also it is easily visible that under $h \to 0$ all periodic polynomials in Theorem 2 transform to ordinary polynomials with respect to the variable ξ_m. It is very correlated with a continuous case [3].

Unfortunately, for this case the estimates for a comparison of u and u_d are not so simple as in the theorem 1; now we can assert that $\tilde{u}_d$ converges to $\tilde{u}$ under $h \to 0$.

0.3.2 An error estimate

If we put strong enough restrictions on a right-hand side and factorization elements then one can give a comparison between discrete and continuous solutions.

Lemma 2. If $u \in S(\mathbb{R}^m)$ then the following estimate

$$|(F^{-1}P_{\xi'}\tilde{u})(\tilde{x}) - (F_d^{-1}P_{\xi'}^{per}\widetilde{Q_h u})(\tilde{x})| \leq Ch^{\beta}, \quad \tilde{x} \in h\mathbb{Z}_+^m$$

holds for $\forall \beta > 0$, and the constant C depends on u only.

Proof. Here we need the description and comparison for two projectors related to the Hilbert transform, both standard and periodic. Let us denote by $\chi(x)$ an indicator of the half-space $\mathbb{R}_+^m$ and by $\chi_d(\tilde{x})$ an indicator of the discrete half-space $h\mathbb{Z}_+^m$. Then according to structural properties of two mentioned transforms we have the following equalities

$$F^{-1}P_{\xi'}\tilde{u} = \chi \cdot u, \quad F_d^{-1}P_{\xi'}^{per}\widetilde{Q_h u} = \chi_d \cdot (Q_h u).$$

Further one can apply Lemma 1. □

Starting from Lemma 2 and the Theorem 1 we are able to compare discrete and continuous solutions in a half-space. Below we give this comparison under conditions of the Theorem 2 when a unique solution exists.

Theorem 5. If the symbol $A(\xi)$ satisfies the condition (1) and is infinitely differentiable in $\mathbb{R}^m$ with the factors $A_{\pm}(\xi)$, u is a solution of the equation (5), u_d is a solution of the equation (7) then for $v \in S(\mathbb{R}^m)$ we have the following error estimate

$$|u(\tilde{x}) - u_d(\tilde{x})| \leq Ch^{\beta}, \quad \forall \tilde{x} \in h\mathbb{Z}_+^m,$$

for arbitrary $\beta > 0$.

Remark 4. To refine this theorem we will describe how we need to choose a right-hand side for solving the equation (7). The solution of the equation (5) in Fourier images has the form

$$\tilde{u}(\xi) = A_+^{-1}(\xi)P_{\xi'}A_-^{-1}(\xi)\widetilde{\ell v}(\xi),$$

where $P_{\xi'} = \frac{1}{2}(I + H_{\xi'})$ is a projector defined by the classical Hilbert transform with respect to a variable ξ_m [3]:

$$(H_{\xi'}\tilde{u})(\xi) = \frac{1}{\pi i}\ v.p. \int_{-\infty}^{+\infty} \frac{\tilde{u}(\xi', \eta_m)d\eta_m}{\xi_m - \eta_m},$$

ℓv is an arbitrary continuation of v from $\mathbb{R}_+^m$ onto a whole $\mathbb{R}^m$ in corresponding functional space. Since the right-hand side in the equation (6) is defined in $h\mathbb{Z}_+^m$ only then one needs to choose $Q_h(\ell v)$ instead of ℓv_d to obtain the required estimate.

0.4 Equations in a cone

Here we will consider briefly more complicated case than a half-space.

Let D be a sharp convex cone, and let $\overset{*}{D}$ be a conjugate cone for D, i.e.,

$$\overset{*}{D}=\{x\in\mathbb{R}^m : x\cdot y>0,\ y\in D\}.$$

Let $T(\overset{*}{D})\subset\mathbb{C}^m$ be a set of the type $\mathbb{T}^m+i\overset{*}{D}$. For $\mathbb{T}^m\equiv\mathbb{R}^m$ such a domain of multidimensional complex space is called a radial tube domain over the cone $\overset{*}{D}$ [9–11]. We introduce the function

$$B_d(z)=\sum_{\tilde{x}\in D_d}e^{i\tilde{x}\cdot z},\quad z=\xi+i\tau,\quad \xi\in\mathbb{T}^m,\quad \tau\in\overset{*}{D},$$

and define the operator

$$(B_d u)(\xi)=\lim_{\tau\to 0}\int\limits_{\mathbb{T}^m}B_d(z-\eta)u_d(\eta)d\eta.$$

This operator is roughly speaking a conical analogue of the periodic Hilbert transform $H_{\xi'}^{per}$.

To describe solvability conditions for the equation (6) we introduce the following concept.

Definition 5. The periodic wave factorization for an elliptic symbol $A_d(\xi)$ is called its representation in the form

$$A_d(\xi)=A_{d,\neq}(\xi)A_{d,=}(\xi),$$

where the factors $A_{d,\neq}(\xi),A_{d,=}(\xi)$ admit an analytical continuation into domains $T(\overset{*}{D}),T(-\overset{*}{D})$ respectively and satisfy the estimates

$$|A_{d,\neq}^{\pm 1}(\xi)|\leq c_1(1+|\hat{\zeta}^2|)^{\pm\frac{æ}{2}},\quad |A_{d,=}^{\pm 1}(\xi)|\leq c_2(1+|\hat{\zeta}^2|)^{\pm\frac{\alpha-æ}{2}},$$

with constants c_1,c_2 non-depending on h,

$$\hat{\zeta}^2\equiv\hbar^2\left(\sum_{k=1}^{m}(e^{-ih(\xi_k+\tau_k)}-1)^2\right),\quad \xi\in\mathbb{T}^m,\tau\in\pm\overset{*}{D}.$$

The number $æ\in\mathbb{R}$ is called an index of the periodic wave factorization.

Theorem 6. If the elliptic symbol $\tilde{A}_d(\xi)$ admits periodic wave factorization with the index æ so that $|æ-s|<1/2$ then the operator $A_d:H^s(D_d)\to H^{s-\alpha}(D_d)$ is invertible and a solution of the equation (6) for arbitrary right-hand side $v_d\in H_0^s(D_d)$ in Fourier images is given by the formula

$$\tilde{u}_d(\xi)=A_{d,\neq}^{-1}(\xi)B_d(A_{d,=}^{-1}(\xi)\widetilde{\ell v}_d(\xi)),\tag{10}$$

where ℓv_d is an arbitrary continuation of v_d into $H^s(h\mathbb{Z}^m)$.

Using the latter formula (10) for the solution of the equation (6) and the previous considerations one can obtain conical analogues of theorems 1 and 5. We hope to give more detailed analysis in forthcoming papers.

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В.Б. Васильев

Арнайы псевдодифференциалды теңдеулер үшін кейбір жуықтау есептеулер

Мақалада дискретті псевдодифференциалды операторлар мен теңдеулерді олардың үздіксіз аналогтары үшін жуықтау операторлар мен теңдеулер ретінде қарастырылды. Осы мақсатпен мұндай теңдеулердің шешімділігіне сәйкес дискретті кеңістіктер зерттелді, дискретті және үздіксіз шешімдердің қателіктерін бағаланды. Бұл тәсіл Евклид кеңістігінің белгілі бір қарапайым облыстарында дискретті Фурье түрлендіруі және факторизациялау әдісіне негізделген.

Кілт сөздер: дискретті псевдодифференциалды оператор, периодты факторизация, шешімділік, жуықтау шешімі, қателерді бағалау.

В.Б. Васильев

О некоторых приближенных вычислениях для специальных псевдодифференциальных уравнений

В статье рассмотрены дискретные псевдодифференциальные операторы и уравнения как приближенные операторы и уравнения для их непрерывных аналогов. Изучена разрешимость таких уравнений в соответствующих дискретных пространствах, и даны некоторые оценки погрешности для дискретных и непрерывных решений. Этот подход основан на дискретном преобразовании Фурье и технике факторизации, которая используется для специальных канонических областей в евклидовом пространстве.

Ключевые слова: дискретный псевдодифференциальный оператор, периодическая факторизация, разрешимость, приближенное решение, оценки погрешности.

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Some approximations of second order derivatives complex-valued functions

In this paper, we generalize the well known finite difference method to compute derivatives of real valued function to approximate of second order complex derivatives w_{zz} and $w_{\overline{z}\overline{z}}$ for complex-valued function w. Exploring different combinations of terms, we derive several approximations to compute the second order derivatives of complex-valued function. Several second order of accuracy finite differences to calculate derivatives are proposed. Error analysis in test examples is carried out by using Matlab program.

Keywords: finite difference, approximation, complex-valued function, approximation formulas.

Introduction

Boundary value problems for equations with complex-valued functions and partial derivatives with respect to complex variables have important applications in various areas of mathematical modeling of real physical processes [1–6]. The theory of finite difference method in case of real valued function and its applications to solve boundary value problems for partial differential equations is described in [7]. In [8–10], a complex step method for computing derivatives of real valued functions by introducing a complex step in a strict sense is developed. Several finite differences to compute first order derivatives of complex valued function discussed in [7].

Let C be a set of complex numbers, let $\Omega_1, \Omega_2 \subset C$, let $\omega : \Omega_1 \to \Omega_2$ be a complex-valued function. For each $z = x + iy \in \Omega_1$ its image $\omega(z) = \omega(x, y) \in \Omega_2$ can be rewritten as $u(x, y) + iv(x, y)$ by introducing pair of real-valued two-dimensional functions u and v. Second derivatives w_{zz} and $w_{\overline{z}\overline{z}}$ at point $z = x + iy$ are defined by

$$\begin{aligned} \omega_{zz}(x,y) &= \tfrac{1}{2}\left(\tfrac{\partial\omega_z}{\partial x} - i\tfrac{\partial\omega_z}{\partial y}\right) = \tfrac{1}{4}\left(\tfrac{\partial^2\omega}{\partial x^2} - i\tfrac{\partial^2\omega}{\partial y\partial x} - i\tfrac{\partial^2\omega}{\partial x\partial y} - \tfrac{\partial^2\omega}{\partial y^2}\right); \\ \omega_{\bar z\bar z}(x,y) &= \tfrac{1}{2}\left(\tfrac{\partial\omega_{\bar z}}{\partial x} + i\tfrac{\partial\omega_{\bar z}}{\partial y}\right) = \tfrac{1}{4}\left(\tfrac{\partial^2\omega}{\partial x^2} + i\tfrac{\partial^2\omega}{\partial y\partial x} + i\tfrac{\partial^2\omega}{\partial x\partial y} - \tfrac{\partial^2\omega}{\partial y^2}\right). \end{aligned} \tag{1}$$

Approximation of second order derivatives

Theorem 1. Assume that the functions $\frac{\partial^4 u}{\partial y\partial x^3}, \frac{\partial^6 u}{\partial y^3\partial x^3}, \frac{\partial^4 v}{\partial y\partial x^3}, \frac{\partial^6 v}{\partial y^3\partial x^3}$ are continuous and bounded on Ω_1, h and τ are positive and sufficiently small numbers. Then, the following second order of accuracy approximate formulas for ω_{zz} are valid:

$$\begin{aligned} &\omega_{zz}(x,y) = \tfrac{1}{4h^2}\omega(x+h,y) + \left(-\tfrac{1}{2h^2} + \tfrac{1}{2\tau^2}\right)\omega(x,y) + \tfrac{1}{4h^2}\omega(x-h,y) - \\ &-\tfrac{i}{8h\tau}\omega(x+h,y+\tau) + \tfrac{i}{8h\tau}\omega(x-h,y+\tau) + \tfrac{i}{8h\tau}\omega(x+h,y-\tau) - \\ &-\tfrac{i}{8h\tau}\omega(x-h,y-\tau) - \tfrac{1}{4\tau^2}\omega(x,y+\tau) - \tfrac{1}{4\tau^2}\omega(x,y-\tau) + O\left(h^2+\tau^2\right); \\ &x+iy, x\pm h+iy, x+h+i(y\pm\tau), x-h+i(y\pm\tau) \in\Omega_1; \end{aligned} \tag{2}$$

$$\begin{aligned}
&\omega_{zz}(x,y) = -\tfrac{1}{4h^2}\omega(x+3h,y) + \tfrac{1}{h^2}\omega(x+2h,y) - \tfrac{5}{4h^2}\omega(x+h,y) + \\
&+\left(\tfrac{1}{2h^2} - \tfrac{1}{2\tau^2}\right)\omega(x,y) - \tfrac{i}{8h\tau}\omega(x+h,y+\tau) + \tfrac{i}{8h\tau}\omega(x-h,y+\tau) + \\
&+\tfrac{i}{8h\tau}\omega(x+h,y-\tau) - \tfrac{i}{8h\tau}\omega(x-h,y-\tau) + \tfrac{1}{4\tau^2}\omega(x,y+3\tau) - \\
&-\tfrac{1}{\tau^2}\omega(x,y+2\tau) + \tfrac{5}{4\tau^2}\omega(x,y+\tau) + O\left(h^2+\tau^2\right); \\
&x+3h+iy, x+2h+iy, x+h+iy, x+iy \in \Omega_1; \\
&x+h+i(y\pm\tau), x+i(y+3\tau), x+i(y+2\tau), x+i(y+\tau) \in \Omega_1;
\end{aligned} \tag{3}$$

$$\begin{aligned}
&\omega_{zz}(x,y) = -\tfrac{1}{4h^2}\omega(x-3h,y) + \tfrac{1}{h^2}\omega(x-2h,y) - \tfrac{5}{4h^2}\omega(x-h,y) + \\
&+\left(\tfrac{1}{2h^2} - \tfrac{1}{2\tau^2}\right)\omega(x,y) - \tfrac{i}{8h\tau}\omega(x+h,y+\tau) + \tfrac{i}{8h_1\tau}\omega(x-h,y+\tau) + \\
&+\tfrac{i}{8h\tau}\omega(x+h,y-\tau) - \tfrac{i}{8h\tau}\omega(x-h,y-\tau) + \tfrac{1}{4\tau^2}\omega(x,y-3\tau) - \\
&-\tfrac{1}{\tau^2}\omega(x,y-2\tau) + \tfrac{5}{4\tau^2}\omega(x,y-\tau) + O\left(h^2+\tau^2\right); \\
&x-3h+iy, x-2h+iy, x-h+iy, x+iy, x-h+i\ (y+\tau), \in \Omega_1; \\
&x+h+i(y\pm\tau), x+i(y-3\tau), x+i(y-2\tau), x+i(y-\tau) \in \Omega_1;
\end{aligned} \tag{4}$$

$$\begin{aligned}
&\omega_{zz}(x,y) = -\tfrac{1}{4h^2}\omega(x+3h,y) + \tfrac{1}{h^2}\omega(x+2h,y) - \tfrac{5}{4h^2}\omega(x+h,y) + \\
&+\left(\tfrac{1}{2h^2} - \tfrac{1}{2\tau^2}\right)\omega(x,y) - \tfrac{i}{8h\tau}\omega(x+h,y+\tau) + \tfrac{i}{8h\tau}\omega(x-h,y+\tau) + \\
&+\tfrac{i}{8h\tau}\omega(x+h,y-\tau) - \tfrac{i}{8h\tau}\omega(x-h,y-\tau) + \tfrac{1}{4\tau^2}\omega(x,y-3\tau) - \\
&-\tfrac{1}{\tau^2}\omega(x,y-2\tau) + \tfrac{5}{4\tau^2}\omega(x,y-\tau) + O\left(h^2+\tau^2\right); \\
&x+3h+iy, x+2h+iy, x+h+iy, x+iy, x+h+i(y\pm\tau) \in \Omega_1; \\
&x-h+i(y\pm\tau), x+i(y-\tau), x+i(y-2\tau), x+i(y-3\tau) \in \Omega_1;
\end{aligned} \tag{5}$$

$$\begin{aligned}
&\omega_{zz}(x,y) = -\tfrac{1}{4h^2}\omega(x-3h,y) + \tfrac{1}{h^2}\omega(x-2h,y) - \tfrac{5}{4h^2}\omega(x-h,y) + \\
&+\left(\tfrac{1}{2h^2} - \tfrac{1}{2\tau^2}\right)\omega(x,y) - \tfrac{i}{8h\tau}\omega(x+h,y+\tau) + \tfrac{i}{8h_1\tau}\omega(x-h,y+\tau) + \\
&+\tfrac{i}{8h\tau}\omega(x+h,y-\tau) - \tfrac{i}{8h\tau}\omega(x-h,y-\tau) + \tfrac{1}{4\tau^2}\omega(x,y+3\tau) - \\
&-\tfrac{1}{\tau^2}\omega(x,y+2\tau) + \tfrac{5}{4\tau^2}\omega(x,y+\tau) + O\left(h^2+\tau^2\right); \\
&x-3h+iy, x-2h+iy, x-h+iy, x+iy, x+h+i(y\pm\tau) \in \Omega_1; \\
&x-h+i(y\pm\tau), x+i(y+\tau), x+i(y+2\tau), x+i(y+3\tau) \in \Omega_1.
\end{aligned} \tag{6}$$

Proof. By using Taylor decomposition formula for $\frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 v}{\partial x^2}, \frac{\partial^2 u}{\partial y^2}, \frac{\partial^2 v}{\partial y^2}$ at point $(x,y) \in \Omega_1$ we have that there exist real numbers c_1, c_2, d_1, d_2 such that

$$\frac{\partial^2 u}{\partial x^2}(x,y) = \frac{u(x+h,y)-2u(x,y)+u(x-h,y)}{h^2} + \frac{\partial^3 u}{\partial x^3}(c_1,y)\frac{h^2}{6};$$

$$\frac{\partial^2 v}{\partial x^2}(x,y) = \frac{v(x+h,y)-2v(x,y)+v(x-h,y)}{h^2} + \frac{\partial^3 v}{\partial x^3}(c_2,y)\frac{h^2}{6};$$

$$\frac{\partial^2 u}{\partial y^2}(x,y) = \frac{u(x,y+\tau)-2u(x,y)+u(x,y-\tau)}{\tau^2} + \frac{\partial^3 u}{\partial y^3}(x,d_1)\frac{\tau^2}{6};$$

$$\frac{\partial^2 v}{\partial y^2}(x,y) = \frac{v(x,y+\tau)-2v(x,y)+v(x,y-\tau)}{\tau^2} + \frac{\partial^3 v}{\partial y^3}(x,d_2)\frac{\tau^2}{6}.$$

Thus

$$\begin{aligned}\frac{\partial^2 \omega}{\partial x^2}(x,y) &= \frac{\omega(x+h,y)-2\omega(x,y)+\omega(x-h,y)}{h^2} + \left(\frac{\partial^3 u}{\partial x^3}(c_1,y) + i\frac{\partial^3 v}{\partial x^3}(c_2,y)\right)\frac{h^2}{6};\\ \frac{\partial^2 \omega}{\partial y^2}(x,y) &= \frac{\omega(x,y+\tau)-2\omega(x,y)+\omega(x,y-\tau)}{\tau^2} + \left(\frac{\partial^3 u}{\partial y^3}(x,d_1) + i\frac{\partial^3 v}{\partial y^3}(x,d_2)\right)\frac{\tau^2}{6}.\end{aligned} \tag{7}$$

Applying Taylor decomposition formula for $\frac{\partial u}{\partial x}$ at points $(x, y+\tau)$ and $(x, y-\tau)$, we have that there exists c_3 such that

$$\begin{aligned}\frac{\partial u}{\partial x}(x,y+\tau) &= \frac{u(x+h,y+\tau)-u(x-h,y+\tau)}{2h} + \frac{\partial^3 u}{\partial x^3}(c_3,y+\tau)\frac{h^2}{6};\\ \frac{\partial u}{\partial x}(x,y-\tau) &= \frac{u(x+h,y-\tau)-u(x-h,y-\tau)}{2h} + \frac{\partial^3 u}{\partial x^3}(c_3,y-\tau)\frac{h^2}{6}.\end{aligned} \tag{8}$$

By Taylor decomposition formula for $\frac{\partial^2 u}{\partial y \partial x}$ at point (x,y), we have

$$\frac{\partial^2 u}{\partial y \partial x}(x,y) = \frac{\frac{\partial u}{\partial x}(x,y+\tau) - \frac{\partial u}{\partial x}(x,y-\tau)}{2\tau} + \frac{\partial^4 u}{\partial y^3 \partial x}(x,d_3)\ \frac{\tau^2}{6} \tag{9}$$

for some constant d_3 between $y-\tau$ and $y+\tau$. From (8) and (9) it follows that

$$\begin{aligned}\frac{\partial^2 u}{\partial y \partial x}(x,y) &= \frac{u(x+h,y+\tau)-u(x-h,y+\tau)-\ u(x+h,y-\tau)+u(x-h,y-\tau)}{4h\tau}+\\ &+\frac{1}{2\tau}\left[\frac{\partial^3 u}{\partial x^3}(c_3,y+\tau)\frac{h^2}{6} - \frac{\partial^3 u}{\partial x^3}(c_3,y-\tau)\frac{h^2}{6}\right] + \frac{\partial^4 u}{\partial y^3 \partial x}(x,d_3)\ \frac{\tau^2}{6}.\end{aligned} \tag{10}$$

Since

$$\frac{1}{2\tau}\left[\frac{\partial^3 u}{\partial x^3}(c_3,y+\tau)\ - \frac{\partial^3 u}{\partial x^3}(c_3,y-\tau)\ \right] = \frac{\partial^4 u}{\partial y \partial x^3}(c_3,y) + \frac{\partial^6 u}{\partial y^3 \partial x^3}(c_3,d_3)\ \frac{\tau^2}{6},$$

we have

$$\frac{\partial^2 u}{\partial y \partial x}(x,y) \quad = \quad \frac{u(x+h,y+\tau)-u(x-h,y+\tau)-\ u(x+h,y-\tau)+u(x-h,y-\tau)}{4h\tau}+ \tag{11}$$

$$+O(h^2+\tau^2).$$

In the similar manner it can be obtained that

$$\frac{\partial^2 v}{\partial y \partial x}(x,y) \quad = \quad \frac{v(x+h,y+\tau)-v(x-h,y+\tau)-\ v(x+h,y-\tau)+v(x-h,y-\tau)}{4h\tau}+ \tag{12}$$

$$+O(h^2+\tau^2).$$

Therefore, from (1), (7), (11), and (12) we get (2).

From Taylor decomposition formula it can be showed that there exist numbers c_4, c_5, d_4, d_5 such that

$$x-h < c_4,\ c_5 < x+3h,\ y-\tau < d_4, d_5 < y+3\tau;$$

$$\begin{aligned}\frac{\partial^2 \omega}{\partial x^2}(x,y) &= \frac{-\omega(x+3h,y)+4\omega(x+2h,y)-5\omega(x+h,y)+2\omega(x,y)}{h^2}+\\ &+\left(\frac{\partial^3 u}{\partial x^3}(c_4,y) + i\frac{\partial^3 v}{\partial x^3}(c_5,y)\right)\frac{h^2}{6};\\ \frac{\partial^2 \omega}{\partial y^2}(x,y) &= \frac{-\omega(x,y+3\tau)+4\omega(x,y+2\tau)-5\omega(x,y+\tau)+2\omega(x,y)}{\tau^2}+\\ &+\left(\frac{\partial^3 u}{\partial y^3}(x,d_4) + i\frac{\partial^3 v}{\partial y^3}(x,d_5)\right)\frac{\tau^2}{6}.\end{aligned} \tag{13}$$

Formula (3) follows from (1), (13), (11), and (12).

By Taylor decomposition formula, we can get that there exist numbers c_6, c_7, d_6, d_7 such that

$$x-3h<c_6,\ c_7<x+h,\ y-3\tau<d_6,d_7<y+\tau;$$

$$\begin{aligned} \frac{\partial^2\omega}{\partial x^2}(x,y) &= \frac{-\omega(x-3h,y)+4\omega(x-2h,y)-5\omega(x-h,y)+2\omega(x,y)}{h^2}+\\ &+\left(\frac{\partial^3 u}{\partial x^3}(c_6,y)+i\frac{\partial^3 v}{\partial x^3}(c_7,y)\right)\frac{h^2}{6};\\ \frac{\partial^2\omega}{\partial y^2}(x,y) &= \frac{-\omega(x,y-3\tau)+4\omega(x,y-2\tau)-5\omega(x,y-\tau)+2\omega(x,y)}{\tau^2}+\\ &+\left(\frac{\partial^3 u}{\partial y^3}(x,d_6)+i\frac{\partial^3 v}{\partial y^3}(x,d_7)\right)\frac{\tau^2}{6}. \end{aligned} \tag{14}$$

So, from (1), (14), (11), and (12) we can obtain (4).

By Taylor decomposition formula, we can prove that there exist numbers c_8, c_9, d_8, d_9 such that

$$\begin{aligned} &x-h<c_8,\ c_9<x+3h,\ y-3\tau<d_8,d_9<y+\tau,\\ \frac{\partial^2\omega}{\partial x^2}(x,y) &= \frac{-\omega(x+3h,y)+4\omega(x+2h,y)-5\omega(x+h,y)+2\omega(x,y)}{h^2}+\\ &+\left(\frac{\partial^3 u}{\partial x^3}(c_8,y)+i\frac{\partial^3 v}{\partial x^3}(c_9,y)\right)\frac{h^2}{6};\\ \frac{\partial^2\omega}{\partial y^2}(x,y) &= \frac{-\omega(x,y-3\tau)+4\omega(x,y-2\tau)-5\omega(x,y-\tau)+2\omega(x,y)}{\tau^2}+\\ &+\left(\frac{\partial^3 u}{\partial y^3}(x,d_1)+i\frac{\partial^3 v}{\partial y^3}(x,d_2)\right)\frac{\tau^2}{6}. \end{aligned} \tag{15}$$

Hence, (1), (15), (11), and (12) give us (5).

In the similar manner we show that there exist c_{10}, c_{11}, d_{10}, d_{11} such that

$$x-3h<c_{10},\ c_{11}<x,\ y<d_{10},d_{11}<y+3\tau;$$

$$\begin{aligned} \frac{\partial^2\omega}{\partial x^2}(x,y) &= \frac{-\omega(x-3h,y)+4\omega(x-2h,y)-5\omega(x-h,y)+2\omega(x,y)}{h^2}+\\ &+\left(\frac{\partial^3 u}{\partial x^3}(c_{10},y)+i\frac{\partial^3 v}{\partial x^3}(c_{11},y)\right)\frac{h^2}{6};\\ \frac{\partial^2\omega}{\partial y^2}(x,y) &= \frac{-\omega(x,y+3\tau)+4\omega(x,y+2\tau)-5\omega(x,y+\tau)+2\omega(x,y)}{\tau^2}+\\ &+\left(\frac{\partial^3 u}{\partial y^3}(x,d_{10})+i\frac{\partial^3 v}{\partial y^3}(x,d_{11})\right)\frac{\tau^2}{6}. \end{aligned} \tag{16}$$

Finally, (1), (16), (11), and (12) give us (6). The proof of Theorem 1 is complete.

In similar manner it can established the following statement.

Theorem 2. Assume that the functions $\frac{\partial^4 u}{\partial y\partial x^3}, \frac{\partial^6 u}{\partial y^3\partial x^3}, \frac{\partial^4 v}{\partial y\partial x^3}, \frac{\partial^6 v}{\partial y^3\partial x^3}$ are continuous and bounded on Ω_1, h and τ are positive numbers. Then, the following second order of accuracy approximate formulas for $\omega_{\bar z\bar z}$ are valid:

$$\begin{aligned} &\omega_{\bar z\bar z}(x,y)=\frac{1}{4h^2}\omega(x+h,y)+\left(-\frac{1}{2h^2}+\frac{1}{2\tau^2}\right)\omega(x,y)+\frac{1}{4h^2}\omega(x-h,y)+\\ &+\frac{i}{8h\tau}\omega(x+h,y+\tau)-\frac{i}{8h\tau}\omega(x-h,y+\tau)-\frac{i}{8h\tau}\omega(x+h,y-\tau)+\\ &+\frac{i}{8h\tau}\omega(x-h,y-\tau)-\frac{1}{4\tau^2}\omega(x,y+\tau)-\frac{1}{4\tau^2}\omega(x,y-\tau)+O\left(h^2+\tau^2\right);\\ &x+iy,x\pm h+iy,x+h+i(y\pm\tau),x-h+i(y\pm\tau)\ \in\Omega_1; \end{aligned} \tag{17}$$

$$
\begin{gathered}
\omega_{\bar{z}\bar{z}}(x,y) = -\tfrac{1}{4h^2}\omega(x+3h,y) + \tfrac{1}{h^2}\omega(x+2h,y) - \tfrac{5}{4h^2}\omega(x+h,y) + \\
+\left(\tfrac{1}{2h^2} - \tfrac{1}{2\tau^2}\right)\omega(x,y) + \tfrac{i}{8h\tau}\omega(x+h,y+\tau) - \tfrac{i}{8h\tau}\omega(x-h,y+\tau) - \\
-\tfrac{i}{8h\tau}\omega(x+h,y-\tau) + \tfrac{i}{8h\tau}\omega(x-h,y-\tau) + \tfrac{1}{4\tau^2}\omega(x,y+3\tau) - \\
-\tfrac{1}{\tau^2}\omega(x,y+2\tau) + \tfrac{5}{4\tau^2}\omega(x,y+\tau) + O\left(h^2+\tau^2\right); \\
x+3h+iy, x+2h+iy, x+h+iy, x+iy \in\Omega_1; \\
x+h+i(y\pm\tau), x+i(y+3\tau), x+i(y+2\tau), x+i(y+\tau) \in\Omega_1;
\end{gathered}
\tag{18}
$$

$$
\begin{gathered}
\omega_{\bar{z}\bar{z}}(x,y) = -\tfrac{1}{4h^2}\omega(x-3h,y) + \tfrac{1}{h^2}\omega(x-2h,y) - \tfrac{5}{4h^2}\omega(x-h,y) + \\
+\left(\tfrac{1}{2h^2} - \tfrac{1}{2\tau^2}\right)\omega(x,y) + \tfrac{i}{8h\tau}\omega(x+h,y+\tau) - \tfrac{i}{8h_1\tau}\omega(x-h,y+\tau) - \\
-\tfrac{i}{8h\tau}\omega(x+h,y-\tau) + \tfrac{i}{8h\tau}\omega(x-h,y-\tau) + \tfrac{1}{4\tau^2}\omega(x,y-3\tau) - \\
-\tfrac{1}{\tau^2}\omega(x,y-2\tau) + \tfrac{5}{4\tau^2}\omega(x,y-\tau) + O\left(h^2+\tau^2\right); \\
x-3h+iy, x-2h+iy, x-h+iy, x+iy, x-h+i\ (y+\tau), \in\Omega_1; \\
x+h+i(y\pm\tau), x+i(y-3\tau), x+i(y-2\tau), x+i(y-\tau) \in\Omega_1;
\end{gathered}
\tag{19}
$$

$$
\begin{gathered}
\omega_{\bar{z}\bar{z}}(x,y) = -\tfrac{1}{4h^2}\omega(x+3h,y) + \tfrac{1}{h^2}\omega(x+2h,y) - \tfrac{5}{4h^2}\omega(x+h,y) + \\
+\left(\tfrac{1}{2h^2} - \tfrac{1}{2\tau^2}\right)\omega(x,y) + \tfrac{i}{8h\tau}\omega(x+h,y+\tau) - \tfrac{i}{8h\tau}\omega(x-h,y+\tau) - \\
-\tfrac{i}{8h\tau}\omega(x+h,y-\tau) + \tfrac{i}{8h\tau}\omega(x-h,y-\tau) + \tfrac{1}{4\tau^2}\omega(x,y-3\tau) - \\
-\tfrac{1}{\tau^2}\omega(x,y-2\tau) + \tfrac{5}{4\tau^2}\omega(x,y-\tau) + O\left(h^2+\tau^2\right); \\
x+3h+iy, x+2h+iy, x+h+iy, x+iy, x+h+i(y\pm\tau) \in\Omega_1; \\
x-h+i(y\pm\tau), x+i(y-\tau), x+i(y-2\tau), x+i(y-3\tau) \in\Omega_1;
\end{gathered}
\tag{20}
$$

$$
\begin{gathered}
\omega_{\bar{z}\bar{z}}(x,y) = -\tfrac{1}{4h^2}\omega(x-3h,y) + \tfrac{1}{h^2}\omega(x-2h,y) - \tfrac{5}{4h^2}\omega(x-h,y) + \\
+\left(\tfrac{1}{2h^2} - \tfrac{1}{2\tau^2}\right)\omega(x,y) + \tfrac{i}{8h\tau}\omega(x+h,y+\tau) - \tfrac{i}{8h_1\tau}\omega(x-h,y+\tau) - \\
-\tfrac{i}{8h\tau}\omega(x+h,y-\tau) + \tfrac{i}{8h\tau}\omega(x-h,y-\tau) + \tfrac{1}{4\tau^2}\omega(x,y+3\tau) - \\
-\tfrac{1}{\tau^2}\omega(x,y+2\tau) + \tfrac{5}{4\tau^2}\omega(x,y+h_2) + O\left(h^2+\tau^2\right); \\
x-3h+iy, x-2h+iy, x-h+iy, x+iy, x+h+i(y\pm\tau) \in\Omega_1; \\
x-h+i(y\pm\tau), x+i(y+\tau), x+i(y+2\tau), x+i(y+3\tau) \in\Omega_1.
\end{gathered}
\tag{21}
$$

Numerical results

In this section, we give numerical results for the second order of accuracy finite differences to calculate the second derivatives with respect to complex variables in test example by using Matlab program. Let $\Omega = \{z \mid z = x + iy, -1 \le x \le 1, -1 \le y \le 1\}, w(z) = z^2\overline{z} + \cos(z) + \sin(\overline{z})$. The set of grid points are defined by

$$\Omega_{h,\tau} = \left\{z_{k,m} = x_k + iy_m,\ x_k = (k-1)h, k = \overline{1, N+1};\quad y_m = (m-1)\tau, m = \overline{1, M+1}\right\},$$

$h = \frac{2}{N}, \quad \tau = \frac{2}{M}$.

Let $S = \{0, 1, ...N\}$, $Q = \{0, 1, ...M\}$. Denote by

$$I^{(1)} = S - \{0\}, J^{(1)} = Q - \{0\};$$

$$I^{(2)} = S - \{0, N-1, N\}, J^{(2)} = Q - \{0, M-1, M\};$$

$$I^{(3)} = S - \{0, 1, 2, N\}, J^{(3)} = Q - \{0, 1, 2, M\};$$

$$I^{(4)} = S - \{0, N-1, N\}, J^{(4)} = Q - \{0, 1, 2\};$$

$$I^{(5)} = S - \{0, 1, 2\}, J^{(5)} = Q - \{0, 1, 2, M-1, M\},$$

a set of indices.

In Table 1 an error of corresponding value of the derivative ω_{zz} is calculated by

$$\left\| w_{zz} - w_{zz}^{(n)} \right\|_{C(\Omega_{h,\tau})} = \max_{k \in I^{(n)},\ m \in J^{(n)}} \left| w_{zz}(z_{k,m}) - w_{zz}^{(n)}(z_{k,m}) \right|,\ n = 1, 2, 3, 4, 5.$$

Here $w_{zz}^{(1)}$, $w_{zz}^{(2)}$, $w_{zz}^{(3)}$, $w_{zz}^{(4)}$, $w_{zz}^{(5)}$ are approximate value of ω_{zz} by formulas (2), (3), (4), (5), (6), respectively.

In Table 2 an error of corresponding value of the derivative $w_{\overline{z}\overline{z}}$ is calculated by

$$\left\| w_{\overline{z}\overline{z}} - w_{\overline{z}\overline{z}}^{(n)} \right\|_{C(\Omega_{h,\tau})} = \max_{k \in I^{(n)},\ m \in J^{(n)}} \left| w_{\overline{z}\overline{z}}(z_{k,m}) - w_{\overline{z}\overline{z}}^{(n)}(z_{k,m}) \right|,\ n = 1, 2, 3, 4, 5,$$

where $w_{\overline{z}\overline{z}}^{(1)}$, $w_{\overline{z}\overline{z}}^{(2)}$, $w_{\overline{z}\overline{z}}^{(3)}$, $w_{\overline{z}\overline{z}}^{(4)}$, $w_{\overline{z}\overline{z}}^{(5)}$ are approximately value of $\omega_{\overline{z}\overline{z}}$ by formulas (17), (18), (19), (20), (21), respectively.

Table 1

Error analysis for ω_{zz}

Approximation formula	N=10 M=10	N=20 M=20	N=40 M=40	N=80 M=80	N=160 M=160
(2)	2.02×10^{-5}	1.39×10^{-6}	9.10×10^{-8}	5.83×10^{-9}	3.70×10^{-10}
(3)	5.49×10^{-3}	8.13×10^{-4}	1.10×10^{-4}	1.44×10^{-5}	1.83×10^{-6}
(4)	4.81×10^{-3}	7.60×10^{-4}	1.06×10^{-4}	1.41×10^{-5}	1.82×10^{-6}
(5)	5.49×10^{-3}	8.13×10^{-4}	1.10×10^{-4}	1.44×10^{-5}	1.83×10^{-6}
(6)	4.81×10^{-3}	7.60×10^{-4}	1.07×10^{-4}	1.41×10^{-5}	1.82×10^{-6}

Table 2

Error analysis for $\omega_{\overline{z}\overline{z}}$

Approximation formula	N=10 M=10	N=20 M=20	N=40 M=40	N=80 M=80	N=160 M=160
(17)	2.02×10^{-5}	1.39×10^{-6}	9.10×10^{-8}	5.83×10^{-9}	3.70×10^{-10}
(18)	5.49×10^{-3}	8.13×10^{-4}	1.10×10^{-4}	1.44×10^{-5}	1.83×10^{-6}
(19)	4.81×10^{-3}	7.60×10^{-4}	1.06×10^{-4}	1.41×10^{-5}	1.82×10^{-6}
(20)	5.49×10^{-3}	8.13×10^{-4}	1.10×10^{-4}	1.44×10^{-5}	1.83×10^{-6}
(21)	4.81×10^{-3}	7.60×10^{-4}	1.07×10^{-4}	1.41×10^{-5}	1.82×10^{-6}

Conclusion

In the present work, we have generalized the finite difference method to compute derivatives of real valued function to approximate the second order complex derivatives ω_{zz} and $\omega_{\bar{z}\bar{z}}$ for the complex-valued function ω. Exploring different combinations of terms, we derive several approximations to compute the second order derivatives of complex-valued function. Several second order of accuracy finite differences to calculate derivatives are proposed. The error analysis in test examples is carried out by using Matlab program.

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Комплексмәнді функциялардың екінші ретті туындылары үшін кейбір жуықтаулар

Мақалада нақты мәнді функциялардың екінші ретті туындыларын есептеуге арналған белгілі ақырлы-айырымдар тәсілі жалпы жағдайда дамытылған. Сол арқылы комплекс мәнді w функциясының екінші ретті туындыларын w_{zz}, $w_{\overline{z}\overline{z}}$ аппроксимациялауға болады. Терминдердің әртүрлі комбинацияларын зерттеу нәтижесінде комплекс мәнді функцияның екінші ретті туындыларын есептеу үшін бірнеші жуықтау формулалары алынды. Туындыларды есептеу үшін дәлдігі екінші ретті бірнеше ақырлы-айырымдар тәсілі ұсынылды. Тест түріндегі мысалдардағы қателіктерге талдау жасау үшін Matlab бағдарламасы пайдаланылды.

Кілт сөздер: ақырлы-айырымдар тәсілі, аппроксимация, комплекс мәнді функция, жуықтау формулалар.

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Некоторые приближения производных второго порядка комплекснозначных функций

В статье обобщен известный метод конечных разностей для вычисления производных вещественной функции на аппроксимацию комплексных производных второго порядка w_{zz} и $w_{\overline{z}\overline{z}}$ для комплекснозначной функции w. Изучая различные комбинации терминов, получено несколько приближений для вычисления производных второго порядка комплекснозначной функции. Предложены несколько конечных разностей второго порядка точности для вычисления производных. Анализ ошибок в тестовых примерах выполнен с использованием программы Matlab.

Ключевые слова: конечная разность, аппроксимация, комплекснозначная функция, формулы приближений.

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On the solvability of the boundary value problems for the elliptic equation of high order on a plane

For the elliptic equation of $2l-$th order with of constant (and only) real coefficients we consider boundary value problem of the normal derivatives (k_j-1) order, $j=1,\ldots,l$, where $1\le k_1<\ldots<k_l\le 2l-1$. When $k_j=j$ it moves into the Dirichlet problem, and when $k_j=j+1$ it moves into the Neumann problem. In this paper, the study is carried out in space $C^{2l,\mu}(\overline{D})$. We found the condition for Fredholm solvability of this problem and computed the index of this problem.

Keywords: elliptic equation, boundary value problem, Dirichlet problem, Neumann problem, solvability of BVP.

Introduction

From the viewpoint of an explicit description of the conditions of solvability of Fredholm and of index for this problem has been studied [1] in the class

$$u\in C^{2l}(D)\cap C^{2l-1,\mu}(\overline{D}),\qquad \sum_{0\le r\le 2l} a_{r,2l}\frac{\partial^{2l}u}{\partial x^{2l-r}\partial y^r}\in C^{\mu}(\overline{D}).$$

In this paper, under the assumption that $\Gamma\in C^{2l,\mu}$ obtained in the paper [1] results extend to a standard class $C^{2l,\mu}(\overline{D})$, which no longer depends on the equation (1).

In [2–8], an explicit form of the Green function of the Dirichlet problem for a polyharmonic equation in a multidimensional ball is constructed. The paper [9, 10] is devoted to the investigation of the solvability of various boundary value problems for a polyharmonic equation in a multidimensional ball. In this paper we obtain a necessary and sufficient condition for the problem to be Fredholm in terms of the original data, that is, from the right-hand side of the inhomogeneous polyharmonic equation and from the right-hand sides of the inhomogeneous boundary conditions. The correct restrictions of the stationary Navier-Stokes equation in a three-dimensional cube are described in [11], and the correct boundary conditions for the pressure in the medium are determined. In [12], initial-boundary value problems for the equations of motion of a viscous heat-conducting gas are studied with allowance for a magnetic field with cylindrical and spherical symmetry. In this paper, we prove theorems on the existence and uniqueness of solutions as a whole with respect to the time of initial-boundary value problems. In [13], a brief summary of the theory of extensions and contractions of operators in Hilbert space is given, and certain classes of well-posed boundary value problems for the bi-Laplace operator are written out. The Green function of the Neumann problem for the Poisson equation in a multidimensional ball is constructed in [14].

Formulation of the problem

In simply connected region D in the plane bounded by a simple smooth contour Γ, we consider the elliptic equation

$$\sum_{0\le r\le k\le 2l} a_{rk}(z)\frac{\partial^k u}{\partial x^{k-r}\partial y^r}=g(z),\quad z=x+iy\in D, \tag{1}$$

with real coefficients $a_{rk}\in C^{\mu}(\overline{D}),\ 0<\mu<1$, constant at $k=2l$. Without loss of generality we can assume that $a_{2l,2l}=1$.

The Generalized Dirichlet - Neumann problem for this equation is determined by the boundary conditions

$$\left.\frac{\partial^{k_j-1}u}{\partial n^{k_j-1}}\right|_\Gamma = f_j, \quad j=1,\ldots,l, \tag{2}$$

where $1\le k_1<k_2<\ldots<k_l\le 2l$, $n=n_1+in_2$ means the unit external normal and under normal derivative $k-$th order we mean the expression

$$\left(n_1\frac{\partial}{\partial x}+n_2\frac{\partial}{\partial y}\right)^k u=\sum_{r=0}^{k}\binom{k}{r}n_1^r n_2^{k-r}\frac{\partial^k u}{\partial x^r\partial y^{k-r}}.$$

Fredholm solvability of the problem

As usual Fredholm property and the index of the problem are understood in relation toward its restricted operator

$$C^{2l,\mu}(\overline{D})\to C^{\mu}(\overline{D})\times\prod_{j=1}^{l}C^{2l-k_j+1,\mu}(\Gamma). \tag{3}$$

For derivatives of $v\in C^{r,\mu}(\Gamma),\ 1\le r\le 2l-1$, with respect to the parameter arc length we have the expression

$$\left(\frac{d}{ds}\right)^r v=\frac{\partial^r v}{\partial e^r}+...$$

where $e=e_1+ie_2=-in$ is the unit tangent vector to the contour Γ, tangential derivative of $r-$ order $\partial^r v/\partial e^r$ is understood as analogous (2) and the dots denote a linear differential operator of order $r-1$, whose coefficients are expressed through the function e_1,e_2 and derivatives of order $r-1$ inclusive. In virtue of the assumptions about the smoothness of the contour Γ coefficients of the operator belong to the class $C^{2l-r,\mu}(\Gamma)$. Therefore, similar to [1] boundary conditions (2) can be rewritten in the equivalent form

$$\left(e_1\frac{\partial}{\partial x}+e_2\frac{\partial}{\partial y}\right)^{2l-k_j}\left(n_1\frac{\partial}{\partial x}+n_2\frac{\partial}{\partial y}\right)^{k_j-1}u+L_j^0u=f_j^0, \quad 1\le j\le l, \tag{4}$$

with the right-hard side

$$f_j^0=f_j^{(2l-k_j)}+\int_\Gamma f_j(t)d_1t,$$

where the symbol d_1t is an element of arc length, and operators

$$L_j^0u=\sum_{0\le r\le s\le 2l-2}a_{j,rs}\frac{\partial^s u}{\partial x^{s-r}\partial y^r}+\int_\Gamma\frac{\partial^{k_j-1}u}{\partial n^{k_j-1}}d_1t,$$

with some coefficients $a_{j,rs}(z)\in C^{1,\mu}(\Gamma)$. It is clear that the operator $L^0=(L_1^0,\ldots,L_l^0)$ is compact $C^{2l,\mu}(\overline{D})\to C^{1,\mu}(\Gamma)$.

Consider the map

$$\mathcal{D}u=(U_1,\ldots,U_{2l}), \quad U_j=\frac{\partial^{2l-1}u}{\partial x^{2l-j}\partial y^{j-1}},$$

that acts from $C^{2l,\mu}(\overline{D})$ in the space $C^{1,\mu}(\overline{D})$ of vector-functions satisfying the relations

$$\frac{\partial U_j}{\partial y}=\frac{\partial U_{j+1}}{\partial x}, \quad 1\le j\le 2l-1. \tag{5}$$

The core of this operators $\ker\mathcal{D}$ is the class P_{2l-2} of all polynomials of degree at most $2l-2$, which is equal to the dimension of $l(2l-1)$.

As in [1] introduce the right-hand operator $\mathcal{D}^{(-1)}$, so that any function $u\in C^{2l,\mu}(\overline{D})$ uniquely represented in the form

$$u=\mathcal{D}^{(-1)}U+p, \quad p\in P_{2l-2}, \tag{6}$$

where the vector-function $U\in C^{1,\mu}(\overline{D})$ satisfying the relations (5).

Substituting this representation in (1) and using (4), from the elliptic equation can come to the equivalent first order system

$$\frac{\partial U}{\partial y} - A\frac{\partial U}{\partial x} + L^1(\mathcal{D}^{(-1)}U + p) = g^1 \tag{7}$$

with $2l \times 2l-$ matrix

$$A = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \cdot & \cdot & \cdot & \dots & \cdot \\ 0 & 0 & 0 & \dots & 1 \\ -a_0 & -a_1 & -a_2 & \dots & -a_{2l-1} \end{pmatrix}, \quad a_r = a_{r,2l},$$

with the right-hand side $g^1 = (0, \dots, 0, g)$ and the operator

$$L^1 v = (0, \dots, 0, L^1_{2l} v), \quad L^1_{2l} v = \sum_{0 \le r \le k \le 2l-1} a_{rk} \frac{\partial^k v}{\partial x^{k-r} \partial y^r}.$$

Note that the operator L^1 is compact $C^{2l,\mu}(\overline{D}) \to C^{\mu}(\overline{D})$.

With respect to the matrix $C = (C_{jk}) \in C^{2l-1,\mu}(\Gamma)$, the elements of which are defined by the relations

$$\sum\nolimits_{k=1}^{2l} C_{jk}(t) z^{k-1} = [e_1(t) + e_2(t)z]^{2l-k_j}[-e_2(t) + e_1(t)z]^{k_j - 1}, \ 1 \le j \le l, \tag{8}$$

the boundary conditions (4) can be written in the form

$$CU^+ + L^0(\mathcal{D}^{(-1)}U + p) = f^0, \tag{9}$$

where the symbol + indicates the limit value functions. Recall that appearing here the operator $L\mathcal{D}^{(-1)}$ is compact $C^{1,\mu}(\overline{D}) \to C^{1,\mu}(\Gamma)$.

We write the characteristic polynomial equation (1) in the form

$$\sum\nolimits_{r=0}^{2l} a_{r,2l} z^r = \prod\nolimits_{k=1}^{m} [(z - \nu_k)(z - \overline{\nu_k})]^{l_k}, \quad \operatorname{Im} \nu_k > 0, \tag{10}$$

and with each vector-function $g(z) = (g_1(z), \dots, g_n(z))$, analytic in the neighborhood of the point $\nu_1, \dots, \nu_m$. We introduce block $n \times l-$ matrix

$$W_g(\nu_1, \dots, \nu_m) = (W_g(\nu_1), \dots, W_g(\nu_m)),$$

where the matrix $W_g(\nu_k) \in \mathbb{C}^{n \times l_k}$ is composed of column - vectors

$$g(\nu_k), g'(\nu_k), \dots, \frac{1}{(l_k - 1)!} g^{(l_k - 1)}(\nu_k).$$

We introduce block $2l \times 2l-$ matrix

$$\widetilde{B} = (B, \overline{B}), \quad B = W_h(\nu_1, \dots, \nu_m) \in \mathbb{C}^{2l \times l},$$
$$\widetilde{J} = \operatorname{diag}(J, \overline{J}) \quad J = \operatorname{diag}(J_1, \dots, J_m), \tag{11}$$

where $h(z) = (1, z, \dots, z^{2l-1})$ and

$$J_k = \begin{pmatrix} \nu_k & 1 & 0 & \dots & 0 \\ 0 & \nu_k & 1 & \dots & 0 \\ \cdot & \cdot & \cdot & \dots & \cdot \\ 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & \dots & \nu_k \end{pmatrix} \in \mathbb{C}^{l_k \times l_k}$$

is a Jordan cell, corresponding to the eigenvalue ν_k.

As shown in [1], the matrix $\widetilde{B}$ is reversible and transfers in A to Jordan form $\widetilde{J}$, i.e. we have the equality

$$\widetilde{B}^{-1} A \widetilde{B} = \widetilde{J}.$$

Obviously, the operation of multiplication by a matrix $\widetilde{B}^{-1}$ transforms real $2l-$ vector-functions U in the complex vector-function $\widetilde{\phi}$ block form $(\phi, \overline{\phi})$. Wherein

$$(\widetilde{B}^{-1} L_A \widetilde{B})\widetilde{\phi} = (L_J \phi, \overline{L_J \phi}), \tag{12}$$

where for brevity

$$L_A = \frac{\partial}{\partial y} - A\frac{\partial}{\partial x}, \quad L_J = \frac{\partial}{\partial y} - J\frac{\partial}{\partial x}.$$

Recall that the operator $\mathcal{D}^{-1}$, appearing in (7), (8), is defined on $2l-$ the vector-functions $U \in C^{1,\mu}(\overline{D})$, satisfying the conditions (5). In terms of projector Q, acting according to the formula

$$(QU)_j = \begin{cases} U_j, & 1 \le j \le 2l-1; \\ 0, & j = 2l, \end{cases}$$

these conditions can be described in the form $QL_A U = 0$. As shown in [1], there is limited to $C^{1,\mu}(\overline{D})$ projector P with the image $\operatorname{im} P = \{U \in C^{1,\mu}(\overline{D}),\ QL_A U = 0\}$. This operator is constructed as follows [1].

We choose ρ so large that the closed region $\overline{D}$ is contained in the disc $D_0 = \{|z| < \rho\}$. Then there is a bounded operator $C^\mu(\overline{D}) \to C^\mu(\overline{D}_0)$ continuation, denoted by $\varphi \to \widehat{\varphi}$, with properties

$$\widehat{\varphi}\big|_D = \varphi, \quad \widehat{\varphi}\big|_{\partial D_0} = 0.$$

To every non-zero complex number $z = x + iy$ we associate an invertible matrix $z_J = x1 + yJ$, where 1 is a single $l \times l-$ matrix. We introduce the integral operator

$$(I^1\varphi)(z) = \frac{1}{\pi i}\int_D (t-z)_J^{-1}\widehat{\varphi}(t)d_2t, \quad z \in D,$$

where d_2t is the area element. This expression is the bounded mapping $C^\mu(\overline{D}) \to C^{1,\mu}(\overline{D})$ and is a right-hard inverse of L_J, i.e.

$$L_J I^1 \varphi = \varphi. \tag{13}$$

Taking into account

$$(\widetilde{B}^{-1} I \widetilde{B})\widetilde{\varphi} = (I^1\varphi, \overline{I^1\varphi}), \quad \widetilde{\varphi} = (\varphi, \overline{\varphi}),$$

obtain an operator I, acting in the space $C^\mu(\overline{D})$ of real $2l-$ vector-functions, which in view of (12) has a similar property in relation to L_A. In our notation the desired projector P is defined by $P = 1 - IQL_A$.

As in [1] via this projector from (7), (8) we can move on to the problem

$$L_A U + L^1(\mathcal{D}^{(-1)}PU + p) = g^1, \quad CU^+ + L^0(\mathcal{D}^{(-1)}PU + p) = f^0, \tag{14}$$

which is already considered in the whole space $C^{1,\mu}(\overline{D})$. Since $QL^0 = 0$, from the first equation of this problem it follows $QL_AU = Qf^1$. Therefore, if the right side f^1 has the property $Qf^1 = 0$, i.e. $f^1_j = 0,\ 1 \le j \le 2l-1$, then any solution U problems (14) satisfies the condition (5). In other words, for the given right-hand side f^1 problem (14) is equivalent to (7), (8).

We use further substitution

$$U = \widetilde{B}\widetilde{\phi}, \quad \widetilde{\phi} = (\phi, \overline{\phi}), \tag{15}$$

according to which we introduce the operators $L^{(1)} : C^{1,\mu}(\overline{D}) \times P_{2l-2} \to C^\mu(\overline{D})$ and $L^{(0)} : C^{1,\mu}(\overline{D}) \times P_{2l-2} \to$ $\to C^\mu(\Gamma)$, acting according to the formulas

$$(L^{(1)}(\phi, p), \overline{L^{(1)}(\phi, p)}) = \widetilde{B}^{-1}L^1(\mathcal{D}^{(-1)}P\widetilde{B}\widetilde{\phi} + p), \quad L^{(0)}(\phi, p) = L^0(\mathcal{D}^{(-1)}P\widetilde{B}\widetilde{\phi} + p).$$

Then, taking into account (11), (12) the substitution of (15) leads (14) to the following equivalent problem

$$L_J\phi + L^{(1)}(\phi, p) = f^1, \quad 2\mathrm{Re}\,(CB\phi) + L^{(0)}(\phi, p) = f^0, \tag{16}$$

where we put $\widetilde{B}^{-1}g^1 = (f^1, \overline{f^1})$, which is considered in the class $C^{1,\mu}(\overline{D})$ $l-$ complex vector-functions ϕ.

So far all reviews have been carried out in the same way as [1] with the difference that in this work problem (16) is considered in the class of functions $\phi \in C^{\mu}(\overline{D}) \cap C^1(D)$, for which $L_J\phi \in C^{\mu}(\overline{D})$. Following [2], we introduce the generalized Cauchy type integrals

$$(I^0\psi)(z) = \frac{1}{2\pi i}\int_\Gamma (t-z)_J^{-1} dt_J \psi(t), \quad z \in D,$$

with a density $\varphi \in C^{1,\mu}(\Gamma)$, where witch respect to the point $t = t_1 + it_2$ on the curve dt_J is a complex matrix differential $dt_1 1 + dt_2 J$ and contour Γ positively oriented with respect to D. It is important to note that it has the property

$$L_J I^0 \varphi^0 = 0. \tag{17}$$

The Cauchy type integrals answer corresponding singular integral

$$(S^0\psi)(t_0) = \frac{1}{\pi i}\int_\Gamma (t-t_0)_J^{-1} dt_J \psi(t), \quad t_0 \in \Gamma,$$

which is understood in the sense of the Cauchy principal value. Note that in the case of a scalar matrix $J = i$ the operator S^0 becomes classic singular Cauchy operator, denoted by S. As shown in [3], operators S and S^0 are bounded in the spaces $C^{\mu}(\Gamma)$, $C^{1,\mu}(\Gamma)$, and the difference $S^0 - S$ is a compact operator. In addition, by the differentiation formulas given in [3] the operator L^0 is bounded $C^{1,\mu}(\Gamma) \to C^{1,\mu}(\overline{D})$ and just corresponds to an analogue of Sokhotskii - Plemelj

$$(I^1\varphi)^+ = (\varphi + S^1\varphi)/2. \tag{18}$$

Based on these results, similarly to the classical theory of singular operators [4] we show that under the assumption of

$$\det[C(t)B] \neq 0, \quad t \in \Gamma, \tag{19}$$

the operator

$$N^0\psi = \mathrm{Re}\,[CB(\psi + S^0\psi)], \tag{20}$$

acting in the space of real $l-$ of vector-functions $\psi \in C^{1,\mu}(\Gamma)$, is Fredholm and its index is given by

$$\mathrm{ind}\, N^0 = -\frac{1}{\pi}[\arg\det(CB)]\big|_\Gamma. \tag{21}$$

Further arguments are similar to those given in [1]. As this paper shows any function $\phi \in C^{1,\mu}(\overline{D})$ can by uniquely represented in the form

$$\phi = I^1\varphi^1 + I^0\varphi^0 + i\xi, \quad \xi \in \mathbb{R}^l,$$

with some complex $l-$ vector-function $\varphi^1 \in C^{\mu}(\overline{D})$ and real $\varphi^0 \in C^{1,\mu}(\Gamma)$. The substitution of this representation in (16) given (13), (17), (18) reduces the problem to an equivalent system of integral equations

$$\varphi^1 + L_J(I^0\varphi^0 + i\xi) + L^{(1)}(I^1\varphi^1 + I^0\varphi^0 + i\xi, p) = f^1;$$

$$\mathrm{Re}\,[CB(\varphi^0 + S^0\varphi^0)] + 2\mathrm{Re}\,[CB(I^1\varphi^1 + i\xi)] + L^{(0)}(I^1\varphi^1 + i\xi, p) = f^0.$$

In the notation (20) we write it briefly in the operator form

$$N^0\varphi^0 + M^{00}\varphi^0 + M^{01}\varphi^1 + T^0(p,\xi) = f^0, \quad \varphi^1 + M^{10}\varphi^0 + M^{11}\varphi^1 + T^1(p,\xi) = f^1, \tag{22}$$

with the relevant operators T^i and

$$M^{00}\varphi^0 = L^{(0)}I^0\varphi^0, \quad M^{01}\varphi^1 = 2\mathrm{Re}\,(CBI^1\varphi^1) + L^{(0)}I^1\varphi^1,$$

$$M^{10}\varphi^0 = L^{(1)}I^0\varphi^0, \quad M^{11}\varphi^1 = L^{(1)}I^1\varphi^1.$$

Since the operators $L^{(0)}$ and $L^{(1)}$ are compact, in the operator matrix

$$M = \begin{pmatrix} M^{00} & M^{01} \\ M^{10} & M^{11} \end{pmatrix},$$

acting in the space $C^{1,\mu}(\Gamma) \times C^{\mu}(\overline{D})$, all elements except M^{01} are compact. Therefore, by the general theory of Fredholm operators [5] the operators $N = \mathrm{diag}\,(N^0, 1)$ and $N + M$ are Fredholm equivalent and their indices coincide. Recalling that $\dim P_{2l-2} = l(2l-1)$ and $\xi \in \mathbb{R}^l$, taking into account (20) and the corresponding properties of Fredholm operators we conclude that the next theorem is proved.

Theorem. Suppose that condition

$$\det[C(t)B] \neq 0, \quad t \in \Gamma$$

is satisfied. Then the problem (1), (2) is Fredholm in the class $C^{2l,\mu}(\overline{D})$, and its index æ is calculated by the formula

$$æ = -\frac{1}{\pi}[\arg\det(CB)]\big|_{\Gamma} + 2l^2,$$

where the increment of a continuous branch of the argument on the contour Γ is taken in the counterclockwise direction.

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Б.Д. Қошанов, А.П. Солдатов

Жазықтықта жоғары дәрежелі эллиптикалық теңдеулер үшін шеттік есептердің шешімділігі туралы

Мақалада тұрақты (тек жоғары дәрежелері) нақты коэффициентті $2l$-дәрежелі, шекарада (k_j-1)-дәрежелі нормал туындылары берілген шеттік есептер қарастырылған, $j = 1, \ldots, l$, $1 \leq k_1 < \ldots < k_l \leq 2l-1$. Бұл есеп $k_j = j$ болған кезде — Дирихле есебі, ал $k_j = j+1$ кезде Нейман есебі болады. Авторлар осы есептің фредгольмді шешімділігінің шартын тауып, индексін есептеген.

Кілт сөздер: эллиптикалық теңдеулер, шеттік есептер, Дирихле есебі, Нейман есебі, шеттік есептердің шешімділігі.

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О разрешимости краевых задач для эллиптического уравнения высокого порядка на плоскости

В статье для эллиптического уравнения 2 l-го порядка с постоянными (и только старшими) вещественными коэффициентами рассмотрена краевая задача, заключающаяся в задании нормальных производных $(k_j - 1)$−го порядка, $j = 1, \dots, l$, где $1 \leq k_1 < \dots < k_l \leq 2l - 1$. При $k_j = j$ она переходит в задачу Дирихле, а при $k_j = j + 1$ – в задачу Неймана. Авторами найдено условие фредгольмовой разрешимости этой задачи и вычислен индекс.

Ключевые слова: эллиптическое уравнение, краевые задачи, задача Дирихле, задача Неймана, разрешимость краевых задач.

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On one problem for restoring the density of sources of the fractional heat conductivity process with respect to initial and final temperatures

In this paper we consider inverse problems for a fractional heat equation, where the fractional time derivative is taken into account in Riemann–Liouville sense. For the solution of this equation, we have to find an unknown right-hand side depending only on a spatial variable. The problem modeling the process of determining the temperature and density of sources in the process of fractional heat conductivity with respect to given initial and final temperatures is considered. Problems with general boundary conditions with respect to the spatial variable that are not strongly regular are investigated. The existence and uniqueness of classical solution to the problem are proved. The problem is considered independent from a corresponding spectral problem for an operator of multiple differentiation with not strongly regular boundary conditions has the basis property of root functions.

Keywords: Inverse problem, heat equation, fractional heat conductivity, not strongly regular boundary conditions, method of separation of variables.

1 Introduction

It is well-known that problems of determining coefficients or the right-hand side of a differential equation simultaneously with its solution are called inverse problems of mathematical physics. These problems often arise in various areas (seismology, exploration of minerals, biology, medicine, quality control of industrial products etc.) that place them among the current problems of modern mathematics.

In this article, we consider a class of problems which model the process of determining the temperature and density of heat sources with respect to given initial and final temperatures. Their mathematical statement leads to the inverse problems for a fractional heat equation in which along with solving the equation we have to find an unknown right-hand side depending only on a spatial variable.

The questions of solvability of various inverse problems for parabolic equations were studied in many articles. The closest to the subject of this paper is [1], in which one case of regular but not strongly regular boundary conditions was considered. The analysis was carried out by the Fourier method using a basis of eigenfunctions and associated functions. In contrast to this (and other) article, we study the inverse problems for the fractional heat equation with general boundary conditions with respect to the spatial variable which are regular but not strongly regular.

Let $\Omega = \{(x,t)\,,\ 0<x<1,\ 0<t<T\}$. In Ω we consider a problem of finding the right-hand side $f(x)$ of the fractional heat equation

$$D_{0+}^{\alpha}\left(u\left(x,t\right)\ -\ u\left(x,0\right)\right) - u_{xx}\left(x,t\right) = f\left(x\right) + F\left(x,t\right),\ \left(x,t\right)\in\Omega \tag{1}$$

and its solutions $u\left(x,t\right)$ satisfying the initial and final conditions

$$u\left(x,0\right)=\varphi\left(x\right),\quad u\left(x,T\right)=\psi\left(x\right),\ 0\le x\le 1, \tag{2}$$

and the boundary conditions

$$\begin{cases} a_1u_x\left(0,t\right)+b_1u_x\left(1,t\right)+a_0u\left(0,t\right)+b_0u\left(1,t\right)=0; \\ c_1u_x\left(0,t\right)+d_1u_x\left(1,t\right)+c_0u\left(0,t\right)+d_0u\left(1,t\right)=0. \end{cases} \tag{3}$$

The coefficients $a_k,\ b_k,\ c_k,\ d_k$ with $k=0,1$ in (3) are real numbers, D^{α}_{0+} stands for the Riemann-Liouville fractional derivative of order $0<\alpha<1$:

$$D^{\alpha}_{0+}y\left(t\right)=\frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_0^t\frac{y(s)ds}{\left(t-s\right)^{\alpha}},$$

while $\varphi\left(x\right)$, $\psi\left(x\right)$ and $F\left(x,t\right)$ are given functions.

Definition. By a *regular solution* of the inverse problem (1)–(3) we mean a pair of functions $\left(u(x,t),\ f(x)\right)$ of the class $u(x,t)\in C^{2,1}_{x,t}\left(\overline{\Omega}\right)$, $f(x)\in C[0,1]$ that inverts equation (1) and conditions (2)–(3) into an identity.

The use of the Fourier method for solving problem (1)–(3) leads to the spectral problem for the operator ℓ given by the differential expression $\ell\left(y\right)=-y''\left(x\right),\quad 0<x<1$ and boundary conditions

$$\begin{cases} a_1y'\left(0\right)+b_1y'\left(1\right)+a_0y\left(0\right)+b_0y\left(1\right)=0; \\ c_1y'\left(0\right)+d_1y'\left(1\right)+c_0y\left(0\right)+d_0y\left(1\right)=0. \end{cases} \tag{4}$$

These boundary conditions are called *regular* [2] if one of the following three conditions

$$\begin{aligned} &i.\ \ a_1d_1-b_1c_1\neq 0; \\ &ii.\ \ a_1d_1-b_1c_1=0,\quad |a_1|+|b_1|>0,\ a_1d_0+b_1c_0\neq 0; \\ &iii.\ \ a_1=b_1=c_1=d_1=0,\ a_0d_0-b_0c_0\neq 0 \end{aligned} \tag{5}$$

is satisfied. Regular boundary conditions are strongly regular in the first and third cases, while in the second case this requires the additional condition

$$a_1c_0+b_1d_0\neq\pm\left[a_1d_0+b_1c_0\right]. \tag{6}$$

Particular cases of (1)–(3) were considered in [1] with boundary conditions (3) which are not strongly regular: the case of conditions of Samarskii–Ionkin type

$$u\left(1,t\right)=0,\quad u_x\left(0,t\right)=u_x\left(1,t\right)$$

and the case of periodic boundary conditions

$$u\left(0,t\right)=u\left(1,t\right),\quad u_x\left(0,t\right)=u_x\left(1,t\right).$$

However, the method of proof of [1] does not automatically carry over to problems with arbitrary not strongly regular boundary conditions (3). This has essentially to do with the use in [1] of a basis of eigenfunctions and generalized eigenfunctions of the corresponding problem (4) for the operator of multiple differentiation. Unfortunately, not all problems of this type have the basis property. Therefore, in order to study the formulated problem, regardless of the basis properties of the system of root vectors of the operator ℓ, we use the method first substantiated in our work [3]. In [3] a class of problems modeling the process of determining the temperature and density of heat sources with respect to given initial and final temperature is considered. To solve direct heat conductivity problems with general not strongly regular boundary conditions with respect to the spatial variable, this method is described in detail in [4].

The solvability of various inverse problems for parabolic equations was studied in papers of Yu.E. Anikonov and Yu.Ya. Belov, B.A. Bubnov, A.I. Prilepko and A.B. Kostin, V.N. Monakhov, A.I. Kozhanov, I.A. Kaliev, K.B. Sabitov and many others.

These citations can be seen in our papers [3] and [5]. We note [6–28] as recent papers close to the theme of our article. In these papers different variants of direct and inverse initial-boundary value problems for evolutionary equations are considered, including problems with nonlocal boundary conditions and problems for equations with fractional derivatives.

We solve the problem by the Fourier method. Some new variants for solving nonlocal boundary value problems by the method of separation of variables were used in our papers [29–35].

2 Case of Sturm-type boundary conditions

A particular case of strongly regular boundary conditions are Sturm-type conditions: $b_0 = b_1 = c_0 = c_1 = 0$:

$$\begin{cases} a_1 u_x(0,t) + a_0 u(0,t) = 0; \\ d_1 u_x(1,t) + d_0 u(1,t) = 0. \end{cases} \tag{7}$$

By ℓ_1 let us denote a corresponding ordinary differential operator arising when applying the method of separation of variables to problem (1), (2), (7). Spectral problem $\ell_1 y = \lambda y$ has the form

$$\begin{gathered} \ell_1(y) \equiv -y''(x) = \lambda y(x), \ 0 < x < 1; \\ a_1 y(0) + a_0 y(0) = 0, \quad d_1 y(1) + d_0 y(1) = 0. \end{gathered} \tag{8}$$

Denote by λ_k the eigenvalues of the operator ℓ_1 enumerated in the increasing order of their absolute values, and by $y_k(x)$, for $k = 1, 2, \ldots$, denote corresponding normalized eigenfunctions. It is known [2] that the eigenvalues of these problems are real and simple, while the system of their eigenfunctions forms an orthonormal basis in $L_2(0,1)$. Thus, we can represent the solution $u(x,t)$, $f(x)$ to (1), (2), (7) as the series:

$$u(x,t) = \sum_{k=1}^{\infty} u_k(t)\, y_k(x), \ f(x) = \sum_{k=1}^{\infty} f_k y_k(x). \tag{9}$$

Substituting (9) into (1) and (2), we obtain the problems

$$D_{0+}^{\alpha}(u_k(t) - u_k(0)) + \lambda_k u_k(t) = f_k + F_k(t), \quad u_k(0) = \varphi_k, \quad u_k(T) = \psi_k \tag{10}$$

for finding the unknown functions $u_k(t)$ and coefficients f_k. Here $F_k(t)$, φ_k and ψ_k are the Fourier coefficients of $F(x,t)$, φ and ψ with respect to the system $\{y_k(x)\}$. Then we get

$$F_k(t) = (F(x,t), \ y_k(x)), \ \varphi_k = (\varphi(x), \ y_k(x)), \ \text{and} \ \psi_k = (\psi(x), \ y_k(x)).$$

The inverse problem (10) is investigated similarly, as in [1]. A solution to (10) exists, is unique, and can be written explicitly. Without dwelling on the details, we write out its solution:

$$u_k = \frac{\psi_k - U_k(T) - \varphi_k e_\alpha(T, \ \lambda_k)}{\gamma_k} \int_0^t (t-\tau)^{\alpha-1} e_\alpha(\tau, \ \lambda_k) d\tau + \varphi_k e_\alpha(t, \ \lambda_k) + U_k(t), \tag{11}$$

$$f_k = \Gamma(1+\alpha) \frac{\psi_k - U_k(T) - \varphi_k e_\alpha(T, \ \lambda_k)}{\alpha \gamma_k}, \tag{12}$$

where $U_k(t)$ is a solution of problem

$$D_{0+}^{\alpha}(U_k(t) - U_k(0)) + \lambda_k U_k(t) = F_k(t), \quad U_k(0) = 0.$$

In (11) and (12) function $e_\alpha(\tau, \mu)$ is expressed by the function of Mittag–Leffler:

$$e_\alpha(\tau, \mu) := E_\alpha(-\mu \tau^\alpha), \quad E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(1+\alpha k)}, \ \alpha \in [0, +\infty),$$

$$\gamma_k = \int_0^T (T-\tau)^{\alpha-1} e_\alpha(\tau, \ \lambda_k) d\tau. \tag{13}$$

The Mittag–Leffler function $e_\alpha(\tau, \mu)$ for $\mu > 0$ and $0 < \alpha \le 1$ is absolutely monotone function with respect to τ (see [36; 268]). Since $e_\alpha(0, \ \lambda_k) = 1$, then from (13) it is easy to see that there exists a constant $\widehat{\gamma} > 0$ such that

$$\gamma_k \ge \widehat{\gamma} > 0, \ \forall \ k = 1, 2, \ldots. \tag{14}$$

Inserting (11) and (12) into (9), we arrive at a formal solution to the problem. In order to complete our study, it is necessary, as in the Fourier method, to justify the smoothness of the resulting formal solutions and the convergence of all appearing series. Let us state the main result of this section.

Theorem 1. If $F(x,t) \in C^2\left(\overline{\Omega}\right)$, $\varphi(x)$, $\psi(x) \in C^4[0,1]$ *and functions* $F(x,t)$, $\varphi(x)$, $\psi(x)$, $\varphi''(x)$ *and* $\psi''(x)$ *satisfy (7), then there exists a unique classical solution* $u(x,t) \in C^{2,1}_{x,t}\left(\overline{\Omega}\right)$, $f(x) \in C[0,1]$ *to the inverse problem (1), (2), (7).*

Proof. Since $\varphi''(x)$, $\psi''(x) \in C^2[0,1]$ and satisfy (7), by Steklov's theorem [37; 41] they admit expansions into absolutely and uniformly converging Fourier series in the eigenfunctions $\{y_k(x)\}$.

Thus, the series

$$\varphi''(x) = -\sum_{k=1}^{\infty} \lambda_k \varphi_k y_k(x),\ \psi''(x) = -\sum_{k=1}^{\infty} \lambda_k \psi_k y_k(x) \tag{15}$$

converges absolutely and uniformly.

From (11), (12), taking into account (14), since

$$\lim_{k\to\infty} \lambda_k = +\infty,\ |e_\alpha(T,\lambda_k)| \le M_1,\ |e_\alpha(t,\lambda_k)| \le M_2,$$

it is easy to get uniform estimates with respect to k

$$|u_k(t)| \le C\left(|\varphi_k| + |\psi_k| + |U_k(t)|\right);$$

$$\left|D^\alpha_{0+} u_k(t)\right| \le C\left(|\varphi_k| + |\psi_k| + |U_k(t)|\right)|\lambda_k|;$$

$$|f_k| \le C\left(|\varphi_k| + |\psi_k| + |U_k(T)|\right).$$

Hence, from the uniform and absolute convergence of series (15) there follow the convergence of series (9) and the belonging of the solution of (1), (2), (7) to the classes $u(x,t) \in C^{2,1}_{x,t}\left(\overline{\Omega}\right)$, $f(x) \in C[0,1]$.

Let us prove the uniqueness of the solution. Suppose that there are two generalized solutions of the inverse problem (1), (2), (7): $(u_1(x,t),\ f_1(x))$ and $(u_2(x,t),\ f_2(x))$. Denote

$$u(x,t) = u_1(x,t) - u_2(x,t),\quad f(x) = f_1(x) - f_2(x).$$

Then the functions $(u(x,t),\ f(x))$ satisfy equation (1), the boundary conditions (7) and the homogeneous conditions (2):

$$u(x,0) = 0,\quad u(x,T) = 0,\ 0 \le x \le 1. \tag{16}$$

Let us show that the inverse problem (1), (7), (16) has only zero solution. Let us introduce notations

$$u_k(t) = \int_0^1 u(x,t) y_k(x)\, dx,\quad f_k = \int_0^1 f(x) y_k(x)\, dx,\quad (k = 1,2,\dots). \tag{17}$$

We apply the operator D^α_{0+} to $u_k(t)$. Then, using equation (1), by integrating by parts, we obtain a problem given by
the equation

$$D^\alpha_{0+} u_k(t) + \lambda_k u_k(t) = f_k, \tag{18}$$

and the boundary conditions

$$u_k(0) = 0,\qquad u_k(T) = 0. \tag{19}$$

General solution of equation (18) has the form (see [1], Eq. (25)):

$$u_k(t) = \frac{f_k \alpha}{\Gamma(1+\alpha)} \int_0^t (t-\tau)^{\alpha-1} e_\alpha(\tau,\ \lambda_k) d\tau + u_k(0)\, e_\alpha(t,\ \lambda_k).$$

Using the first of conditions (19), from here we have

$$u_k(t) = \frac{f_k \alpha}{\Gamma(1+\alpha)} \int_0^t (t-\tau)^{\alpha-1} e_\alpha(\tau,\ \lambda_k) d\tau. \tag{20}$$

Substituting this into the second condition of (19), we get

$$\frac{f_k \alpha}{\Gamma(1+\alpha)} \int_0^T (T-\tau)^{\alpha-1} e_\alpha(\tau,\ \lambda_k) d\tau = 0. \tag{21}$$

Since for $\mu > 0$ and $0 < \alpha \leq 1$ the function $e_\alpha(\tau, \mu)$ is absolutely monotone with respect to τ [36] and since $e_\alpha(0,\ \lambda_k) = 1$, then the integral in (21) is a strictly positive value. Consequently equation (21) holds if and only if $f_k = 0$. But then from (20) we get $u_k(t) \equiv 0$.

Therefore, using this result, from (17) we find

$$\int_0^1 u(x,t) y_k(x)\, dx \equiv 0, \quad \int_0^1 f(x) y_k(x)\, dx = 0, \qquad (k = 1, 2, \dots).$$

Further, by the completeness of system $\{y_k(x)\}$ in $L_2(0,1)$ we obtain $u(x,t) \equiv 0$ and $f(x) \equiv 0$ for all $(x,t) \in \overline{\Omega}$. The uniqueness of the generalized solution of the inverse problem (1), (2), (7) is proved. Theorem 1 is completely proved.

3 Regular, but not strongly regular boundary conditions

In [3] a class of regular but not strongly regular boundary conditions was described in a convenient form.

Lemma 1 [3]. *If the boundary conditions (4) are regular but not strongly regular then the boundary conditions (3) reduce to*

$$\begin{cases} a_1 u_x(0,t) + b_1 u_x(1,t) + a_0 u(0,t) + b_0 u(1,t) = 0; \\ c_0 u(0,t) + d_0 u(1,t) = 0, \end{cases} \quad |a_1| + |b_1| > 0; \tag{22}$$

of one of the following four types:

$$\begin{array}{lll} I. & a_1 + b_1 = 0, & c_0 - d_0 \neq 0; \\ II. & a_1 - b_1 = 0, & c_0 + d_0 \neq 0; \\ III. & c_0 + d_0 = 0, & a_1 - b_1 \neq 0; \\ IV. & c_0 - d_0 = 0, & a_1 + b_1 \neq 0. \end{array} \tag{23}$$

Also in [4] the following result was proved.

Lemma 2 [4]. *We can always equivalently reduce the solution of the problem (1)–(3) in the case of regular but not strongly regular conditions to solve successively two problems with strongly regular Sturm boundary conditions.*

Using Lemma 2, we can obtain the existence of the solution of (1)–(3), as well as its uniqueness and smoothness, from Theorem 1 for the corresponding problems with strongly regular Sturm-type boundary conditions. In the next four sections, we will outline this method in more detail.

The method of solution, consisting in reducing the initial problem to a sequential solution of two initial-boundary value problems with homogeneous boundary conditions of the Sturm type with respect to a spatial variable, will be formulated separately for each of types mentioned in Lemma 1.

4 Reduction of the problem of type I to a sequential solution of two problems with homogeneous boundary conditions of the Sturm type

Consider a problem of type I. Since $a_1 + b_1 = 0$, and herewith $|a_1| + |b_1| > 0$, then without loss of generality we can assume $a_1 = -b_1 = 1$. Since $c_0 - d_0 \neq 0$, then without loss of generality we can assume $c_0 - d_0 = -1$. To simplify writing (omitting additional indexes) we denote $c_0 = c$. Then $d_0 = 1 + c$.

Therefore the problem of type I can be formulated in the form:

In $\Omega = \{(x,t) : 0 < x < 1, 0 < t < T\}$ *find a solution* $u(x,t)$ *of the fractional heat equation (1) satisfying the initial condition (2) and boundary conditions of type I:*

$$\begin{cases} u_x(0,t) - u_x(1,t) + a u(0,t) + b u(1,t) = 0, \\ c u(0,t) + (1+c) u(1,t) = 0. \end{cases} \tag{24}$$

Here the coefficients a, b, c of the boundary condition are arbitrary complex numbers.

To solve the problem we introduce the auxiliary functions:

$$v(x,t) = [u(x,t) + u(1-x,t)]/2, \tag{25}$$

$$w(x,t) = u(x,t) - [1 - (1+2c)(2x-1)]\, v(x,t). \tag{26}$$

Note that if the solution has been searched in the form of the sum of even and odd parts $u(x,t) = C(x,t) + S(x,t)$ in the initial version of the method (see [3]), then now in a variant suggested by us:
– the function $v(x,t)$ is even on the interval $0 < x < 1$, and is the even part of the function $u(x,t)$;
– and the function $w(x,t)$ is not the odd part of the function $u(x,t)$, though it is the odd function.

The last follows from the fact that $w(x,t)$ can be represented in the form

$$w(x,t) = \frac{1}{2}\left[u(x,t) - u(1-x,t)\right] + (1+2c)(2x-1)\, v(x,t), \tag{27}$$

that is, in the form of the sum of the odd part $\frac{1}{2}\left[u(x,t) - u(1-x,t)\right]$ of the function $u(x,t)$ and of the summand $(1+2c)(2x-1)\, v(x,t)$, which (it is easy to verify) is also the odd function on the whole interval $0 < x < 1$.

From (26) it is easy to see that if we find the functions $v(x,t)$ and $w(x,t)$, then the solution of the initial problem can be reestablished by the formula

$$u(x,t) = w(x,t) + \left[1 - (1+2c)(2x-1)\right] v(x,t). \tag{28}$$

Thus, if in the previous variant the solution is represented in the form of the sum of even and odd parts of the solution, then in the new variant suggested by us it is not quite so. In representation (28) the first summand is even on the interval $0 < x < 1$, and the second summand is neither even, nor odd for $1 + 2c \neq 0$.

It is easy to make sure that the functions $v(x,t)$ and $w(x,t)$ are solutions of the fractional heat equations, satisfy the initial and homogeneous boundary conditions in Ω.

For the function $v(x,t)$ we obtain the initial-boundary value problem which we need to solve first:

$$D^{\alpha}_{0+}\left(v(x,t) - v(x,0)\right) - v_{xx}(x,t) = f_0(x); \tag{29}$$

$$v(x,0) = \varphi_0(x), \quad v(x,T) = \psi_0(x) \quad 0 \le x \le 1; \tag{30}$$

$$v_x(0,t) + \left[a(1+c) - bc\right] v(0,t) = 0, \quad 0 \le t \le T; \tag{31}$$

$$v_x(1,t) - \left[a(1+c) - bc\right] v(1,t) = 0, \quad 0 \le t \le T. \tag{32}$$

Here we use the notations

$$\begin{gathered} f_0(x) = \tfrac{1}{2}\left[f(x) + f(1-x)\right], \\ \varphi_0(x) = \tfrac{1}{2}\left[\varphi(x) + \varphi(1-x)\right], \quad \psi_0(x) = \tfrac{1}{2}\left[\psi(x) + \psi(1-x)\right]. \end{gathered} \tag{33}$$

Having the solution $v(x,t)$ of this problem, for the function $w(x,t)$ we get the initial-boundary value problem which we need to solve second:

$$D^{\alpha}_{0+}\left(w(x,t) - w(x,0)\right) - w_{xx}(x,t) = f_1(x) + F_1(x,t); \tag{34}$$

$$w(x,0) = \varphi_1(x), \quad w(x,T) = \psi_1(x), \quad 0 \le x \le 1; \tag{35}$$

$$w(0,t) = 0, \quad 0 \le t \le T; \tag{36}$$

$$w(1,t) = 0, \quad 0 \le t \le T. \tag{37}$$

Here we use the notations

$$f_1(x) = f(x) - \left[1 - (1+2c)(2x-1)\right] f_0(x), \quad F_1(x,t) = -4(1+2c)\, v_x(x,t); \tag{38}$$

$$\begin{gathered} \varphi_1(x) = \varphi(x) - \left[1 - (1+2c)(2x-1)\right] \varphi_0(x); \\ \psi_1(x) = \psi(x) - \left[1 - (1+2c)(2x-1)\right] \psi_0(x). \end{gathered} \tag{39}$$

By direct checking from (33) and (39) it is easy to make sure that if the initial and final data $\varphi(x)$ and $\psi(x)$ of problem (1), (2), (24) satisfy necessary (classical and well-known) consistency conditions, then the initial and final data $\varphi_0(x)$, $\varphi_1(x)$ and $\psi_0(x)$, $\psi_1(x)$ also satisfy the necessary consistency conditions of their corresponding problems.

Thus the solution of the problem of type I (1), (2), (24) is reduced to the sequential solution of two problems *with homogeneous* boundary conditions of the Sturm type with respect to the spatial variable:

– At first for the function $v(x,t)$ we solve the initial-boundary value problem (29)–(32) with the homogeneous boundary conditions of the Sturm type with respect to the spatial variable;

– Then, using the obtained value $v(x,t)$, for the function $w(x,t)$ we solve the initial-boundary value problem (34)–(37) with the homogeneous boundary conditions of the Sturm type (in this particular case they are the Dirichlet conditions) with respect to the spatial variable.

Therefore the main result on the existence and uniqueness of the solution of the problem of type I (1), (2), (24) in classical and generalized senses follows from Theorem 1 on corresponding solvability of boundary value problems with conditions of the Sturm type. We will formulate this main result at once for all the four types of not strongly regular boundary conditions at the end of the paper.

5 Reduction of the problem of type II to a sequential solution of two problems with homogeneous boundary conditions of the Sturm type

Consider a problem of type II. Since $a_1 - b_1 = 0$, and herewith $|a_1| + |b_1| > 0$, then without loss of generality we can assume $a_1 = b_1 = 1$. Since $c_0 + d_0 \neq 0$, then without loss of generality we can assume $c_0 + d_0 = 1$. To simplify writing (omitting additional indexes) we denote $c_0 = c$. Then $d_0 = 1 - c$.

Therefore the problem of type I can be formulated in the form:

In $\Omega = \{(x,t) : 0 < x < 1, 0 < t < T\}$ find a solution $u(x,t)$ of the fractional heat equation (1) satisfying the initial condition (2) and boundary conditions of type II:

$$\begin{cases} u_x(0,t) + u_x(1,t) + au(0,t) + bu(1,t) = 0; \\ cu(0,t) + (1-c)\,u(1,t) = 0. \end{cases} \tag{40}$$

Here the coefficients $a,\ b,\ c$ of the boundary condition are arbitrary complex numbers.

We introduce the auxiliary functions:

$$v(x,t) = \frac{1}{2}\left[u(x,t) - u(1-x,t)\right], \tag{41}$$

$$w(x,t) = u(x,t) - \left[1 - (1-2c)(2x-1)\right] v(x,t). \tag{42}$$

Note that if the solution has been searched in the form of the sum of even and odd parts $u(x,t) = C(x,t) + S(x,t)$ in the initial version of the method (see [3]), then in a new variant suggested by us:

– the function $v(x,t)$ is odd on the interval $0 < x < 1$, and is the odd part of the function $u(x,t)$;

– and the function $w(x,t)$ is not the even part of the function $u(x,t)$, though it is the even function.

The last follows from the fact that $w(x,t)$ can be represented in the form

$$w(x,t) = \frac{1}{2}\left[u(x,t) + u(1-x,t)\right] + (1-2c)(2x-1)\,v(x,t), \tag{43}$$

that is, in the form of the sum of the even part $\frac{1}{2}\left[u(x,t) - u(1-x,t)\right]$ of the function $u(x,t)$ and the summand $(1-2c)(2x-1)\,v(x,t)$, which (it is easy to verify) is also the even function on the interval $0 < x < 1$.

From (42) it is easy to find the functions $v(x,t)$ and $w(x,t)$, then the solution of the initial problem can be reestablished by the formula

$$u(x,t) = w(x,t) + \left[1 - (1-2c)(2x-1)\right] v(x,t). \tag{44}$$

Thus if in the previous variant of the method the solution is represented in the form of the sum of the even and odd parts of the solution, then in the new variant suggested by us it is not quite so. In representation (44) the first summand is even on the interval $0 < x < 1$, and the second summand is neither even, nor odd for $1 - 2c \neq 0$.

For the function $v(x,t)$ we obtain the initial-boundary value problem which we need to solve first:

$$D_{0+}^{\alpha}\left(v(x,t)\ -\ v(x,0)\right) - v_{xx}(x,t) = f_0(x), \tag{45}$$

$$v(x,0) = \varphi_0(x), \quad v(x,T) = \psi_0(x) \quad 0 \le x \le 1, \tag{46}$$

$$v_x(0,t) + \left[a(1-c) - bc\right] v(0,t) = 0, \quad 0 \le t \le T, \tag{47}$$

$$v_x(1,t) - \left[a(1-c) - bc\right] v(1,t) = 0, \quad 0 \le t \le T. \tag{48}$$

Here we use the notations

$$f_0(x) = \tfrac{1}{2}\left[f(x) - f(1-x)\right],$$
$$\varphi_0(x) = \tfrac{1}{2}\left[\varphi(x) - \varphi(1-x)\right], \quad \psi_0(x) = \tfrac{1}{2}\left[\psi(x) - \psi(1-x)\right]. \tag{49}$$

Having the solution $v(x,t)$ of this problem, for the function $w(x,t)$ we get the initial-boundary value problem which we need to solve second:

$$D_{0+}^{\alpha}\left(w(x,t) \; - \; w(x,0)\right) - w_{xx}(x,t) = f_1(x) + F_1(x,t), \tag{50}$$

$$w(x,0) = \varphi_1(x), \quad w(x,T) = \psi_1(x), \quad 0 \le x \le 1, \tag{51}$$

$$w(0,t) = 0, \quad 0 \le t \le T, \tag{52}$$

$$w(1,t) = 0, \quad 0 \le t \le T. \tag{53}$$

Here we use the notations

$$f_1(x) = f(x) - \left[1 - (1-2c)(2x-1)\right] f_0(x), \quad F_1(x,t) = -4(1-2c)\, v_x(x,t), \tag{54}$$

$$\varphi_1(x) = \varphi(x) - \left[1 - (1-2c)(2x-1)\right]\varphi_0(x)$$
$$\psi_1(x) = \psi(x) - \left[1 - (1-2c)(2x-1)\right]\psi_0(x). \tag{55}$$

By direct checking from (49) and (55) it is easy to make sure that if the initial and final data $\varphi(x)$ and $\psi(x)$ of problem (1), (2), (40) satisfy necessary (classical and well-known) consistency conditions, then the initial and final data $\varphi_0(x)$, $\varphi_1(x)$ and $\psi_0(x)$, $\psi_1(x)$ also satisfy the necessary consistency conditions of their corresponding problems.

Thus the solution of the problem of type II (1), (2), (40) is reduced to the sequential solution of two problems *with homogeneous* boundary conditions of the Sturm type with respect to the spatial variable:

– At first for the function $v(x,t)$ we solve the initial-boundary value problem (45)–(48) with the homogeneous boundary conditions of the Sturm type (in this case they are the Dirichlet conditions) with respect to the spatial variable;

– Then, using the obtained value $v(x,t)$, for the function $w(x,t)$ we solve the initial-boundary value problem (50)–(53) with the homogeneous boundary conditions of the Sturm type (in this case with conditions of the Dirichlet problem) with respect to the spatial variable.

Therefore the main result on the existence and uniqueness of the solution of the problem of type II (1), (2), (40) in classical and generalized senses follows from Theorem 1 on corresponding solvability of boundary value problems with conditions of the Sturm type. We will formulate this main result at once for all the four types of not strongly regular boundary conditions at the end of the paper.

6 Reduction of the problem of type III to a sequential solution of two problems with homogeneous boundary conditions of the Sturm type

Consider a problem of type III. Since $c_0 + d_0 = 0$, and herewith $|c_0| + |d_0| > 0$, then without loss of generality we can assume $c_0 = -d_0 = 1$. Since $a_1 - b_1 \ne 0$, then without loss of generality we can assume $a_1 - b_1 = -1$. To simplify writing (omitting additional indexes) we denote $a_1 = c$. Then $b_1 = 1 + c$.

Therefore the problem of type III can be formulated in the form:

In $\Omega = \{(x,t) : 0 < x < 1, 0 < t < T\}$ find a solution $u(x,t)$ of the fractional heat equation (1) satisfying the initial condition (2) and the boundary condition of type III:

$$\begin{cases} c u_x(0,t) + (1+c)\, u_x(1,t) + a u(0,t) = 0; \\ u(0,t) - u(1,t) = 0. \end{cases} \tag{56}$$

Here the coefficients $a,\ b,\ c$ of the boundary condition are arbitrary complex numbers.

We introduce the auxiliary functions:

$$v(x,t) = \frac{1}{2}\left[u(x,t) - u(1-x,t)\right]; \tag{57}$$

$$w(x,t) = u(x,t) - \left[1 - (1+2c)(2x-1)\right] v(x,t). \tag{58}$$

Note that if the solution has been searched in the form of a sum of even and odd parts $u(x,t) = C(x,t) + S(x,t)$ in the initial version of the method (see [3]), then in a variant suggested by us:

– the function $v(x,t)$ is odd on the interval $0 < x < 1$, and is the odd part of the function $u(x,t)$;

– and the function $w(x,t)$ is not the even part of the function $u(x,t)$, though it is the even function.

The last follows from the fact that $w(x,t)$ can be represented in the form

$$w(x,t) = \frac{1}{2}\left[u(x,t) + u(1-x,t)\right] + (1+2c)(2x-1)\,v(x,t), \tag{59}$$

that is, in the form of the sum of the even part $\frac{1}{2}\left[u(x,t) + u(1-x,t)\right]$ of the function $u(x,t)$ and the summand $(1+2c)(2x-1)\,v(x,t)$, which (it is easy to verify) is also the even function on the interval $0 < x < 1$.

From (58) it is easy to see that if we find the functions $v(x,t)$ and $w(x,t)$, then the solution of the initial problem can be reestablished by the formula

$$u(x,t) = w(x,t) + \left[1 - (1+2c)(2x-1)\right] v(x,t). \tag{60}$$

Thus if in the previous variant of the method the solution is represented in the form of the sum of the even and odd parts of the solution, then in the new variant suggested by us it is not quite so. In representation (60) the first summand is even on the interval $0 < x < 1$, and the second summand is neither even, nor odd for $(1+2c) \neq 0$.

For the function $v(x,t)$ we obtain the initial-boundary value problem which we need to solve first:

$$D^{\alpha}_{0+}\left(v(x,t) \ - \ v(x,0)\right) - v_{xx}(x,t) = f_0(x); \tag{61}$$

$$v(x,0) = \varphi_0(x), \quad v(x,T) = \psi_0(x) \quad 0 \le x \le 1; \tag{62}$$

$$v(0,t) = 0, \quad 0 \le t \le T; \tag{63}$$

$$v(1,t) = 0, \quad 0 \le t \le T. \tag{64}$$

Here we use the notations

$$\begin{gathered} f_0(x) = \tfrac{1}{2}\left[f(x) - f(1-x)\right]; \\ \varphi_0(x) = \tfrac{1}{2}\left[\varphi(x) - \varphi(1-x)\right], \quad \psi_0(x) = \tfrac{1}{2}\left[\psi(x) - \psi(1-x)\right]. \end{gathered} \tag{65}$$

Having the solution $v(x,t)$ of this problem, for the function $w(x,t)$ we get the initial-boundary value problem which we need to solve second:

$$D^{\alpha}_{0+}\left(w(x,t) \ - \ w(x,0)\right) - w_{xx}(x,t) = f_1(x) + F_1(x,t); \tag{66}$$

$$w(x,0) = \varphi_1(x), \quad w(x,T) = \psi_1(x), \quad 0 \le x \le 1; \tag{67}$$

$$w_x(0,t) - aw(0,t) = 0, \quad 0 \le t \le T; \tag{68}$$

$$w_x(1,t) + aw(1,t) = 0, \quad 0 \le t \le T. \tag{69}$$

Here we use the notations

$$f_1(x) = f(x) - \left[1 - (1+2c)(2x-1)\right] f_0(x), \quad F_1(x,t) = -4(1+2c)v_x(x,t); \tag{70}$$

$$\begin{gathered} \varphi_1(x) = \varphi(x) - \left[1 - (1+2c)(2x-1)\right] \varphi_0(x); \\ \psi_1(x) = \psi(x) - \left[1 - (1+2c)(2x-1)\right] \psi_0(x). \end{gathered} \tag{71}$$

By direct checking from (65) and (71) it is easy to make sure that if the initial and final data $\varphi(x)$ and $\psi(x)$ of problem (1), (2), (56) satisfy necessary (classical and well-known) consistency conditions, then the initial and final data $\varphi_0(x)$, $\varphi_1(x)$ and $\psi_0(x)$, $\psi_1(x)$ also satisfy the necessary consistency conditions of their corresponding problems.

Thus the solution of the problem of type III (1), (2), (56) is reduced to the sequential solution of two problems with homogeneous boundary conditions of the Sturm type with respect to the spatial variable:

– At first for the function $v(x,t)$ we solve the initial-boundary value problem (61)–(94) with the homogeneous boundary conditions of the Sturm type (in this case with conditions of the Dirichlet problem) with respect to the spatial variable;

– Then, using the obtained value $v(x,t)$, for the function $w(x,t)$ we solve the initial-boundary value problem (66)–(69) with the homogeneous boundary conditions of the Sturm type with respect to the spatial variable.

Therefore the main result on the existence and uniqueness of the solution of the problem of type III (1), (2), (56) in classical and generalized senses follows from the Theorem 1 on corresponding solvability of boundary value problems with conditions of the Sturm type. We will formulate this main result at once for all the four types of not strongly regular conditions at the end of the paper.

7 Reduction of the problem of type IV to a sequential solution of two problems with homogeneous boundary conditions of the Sturm type

Consider a problem of type IV. Since $c_0 - d_0 = 0$, and herewith $|c_0| + |d_0| > 0$, then without loss of generality we can assume $c_0 = d_0 = 1$. Since $a_1 + b_1 \neq 0$, then without loss of generality we can assume $a_1 + b_1 = 1$. To simplify writing (omitting additional indexes) we denote $a_1 = c$. Then $b_1 = 1 - c$.

Therefore the problem of type IV can be formulated in the form:

In$\Omega = \{(x,t), 0 < x < 1, 0 < t < T\}$ *find a solution* $u(x,t)$ *of the fractional heat equation (1) satisfying the initial condition (2) and the boundary conditions of type IV:*

$$\begin{cases} cu_x(0,t) + (1-c)\,u_x(1,t) + au(0,t) = 0; \\ u(0,t) + u(1,t) = 0. \end{cases} \tag{72}$$

Here the coefficients $a,\ b,\ c$ of the boundary condition are arbitrary complex numbers.

We introduce the auxiliary functions:

$$v(x,t) = \frac{1}{2}\left[u(x,t) + u(1-x,t)\right]; \tag{73}$$

$$w(x,t) = u(x,t) - \left[1 - (1-2c)(2x-1)\right] v(x,t). \tag{74}$$

Note that if the solution has been searched in the form of the sum of the even and odd parts $u(x,t) = C(x,t) + S(x,t)$ in the initial version of the method (see [3]), then in the variant suggested by us:

– the function $v(x,t)$ is even on the interval $0 < x < 1$, and is the even part of the function $u(x,t)$;

– and the function $w(x,t)$ is not the odd part of the function $u(x,t)$, though it is the odd function.

The last follows from the fact that $w(x,t)$ can be represented in the form

$$w(x,t) = \frac{1}{2}\left[u(x,t) - u(1-x,t)\right] + (1-2c)(2x-1)\,v(x,t), \tag{75}$$

that is, in the form of the sum of the odd part $\frac{1}{2}\left[u(x,t) - u(1-x,t)\right]$ of the function $u(x,t)$ and the summand $(1-2c)(2x-1)\,v(x,t)$, which (it is easy to verify) is also the odd function on the interval $0 < x < 1$.

From (74) it is easy to see that if we find the functions $v(x,t)$ and $w(x,t)$, then the solution of the initial problem can be reestablished by the formula

$$u(x,t) = w(x,t) + \left[1 - (1-2c)(2x-1)\right] v(x,t). \tag{76}$$

Thus if in the previous variant of the method the solution is represented in the form of the sum of the even and odd parts of the solution, then in the new variant suggested by us it is not quite so. In representation (76) the first summand is odd on the interval $0 < x < 1$, and the second summand is neither even, nor odd for $(1-2c) \neq 0$.

For the function $v(x,t)$ we obtain the initial-boundary value problem which we need to solve first:

$$D^{\alpha}_{0+}\left(v(x,t) - v(x,0)\right) - v_{xx}(x,t) = f_0(x); \tag{77}$$

$$v(x,0) = \varphi_0(x), \quad v(x,T) = \psi_0(x) \quad 0 \le x \le 1; \tag{78}$$

$$v(0,t) = 0, \quad 0 \le t \le T, \tag{79}$$

$$v(1,t) = 0, \quad 0 \le t \le T. \tag{80}$$

Here we use the notations

$$f_0(x) = \frac{1}{2}\left[f(x) + f(1-x)\right],$$
$$\varphi_0(x) = \frac{1}{2}\left[\varphi(x) + \varphi(1-x)\right], \quad \psi_0(x) = \frac{1}{2}\left[\psi(x) + \psi(1-x)\right]. \tag{81}$$

Having the solution $v(x,t)$ of this problem, for the function $w(x,t)$ we get the initial-boundary value problem which we need to solve second:

$$D_{0+}^{\alpha}\left(w(x,t) \ - \ w(x,0)\right) - w_{xx}(x,t) = f_1(x) + F_1(x,t); \tag{82}$$

$$w(x,0) = \varphi_1(x), \quad w(x,T) = \psi_1(x), \quad 0 \le x \le 1; \tag{83}$$

$$w_x(0,t) + aw(0,t) = 0, \quad 0 \le t \le T; \tag{84}$$

$$w_x(1,t) - aw(1,t) = 0, \quad 0 \le t \le T. \tag{85}$$

Here we use the notations

$$f_1(x) = f(x) - \left[1 - (1-2c)(2x-1)\right] f_0(x), \quad F_1(x,t) = -4(1-2c)v_x(x,t); \tag{86}$$

$$\varphi_1(x) = \varphi(x) - \left[1 - (1-2c)(2x-1)\right]\varphi_0(x);$$
$$\psi_1(x) = \psi(x) - \left[1 - (1-2c)(2x-1)\right]\psi_0(x). \tag{87}$$

By direct checking from (81) and (87) it is easy to make sure that if the initial and final data $\varphi(x)$ and $\psi(x)$ of problem (1), (2), (72) satisfy necessary (classical and well-known) consistency conditions, then the initial and final data $\varphi_0(x)$, $\varphi_1(x)$ and $\psi_0(x)$, $\psi_1(x)$ also satisfy the necessary consistency conditions of their corresponding problems.

Thus the solution of the problem of type IV (1), (2), (72) is reduced to the sequential solution of two problems with homogeneous boundary conditions of the Sturm type with respect to the spatial variable:

– At first for the function $v(x,t)$ we solve the initial-boundary value problem (77)–(80) with the homogeneous boundary conditions of the Sturm type (in this case with boundary conditions of Dirichlet) with respect to the spatial variable;

– Then using the obtained value $v(x,t)$, for the function $w(x,t)$ we solve the initial-boundary value problem (82)–(85) with the homogeneous boundary conditions of the Sturm type with respect to the spatial variable.

Therefore the main result on the existence and uniqueness of the solution of the problem of type IV (1), (2), (72) in classical and generalized senses follows from the Theorem 1 on corresponding solvability of boundary value problems with conditions of the Sturm type. We will formulate this result as well as the results of sections 4, 5 and 6 at once for all the four types of not strongly regular boundary conditions in the next section.

8 Formulation of the main result on solvability of the fractional heat equation with not strongly regular boundary conditions

For completeness of exposition we once again formulate the problem under consideration:

In $\Omega = \{(x,t), 0 < x < 1, 0 < t < T\}$ *find a right-hand side* $f(x)$ *of the fractional heat equation*

$$D_{0+}^{\alpha}\left(u(x,t) \ - \ u(x,0)\right) - u_{xx}(x,t) = f(x) + F(x,t), \tag{88}$$

and its solutions $u(x,t)$ *satisfying the initial and final conditions*

$$u(x,0) = \varphi(x), \quad u(x,T) = \psi(x), \ 0 \le x \le 1, \tag{89}$$

and not strongly regular boundary conditions of the general form

$$\begin{cases} a_1 u_x(0,t) + b_1 u_x(1,t) + a_0 u(0,t) + b_0 u(1,t) = 0; \\ c_0 u(0,t) + d_0 u(1,t) = 0. \end{cases} \tag{90}$$

The coefficients $a_k, \ b_k, \ c_k, \ d_k \ (k = 0,1)$ of the boundary condition (90) are arbitrary real numbers, and $\varphi(x)$, $\psi(x)$ and $F(x,t)$ are given functions.

We consider boundary conditions which are regular, but not strongly regular, that is, cases when one of the conditions holds:

$$\begin{array}{lll} I. & a_1 + b_1 = 0, & c_0 - d_0 \neq 0; \\ II. & a_1 - b_1 = 0, & c_0 + d_0 \neq 0; \\ III. & c_0 - d_0 = 0, & a_1 + b_1 \neq 0; \\ IV. & c_0 + d_0 = 0, & a_1 - b_1 \neq 0. \end{array} \tag{91}$$

As shown in sections 4 – 8, the solution to the problem with the not strongly regular boundary conditions of all the four types has been reduced to the sequential solution of two problems with the homogeneous boundary conditions of the Sturm type with respect to the spatial variable. Herewith one of these problems has the Dirichlet boundary conditions with respect to the spatial variable, that is, it is a classical first initial-boundary value problem.

On the basis of this fact, using the results from Theorem 1, now we can easily formulate a theorem on well-posedness of the general problem with the not strongly regular boundary conditions with respect to the spatial variable.

Theorem 2. Let one of conditions (91) hold. That is, the boundary conditions (90) are regular, but not strongly regular. If $F(x,t) \in C^2\left(\overline{\Omega}\right)$, $\varphi(x)$, $\psi(x) \in C^4[0,1]$ *and the functions* $F(x,t)$, $\varphi(x)$, $\psi(x)$, $\varphi''(x)$ *and* $\psi''(x)$ *satisfy (4) then there exists the unique classical solution* $u(x,t) \in C^{2,1}_{x,t}\left(\overline{\Omega}\right)$, $f(x) \in C[0,1]$ *to the inverse problem (1), (2), (90).*

Note that by this method, problem (1), (2), (90) has been solved regardless whether the corresponding spectral problem for the operator of twofold differentiation with the not strongly regular boundary conditions (4) has the basis property of root functions.

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Бөлшек жылуөткізгіштік үрдісі көзінің тығыздығын бастапқы және ақырғы температуралары бойынша қалпына келтіру есебі туралы

Мақалада жұмыста бөлшек жылуөткізгіштік теңдеуі үшін кері есептер қарастырылған. Уақыт бойынша Риман-Лиувилл мағынасындағы бөлшек ретті туындылар пайдаланылды. Берілген теңдеудің шешімімен қатар, теңдеудің оң жағындағы белгісіз болып отырған функцияны анықтау мәселесі шешімін тапқан. Бұл жерде теңдеудің оң жағындағы белгісіз функция уақыт айнымалысынан тәуелсіз болады. Бастапқы және ақырғы температураларға қатысты бөлшек жылуөткізгіштік үрдісі көзінің тығыздығын және температурасын анықтау мәселесін модельдейтін есеп зерттелген. Қатаң регулярлы болмайтын кеңістіктегі айнымалылар бойынша жалпы түрдегі шеттік есептерге қатысты мәселелер қарастырылған. Есептің классикалық шешімінің бар және жалғыз болатындығы көрсетілген. Есепке қатысты еселеп дифференциалдау операторы үшін шеттік шарттары қатаң регулярлы емес спектралдық есептің меншікті функциялары базис болмайтын болса да, есептің шешімі табылған.

Кілт сөздер: кері есеп, жылуөткізгіштік теңдеуі, бөлшек жылуөткізгіштік, қатаң регулярлы емес шеттік шарттар, айнымалыларды айыру тәсілі.

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Об одной задаче восстановления плотности источников процесса дробной теплопроводности по начальной и конечной температурам

В статье рассмотрены обратные задачи для дробного уравнения теплопроводности, где дробная производная по времени понимается в смысле Римана-Лиувилля. Вместе с решением этого уравнения необходимо найти неизвестную правую часть, зависящую только от пространственной переменной. Рассмотрена задача, моделирующая процесс определения температуры и плотности источников в процессе дробной теплопроводности относительно заданных начальных и конечных температур. Исследованы проблемы с общими граничными условиями относительно пространственной переменной, которые не являются усиленно регулярными. Доказаны существование и единственность классического решения задачи. Задача решается независимо от того, что соответствующая спектральная задача для оператора кратного дифференцирования с неусиленно регулярными граничными условиями может не иметь свойства базисности корневых функций.

Ключевые слова: обратная задача, уравнение теплопроводности, дробная теплопроводность, неусиленно регулярные граничные условия, метод разделения переменных.

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A criterion for the existence of soliton solutions of telegraph equation

In this paper we consider a telegraph equation. In the case of a rectangular domain for the Cauchy potential the lateral boundary conditions obtained. When considering the equation in the first quadrant a criterion for the existence of soliton solutions is obtained.

Keywords: telegraph equation, telegraph potential, fundamental solution, soliton solution, nonlocal boundary conditions, convolution.

Introduction

Many studies have been devoted to the study of classical potentials: the Newton potential, the volume heat potential and the wave potential. In the equations of mathematical physics Newton's potentials are used to solve classical problems (the Dirichlet problem, Neumann problem, Robin problem) for the Laplace equation and other elliptic equations. It should be noted that for the first time the exact nonlocal boundary conditions of the Newton potential, the volume heat potential and the wave potential have been found recently [1–3]. After, boundary conditions of surface potentials that satisfy homogeneous equations were studied [4, 5]. Further applying these results, the boundary conditions for the volume elliptic-parabolic potential were found and so on [6–19].

Boundary conditions of Telegraph Equation

In the band $\Omega = \left\{t \geq 0, 0 \leq x \leq \frac{1}{2}\right\}$ we select a limited subdomain $\Omega_1 = \left\{0 \leq t \leq \frac{1}{2}, 0 \leq x \leq \frac{1}{2}\right\}$ and consider the Cauchy problem for a one-dimensional telegraph equation

$$Lu(x,t) = \frac{\partial^2 u(x,t)}{\partial t^2} - \frac{\partial^2 u(x,t)}{\partial x^2} - \lambda u(x,t) = f(x,t); \tag{1}$$

$$u(x,t)|_{t=0} = 0, \tag{2}$$

$$\frac{\partial u(x,t)}{\partial t}|_{t=0} = 0. \tag{3}$$

In the characteristic coordinates $\xi = x + t,\ \eta = x - t$ the band Ω turns into a band $\widetilde{\Omega}$, and the subdomain Ω_1 turns into a subdomain $\widetilde{\Omega}_1$ with bounded segments:

$$A_0B_0 : \eta = \xi,\quad 0 \leq \xi \leq \frac{1}{2};\quad A_0A : \eta = -\xi,\quad 0 \leq \xi \leq \frac{1}{2};$$

$$B_0B : \eta = 1 - \xi,\quad \frac{1}{2} \leq \xi \leq 1;\quad AB : \eta = \xi - 1,\quad \frac{1}{2} \leq \xi \leq 1,$$

as $t \geq 0$, then $\eta \leq \xi$. Equation (1) also turns into equation

$$Lu(\xi,\eta) = \frac{\partial^2 u(\xi,\eta)}{\partial\xi\partial\eta} + \frac{\lambda}{4}u(\xi,\eta) = f_1(\xi,\eta) \tag{4}$$

and the Cauchy data (2) and (3) are expressed in

$$u(\xi,\eta)|_{\eta=\xi} = 0, \tag{5}$$

$$\left(\frac{\partial u(\xi,\eta)}{\partial\xi} - \frac{\partial u(\xi,\eta)}{\partial\eta}\right)\Big|_{\eta=\xi} = 0. \tag{6}$$

It is well known that the Riemann function $R(\xi,\eta,\xi_1,\eta_1)$ (see [20; 92]) of the telegraph equation (4) is representable in the form

$$R(\xi,\eta,\xi_1,\eta_1) = J_0\left(\sqrt{\lambda(\xi-\xi_1)(\eta-\eta_1)}\right), \tag{7}$$

where $J_0(z)$ is a zero-order Bessel function (see [20; 91]).

A fundamental solution of the Cauchy problem (4)–(6) in the domain $\widetilde{\Omega}$ is given by the formula:

$$\varepsilon(\xi-\xi_1,\eta_1-\eta) = -\theta(\xi-\xi_1)\theta(\eta_1-\eta)\times R(\xi,\eta,\xi_1,\eta_1). \tag{8}$$

In contrast to [21; 256], here the fundamental solution is taken with a negative sign and the second argument $(\eta_1-\eta)$ in accordance with the domain $\widetilde{\Omega}$.

The telegraph potential in the domain $\widetilde{\Omega}$ is called the integral

$$u(\xi,\eta) = \varepsilon * f = \int_{\widetilde{\Omega}} \varepsilon(\xi-\xi_1,\eta_1-\eta) f(\xi_1,\eta_1)d\xi_1 d\eta_1 =$$

$$= \int_{\eta}^{\xi} d\xi_1 \int_{\xi_1}^{\eta} R(\xi,\eta,\xi_1,\eta_1) f(\xi_1,\eta_1) d\eta_1 =$$

$$= \int_{\eta}^{\xi} d\xi_1 \int_{\xi_1}^{\eta} J_0(\sqrt{\lambda(\xi-\xi_1)(\eta-\eta_1)}) f(\xi_1,\eta_1) d\eta_1. \tag{9}$$

It is easy to verify that the telegraph potential (9) satisfies the homogeneous initial Cauchy conditions for $\eta=\xi$, $0\le\xi\le 1/2$, i.e.

$$u(\xi,\eta)|_{\eta=\xi} = 0, \tag{10}$$

$$\left(\frac{\partial u(\xi,\eta)}{\partial\xi} - \frac{\partial u(\xi,\eta)}{\partial\eta}\right)\Big|_{\eta=\xi} = 0, \tag{11}$$

and equation (4) (see [21]).

Let us find a lateral boundary conditions of the telegraph potential (9) for $A_0A: \xi=-\eta,\ \ 0\le\xi\le\frac{1}{2}$ and $B_0B: \xi = 1-\eta,\ \ \frac{1}{2}\le\xi\le 1$, which is equivalent to $x=0$, and $x=\frac{1}{2}$ in the original coordinates (x,t).

Theorem 1. Let $f(\xi,\eta)\in C^1\left(\overline{\widetilde{\Omega}}\right)$, then the telegraph potential $u(\xi,\eta)\in C^2\left(\overline{\widetilde{\Omega}}\right)$ satisfies the following lateral boundary conditions:

$$N[u]|_{A_0A} = N[u]|_{\xi=-\eta} = \int_0^{\xi} J_0(\sqrt{4\lambda(\xi-\xi_1)^2})\frac{\partial u}{\partial\eta_1}(\xi_1,-\eta_1)d\xi_1 +$$

$$+2\lambda\int_0^{\xi}\frac{J_1(\sqrt{-4\lambda(\xi-\xi_1)^2})}{(\sqrt{-4\lambda(\xi-\xi_1)^2})}u(\xi_1,-\xi_1)d\xi_1 = 0; \tag{12}$$

$$N[u]|_{B_0B} = N[u]|_{\xi=1-\eta} = -\int_{\frac{1}{2}}^{\xi} J_0(\sqrt{-4\lambda(\xi-\xi_1)^2})\frac{\partial u}{\partial\xi_1}(\xi_1,1-\xi_1)d\xi_1 +$$

$$+2\lambda\int_{\frac{1}{2}}^{\xi}\frac{J_1((\sqrt{-4\lambda(\xi-\xi_1)^2}))}{(\sqrt{-4\lambda(\xi-\xi_1)^2})}(\xi-\xi_1)u(\xi_1,1-\xi_1)d\xi_1 = 0, \tag{13}$$

where $J_1(z)$ is a Bessel function of the first order.

Conversely, if $u(\xi,\eta)\in C^2\left(\overline{\widetilde{\Omega}}\right)$ is a solution of the telegraph equation (4), satisfying the initial conditions (5)-(6) and the lateral boundary conditions (12)-(13), then $u(\xi,\eta)$ is given by the telegraph potential (9).

We note that for $\lambda=0$ the lateral boundary conditions of the telegraph potential coincide with the boundary conditions of the one-dimensional wave potential which is given in [3].

Proof. We continue the function $f(\xi,\eta)$ outside of the square $\widetilde{\Omega}_1$ with zero, i.e. $f(\xi,\eta)\equiv 0$ in $R^2/\widetilde{\Omega}_1$. Then the telegraph potential

$$u(\xi,\eta) = \varepsilon * f = \int_{\eta}^{\xi} d\xi_1 \int_{\xi_1}^{\eta} R(\xi,\eta,\xi_1,\eta_1) f(\xi_1,\eta_1) d\eta_1$$

gives a solution of equation (4) for all $(\xi,\eta)\in R^2$ and $u(\xi,\eta)\in C^2(\overline{\overline{\Omega}})$ satisfies the homogeneous initial Cauchy conditions (5)–(6) for the whole straight line $\xi=\eta,\ -\infty<\xi<+\infty$. The value of the function $u(\xi,\eta)$ at the point (ξ,η) is determined by the value of $f(\xi_1,\eta_1)$ in the characteristic triangle, i.e. $\triangle_{\xi,\eta}=\{\eta\le\xi_1\le\xi,\ \ \eta\le\eta_1\le\xi_1\}$.

Therefore the value of the function $u(\xi,\eta)$ on $u|_{A_0A=\eta=-\xi}=u(\xi,-\xi)$ is defined by the formula

$$u(\xi,-\xi)=\int_{-\xi}^{\xi}d\xi_1\int_{\xi_1}^{-\xi}R(\xi,-\xi,\xi_1,\eta_1)f(\xi_1,\eta_1)d\eta_1. \tag{14}$$

As $f\equiv 0$ outside $\xi_1\le 0$ and $\eta_1\le-\xi_1$, then the integral (14) takes the form

$$u|_{A_0A}=u(\xi,-\xi)=\int_0^{\xi}d\xi_1\int_{\xi_1}^{-\xi_1}R(\xi,-\xi,\xi_1,\eta_1)f(\xi_1,\eta_1)d\eta_1=$$

$$=\int_0^{\xi}d\xi_1\int_{\xi_1}^{-\xi_1}J_0(\sqrt{\lambda(\xi-\xi_1)(-\xi-\eta_1)})f(\xi_1,\eta_1)d\eta_1. \tag{15}$$

Now, in (15) instead of the function $f(\xi_1,\eta_1)$ we put $\frac{\partial^2u(\xi_1,\eta_1)}{\partial\xi_1\partial\eta_1}+\frac{\lambda}{4}u(\xi_1,\eta_1)$, i.e.

$$u(\xi,-\xi)=\int_0^{\xi}d\xi_1\int_{\xi_1}^{-\xi_1}J_0(\sqrt{\lambda(\xi-\xi_1)(-\xi-\eta_1)})\left(\frac{\partial^2u(\xi_1,\eta_1)}{\partial\xi_1\partial\eta_1}+\frac{\lambda}{4}u(\xi_1,\eta_1)\right)d\eta_1=$$

$$=I_1+I_2, \tag{16}$$

where

$$I_1=\frac{\lambda}{4}\int_0^{\xi}d\xi_1\int_{\xi_1}^{-\xi_1}J_0(\sqrt{\lambda(\xi-\xi_1)(-\xi-\eta_1)})u(\xi_1,\eta_1)d\eta_1; \tag{17}$$

$$I_2=\int_0^{\xi}d\xi_1\int_{\xi_1}^{-\xi_1}J_0(\sqrt{\lambda(\xi-\xi_1)(-\xi-\eta_1)})\frac{\partial^2u(\xi_1,\eta_1)}{\partial\xi_1\partial\eta_1}d\eta_1. \tag{18}$$

Integrating by parts the integral I_2, we obtain

$$I_2=\int_0^{\xi}\left(J_0(\sqrt{\lambda(\xi-\xi_1)(-\xi-\eta_1)}\frac{\partial u(\xi_1,\eta_1)}{\partial\xi_1}\right)|_{\xi_1}^{-\xi_1}d\xi_1-$$

$$-\int_0^{\xi}d\xi_1\int_{\xi_1}^{-\xi_1}\frac{\partial}{\partial\eta_1}J_0(\sqrt{\lambda(\xi-\xi_1)(-\xi-\eta_1)})\frac{\partial u(\xi_1,\eta_1)}{\partial\xi_1}d\eta_1=$$

$$=\int_0^{\xi}J_0(\sqrt{\lambda(\xi-\xi_1)(\xi_1-\xi)})\frac{\partial u(\xi_1,-\xi_1)}{\partial\xi_1}d\xi_1-$$

$$-\int_0^{\xi}J_0(\sqrt{\lambda(\xi-\xi_1)(-\xi-\xi_1)})\frac{\partial u(\xi_1,\xi_1)}{\partial\xi_1}d\xi_1-$$

$$-\int_0^{\xi}d\xi_1\int_{\xi_1}^{-\xi_1}\frac{\partial}{\partial\eta_1}J_0(\sqrt{\lambda(\xi-\xi_1)(-\xi-\eta_1)})\frac{\partial u(\xi_1,\eta_1)}{\partial\xi_1}d\eta_1. \tag{19}$$

From the initial Cauchy condition (5) it follows that $\frac{\partial u(\xi_1,\xi_1)}{\partial\xi_1}=0$. Taking this into account, the integral I_2 can be rewritten as

$$I_2=I_{2,1}-I_{2,2}=\int_0^{\xi}J_0(\sqrt{\lambda(\xi-\xi_1)(\xi_1-\xi)})\frac{\partial u(\xi_1,-\xi_1)}{\partial\xi_1}d\xi_1-$$

$$-\int_0^{\xi}d\xi_1\int_{\xi_1}^{-\xi_1}\frac{\partial}{\partial\eta_1}J_0(\sqrt{\lambda(\xi-\xi_1)(-\xi-\eta_1)})\frac{\partial u(\xi_1,\eta_1)}{\partial\xi_1}d\eta_1. \tag{20}$$

In the integral $I_{2,2}$ we change the order of integration and the limits in the domain $\eta\ge 0: 0\le\eta_1\le\xi$ and in the domain $\eta\le 0: -\xi\le\eta_1\le 0$, then we obtain

$$I_{2,2} = \int_0^\xi d\xi_1 \int_{\xi_1}^{-\xi_1} \frac{\partial}{\partial\eta_1} J_0(\sqrt{\lambda(\xi-\xi_1)(-\xi-\eta_1)}) \frac{\partial u(\xi_1,\eta_1)}{\partial\xi_1} d\eta_1 =$$

$$= \int_{-\xi}^0 d\eta_1 \int_\xi^{-\eta_1} \frac{\partial}{\partial\eta_1} J_0(\sqrt{\lambda(\xi-\xi_1)(-\xi-\eta_1)}) \frac{\partial u(\xi_1,\eta_1)}{\partial\xi_1} d\xi_1 +$$

$$+ \int_0^\xi d\eta_1 \int_\xi^{\eta_1} \frac{\partial}{\partial\eta_1} J_0(\sqrt{\lambda(\xi-\xi_1)(-\xi-\eta_1)}) \frac{\partial u(\xi_1,\eta_1)}{\partial\xi_1} d\xi_1.$$

Using the formula $\frac{\partial}{\partial z} J_0(z) = -J_1(z)$ (see [21]), we conclude that

$$\frac{\partial}{\partial\eta_1} J_0(\sqrt{\lambda(\xi-\xi_1)(-\xi-\eta_1)}) = \frac{\partial}{\partial z} J_0(z) \frac{\partial z}{\partial\eta_1} =$$

$$= -J_1(z) \frac{\partial}{\partial\eta_1} (\sqrt{\lambda(\xi-\xi_1)(-\xi-\eta_1)}) = J_1(z) \frac{\lambda(\xi-\xi_1)}{2\sqrt{\lambda(\xi-\xi_1)(-\xi-\eta_1)}}. \quad (21)$$

Taking into account the last relation from (21), integrating by parts the integral $I_{2,2}$, we find

$$I_{2,2} = \frac{\lambda}{2} \int_{-\xi}^0 \frac{J_1(\sqrt{\lambda(\xi-\xi_1)(-\xi-\eta_1)})(\xi-\xi_1)}{\sqrt{\lambda(\xi-\xi_1)(-\xi-\eta_1)}} u(\xi_1,\eta_1)|_\xi^{-\eta_1} d\eta_1 +$$

$$+ \frac{\lambda}{2} \int_0^\xi \frac{J_1(\sqrt{\lambda(\xi-\xi_1)(-\xi-\eta_1)})(\xi-\xi_1)}{\sqrt{\lambda(\xi-\xi_1)(-\xi-\eta_1)}} u(\xi_1,\eta_1)|_\xi^{\eta_1} d\eta_1 -$$

$$- \int_{-\xi}^0 d\eta_1 \int_\xi^{-\eta_1} \frac{\partial^2}{\partial\xi_1\partial\eta_1} J_0(\sqrt{\lambda(\xi-\xi_1)(-\xi-\eta_1)}) u(\xi_1,\eta_1) d\eta_1 +$$

$$- \int_0^\xi d\eta_1 \int_\xi^{\eta_1} \frac{\partial^2}{\partial\xi_1\partial\eta_1} J_0(\sqrt{\lambda(\xi-\xi_1)(-\xi-\eta_1)}) u(\xi_1,\eta_1) d\eta_1 =$$

$$= \frac{\lambda}{2} \int_{-\xi}^0 \frac{J_1(\sqrt{-\lambda(\xi+\eta_1)(\xi+\eta_1)})(\xi+\eta_1)}{\sqrt{-\lambda(\xi+\eta_1)(\xi+\eta_1)}} u(-\eta_1,\eta_1) d\eta_1 +$$

$$+ \int_{-\xi}^0 d\eta_1 \int_\xi^{-\eta_1} \frac{\partial^2}{\partial\xi_1\partial\eta_1} J_0(\sqrt{\lambda(\xi-\xi_1)(-\xi-\eta_1)}) u(\xi_1,\eta_1) d\eta_1 -$$

$$- \int_0^\xi d\eta_1 \int_\xi^{\eta_1} \frac{\partial^2}{\partial\xi_1\partial\eta_1} J_0(\sqrt{\lambda(\xi-\xi_1)(-\xi-\eta_1)}) u(\xi_1,\eta_1) d\eta_1. \quad (22)$$

In the first integral of (22) replacing the variables $-\eta_1 = \xi_1, \eta_1 = \xi$, we have

$$I_{2,2}^1 = \frac{\lambda}{2} \int_0^\xi \frac{J_1(\sqrt{-\lambda(\xi-\xi_1)^2})}{\sqrt{-\lambda(\xi-\xi_1)^2}} (\xi-\xi_1) u(\xi_1,-\xi_1) d\xi_1. \quad (23)$$

Taking into account the integrals $I_{2,1}$, $I_{2,2}^1$ and formulas (17), (22), we have that

$$u|_{A_0A} = u(\xi,-\xi) = \int_0^\xi J_0 \sqrt{\lambda(\xi-\xi_1)(\xi_1-\xi)} \frac{\partial u(\xi_1,-\xi_1)}{\partial\xi_1} d\xi_1 -$$

$$- \frac{\lambda}{2} \int_0^\xi \frac{J_1(\sqrt{-\lambda(\xi-\xi_1)^2})}{\sqrt{-\lambda(\xi-\xi_1)^2}} (\xi-\xi_1) u(\xi_1,-\xi_1) d\xi_1 +$$

$$+ \int_0^\xi d\xi_1 \int_{\xi_1}^{-\xi_1} \left(\frac{\partial^2}{\partial\xi_1\partial\eta_1} J_0(\sqrt{\lambda(\xi-\xi_1)(\eta-\eta_1)}) + \frac{\lambda}{4} J_0(\sqrt{\lambda(\xi-\xi_1)(\eta-\eta_1)}) \right) u(\xi_1,\eta_1) d\eta_1. \quad (24)$$

As $J_0(\sqrt{\lambda(\xi-\xi_1)(\eta-\eta_1)})$ is the Riemann function of the telegraph equation (4), then

$$\frac{\partial^2}{\partial\xi_1\partial\eta_1}J_0(\sqrt{\lambda(\xi-\xi_1)(\eta-\eta_1)})+\frac{\lambda}{4}J_0(\sqrt{\lambda(\xi-\xi_1)(\eta-\eta_1)})=0. \tag{25}$$

Therefore, we have

$$u|_{A_0A}=u(\xi,-\xi)=\int_0^\xi J_0\sqrt{\lambda(\xi-\xi_1)(\xi_1-\xi)}\frac{\partial u(\xi_1,-\xi_1)}{\partial\xi_1}d\xi_1-$$
$$-\frac{\lambda}{2}\int_0^\xi \frac{J_1(\sqrt{-\lambda(\xi-\xi_1)^2})}{\sqrt{-\lambda(\xi-\xi_1)^2}}(\xi-\xi_1)u(\xi_1,-\xi_1)d\xi_1. \tag{26}$$

It is easy to verify that the total derivative

$$\frac{d}{d\xi_1}u(\xi_1,-\xi_1)=\left(\frac{\partial u(\xi_1,-\xi_1)}{\partial\xi_1}-\frac{\partial u(\xi_1,-\xi_1)}{\partial\eta_1}\right)|_{\eta=-\xi_1},$$

then

$$\frac{\partial u(\xi_1,-\xi_1)}{\partial\xi_1}=\frac{du(\xi_1,-\xi_1)}{d\xi}+\frac{\partial u(\xi_1,-\xi_1)}{\partial\eta_1}. \tag{27}$$

Taking this into account, from (25) integrating by parts, we obtain

$$\int_0^\xi J_0(\sqrt{\lambda(\xi-\xi_1)(\xi_1-\xi)})\left(\frac{du(\xi_1,-\xi_1)}{d\xi}+\frac{\partial u(\xi_1,-\xi_1)}{\partial\eta_1}\right)|_{\eta_1=-\xi_1}d\xi_1=$$
$$=\int_0^\xi J_0(\sqrt{\lambda(\xi-\xi_1)^2})\frac{\partial u(\xi_1,\eta_1)}{\partial\eta_1}|_{\eta_1=-\xi_1}d\xi_1+$$
$$+J_0(\sqrt{\lambda(\xi-\xi_1)^2})u(\xi_1,-\xi_1)|_{\xi_1=\xi}-\int_0^\xi\frac{\partial}{\partial\xi_1}J_0(\sqrt{\lambda(\xi-\xi_1)^2})u(\xi_1,-\xi_1)d\xi_1=$$
$$=\frac{1}{2}\lambda\frac{J_1(\sqrt{-\lambda(\xi-\xi_1)^2})}{\sqrt{-\lambda(\xi-\xi_1)^2}}u(\xi_1,-\xi_1) \tag{28}$$

as $J_0(0)=1$.

From the last relation it follows that

$$u(\xi,-\xi)=u|_{A_0A}+N[u]|_{A_0A}=u(\xi,-\xi)+N[u]|_{\eta=-\xi}, \tag{29}$$

i.e.

$$N[u]=\int_0^\xi J_0(\sqrt{4\lambda(\xi-\xi_1)^2})\frac{\partial u}{\partial\eta_1}u(\xi,-\eta_1)|_{\eta_1=\xi}d\xi_1+$$
$$+2\lambda\int_0^\xi\frac{J_1(\sqrt{-4\lambda(\xi-\xi_1)^2})}{\sqrt{-4\lambda(\xi-\xi_1)^2}}u(\xi_1,-\xi_1)d\xi_1=0. \tag{30}$$

The boundary condition (29) is the lateral boundary condition of the telegraph potential on A_0A: $\eta=-\xi$, $0<\xi<1/2$.

If $\lambda=0$, then from (29) by differentiating by parts, we obtain the lateral boundary conditions for the one-dimensional Cauchy wave potential in the case of T.Sh. Kalmenov, D. Suragan [3].

Similarly, we find the boundary conditions on B_0B:

$$N[u]|_{B_0B}=-\int_{\frac{1}{2}}^\xi J_0(\sqrt{4\lambda(\xi-\xi_1)^2})\frac{\partial u}{\partial\xi}(\xi_1,1-\xi_1)d\xi_1+$$
$$+2\lambda\int_{\frac{1}{2}}^\xi\frac{J_1(\sqrt{-4\lambda(\xi-\xi_1)^2})}{\sqrt{-4\lambda(\xi-\xi_1)^2}}(\xi-\xi_1)u(\xi_1,1-\xi_1)d\xi_1=0. \tag{31}$$

Thus, the lateral boundary conditions $N[u]$ on A_0A and B_0B are given by formulas (29), (30), respectively.

Conversely. Let $\vartheta \in C^2(\overline{\Omega})$ satisfy equation (4), homogeneous initial conditions (5)–(6) and the lateral boundary conditions(29)–(30). Let $u(\xi,\eta)$ be the telegraph potential defined by (9), then $\omega = u - \vartheta$ satisfies the homogeneous equation (4) and the homogeneous initial conditions (5)–(6).

By virtue of (24) and (31), we have

$$\omega + N[\omega]|_{A_0A} = \omega|_{A_0A} = 0,$$

$$\omega + N[\omega]|_{B_0B} = \omega|_{B_0B} = 0.$$

From the uniqueness of the solution of the mixed Cauchy problem we have $\omega = u - \vartheta \equiv 0$, i.e. $u = \vartheta$. By continuation of the solution outside the square under consideration, we see that $N[u]|_{x=0} = 0$, and $N[u]|_{x=1} = 0$.

Theorem 1 is completely proved.

Telegraph potential solitons

In a quarter of the plane $\Omega = \{x \geq 0,\ t \geq 0\}$ we consider the Cauchy problem for a homogeneous telegraph equation

$$Lu(x,t) = \Box u(x,t) - \lambda u(x,t) = \left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}\right) u(x,t) - \lambda u(x,t) = 0; \tag{32}$$

$$u(x,t)|_{t=0} = \tau(x), \tag{33}$$

$$\frac{\partial u(x,t)}{\partial t}|_{t=0} = \nu(x). \tag{34}$$

We assume that $\tau(x) \equiv \nu(x) \equiv 0$ at $x \leq 0$ and we seek a solution in the whole half-space $t \geq 0$. It is natural, that it is determined by the Cauchy data $\tau(x)$ and $\nu(x)$ at $x \geq 0$.

By $\widetilde{\tau}(x)$ and $\widetilde{\nu}(x)$ we denote the functions

$$\widetilde{\tau}(x) = \begin{cases} \tau(x), & x \geq 0, \\ 0, & x \leq 0, \end{cases} \tag{35}$$

$$\widetilde{\nu}(x) = \begin{cases} \nu(x), & x \geq 0, \\ 0, & x \leq 0. \end{cases} \tag{36}$$

The solution of the Cauchy problem $u(x,t)$ is determined by the Riemann formula (see [22; 174])

$$u(x,t) = \frac{1}{2}u(x_0,0)R(x_0,0,x,t) + \frac{1}{2}u(x_1,0)R(x_1,0,x,t) +$$

$$+\frac{1}{2}\int_{x_0}^{x_1}\left(\frac{\partial}{\partial\eta_1}u(\xi_1,0)R(\xi_1,0,x,t) - u(\xi_1,0)\frac{\partial}{\partial\eta_1}R(\xi_1,0,x,t)\right)d\xi_1 =$$

$$= \frac{1}{2}\tau(x_0)R(x_0,0,x,t) + \frac{1}{2}\tau(x_1)R(x_1,0,x,t) +$$

$$+\frac{1}{2}\int_{x_0}^{x_1}\left(\nu(\xi_1)R(\xi_1,0,x,t) - \tau(\xi_1)\frac{\partial}{\partial\eta_1}R(\xi_1,0,x,t)\right)d\xi_1, \tag{37}$$

where $x_0 = x - t,\quad x_1 = x + t$.

At $x_1 = 0$, $x_0 = -t < 0$, taking into account that $\tau(x_0) = 0$ and $\nu(x) = 0$, from (37) it follows that

$$\tau(t) = u(0,t) = \frac{1}{2}\tau(x+t)R(x+t,0,x,t)|_{x=0} +$$

$$+\frac{1}{2}\int_0^{x_1}\left(\nu(\xi_1)R(\xi_1,0,x,t) - \tau(\xi_1)\frac{\partial}{\partial\eta_1}R(\xi_1,0,x,t)\right)d\xi_1|_{x=0} =$$

$$= \frac{1}{2}\tau(t)R(t,0,0,t) + \frac{1}{2}\int_0^t\left(\nu(\xi_1)R(\xi_1,0,0,t) - \tau(\xi_1)\frac{\partial}{\partial\eta_1}R(\xi_1,0,0,t)\right)d\xi_1. \tag{38}$$

Equality (38) is the lateral boundary condition for the surface wave potential.

It follows

Lemma. Suppose that the Cauchy data $\tau(x)$, $\nu(x) \in C^2(-\infty,\infty)$ and $\tau(x) = u(x,0) \equiv 0$, $\nu(x) = \frac{\partial u(x,0)}{\partial t} \equiv 0$ at $x \leq 0$. Then the surface telegraph potential $u(x,t)$ for $x = 0$ satisfies the lateral boundary condition (38).

It is easy to verify that for $\lambda = 0$ the condition (37) becomes a boundary condition of the surface wave potential.

Theorem 2. Suppose that the hypothesis of Lemma holds and $\lambda = 0$ for $x \geq d > 0$. Then the surface telegraph potential $u(x,t)$ at $x \to \infty$ turns into a soliton solution, i.e. $\lim_{x\to\infty} u(x,t) = \varphi(x-t)$ if and only if the condition is fulfilled

$$\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right) u(x,t)|_{x=d} = 0. \tag{39}$$

Proof. It is not difficult to show that if $\lambda = 0$ at $x \geq d > 0$, then the solution of the homogeneous telegraph equation given by (37) can be represented in the form

$$u(x,t) \equiv \psi(x+t) + \varphi(x-t). \tag{40}$$

Then

$$\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right) u(x,t)|_{x=d} = \psi'(d+t)|_{x=d} = 0. \tag{41}$$

Taking this into account, from (40) it follows that $\lim_{x\to\infty} u(x,t) = \varphi(x-t)$.

Theorem 2 is proved.

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Т.Ш. Кәлменов, Г.Д. Арепова

Телеграф теңдеуінің солитон шешімдерінің бар болуының қажетті және жеткілікті шарты

Мақалада телеграф теңдеуі қарастырылған. Төртбұрышты облыс жағдайында Коши потенциалының бүйір шекараларында шекаралық шарттары табылған. Теңдеуді бірінші квадрантта қарастырғанда солитон шешімдерінің бар болуының қажетті және жеткілікті шарты алынған.

Кілт сөздер: телеграф теңдеуі, телеграф потенциалы, іргелі шешім, солитон шешімі, локалды емес шекаралық шарты, үйірткі.

Т.Ш. Кальменов, Г.Д. Арепова

Критерий существования солитонных решений телеграфного уравнения

В статье рассмотрено телеграфное уравнение. Для случая прямоугольной области найдены краевые условия потенциала Коши на боковых границах. При рассмотрении уравнения в первом квадранте получен критерий существования солитонных решений.

Ключевые слова: телеграфное уравнение, телеграфный потенциал, фундаментальное решение, солитонное решение, нелокальные граничные условия, свертка.

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Nonlocal spectral problem for a second-order differential equation with an involution

For the spectral problem $-u''(x) + \alpha u''(-x) = \lambda u(x)$, $-1 < x < 1$, with nonlocal boundary conditions $u(-1) = \beta u(1)$, $u'(-1) = u'(1)$, where $\alpha \in (-1, 1)$, $\beta^2 \neq 1$, we study the spectral properties. We show that if $r = \sqrt{(1-\alpha)/(1+\alpha)}$ is irrational, then the system of eigenfunctions is complete and minimal in $L_2(-1,1)$ but is not a basis. In the case of a rational number r, the root subspace of the problem consists of eigenvectors and an infinite number of associated vectors. In this case, we indicated a method for choosing associated functions that provides the system of root functions of the problem is an unconditional basis in $L_2(-1,1)$.

Keywords: ODE with involution, nonlocal boundary-value problem, spectral problem, basicity of root functions

1 Introduction

In the present paper, we carry out a complete spectral analysis of the problem

$$Lu = -u''(x) + \alpha u''(-x), \ -1 < x < 1; \qquad u'(-1) = u'(1), \quad u(-1) = \beta u(1), \tag{1}$$

where the differential expression contains an involution transformation of the independent variable in the highest derivative and the boundary conditions are nonlocal.

Throughout the following, the parameter α in problem (1) is an arbitrary number in the interval $(-1,1)$. The case $\beta = 1$ (when the boundary conditions of the problem are periodic) was investigated in [1]. The case when $\beta = -1$ leads to a degenerate problem. In this case, as it is easy to see, any number λ is an eigenvalue. Therefore, in this paper we assume that β is an arbitrary real number for which $\beta^2 \neq 1$.

If $\alpha = \beta = 0$, then problem (1) becomes the well-known nonlocal problem of the Samarskii- Ionkin type [2], which is an example of a nonself-adjoint problem whose set of root functions contains, in addition to eigenfunctions, infinitely many associated functions. Il'in [3] dubbed such problems essentially nonself-adjoint and pointed out their typical instability both under the choice of associated functions and under small perturbations of the operator. For details, see [4] and also [5–9].

We show that problem (1) has all specific features of essentially nonself-adjoint problems and that its spectral properties can change fundamentally under arbitrarily small variations of the parameter α.

We note that the case $\beta = 0$ was investigated in detail in [10] for the space L_2 and in [11] for the space L_p, $1 < p < \infty$.

The main result of the present paper is stated in the following theorems.

Theorem 1. Let $r = \sqrt{(1-\alpha)/(1+\alpha)}$ be irrational. Then the system of root functions of problem (1) contains only eigenfunctions; moreover, it is complete and minimal in $L_2(-1,1)$ but is not a basis.

Theorem 2. Let $r = \sqrt{(1-\alpha)/(1+\alpha)}$ be rational. Then the spectrum of problem (1) splits into two sequences $\{\lambda_n\} \cup \{\lambda_n^*\}$. For each $\lambda = \lambda_n$, there exists only one eigenfunction, and for each $\lambda = \lambda_n^*$, there exists one eigenfunction and one associated function. The system of root functions is complete and minimal in $L_2(-1,1)$, and the associated functions can be chosen in such a way that the entire system is an unconditional basis in $L_2(-1,1)$.

Note that functional-differential equations similar to the equation in (1) were studied by numerous authors. The algebraic and analytic aspects of the theory of ordinary differential equations with involution were discussed in the monographs [12, 13]. Spectral problems arising in connection with differential operators with involution were considered in [14–18] for first-order operators and in [19, 20] for second-order operators. Spectral problems for ordinary differential operators with non-strongly regular boundary conditions and their applications for parabolic problems were investigated in [21-25].

2 Case of irrational r

The problem adjoint to (1) has the form

$$\begin{gathered} Lv(x) = -v''(x) + \alpha v''(-x), \ -1 < x < 1; \\ (\alpha - \beta)v'(-1) = (\alpha\beta - 1)v'(1), \ \ v(-1) = v(1). \end{gathered} \tag{2}$$

By a straightforward computation, one can readily show that the spectra of problems (1) and (2) coincide and

$$\sigma(L) = \{0; \ (1 \pm \alpha)\pi^2 n^2 | \ n \in \mathbb{N}\}, \tag{3}$$

while the eigenfunctions of the direct problem (1) have the form (here and in what follows, μ is the arithmetic value of the root $\sqrt{\lambda}$)

$$\begin{gathered} \mu_0 = 0: \ u_0 = (1-\beta)x + 1 + \beta; \\ \mu'_l = \sqrt{1+\alpha}\pi l: \ u_l^{(1)}(x) = \sin(\pi l x), \ \ l \in \mathbb{N}; \\ \mu''_k = \sqrt{1-\alpha}\pi k: \ u_k^{(2)}(x) = (1+\beta)\sin(\pi r k)\cos(\pi k x) + (1-\beta)\cos(\pi k)\sin(\pi r k x), \ \ k \in \mathbb{N} \end{gathered} \tag{4}$$

and eigenfunctions of the adjoint problem (2) have the form

$$\begin{gathered} \mu_0 = 0: \ v_0(x) = 1; \\ \mu''_k = \sqrt{1-\alpha}\pi k: \ v_k^{(2)}(x) = \cos(\pi k x), \ \ k \in \mathbb{N}; \\ \mu'_l = \sqrt{1+\alpha}\pi l: \ v_l^{(1)}(x) = (1+\beta) r \sin\tfrac{\pi l}{r} \sin(\pi l x) + (1-\beta)\cos(\pi l)\cos\tfrac{\pi l x}{r}, \ \ l \in \mathbb{N}. \end{gathered} \tag{5}$$

Lemma 1. Let r be irrational. Then each of systems (4) and (5) is complete and minimal in $L_2(-1,1)$.

Proof. Let us carry out the proof, say, for system (4). Consider an arbitrary function $f(x) \in L_2(-1,1)$ orthogonal to all functions of system (4). Since it is orthogonal to the functions $u_l^{(1)}(x), l \in \mathbb{N}$, we see that it coincides almost everywhere with an even function. Thus,

$$0 = \int_{-1}^{1} f(x) u_k^{(2)}(x) dx = (1+\beta)\sin(\pi r k) \int_{-1}^{1} f(x)\cos(\pi k x) dx.$$

Since $r \notin Q$ and $(1+\beta) \neq 0$, it follows that the function $f(x)$ is orthogonal to the functions $\cos(\pi k x)$, $k \in \mathbb{N}$, and hence $f(x) = const$ almost everywhere on $[-1,1]$. Finally, from the relation $(f, u_0) = 0$, since $(1+\beta) \neq 0$, it follows that $f(x) = 0$ almost everywhere on $[-1,1]$.

Since systems (4) and (5) are complete, it follows that they are closed in $L_2(-1,1)$; and since they correspond to mutually adjoint problems, we find that they are minimal. The proof of the lemma is complete.

Let us modify the eigenfunctions so as to ensure that systems (4) and (5) form a biorthonormal pair in $L_2(-1,1)$. Since

$$(u_0, v_0) = 2(1+\beta), \ \left(u_l^{(1)}, v_l^{(1)}\right) = (1+\beta) r \sin\frac{\pi l}{r}, \ \left(u_l^{(2)}, v_l^{(2)}\right) = (1+\beta)\sin(\pi r k), \tag{6}$$

it follows that the modification should have the form

$$\tilde{u}_0(x) = \frac{1-\beta}{1+\beta} x + 1, \ \tilde{v}_0(x) = \frac{1}{2};$$

$$\tilde{u}_l^{(1)}(x) = \sin(\pi l x), \quad \tilde{v}_l^{(1)}(x) = \sin(\pi l x) + \frac{1-\beta}{1+\beta}\frac{\cos(\pi l)}{r\sin\frac{\pi l}{r}}\cos\frac{\pi l x}{r};$$

$$\tilde{u}_k^{(2)}(x) = \cos(\pi k x) + \frac{1-\beta}{1+\beta}\frac{\cos(\pi k)}{\sin(\pi r k)}\sin(\pi r k x), \quad \tilde{v}_k^{(2)}(x) = \cos(\pi k x).$$

Let us compute the norms of these functions in $L_2(-1,1)$. We have

$$\left\|\tilde{u}_l^{(1)}\right\| = 1, \quad \left\|\tilde{v}_l^{(1)}\right\|^2 = 1 + \left(\frac{1-\beta}{1+\beta}\right)^2\left(r\sin\frac{\pi l}{r}\right)^{-2}\left(1+\frac{r}{2\pi l}\sin\frac{2\pi l}{r}\right);$$

$$\left\|\tilde{u}_k^{(2)}\right\|^2 = 1 + \left(\frac{1-\beta}{1+\beta}\right)^2(\sin\pi r k)^{-2}\left(1+\frac{1}{2\pi r k}\sin(2\pi r k)\right), \quad \left\|\tilde{v}_k^{(2)}\right\| = 1.$$

Lemma 2. Let r be irrational. Then there exist sequences $\{l_n\}$ and $\{k_n\}$ such that $\left\|\tilde{v}_{l_n}^{(1)}\right\| \to \infty$ and $\left\|\tilde{u}_{k_n}^{(2)}\right\| \to \infty$, as $n \to \infty$.

Proof. By virtue of the theorem on the approximation of real numbers by rational fractions [26; 25], the inequalities

$$\left|r - \frac{l}{s}\right| < \frac{1}{s^2}, \quad \left|\frac{1}{r} - \frac{k}{q}\right| < \frac{1}{q^2} \tag{7}$$

have infinitely many solutions $l, k, s, q \in \mathbb{N}$. We denote these solutions by l_n, k_n, s_n and q_n. Then from inequality (7) we have $\left|\frac{\pi l_n}{r} - \pi s_n\right| < \frac{\pi}{r s_n}$ and hence

$$\sin^2\frac{\pi l_n}{r} = \sin^2\left(\frac{\pi l_n}{r} - \pi s_n\right) < \sin^2\frac{\pi}{r s_n}.$$

In a similar way, we obtain the inequalities $|\pi r k_n - \pi q_n| < \frac{\pi r}{q_n}$ and

$$\sin^2(\pi r k_n) < \sin^2\frac{\pi r}{q_n}.$$

Therefore,

$$\left\|\tilde{v}_{l_n}^{(1)}\right\|^2 > 1 + \left(\frac{1-\beta}{1+\beta}\right)^2\frac{2}{3}\left(r\sin\frac{\pi}{r s_n}\right)^{-2}, \quad \left\|\tilde{u}_{k_n}^{(2)}\right\|^2 > 1 + \left(\frac{1-\beta}{1+\beta}\right)^2\frac{2}{3}\left(r\sin\frac{\pi r}{q_n}\right)^{-2}.$$

The proof of the lemma is complete, because the right-hand sides of these inequalities infinitely increase as $n \to \infty$.

Lemma 2 essentially completes the proof of Theorem 1, because it follows from the lemma that the considered biorthonormal pair of function of these systems does not satisfy the condition of uniform boundedness for the product of norms:

$$\left\|\tilde{u}_l^{(1)}\right\| \cdot \left\|\tilde{v}_l^{(1)}\right\| \le c_1, \quad \left\|\tilde{u}_k^{(2)}\right\| \cdot \left\|\tilde{v}_k^{(2)}\right\| \le c_2, \tag{8}$$

which is necessary for the basis property [27] in $L_2(-1,1)$.

3 Case of rational r

Let $r = \sqrt{(1-\alpha)/(1+\alpha)}$ be a rational number, which can be represented by an irreducible fraction $r = \frac{m_1}{m_2}$ where $m_1, m_2 \in \mathbb{N}$.

Then a merging effect is observed for the following points of the spectrum $\sigma(L)$

$$\mu'_{m_1 n} = \mu''_{m_2 n}, \quad n \in \mathbb{N}. \tag{9}$$

We denote the sequence extracted in (9) by μ_n^* and note that the eigenfunctions corresponding to the eigenvalues $\lambda = \lambda_n^* \equiv (\mu_n^*)^2$ are linearly dependent,

$$u_{m_1 n}^{(1)}(x) = (-1)^{m_2 n}(1-\beta)^{-1}u_{m_2 n}^{(2)}(x) = \sin(\pi m_1 n x) \equiv u_n^*(x);$$

$$(-1)^{m_1 n}(1-\beta)^{-1}v_{m_1 n}^{(1)}(x) = v_{m_2 n}^{(2)}(x) = \cos(\pi m_2 n x) \equiv v_n^*(x).$$

Therefore, the systems of eigenfunctions (4) and (5) become incomplete in $L_2(-1,1)$.

We supplement the eigenfunctions corresponding to $\lambda = \lambda_n^*$ by associated functions, that is, the solutions of the inhomogeneous problems

$$\begin{aligned} &Lu(x) = \lambda_n^* u(x) + u_n^*(x), \quad -1 < x < 1; \\ &u'(-1) = u'(1), \quad u(-1) = \beta u(1); \end{aligned} \tag{10}$$

$$\begin{aligned} &Lv(x) = \lambda_n^* v(x) + v_n^*(x), \quad -1 < x < 1; \\ &(\alpha-\beta)v'(-1) = (\alpha\beta-1)v'(1), \quad v(-1) = v(1). \end{aligned} \tag{11}$$

By straightforward computations, we find the functions

$$u_{n,1}^*(x) = (2(1+\alpha)\pi m_1 n)^{-1}\left[x\cos(\pi m_1 n x) + \frac{1+\beta}{1-\beta}(-1)^{(m_1+m_2)n}\cos(\pi m_2 n x)\right] + a_n u_n^*(x); \tag{12}$$

$$v_{n,1}^*(x) = (2(1-\alpha)\pi m_2 n)^{-1}\left[-x\sin(\pi m_2 n x) + \frac{1+\beta}{1-\beta} r(-1)^{(m_1+m_2)n}\sin(\pi m_1 n x)\right] - a_n v_n^*(x), \tag{13}$$

which are solutions of problem (10) and (11), respectively, for arbitrary $a_n \in \mathbb{R}$.

Note that if we substitute $u_{n,1}^*(x)$ for $u_n^*(x)$ into the right-hand side of (10) and $v_{n,1}^*(x)$ for $v_n^*(x)$ into (11), then problems (10) and (11) have no solutions. It follows that the corresponding problems have no associated functions of the second or any higher order.

Lemma 3. Let $r = m_1/m_2$ be rational. Then each of the systems of root functions obtained by the following procedures is complete and minimal in $L_2(-1,1)$:

– for problem (1), one takes the union of the eigenfunctions (4) corresponding to $\lambda \neq \lambda_n^*$, the eigenfunctions $u_n^*(x)$, and the associated functions $u_{n,1}^*(x)$, $n \in \mathbb{N}$;

– for problem (2), one takes the union of the eigenfunctions (5) corresponding to $\lambda \neq \lambda_n^*$, the eigenfunctions $v_n^*(x)$, and the associated functions $v_{n,1}^*(x)$, $n \in \mathbb{N}$.

Proof. The proof is similar to that of Lemma 1. Consider the system of root functions of problem (1) and suppose that a function $f(x) \in L_2(-1,1)$ is orthogonal to all functions of that system.

Since the function $f(x)$ is orthogonal to all eigenfunctions $u_l^{(1)}(x)$, $l \in \mathbb{N}$, we find that it coincides almost everywhere with an even function. In addition, the function $f(x)$ is orthogonal to all eigenfunctions $u_k^{(2)}(x)$, $k \equiv 0 \pmod{m_2}$, and all associated functions $u_{n,1}^*(x)$, $n \in \mathbb{N}$. By virtue of its evenness, in this case, the function $f(x)$ is orthogonal to all functions $\cos(\pi k x)$, $k \in \mathbb{N}$, as well. Therefore, it is equal almost everywhere to a constant, which, just as in Lemma 1, implies the assertion of the lemma. The proof of the lemma is complete.

Let us now modify the root functions of problem (4) and (5) so as to ensure that they form a biorthonormal pair.

If $\lambda \neq \lambda_n^*$, then the corresponding eigenfunctions satisfy the same relations (6), where $l \equiv 0 \pmod{m_1}$ and $k \equiv 0 \pmod{m_2}$. Therefore, $l = l_1 m_1 + l_2$, where $l_1, l_2 \in \mathbb{N}$, $1 \leq l_2 \leq m_1 - 1$, and the number $\frac{l}{r} = l_1 m_2 + l_2 \frac{m_2}{m_1}$ is not an integer; consequently,

$$\sin^2\frac{\pi l}{r} = \sin^2\left(\pi\left(\frac{l}{r} - l_1 m_2\right)\right) = \sin^2\left(\pi\frac{l_2 m_2}{m_1}\right) \geq \sin^2\frac{\pi}{m_1}. \tag{14}$$

Likewise, we have $k = k_1 m_2 + k_2$, $k_1, k_2 \in \mathbb{N}$, $1 \leq k_2 \leq m_2 - 1$, and the number $rk = k_1 m_1 + k_2 \frac{m_1}{m_2}$ is not an integer; consequently,

$$\sin^2(\pi r k) = \sin^2(\pi(rk - k_1 m_1))\sin^2\left(\pi\frac{k_2 m_1}{m_2}\right) \geq \sin^2\frac{\pi}{m_2}. \tag{15}$$

Consider the eigenvalues $\lambda = \lambda_n^*$. We have

$$(u_n^*, v_n^*) = (u_{n,1}^*, v_{n,1}^*) = 0, \quad (u_n^*, v_{n,1}^*) = (u_{n,1}^*, v_n^*) = (2(1+\alpha)\pi m_1 n)^{-1}\frac{1+\beta}{1-\beta}(-1)^{(m_1+m_2)n}.$$

Therefore, biorthonormal pairs in $L_2(-1,1)$ are formed by the function systems

$$\begin{aligned} &\tilde{u}_0(x), \quad \tilde{u}_l^{(1)}(x), \ l \neq 0 \pmod{m_1}, \quad \tilde{u}_k^{(2)}(x), \ k \neq 0 \pmod{m_2}; \\ &\tilde{u}_n^*(x) = \sin(\pi m_1 n x), \quad \tilde{u}_{n,1}^*(x) = u_{n,1}^*(x), \quad n \in \mathbb{N}, \end{aligned} \tag{16}$$

for problem (1) and

$$\tilde v_0(x),\ \ \tilde v_l^{(1)}(x),\ l\neq 0 \ (\mathrm{mod}\ m_1),\ \ \tilde v_k^{(2)}(x),\ k\neq 0\ (\mathrm{mod}\ m_2);$$

$$\tilde v_n^*(x) = 2(1+\alpha)\pi m_1 n(-1)^{(m_1+m_2)n}\tfrac{1-\beta}{1+\beta}\cos(\pi m_2 nx); \tag{17}$$

$$\tilde v_{n,1}^*(x) = -r^{-1}(-1)^{(m_1+m_2)n}\tfrac{1-\beta}{1+\beta}x\sin(\pi m_2 nx)+\sin(\pi m_1 nx)-a_n\tilde v_n^*(x),\quad n\in\mathbb{N},$$

for problem (2) with arbitrary constants $a_n\in\mathbb{R}$.

Let us evaluate and estimate the product of norms of root functions. By virtue of the relations presented before Lemma 2 and the estimates (14) and (15), the products of norms of the corresponding eigenfunctions are uniformly bounded for $\lambda\neq\lambda_n^*$. If $\lambda=\lambda_n^*$, then we have

$$\|\tilde u_n^*\|^2=1,\quad \|\tilde v_n^*\|^2=\left(2(1+\alpha)\pi m_1 n\frac{1-\beta}{1+\beta}\right)^2;$$

$$\left\|\tilde u_{n,1}^*\right\|^2=\frac{(1/3)+\frac{1-\beta}{1+\beta}+(2\pi^2m_1^2n^2)^{-1}}{(2(1+\alpha)\pi m_1 n)^2}-\frac{a_n n^{-2}}{2(1+\alpha)\pi^2 m_1^2}+a_n^2;$$

$$\left\|\tilde v_{n,1}^*\right\|^2=\frac{(1/3)-(2\pi^2m_2^2n^2)^{-1}}{r^2}\left(\frac{1-\beta}{1+\beta}\right)^2+1+2(1+\alpha)\frac{1-\beta}{1+\beta}a_n+(2(1+\alpha)\pi m_1)^2a_n^2n^2.$$

Therefore, each of the products $\|\tilde u_n^*\|^2\left\|\tilde v_{n,1}^*\right\|^2$ and $\left\|\tilde u_{n,1}^*\right\|^2\|\tilde v_n^*\|^2$ has the form

$$c_3(n)+c_4a_n+c_5a_n^2n^2, \tag{18}$$

where $c_4, c_5>0$ are constants and $c_3(n)$ satisfies the inequality $0<c'_3\le c_3(n)\le c''_3$ for all $n\in\mathbb{N}$.

Lemma 4. The products of L_2-norms of the respective root functions in the biorthonormal pair are uniformly bounded if and only if

$$a_n=O(n^{-1})\quad as\quad n\to\infty. \tag{19}$$

Proof. Indeed, if $a_n=O(1)$, then (18) is uniformly bounded provided that $a_n n=O(1)$. If the sequence a_n is not bounded, then (18) is equivalent to $c_5a_n^2n^2$ as $n\to\infty$ and hence is again uniformly bounded under condition (19). The proof of the lemma is complete.

Let us show that, in a sense, condition (19) is a rule for the selection of associated functions which provides the basis property of considered systems of root functions in $L_2(-1,1)$.

Lemma 5. If condition (19) is satisfied, then each of systems (16) and (17), after the normalization in $L_2(-1,1)$, satisfies a Bessel type inequality and hence forms an unconditional basis in $L_2(-1,1)$.

Proof. For example, consider system (16) of root functions of problem (1). If $\lambda\neq\lambda_n^*$, then the normalization gives the system

$$\sqrt{\frac{3}{2}}\left[3+\left(\frac{1-\beta}{1+\beta}\right)^2\right]^{-1/2}\left(1+\frac{1-\beta}{1+\beta}x\right),\quad \sin(\pi lx);$$

$$\left\|\tilde u_k^{(2)}\right\|_2^{-1}\left(\cos(\pi kx)+\frac{1-\beta}{1+\beta}\frac{\cos(\pi k)}{\sin(\pi rk)}\sin\left(\pi\frac{m_1}{m_2}kx\right)\right);$$

where $l\neq 0$ (mod m_1), $k\neq 0$ (mod m_2), and, as was shown above, $1\le\left\|\tilde u_k^{(2)}\right\|_2\le c_6$.

For $\lambda=\lambda_n^*$, we have

$$\frac{\tilde u_n^*(x)}{\|\tilde u_n^*\|}=\sin(\pi m_1 nx);$$

$$\left\|\tilde u_{n,1}^*\right\|^{-1}\tilde u_{n,1}^*(x)=A_n^{(1)}\left[x\cos(\pi m_1nx)+\frac{1+\beta}{1-\beta}(-1)^{(m_1+m_2)n}\cos(\pi m_2nx)\right]+A_n^{(2)}\sin(\pi m_1 nx),$$

where $0<c_7\le A_n^{(1)}, A_n^{(2)}\le c_8, n\in\mathbb{N}$.

Thus, to justify the Bessel property, it suffices to prove the Bessel property of the following three systems ($n\in\mathbb{N}$):

$$\sin(\pi nx),\quad \cos(\pi nx), \tag{20}$$

$$\cos\left(\pi\frac{m_1}{m_2}nx\right); \tag{21}$$

$$x\cos(\pi m_1 nx). \tag{22}$$

System (20) is orthonormal in $L_2(-1,1)$ and hence satisfies the Bessel type inequality with constant $B=1$. The Bessel property of system (22) follows from the Bessel property of system (20), because the factor x is bounded. Finally, system (21) is a Bessel system by virtue of the following assertion.

Lemma 6. Let $\{\gamma_k\}$ be a sequence of complex numbers such that

$$\sup_k |Im(\gamma_k)| < \infty, \quad \sup_{t\geq 1} \sum_{k:|Re(\gamma_k)-t|\leq 1} 1 < \infty. \tag{23}$$

Then each of the systems $\{\sin(\gamma_k x)\}$ and $\{\cos(\gamma_k x)\}$ is a Bessel system in $L_2(-1,1)$.

Proof. By virtue of the estimates (23), $\gamma_k = \pi n + \delta_{nk}$, where

$$\sup_{n,k} |\mathrm{Im}(\delta_{nk})| < \infty, \quad \sup_n \sum_{k:|\mathrm{Re}(\delta_{nk})|\leq 1} 1 < \infty.$$

Therefore,

$$\int_{-1}^{1} f(x)\sin(\gamma_k x)dx = \cos(\delta_{nk})\int_{-1}^{1} f(x)\sin(\pi nx)dx + \sin(\delta_{nk})\int_{-1}^{1} f(x)\cos(\pi nx)dx +$$

$$+\delta_{nk}\int_{-1}^{1}\sin(\delta_{nk}x)\int_{-1}^{x} f(\xi)\sin(\pi n\xi)d\xi dx - \delta_{nk}\int_{-1}^{1}\cos(\delta_{nk}x)\int_{-1}^{x} f(\xi)\cos(\pi n\xi)d\xi dx,$$

which implies a Bessel type inequality for the system $\{\sin(\gamma_k x)\}$.

System (21) satisfies condition (23), because

$$\mathrm{Im}(\gamma_k) = 0, \quad \sum_{k:|\mathrm{Re}(\gamma_k)-t|\leq 1} \leq 2m_2 + 1.$$

The unconditional basis property of system (16) follows from the well-known Bari theorem [28]. The proof of the lemma is complete.

Theorem 2 is completely proved.

We note that using the proven basis property of the system of root functions in the case when the parameter r is a rational number, the problems describing the process of heat propagation in a thin closed wire wrapped around a weakly permeable insulation can be considered by the method of separation of variables. Such problems with periodic boundary conditions with respect to the space variable were considered in [1].

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Л.В. Крицков, М.А. Садыбеков, Ә.М. Сәрсенбі

Инволюциясы бар екінші ретті дифференциалды теңдеу үшін локалдық емес шеттік есеп

Шеттік шарттары $u(-1) = \beta u(1)$, $u'(-1) = u'(1)$, $\beta^2 \neq 1$, локалдық емес мынадай спектралдық есептің $-u''(x) + \alpha u''(-x) = \lambda u(x)$, $-1 < x < 1$, $\alpha \in (-1, 1)$, спектралдық қасиеттері зерттелді. Егер $r = \sqrt{(1-\alpha)/(1+\alpha)}$ иррационал сан болса, онда есептің меншікті функциялары толық және минималды жүйе құрайды, алайда базис емес. Осы тұжырым дәлелденген. Егер r рационал сан болса, онда есептің ақырсыз қосымша алынған функциялары бар. Бұл жағдайда қосымша алынған функцияларды таңдап алу жолдары келтірілген. Және таңдап алынған түпкілікті функциялар жүйесі $L_2(-1, 1)$ кеңістігінде базис құрайтыны көрсетілген.

Кілт сөздер: инволюциясы бар жай дифференциалды теңдеулер, локалдық емес шеттік есеп, спектралдық есеп, базис, түпкілікті функциялар.

Л.В. Крицков, М.А. Садыбеков, А.М. Сарсенби

Нелокальная краевая задача для дифференциального уравнения второго порядка с инволюцией

В статье изучены спектральные свойства для спектральной задачи $-u''(x) + \alpha u''(-x) = \lambda u(x)$, $-1 < x < 1$, с нелокальными граничными условиями $u(-1) = \beta u(1)$, $u'(-1) = u'(1)$, где $\alpha \in (-1, 1)$, $\beta^2 \neq 1$. Показано, что если $r = \sqrt{(1-\alpha)/(1+\alpha)}$ иррационально, то система собственных функций полна и минимальна в $L_2(-1, 1)$, но не образует базиса. В случае рационального числа r корневое подпространство задачи состоит из собственных векторов и бесконечного числа присоединенных векторов. В этом случае указан метод выбора присоединенных функций, при котором система корневых функций задачи является безусловным базисом в $L_2(-1, 1)$.

Ключевые слова: ОДУ с инволюцией, нелокальная краевая задача, спектральная задача, базисность корневых функций.

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New singular solutions for the (3+1)-D Protter problem

For the nonhomogeneous wave equation with three space and one time variables we study a boundary value problem that can be regarded as a four-dimensional analogue of the Darboux problem in $\mathbb{R}^2$. Unlike the planar Darboux problem, the $\mathbb{R}^4$-version is not well posed and has an infinite-dimensional cokernel. Therefore the problem is not Fredholm in the framework of classical solvability. On the other hand, it is known that for smooth right-hand side functions, there is a uniquely determined generalized solution that may have a strong power-type singularity at one boundary point. The singularity is isolated at the vertex of the characteristic light cone and does not propagate along the cone. In the present article we announce new singular solutions with exponential growth.

Keywords: wave equation, boundary value problems, generalized solution, singular solutions, propagation of singularities, special functions.

Introduction

In this paper we consider some boundary value problems for the wave equation with three space and one time variables that were proposed by M.H. Protter. From a historical perspective, Protter formulated these problems in connection with BVPs for mixed-type equations that describe transonic flows in fluid dynamics. The topic was extensively studied in the 1950 s and 1960 s with the development of supersonic aircrafts. In particular, the classical two-dimensional Guderley-Morawetz problem for the Gellerstedt equation of hyperbolic-elliptic type models flows around airfoils and is well studied. Regarding 2-D mixed-type boundary value problems and their transonic background we refer to the recent survey by Morawetz [1]. In 1954 Protter [2] formulated some multi-dimensional analogues of the planar Guderley-Morawetz problem. Initially, expectation was that the methods used in the $2D$ case could be applied, with minor modifications, for the problems in higher dimensions. However, the multi-dimensional case turns out to be quite different and the situation there is still not clear. Some of the difficulties and differences with the planar BVPs are illustrated by the related Protter's problems in the hyperbolic part of the domain, also formulated in [2]. In particular, for the wave equation in $\mathbb{R}^4$, with points $(x,t)=(x_1,x_2,x_3,t)$,

$$u_{x_1x_1}+u_{x_2x_2}+u_{x_3x_3}-u_{tt}=f(x,t) \tag{1}$$

the domain is

$$\Omega=\left\{(x,t):0<t<1/2, t<\sqrt{x_1^2+x_2^2+x_3^2}<1-t\right\}.$$

The boundary of Ω consists of two characteristic cones

$$\Sigma_1=\left\{(x,t):0<t<1/2, \sqrt{x_1^2+x_2^2+x_3^2}=1-t\right\},$$

$$\Sigma_2=\left\{(x,t):0<t<1/2, \sqrt{x_1^2+x_2^2+x_3^2}=t\right\}$$

and the ball

$$\Sigma_0=\left\{t=0, \sqrt{x_1^2+x_2^2+x_3^2}<1\right\}.$$

Let us point out that the origin $O:x=0,t=0$ is both the center of the non-characteristic part of the boundary Σ_0, and the vertex of the characteristic cone Σ_2. We will study the following BVPs.

Problem P1. Find a solution of the wave equation (1) in Ω which satisfies the boundary conditions

$$P1: \quad u|_{\Sigma_0}=0, \quad u|_{\Sigma_1}=0.$$

One can regard the domain Ω as a four-dimensional analogue of the characteristic triangle $D = \{(x_1,t) \in \mathbb{R}^2 : 0 < t < x_1 < 1-t\}$ for the string operator $\Box v(x_1,t) := v_{x_1x_1} - v_{tt}$ in $\mathbb{R}^2$ with points (x_1,t). The boundary of D consists of two characteristic – $l_1 = \{x_1 = 1-t, 0 < t < 1/2\}$ and $l_2 = \{x_1 = t, 0 < t < 1/2\}$, and a non-characteristic segment – $l_0 = \{t = 0, 0 < x_1 < 1\}$. In fact, the domain Ω can be constructed by revolving D in $\mathbb{R}^4$ about the t-axis. Then the segments l_0, l_1 and l_2 form Σ_0, Σ_1 and Σ_2, respectively. In this context the Protter problems $P1$ and $P1^*$ are four-dimensional variants of the classical Darboux problems for the string equation in $D \subset \mathbb{R}^2$: the data are prescribed on one of the characteristics and on the non-characteristic part of the boundary. On the other hand, unlike the planar Darboux problem, the Protter's problems in $\mathbb{R}^4$ are not well posed. Actually, the homogeneous adjoint problem $P1^*$ has smooth classical solutions and the linear space they generate is infinite dimensional (see Lemma 1 in the next section). Thus, in the frame of classical solvability the Protter problem $P1$ is not Fredholm, since it has infinite-dimensional cokernel. Naturally, a necessary condition for the existence of a classical solution for the problem $P1$ is the orthogonality of the right-hand side function f to the cokernel. Alternatively, to avoid imposing an infinite number of conditions on f, the notion of generalized solution have been introduced.

Definition 1 [3]. A function $u = u(x,t)$ is called a generalized solution of the problem $P1$ in Ω, if the following conditions are satisfied:

1) $u \in C^1\left(\overline{\Omega}\backslash O\right)$, $u|_{\Sigma_0\backslash O} = 0$, $u|_{\Sigma_1} = 0$, and

2) the identity

$$\int_{\Omega} (u_t w_t - u_{x_1}w_{x_1} - u_{x_2}w_{x_2} - u_{x_3}w_{x_3} - fw)\,dxdt = 0$$

holds for all $w \in C^1(\overline{\Omega})$ such that $w = 0$ on Σ_0 and in a neighborhood of Σ_2.

Notice that this definition allows the generalized solution of the problem $P1$ to have singularity on Σ_2. Now, it is known that when the right-hand side f is smooth, there exists a unique generalized solution of the problem $P1$ and it turns out that its singularity is isolated at only one point, that is, the origin O. In [4] it is shown that for each $n \in \mathbb{N}$ there is a generalized solution that behaves like $|x|^{-n}$ near O. The existence of a solution with exponential growth is announced in [5]. It is interesting that these singularities are isolated at the vertex O and do not propagate along the characteristic cone Σ_2. This differs the conventional case of propagation of singularities, like in Hörmander [6, Chapter 24.5].

In this paper we discuss for right-hand sides $f \in C^1(\overline{\Omega})$ the behavior of the generalized solution of problem $P1$ and the rate of its growth at the point O.

In the special case when the right-hand side function f is a harmonic polynomial, the exact behavior of the generalized solution of problem $P1$ is found in [3]. In [7] the semi-Fredholm solvability of problem $P1$ is discussed. A short historic survey and a comparison of various recent results for Protter problems can be found in [8–10]. Garabedian [11] proved the uniqueness of a classical solution for the problem $P1$. According to the classical and singular solutions let us mention here a series of papers by Aldashev (see [12–15]). Some other multi-dimensional versions of the planar Darboux problem for the wave equation are studied in [16–19]. For Protter problems for the wave equation but with lower order terms see [20, 21] and references therein. The existence of bounded or unbounded solutions for some other connected equations is considered in [13, 22]. Regarding results for degenerated hyperbolic equations we refer to [14, 23, 24] for Keldysh-type equations see [24, 25], and for BVPs for multi-dimensional mixed-type Lavrent'ev-Bitsadze equation see [12, 15]. For the Protter's mixed-type hyperbolic-elliptic problems, uniqueness results for quasi-regular solutions are proved in [26]. There are a recent series of results concerning existence or nonexistence of nontrivial solutions of related quasi-linear problems of mixed hyperbolic-elliptic type in the multi-dimensional case, see [27, 28].

In the present paper new singular solutions of problem $P1$ with exponential growth at the origin O are announced. The main Theorem 6 is formulated in the last section. It is based on some previous results from [10] for the existence and the behavior of the generalized solution, that will be presented and discussed in the next section.

Existence of generalized solutions

Naturally, the behavior of the generalized solution of problem $P1$ is affected by the correlations of the right-hand side function f with the solutions of the homogeneous adjoint problem $P1^*$. In order to construct the latter, we will use in $\mathbb{R}^3$ the orthonormal system of spherical functions Y_n^m ($n \in \mathbb{N}\cup\{0\}$, and $m = 1,\ldots,2n+1$). The spherical functions are introduced commonly on the unit sphere $S^2 := \{(x_1,x_2,x_3) : x_1^2 + x_2^2 + x_3^2 = 1\}$ with spherical polar coordinates (see [29]). Expressed in Cartesian coordinates here, one can define them by

$$Y_n^{2k}(x_1,x_2,x_3)=C_{n,k}\frac{d^k}{dx_3^k}P_n(x_3)\,\mathrm{Im}\left\{(x_1+ix_2)^k\right\},\quad \text{for}\quad k=1,...,n;$$

$$Y_n^{2k+1}(x_1,x_2,x_3)=C_{n,k}\frac{d^k}{dx_3^k}P_n(x_3)\,\mathrm{Re}\left\{(x_1+ix_2)^k\right\},\quad \text{for}\quad k=0,...,n,$$

where $C_{n,k}$ are constants and P_n are the Legendre polynomials. The Legendre polynomials are given by the Rodrigues formula as

$$P_n(s):=\frac{1}{2^n n!}\frac{d^n}{ds^n}(s^2-1)^n=\sum_{k=0}^{\left[\frac{n}{2}\right]}a_{n,2k}s^{n-2k},$$

with coefficients

$$a_{n,2k}=(-1)^k\frac{(2n-2k)!}{2^n k!(n-k)!(n-2k)!}. \tag{2}$$

The constants $C_{n,m}$ are such that functions Y_n^m form a complete orthonormal system in $L_2(S^2)$. For convenience in the discussions that follow, we extend the spherical functions out of S^2 radially, keeping the same notation Y_n^m for the extended function, i.e., $Y_n^m(x):=Y_n^m(x/|x|)$ for $x\in\mathbb{R}^3\backslash O$.

Now, let us define for $n,k\in\mathbb{N}\cup\{0\}$ the functions

$$h_{n,k}(\xi,\eta)=\int_{\eta}^{\xi}s^kP_n\left(\frac{\xi\eta+s^2}{s(\xi+\eta)}\right)ds.$$

Following Lemma 1 from [10] and Lemmas 1.1 and 2.3 from [30] we can construct solutions of the homogeneous adjoint problem.

Lemma 1 [10]. The functions

$$v_{k,m}^n(x,t)=|x|^{-1}h_{n,n-2k-2}\left(\frac{|x|+t}{2},\frac{|x|-t}{2}\right)Y_n^m(x).$$

are classical solutions from $C^\infty(\Omega)\cap C(\overline{\Omega})$ of the homogeneous problem $P1^*$ for $n\in\mathbb{N}$, $m=1,\dots,2n+1$ and $k=0,1,\dots,[(n-1)/2]-2$.

Solutions for the homogenous adjoint problem were first found by Tong Kwang-Chang [31]. Some different representations of the solutions of the homogeneous problem $P1^*$ and the functions $v_{k,m}^n$ are given by Khe Kan Cher [22].

Next we will present some useful conditions from [10] for the function f that are sufficient for the existence of the generalized solution of problem $P1$.

Since the spherical functions form a complete orthonormal system in $L_2(S^2)$, generally, a smooth function $f(x,t)$ can be expanded as a harmonic series

$$f(x,t)=\sum_{n=0}^{\infty}\sum_{m=1}^{2n+1}f_n^m(|x|,t)Y_n^m(x) \tag{3}$$

with Fourier coefficients

$$f_n^m(r,t):=\int_{S(r)}f(x,t)Y_n^m(x)\,d\sigma_r\,, \tag{4}$$

where $S(r)$ is the three-dimensional sphere $S(r):=\{x=(x_1,x_2,x_3)\in\mathbb{R}^3:\ |x|=r\}$. The results from [10] ensure the existence of the generalized solution of problem $P1$ assuming that the Fourier series (3) converges fast enough. They also give a priori estimates for the singularity of the solution. In fact, the behavior of the generalized solution depends strongly on the $L_2(\Omega)$-inner product of the right-hand side function $f(x,t)$ with the functions $v_{k,m}^n(x,t)$ from Lemma 1 (see also [20, 3]). Accordingly, we denote by $\beta_{k,m}^n$ the parameters

$$\beta_{k,m}^n:=\int_{\Omega}v_{k,m}^n(x,t)f(x,t)\,dxdt, \tag{5}$$

where $n = 0, \ldots, l$; $k = 0, \ldots, \left[\frac{n-1}{2}\right]$ and $m = 1, \ldots, 2n+1$. In order to formulate the general existence result, we need also to introduce for $p \geq 0$ and $k \in \mathbb{N}$ the series

$$||f; n^p; C^k|| := \left\| f_0^0(|x|, t) \right\|_{C^0(\Omega)} + \sum_{n=1}^{\infty} n^p \left\| \sum_{m=1}^{2n+1} f_n^m(|x|, t) Y_n^m(x) \right\|_{C^k(\Omega)}$$

and the power series

$$\Phi(s) := \sum_{n=1}^{\infty} \left[\sum_{m=1}^{2n+1} \sum_{k=0}^{[(n-1)/2]} \left| \beta_{k,m}^n \right| \right] s^n .$$

Apparently, the convergence of $||f; n^p; C^k||$ gives information on the rate of convergence of the Fourier series (3).

Theorem 2 [10]. Let the function $f(x,t)$ belong to $C^1(\overline{\Omega})$. Suppose that the series $||f; n^6; C^0||$ and $||f; n^4; C^1||$ are convergent and the power series $\Phi(s)$ has an infinite radius of convergence. Then there exists a unique generalized solution $u(x,t) \in C^1(\overline{\Omega} \backslash O)$ of the Protter problem $P1$ and it satisfies in $\overline{\Omega} \backslash O$ the a priori estimates

$$|u(x,t)| \leq C \left[\Phi\left(\frac{C_1}{|x| + t} \right) + |x|^{-1} ||f; n^4; C^0|| \right];$$

$$|u(x,t)| \leq C \left[\Phi\left(\frac{C_1}{|x| + t} \right) + ||f; n^6; C^0|| + ||f; n^4; C^1|| \right];$$

$$\sum_{i=1}^{3} |u_{x_i}(x,t)| + |u_t(x,t)| \leq C|x|^{-2} \left[\Phi\left(\frac{C_2}{|x| + t} \right) + ||f; n^6; C^0|| \right];$$

where the constants C, C_1 and C_2 are independent of the function $f(x,t)$.

In these estimates, the singularity of the generalized solution at the origin O is controlled by the function $\Phi(s)$, while $||f; n^p; C^k||$ bounds the «regular part» of $u(x,t)$.

Notice that the definition of $\Phi(s)$ involves parameters $\beta_{k,m}^n$ with index $k > \left[\frac{n-1}{2}\right] - 2$ also, and the corresponding functions $v_{k,m}^n$ are not classical solutions of the homogenous problem $P1^*$. Nevertheless, these functions $v_{k,m}^n$ still «control» some discontinuities of the generalized solution and cannot be omitted as seen from the following result from [7]. At the same time, Theorem 3 also suggests that there are no other linearly independent nontrivial classical solutions of the homogenous adjoint problem $P1^*$.

Theorem 3 [7]. Let the function $f(x,t)$ belong to $C^{10}(\overline{\Omega})$. Then the necessary and sufficient conditions for existence of *bounded* generalized solution $u(x,t)$ of the Protter problem $P1$ are

$$\int_{\Omega} v_{k,m}^n(x,t) f(x,t) \, dxdt \; = \; 0,$$

for all $n \in \mathbb{N}$, $k = 0, \ldots, \left[\frac{n-1}{2}\right]$, $m = 1, \ldots, 2n+1$. Moreover, this generalized solution $u(x,t) \in C^1(\overline{\Omega} \backslash O)$ and satisfies the a priori estimates

$$|u(x,t)| \leq C \, \|f\|_{C^{10}(\overline{\Omega})} ;$$

$$\sum_{i=1}^{3} |u_{x_i}(x,t)| + |u_t(x,t)| \leq C(|x|^2 + t^2)^{-1} \, \|f\|_{C^{10}(\overline{\Omega})} ,$$

where the constant C is independent of the function $f(x,t)$.

In practice, it is not always easy to compute all the parameters $\beta_{k,m}^n$ from (5) and therefore to construct and study the behaviour of the series $\Phi(s)$. On the other hand, notice that we have

$$\left| \beta_{k,m}^n \right| \leq C n^{1/2} \, \|f_n^m\|_{C^0(\Omega)} ,$$

since directly from the definition of the functions $v_{k,m}^n(x,t)$ we get the estimate $|v_{k,m}^n| \leq |Y_n^m| \leq C n^{1/2}$. This allow us to formulate the next direct corollary of Theorem 2.

Corollary 4. Let the function $f(x,t)$ belong to $C^1(\overline{\Omega})$. Suppose that the series $||f; n^6; C^0||$ and $||f; n^4; C^1||$ are convergent and the power series

$$\Phi_1(s) := \sum_{n=1}^{\infty} \left[\sum_{m=1}^{2n+1} \|f_n^m\|_{C^0(\Omega)} \right] s^n$$

has an infinite radius of convergence. Then the unique generalized solution $u(x,t) \in C^1(\overline{\Omega}\backslash O)$ of the Protter problem $P1$ satisfies near the origin the estimate

$$|u(x,t)| \le C\Phi_1\left(\frac{C_0}{|x|+t}\right), \tag{6}$$

where the constants C and C_0 are independent of the function $f(x,t)$.

Remark. Although Corollary 4 is somewhat weaker than Theorem 2, it still gives better estimate than the previously known general a priori estimates for the singularity of the solution. In particular, Protter problems in the (2+1)-D case (two space and one time dimensions) were studied in [4]. According to [4, Theorem 5.3] the sufficient condition for the existence of a generalized solution is the convergence of the series

$$\sum_{n=1}^{\infty}\frac{1}{n}I_0\left(\frac{2n}{\varepsilon}\right)\left(\left\|f_n^1\right\|_{C^0(\Omega)}+\left\|f_n^2\right\|_{C^0(\Omega)}\right),\ \text{for all } \varepsilon>0,$$

where I_0 is the modified Bessel function of first kind, and f_n^i are the Fourier coefficients for the right-hand side, and could be viewed as the analogues of the functions f_n^m given by (4). Using the inequality $I_0(s) \le e^s$ for $s \ge 0$, one could paraphrase Theorem 5.3 from 4 in somewhat weaken form as follows. Suppose that the power series

$$\Phi_2(s) := \sum_{n=1}^{\infty}\left(\left\|f_n^1\right\|_{C^0(\Omega)}+\left\|f_n^2\right\|_{C^0(\Omega)}\right)n^{-1}s^n$$

is convergent for all s. Then for the singularity of the unique generalized solution $u(x,t)$ for the (2+1)-D Protter problem $P1$, near the origin we have the estimate

$$|u(x,t)| \le C\Phi_2\left(\exp\left(\frac{4}{|x|+t}\right)\right). \tag{7}$$

Notice that the exponent in the argument of Φ_2 in (7) is replaced now in (6) by simply a linear function.

Evidently, Theorem 2 gives only an upper bound, but the generalized solution does not necessarily grows like $\Phi(C/|x|)$ near the origin. The paper [3] considers the special case when the right-hand side function f is a harmonic polynomial, i.e., (3) is a finite sum ($f_n^m \equiv 0$ for large n), and the function $\Phi(s)$ is simply a polynomial. In [3] the exact asymptotic formula for the generalized solution at O is found. It shows that the a priori estimate is sharp and the solution can indeed have a power-type singularity as $\Phi(C/|x|)$. On the other hand, in the general case $f(x,t) \in C^1(\overline{\Omega})$ stronger singularities are also possible. Actually, a generalized solutions with at least exponential growth at the origin was found in [5]. In the present article the existence of solutions with stronger singularities is announced.

Singular solutions with exponential growth

Regarding the possible singularities of the generalized solution of problem $P1$ the next question naturally arises. Given the function $\phi(s)$, can we find a smooth right-hand side function f such that the corresponding generalized solution grows like $\phi(1/|x|)$ at O? As a possible answer, the following result is given in [10], that provides a method for finding suitable functions f. Recall that $a_{n,2k}$ are the coefficients (2) of the Legendre polynomials.

Theorem 5 [10]. Let the function $f(x,t)$ belong to $C^1(\overline{\Omega})$, the series $||f;n^6;C^0||$, $||f;n^4;C^1||$ are convergent, and the power series $\Phi(s)$ has an infinite radius of convergence. Let the numbers $\alpha_p \ge 0$, $p = 0,1,2,...$, are such that the series

$$\phi(s) := \sum_{p=0}^{\infty}\alpha_p s^p$$

is convergent for all $s \in \mathbb{R}$. Suppose that there is $x^* = (x_1^*, x_2^*, x_3^*) \in \mathbb{R}^3$ such that

$$\sum_{k=0}^{\infty}\sum_{m=1}^{2p+4k+1} p\, a_{n,2k}\beta_{m,k}^{p+2k}Y_{p+2k}^m(x^*) \ge \alpha_p \quad \text{for all } p \in \mathbb{N}\cup\{0\}. \tag{8}$$

Then there exists a number $\delta \in (0, 1/2)$ that the unique generalized solution $u(x,t)$ of problem $P1$ satisfies the estimate

$$|u(tx_1^*, tx_2^*, tx_3^*, t)| \geq \phi\left(\frac{1}{2t}\right)$$

for $t \in (0, \delta)$.

According to Theorem 5 one could try to construct a right-hand side $f(x,t) \in C^1(\overline{\Omega})$ by choosing suitable Fourier coefficients $f_n^m(r,t)$. They have to be «small enough» that the required series $||f; n^p; C^k||$ and $\Phi(s)$ are convergent, but at the same time, satisfy the inequality (8). The main result in the present paper is that it is possible to apply this procedure to build an appropriate function f such that the corresponding solution grows like $\exp(|x|^{-k})$ at O.

Theorem 6. Let $k \in \mathbb{N}$. Then there exist functions $f_k \in C^1(\overline{\Omega})$ and positive numbers $\delta_k \in (0, 1/2)$ and C_k, such that the unique generalized solutions $u_k(x,t) \equiv u_k(x_1, x_2, x_3, t) \in C^1(\overline{\Omega} \backslash O)$ of the problem $P1$ for the wave equation (1) with right-hand function f_k, satisfy the estimates

$$u_k(0,0,t,t) \geq \exp(t^{-k}) \quad \text{for} \quad t \in (0, \delta_k),$$

and

$$|u_k(x,t)| \leq C_k \exp(2|x|^{-k}) \quad \text{for} \quad (x,t) \in \Omega.$$

From [5] it is known that there is a right-hand side function $f \in C^\infty(\overline{\Omega})$ such that the generalized solution grows at least like $\exp(|x|^{-1})$. Obviously this corresponds to the case $k = 1$ in Theorem 6. Unlike [5] here we have also an estimate from above, that shows that the solution behalves «exactly» like $\exp(|x|^{-k})$ at O. On the other hand, the functions f_k are only C^1-smooth, and is not clear whether, like in [5], one could construct functions from $C^\infty(\overline{\Omega})$ with the desired property.

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Т.П. Попов

Проттер (3+1)-D есебінің жаңа сингулярлық шешімдері

Уақытқа байланысты бір айнымалысы бар, үш өлшемді кеңістікте біртекті емес толқын теңдеуі қарастырылды. Бұл теңдеу үшін R^2 кеңістігіндегі Дарбу есебінің аналогы болып табылатын төртөлшемді шеттік есеп зерттелген. Бұл есептің жазықтықтағы Дарбу есебінен мынадай айырмашылығы бар: R^4 кеңістігінде қарастырылатын бұл есеп корректілі емес және оның өлшемі ақырсыз коядросы бар. Классикалық шешімділік тұрғысынан мұндай есеп фредгольмдік емес есеп болып табылады. Екінші жағынан, теңдеудің оң жағындағы функция тегіс функция болған жағдайда теңдеудің белгілі бір жалпылама шешімі бар және де ол шешім бір шеттік нүктеде дәрежелік түрде ерекшеленген болуы мүмкін. Ерекшелік нүктесі характеристикалық конустың төбесінде оқшауланған және конус бойында таралмайды. Бұл мақалада экспонента түрінде өсетін жаңа сингулярлық шешім бар екенін анықталды.

Кілт сөздер: толқын теңдеуі, шеттік есептер, жалпылама шешім, сингулярлық шешімдер, ерекшеліктердің таралуы, арнайы функциялар.

Т.П. Попов

Новые сингулярные решения для (3+1)-D задачи Проттера

Для неоднородного волнового уравнения с тремя пространственными и одной временной переменными изучена краевая задача, которую можно рассматривать как четырехмерный аналог задачи Дарбу в $\mathbb{R}^2$. В отличие от плоской задачи Дарбу, $\mathbb{R}^4$–версия не является корректной и имеет бесконечномерное коядро. Поэтому задача не является фредгольмовой в рамках классической разрешимости. С другой стороны, известно, что для гладких правых частей уравнения есть однозначно определенное обобщенное решение, которое может иметь сильную особенность степенного типа в одной граничной точке. Особенность изолирована в вершине характеристического светового конуса и не распространяется вдоль конуса. В настоящей статье анонсированы новые сингулярные решения с экспоненциальным ростом.

Ключевые слова: волновое уравнение, краевые задачи, обобщенное решение, сингулярные решения, распространение особенностей, специальные функции.

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On the numerical solution of identification hyperbolic-parabolic problems with the Neumann boundary condition

In the present study, a numerical study for source identification problems with the Neumann boundary condition for a one-dimensional hyperbolic-parabolic equation is presented. A first order of accuracy difference scheme for the numerical solution of the identification problems for hyperbolic-parabolic equations with the Neumann boundary condition is presented. This difference scheme is implemented for a simple test problem and the numerical results are presented.

Keywords: source identification problem, hyperbolic-parabolic differential equations, difference schemes.

Introduction

Partial differential equations with unknown source terms are widely used in mathematical modeling of real-life systems in many different fields of science and engineering. They have been studied extensively by many researchers (see [1–15] and the references therein).

Various local and nonlocal boundary value problems for hyperbolic-parabolic equations with unknown sources can be reduced to the boundary value problem for the differential equation with parameter p

$$\begin{cases} u''(t) + Au(t) = p + f(t), \quad 0 < t < 1; \\ u'(t) + Au(t) = p + g(t), \quad -1 < t < 0; \\ u(0+) = u(0-), \quad u'(0+) = u'(0-); \\ u(-1) = \varphi, \quad u(\lambda) = \psi, \quad -1 < \lambda \leq 1 \end{cases} \tag{1}$$

in a Hilbert space H with self-adjoint positive definite operator A. The solvability of problem (1) in the space $C(H)$ of continuous H-valued functions $u(t)$ defined on $[-1, 1]$, equipped with the norm $\|u\|_{C(H)} = \max\limits_{-1 \leq t \leq 1} \|u(t)\|_H$, was investigated in [16]. In applications, the stability inequalities for the solution of three source identification problems for hyperbolic-parabolic equations were obtained.

The first and second order of accuracy stable difference scheme for the approximate solution of problem (1) were constructed and investigated in [17] and [18], respectively. The stability estimates for the approximate solutions of two source identification problems for hyperbolic-parabolic equations were obtained.

In this paper we consider the boundary value problem for hyperbolic-parabolic equations

$$\begin{cases} u_{tt} - \big(a(x)u_x\big)_x + \delta u = p(x) + f(t,x), \quad 0 < x < 1, \ 0 < t < 1; \\ u_t - \big(a(x)u_x\big)_x + \delta u = p(x) + g(t,x), \quad 0 < x < 1, \ -1 < t < 0; \\ u(0+,x) = u(0-,x), \ u_t(0+,x) = u_t(0-,x), \quad 0 \leq x \leq 1; \\ u(-1,x) = \varphi(x), \ u(1,x) = \psi(x), \quad 0 \leq x \leq 1; \\ u_x(t,0) = u_x(t,1) = 0, \quad -1 \leq t \leq 1, \end{cases} \tag{2}$$

where $p(x)$ is an unknown source term. Problem (2) has a unique smooth solution $\{u(t,x), p(x)\}$ for the smooth functions $a(x)$, $\varphi(x)$, $\psi(x)$, $f(t,x)$, $g(t,x)$ and positive constant δ. Note that the boundary value problem (2) can be reduced to the abstract boundary value problem (1) in a Hilbert space $H = L_2[0,1]$ with a self-adjoint positive definite operator A^x defined by formula $A^x u(x) = -\big(a(x)u_x\big)_x + \delta u$ with domain $D(A^x) = \big\{u(x) \ : \ u(x), \ u_x(x), \ \big(a(x)u_x\big)_x \in L_2[0,1], \ u_x(0) = u_x(1) = 0\big\}$.

We construct the first order of accuracy difference schemes for approximate solutions of boundary value problem (2). We discuss the numerical procedure for implementation of this scheme on the computer. We provide with numerical illustration for simple test problem.

Numerical procedure for problem (2)

The solution of problem (2) can be written as following:

$$u(t,x) = v(t,x) + z(x), \quad 0 \le x \le 1, \quad -1 \le t \le 1, \tag{3}$$

where $z(x)$ is the solution of problem

$$\begin{cases} -\left(a(x)z'(x)\right)' + \delta z(x) = p(x), \quad 0 < x < 1; \\ z'(0) = z'(1) = 0 \end{cases} \tag{4}$$

and $v(t,x)$ is the solution of boundary value problem

$$\begin{cases} v_{tt} - \big(a(x)v_x\big)_x + \delta v = f(t,x), \quad 0 < x < 1,\ 0 < t < 1; \\ v_t - \big(a(x)v_x\big)_x + \delta v = g(t,x), \quad 0 < x < 1,\ -1 < t < 0; \\ v(0+,x) = v(0-,x),\ v_t(0+,x) = v_t(0-,x), \quad 0 \le x \le 1; \\ v(1,x) - v(-1,x) = \psi(x) - \varphi(x), \quad 0 \le x \le 1; \\ v_x(t,0) = v_x(t,1) = 0, \quad -1 \le t \le 1. \end{cases} \tag{5}$$

Note that from (2)–(4) we get

$$p(x) = \big(a(x)v_x(1,x)\big)_x - \delta v(1,x) - \left(a(x)\psi'(x)\right)' + \delta\psi(x), \quad 0 < x < 1. \tag{6}$$

Taking into account all of the above, the following numerical algorithm can be used for approximate solutions of the boundary value problem (2):
1. Obtain approximate solutions of the boundary value problem (5);
2. Approximate the source $p(x)$ by using (6);
3. Obtain approximate solutions of the boundary value problem (4);
4. Obtain approximate solutions of the boundary value problem (2) by using (3).

The first step of the algorithm

Let $\tau = 1/N$ and $h = 1/M$. We define the grid points $x_n = nh$, $0 \le n \le M$ and $t_k = k\tau$, $-N \le k \le N$. For the approximate solutions of the boundary value problem (5) we construct the first order of accuracy difference scheme in t

$$\begin{cases} \dfrac{v_n^{k-1} - 2v_n^k + v_n^{k+1}}{\tau^2} - \dfrac{1}{h}\left(a(x_{n+\frac{1}{2}})\dfrac{v_{n+1}^{k+1} - v_n^{k+1}}{h} - a(x_{n-\frac{1}{2}})\dfrac{v_n^{k+1} - v_{n-1}^{k+1}}{h}\right) + \delta v_n^{k+1} = \\ \qquad = f\left(t_{k+1}, x_n\right), \quad 1 \le k \le N-1, \quad 1 \le n \le M-1; \\ \dfrac{v_n^k - v_n^{k-1}}{\tau} - \dfrac{1}{h}\left(a(x_{n+\frac{1}{2}})\dfrac{v_{n+1}^k - v_n^k}{h} - a(x_{n-\frac{1}{2}})\dfrac{v_n^k - v_{n-1}^k}{h}\right) + \delta v_n^k = g\left(t_k, x_n\right); \\ \qquad -N+1 \le k \le 0, \quad 1 \le n \le M-1; \\ \dfrac{v_n^1 - v_n^0}{\tau} - \dfrac{1}{h}\left(a(x_{n+\frac{1}{2}})\dfrac{v_{n+1}^0 - v_n^0}{h} - a(x_{n-\frac{1}{2}})\dfrac{v_n^0 - v_{n-1}^0}{h}\right) + \delta v_n^0 = g\left(t_0, x_n\right),\ 1 \le n \le M-1; \\ v_n^N - v_n^{-N} = \psi(x_n) - \varphi(x_n), \quad 0 \le n \le M; \\ v_1^k = v_0^k, \quad v_M^k = v_{M-1}^k, \quad -N \le k \le N, \end{cases} \tag{7}$$

where v_n^k denotes the numerical approximation of $v(t,x)$ at (t_k, x_n). Note that (7) is the second order of accuracy scheme in x.

The second step of the algorithm

Once the numerical solution of the boundary value problem (5) is computed, we use (6) to approximate the source $p(x)$ at grid points as following:

$$p_n = \frac{1}{h}\left(a(x_{n+\frac{1}{2}})\frac{v_{n+1}^N - v_n^N}{h} - a(x_{n-\frac{1}{2}})\frac{v_n^N - v_{n-1}^N}{h}\right) - \delta v_n^N -$$

$$-\frac{1}{h}\left(a(x_{n+\frac{1}{2}})\frac{\psi(x_{n+1}) - \psi(x_n)}{h} - a(x_{n-\frac{1}{2}})\frac{\psi(x_n) - \psi(x_{n-1})}{h}\right) + \delta\psi(x_n), \quad n = 1, 2, \ldots, M-1.$$

The third step of the algorithm

For the approximate solutions of boundary value problem (4) we have

$$\begin{cases} -\frac{1}{h}\left(a(x_{n+\frac{1}{2}})\frac{z_{n+1} - z_n}{h} - a(x_{n-\frac{1}{2}})\frac{z_n - z_{n-1}}{h}\right) + \delta z_n = p_n, \quad n = 1, 2, \ldots, M-1; \\ z_1 = z_0, \ z_M = z_{M-1}. \end{cases}$$

Solving this system for $z_0, z_1, z_2, \ldots, z_M$ and then using (3), we finally obtain the approximate solutions of boundary value problem (2)

$$u_n^k = v_n^k + z_n, \quad n = 0, 1, \ldots, M, \quad k = -N+1, \ldots, N-1. \tag{8}$$

Numerical Illustration

We consider the initial-boundary value problem

$$\begin{cases} u_{tt} - u_{xx} + u = p(x) + \left((\pi^2 + 2)e^{-t} - 1\right)\cos\pi x, \quad 0 < x < 1, \ 0 < t < 1; \\ u_t - u_{xx} + u = p(x) + \left(\pi^2 e^{-t} - 1\right)\cos\pi x, \quad 0 < x < 1, \ -1 < t < 0; \\ u(0+, x) = u(0-, x), \ u_t(0+, x) = u_t(0-, x), \quad 0 \le x \le 1; \\ u(-1, x) = e^1\cos\pi x, \ u(1, x) = e^{-1}\cos\pi x, \quad 0 \le x \le 1; \\ u_x(t, 0) = u_x(t, 1) = 0, \quad -1 \le t \le 1. \end{cases} \tag{9}$$

The exact solution of the problem (9) is

$$u(t, x) = e^{-t}\cos\pi x, \ \ 0 \le x \le 1, \ -1 \le t \le 1$$

with the source term $p(x) = \cos\pi x, \ 0 < x < 1$.

The first order of accuracy auxiliary difference scheme (7) for the initial-boundary value problem (9) has the following form

$$\begin{cases} \frac{v_n^{k-1} - 2v_n^k + v_n^{k+1}}{\tau^2} - \frac{v_{n-1}^{k+1} - 2v_n^{k+1} + v_{n+1}^{k+1}}{h^2} + v_n^{k+1} = \left((\pi^2 + 2)e^{-t_{k+1}} - 1\right)\cos\pi x_n; \\ \qquad 1 \le k \le N-1, \quad 1 \le n \le M-1; \\ \frac{v_n^k - v_n^{k-1}}{\tau} - \frac{v_{n-1}^k - 2v_n^k + v_{n+1}^k}{h^2} + v_n^k = \left(\pi^2 e^{-t_k} - 1\right)\cos\pi x_n; \\ \qquad -N+1 \le k \le 0, \quad 1 \le n \le M-1; \\ \frac{v_n^1 - v_n^0}{\tau} - \frac{v_{n-1}^0 - 2v_n^0 + v_{n+1}^0}{h^2} + v_n^0 = \left(\pi^2 e^{-t_0} - 1\right)\cos\pi x_n, \quad 1 \le n \le M-1; \\ v_n^N - v_n^{-N} = \left(e^{-1} - e^1\right)\cos\pi x_n, \quad 0 \le n \le M; \\ v_1^k - v_0^k = v_M^k - v_{M-1}^k = 0, \quad -N \le k \le N, \end{cases} \tag{10}$$

which can be written in the matrix form

$$\begin{cases} AV_{n+1} + BV_n + CV_{n-1} = \phi_n, \quad 1 \le n \le M-1; \\ V_1 = V_0, \quad V_M = V_{M-1}, \end{cases} \tag{11}$$

where

$$A = C = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & a & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & a & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & a & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & a & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & b & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & b & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & b \end{bmatrix}_{(2N+1)\times(2N+1)}$$

$$B = \begin{bmatrix} -1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 \\ -1 & c & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & -1 & c & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -1 & c & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & \sigma & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 & -2 & d & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 1 & -2 & d & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & & \ddots & \ddots & \ddots & \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & 1 & -2 & d \end{bmatrix}_{(2N+1)\times(2N+1)}$$

$$V_n = \begin{bmatrix} v_n^{-N} \\ v_n^{-N+1} \\ v_n^{-N+2} \\ \vdots \\ v_n^0 \\ v_n^1 \\ v_n^2 \\ v_n^3 \\ \vdots \\ v_n^N \end{bmatrix}_{(2N+1)\times 1} \qquad \phi_n = \begin{bmatrix} (e^{-1} - e^{1}) \cos \pi x_n \\ \tau\left(\pi^2 e^{-t_{-N+1}} - 1\right) \cos \pi x_n \\ \tau\left(\pi^2 e^{-t_{-N+2}} - 1\right) \cos \pi x_n \\ \vdots \\ \tau\left(\pi^2 e^{-t_0} - 1\right) \cos \pi x_n \\ \tau\left(\pi^2 e^{-t_0} - 1\right) \cos \pi x_n \\ \tau^2\left((\pi^2+2) e^{-t_2} - 1\right) \cos \pi x_n \\ \tau^2\left((\pi^2+2) e^{-t_3} - 1\right) \cos \pi x_n \\ \vdots \\ \tau^2\left((\pi^2+2) e^{-t_N} - 1\right) \cos \pi x_n \end{bmatrix}_{(2N+1)\times 1}$$

with $a = -\dfrac{\tau}{h^2}$, $b = -\dfrac{\tau^2}{h^2}$, $c = 1 + \dfrac{2\tau}{h^2} + \tau$, $d = 1 + \dfrac{2\tau^2}{h^2} + \tau^2$ and $\sigma = -1 + \dfrac{2\tau}{h^2} + \tau$. To solve the matrix equation (11), we use the modified Gauss elimination method [19]. We seek the solution of the matrix equation (11) by the following form:

$$\begin{cases} V_n = \alpha_{n+1} V_{n+1} + \beta_{n+1}, \quad n = M-1, \ldots, 2, 1; \\ V_M = (I - \alpha_M)^{-1} \beta_M, \end{cases}$$

where I is a $(2N+1) \times (2N+1)$ identity matrix, α_n $(1 \leq n \leq M)$ are $(2N+1) \times (2N+1)$ square matrices and β_n $(1 \leq n \leq M)$ are $(2N+1) \times 1$ column vectors, calculated as

$$\begin{cases} \alpha_{n+1} = -\left(B + C\alpha_n\right)^{-1} A; \\ \beta_{n+1} = \left(B + C\alpha_n\right)^{-1} \left(\phi_n - C\beta_n\right) \end{cases}$$

for $n = 1, 2, \ldots, M-1$. Here α_1 is an identity matrix and β_1 is a zero vector.

The numerical solutions are computed using the first order of accuracy scheme (10) for different values of M and N. With the obtained numerical solutions we approximate the source $p(x)$ at grid points as following:

$$p_n = \frac{v_{n-1}^N - 2v_n^N + v_{n+1}^N}{h^2} - v_n^N - e^{-1}\frac{\cos\pi x_{n-1} - 2\cos\pi x_n + \cos\pi x_{n+1}}{h^2} + e^{-1}\cos\pi x_n,\ n = 1,\ldots,M-1.$$

Finally, solving the system

$$\begin{cases} -\dfrac{z_{n-1} - 2z_n + z_{n+1}}{h^2} + z_n = p_n, \quad n = 1,2,\ldots,M-1, \\ z_1 = z_0,\ z_M = z_{M-1}. \end{cases}$$

for $z_0, z_1, z_2, \ldots, z_M$ and then using (8), we obtain the numerical solutions of problem (9).

We compute the error between the exact solution of problem (9) and corresponding numerical solution by

$$\|E_u\|_\infty = \max_{-N<k<N,\ 0\le n\le M} \left|u(t_k,x_n) - u_n^k\right|, \qquad \|E_p\|_\infty = \max_{0<n<M} |p(x_n) - p_n|,$$

where $u(t_k,x_n)$ is the exact value of $u(t,x)$ at (t_k,x_n) and $p(x_n)$ is the exact value of source $p(x)$ at $x = x_n$; u_n^k and p_n represent the corresponding numerical solutions. Table shows the errors between the exact solution of the problem (9) and the numerical solutions computed by using the first order of accuracy scheme for different values of M and N. We observe that the scheme has the first order convergence as it is expected to be.

Table

The errors between the exact solution of the problem (9) and the numerical solutions computed by using the first order of accuracy difference scheme for different values of $h = 1/M$ and $\tau = 1/N$

	$\|E_p\|_\infty$	Order	$\|E_u\|_\infty$	Order
$N = M = 20$	1.3246×10^{-1}	-	2.1520×10^{-1}	-
$N = M = 40$	7.4275×10^{-2}	0.8346	1.1722×10^{-1}	0.8764
$N = M = 80$	4.0124×10^{-2}	0.8884	6.1093×10^{-2}	0.9402
$N = M = 160$	2.1234×10^{-2}	0.9181	3.1190×10^{-2}	0.9699
$N = M = 320$	1.1072×10^{-2}	0.9394	1.5758×10^{-2}	0.9850

Conclusion

In the present study, the numerical study for source identification problems with the Neumann boundary condition for a one-dimensional hyperbolic-parabolic equation has been conducted. In particular, the first order of accuracy difference schemes for the approximate solutions of the boundary value problem (2) has been constructed and the numerical algorithm for implementation of this scheme has been presented. Numerical example has been provided.

Finally, we note that the second order of accuracy difference schemes for the approximate solutions of boundary value problem (2) can be constructed and implemented in the similar way.

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М.А. Ашыралыева, М. Ашыралыев

Шеттік шарты Нейман түрінде болатын идентификациялық гипербола-параболалық есептерді сандық шешу туралы

Шеттік шарты Нейман түріндегі бір өлшемді гипербола-параболалық түрдегі теңдеу үшін көздерді идентификациялау есебін сандық зерттеу нәтижесі ұсынылған. Шеттік шарты Нейман түріндегі гипербола-параболалық түрдегі теңдеулер үшін көздерді идентификациялау есебін сандық шешу үшін дәлдігі бірінші ретті айырымдық формуласы келтірілген. Бұл формула қарапайым есеп үшін пайдаланылған, сонымен қатар сандық есептеулер нәтижесі берілген.

Кілт сөздер: көздерді идентификациялау есебі, гипербола-параболалық дифференциалды теңдеу, айырымдық схема.

М.А. Ашыралыева, М. Ашыралыев

О численном решении идентификационных гиперболо-параболических задач с граничным условием Неймана

В статье представлено численное исследование задачи идентификации источников с граничным условием Неймана для одномерного гиперболо-параболического уравнения. Представлена разностная схема первого порядка точности для численного решения задач идентификации для гиперболо-параболических уравнений с граничным условием Неймана. Эта разностная схема реализована для простой тестовой задачи.

Ключевые слова: задача идентификации источника, гиперболо-параболические дифференциальные уравнения, разностные схемы.

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Inverse source problems for a wave equation with involution

A class of inverse problems for a wave equation with involution is considered for cases of two different boundary conditions, namely, Dirichlet and Neumann boundary conditions. The existence and uniqueness of solutions of these problems are proved. The solutions are obtained in the form of series expansion using a set of appropriate orthogonal basises for each problem. Convergence of the obtained solutions is also justified.

Keywords: inverse problem, involution, nonlocal wave equation, Sturm-Liouville problem, existence of solution, uniqueness of solution.

1 Introduction

In many physical problems, determination of coefficients or right-hand side according to some available information (the source term, in case of a wave equation) in a differential equation is required; these problems are known as inverse problems. These kinds of problems are ill-posed in the sense of Hadamard.

The purpose of this paper is to study inverse problems for a nonlocal wave equation with involution of space variable x. We consider the nonlocal wave equation

$$u_{tt}(x,t) - u_{xx}(x,t) + \varepsilon u_{xx}(\pi - x, t) = f(x), \tag{1}$$

for $(x,t) \in \Omega = \{0 < x < \pi,\ 0 < t < T\}$, where ε is a real number.

Wide opportunities for applying equations with deviating argument in mathematical models have increased the interest of the study of new problems for partial differential equations [1–3].

Among differential equations with deviating arguments, a special place is occupied by equations with a deviation of arguments of alternating character. Such deviations include the so-called deviation of involution type [4]. To describe them, let Γ be an interval in $\mathbb{R}$ and let $X \in \Gamma$ be a real variable.

The homeomorphism

$$\alpha^2(X) = \alpha(\alpha(X)) = X$$

is called a Carleman shift (deviation of involution) [5].

Equations containing Carleman shift are equations with an alternating deviation (at $X^* < X$ being equations with advanced, and at $X^* > X$ being equations with delay, where X^* is a fixed point of the mapping $\alpha(X)$).

Concerning the inverse problems for partial differential equations with involutions, some recent works have been implemented in [6–11].

2 Statement of problems

The paper is devoted to two inverse problems concerning the wave equation with a perturbative term of involution type with respect to the space variable. We obtain existence and uniqueness results for these problems, based on the Fourier method.

Problem D. Find a couple of functions $(u(x,t), f(x))$ satisfying the equation (1), under the conditions

$$u(x,0) = 0,\ x \in [0,\pi], \tag{2}$$

$$u(x,T) = \psi(x),\ x \in [0,\pi], \tag{3}$$

$$u_t(x,0) = 0,\ x \in [0,\pi], \tag{4}$$

and the homogeneous Dirichlet boundary conditions

$$u(0,t) = u(\pi,t) = 0\ ,\ t \in [0,T]\,, \tag{5}$$

where $\psi(x)$ is a given sufficiently smooth function.

Problem N. Find the couple of functions $(u(x,t), f(x))$ in the domain Ω satisfying equation (1), conditions (2), (3), (4) and the homogeneous Neumann boundary conditions

$$u_x(0,t) = u_x(\pi,t) = 0\ ,\ t \in [0,T]\,. \tag{6}$$

A regular solution of the problems D and N is the pair of functions $(u(x,t), f(x))$, where $u \in C^2\left(\bar{\Omega}\right)$ and $f \in C([0,\pi])$.

3 Spectral properties of the perturbed Sturm-Liouville problem

Application of the Fourier method for solving the problems D and N leads to a spectral problem defined by the equation

$$y''(x) - \varepsilon y''(\pi - x) + \lambda y(x) = 0, 0 < x < \pi, \tag{7}$$

and one of the following boundary conditions

$$y(0) = y(\pi) = 0; \tag{8}$$

$$y'(0) = y'(\pi) = 0. \tag{9}$$

It is easy to see that the Sturm-Liouville problem for the equation (7) with one of the boundary conditions (8) and (9) is self-adjoint. It is known that the self-adjoint problem has real eigenvalues and their eigenfunctions form a complete orthonormal basis in $L^2(0,\pi)$ [12]. To further investigate the problems under consideration, we need to calculate the explicit form of the eigenvalues and eigenfunctions.

It is easy to show that for $|\varepsilon| < 1$ the problem (7), (8) has the following eigenvalues

$$\lambda_{2k}^D = (1+\varepsilon)\,4k^2,\ k \in \mathbb{N};$$

$$\lambda_{2k+1}^D = (1-\varepsilon)\,(2k+1)^2\,,\ k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$$

and eigenfunctions

$$\begin{cases} y_{2k}^D = \sqrt{\frac{2}{\pi}} \sin 2kx,\ k \in \mathbb{N}; \\ y_{2k+1}^D = \sqrt{\frac{2}{\pi}} \sin(2k+1)\,x,\ k \in \mathbb{N}_0. \end{cases} \tag{10}$$

Similarly, the problem (7), (9) has the eigenvalues

$$\lambda_{2k+1}^N = (1+\varepsilon)\,(2k+1)^2\,,\ k \in \mathbb{N}_0;$$

$$\lambda_{2k}^N = (1-\varepsilon)\,4k^2,\ k \in \mathbb{N}_0,$$

and corresponding eigenfunctions

$$\begin{cases} y_0^N = \frac{1}{\sqrt{\pi}}; \\ y_{2k+1}^N = \sqrt{\frac{2}{\pi}} \cos(2k+1)\,x,\ k \in \mathbb{N}_0; \\ y_{2k}^N = \sqrt{\frac{2}{\pi}} \cos 2kx,\ k \in \mathbb{N}. \end{cases} \tag{11}$$

The following lemma is proved in [11].

Lemma 1. The systems of functions (10) and (11) are complete and orthonormal in $L^2(0,\pi)$.

4 Main results

For the considered problems D and N, the following theorems are valid.

Theorem 1. Let $|\varepsilon| < 1$, $\psi \in C^4[0,\pi]$ *and* $\psi^{(i)}(0) = \psi^{(i)}(\pi) = 0,\ i = 0,1,2,3,4.$ *If*

$$\cos\sqrt{1-\varepsilon}\,(2k+1)\,T < \delta_1 < 1$$

and

$$\cos\sqrt{1+\varepsilon}2kT < \delta_2 < 1,$$

then the solution of the problem D exists, is unique and it can be written in the form

$$u(x,t) = \sum_{k=0}^{\infty} \frac{\left(1-\cos\sqrt{1-\varepsilon}\,(2k+1)\,t\right)\sin(2k+1)\,x}{\left(1-\cos\sqrt{1-\varepsilon}\,(2k+1)\,T\right)(2k+1)^4}\psi^4_{2k+1}+$$

$$+\sum_{k=1}^{\infty} \frac{\left(1-\cos\sqrt{1+\varepsilon}2kt\right)\sin 2kx}{\left(1-\cos\sqrt{1+\varepsilon}2kT\right)16k^4}\psi^4_{2k}; \tag{12}$$

$$f(x) = \sum_{k=0}^{\infty} \frac{(1-\varepsilon)\,\psi^4_{2k+1}}{\left(1-\cos\sqrt{1-\varepsilon}\,(2k+1)\,T\right)(2k+1)^2}\sin(2k+1)\,x+$$

$$+\sum_{k=1}^{\infty} \frac{(1+\varepsilon)\,\psi^4_{2k}}{\left(1-\cos\sqrt{1+\varepsilon}2kT\right)4k^2}\sin 2kx, \tag{13}$$

where $\psi^{(4)}_{2k+1} = \left(\psi^{(4)}(x)\,,\,y^D_{2k+1}\right)$ and $\psi^{(4)}_{2k} = \left(\psi^{(4)}(x)\,,\,y^D_{2k}\right)$.

Theorem 2. Let $|\varepsilon| < 1$, $\psi \in C^4[0,\pi]$ *and* $\psi^{(i)}(0) = \psi^{(i)}(\pi) = 0, i = 0,1,2,3,4.$ *If*

$$\cos\sqrt{1-\varepsilon}\,(2k+1)\,T < \sigma_1 < 1;$$

and

$$\cos\sqrt{1+\varepsilon}2kT < \sigma_2 < 1,$$

then the solution of the problem N exists, is unique and it can be written in the form

$$u(x,t) = \sum_{k=0}^{\infty} \frac{\left(1-\cos\sqrt{1+\varepsilon}\,(2k+1)\,t\right)\cos(2k+1)\,x}{\left(1-\cos\sqrt{1+\varepsilon}\,(2k+1)\,T\right)(2k+1)^4}\psi^4_{2k+1}+$$

$$+\sum_{k=1}^{\infty} \frac{\left(1-\cos\sqrt{1-\varepsilon}2kt\right)\cos 2kx}{\left(1-\cos\sqrt{1-\varepsilon}2kT\right)16k^4}\psi^4_{2k}; \tag{14}$$

$$f(x) = \sum_{k=0}^{\infty} \frac{(1+\varepsilon)\,\psi^4_{2k+1}}{\left(1-\cos\sqrt{1+\varepsilon}\,(2k+1)\,T\right)(2k+1)^2}\cos(2k+1)\,x+$$

$$+\sum_{k=1}^{\infty} \frac{(1-\varepsilon)\,\psi^4_{2k}}{\left(1-\cos\sqrt{1-\varepsilon}2kT\right)4k^2}\cos 2kx; \tag{15}$$

where $\psi^{(4)}_{2k+1} = \left(\psi^{(4)}(x)\,,\,y^N_{2k+1}\right)$ and $\psi^{(4)}_{2k} = \left(\psi^{(4)}(x)\,,\,y^N_{2k}\right)$.

5 Proof of the uniqueness of the solution

Suppose that there are two solutions $\{u_1(x,t)\,,f_1(x)\}$ and $\{u_2(x,t)\,,f_2(x)\}$ of the problem P. Denote

$$u(x,t) = u_1(x,t) - u_2(x,t)$$

and

$$f(x) = f_1(x) - f_2(x)\,.$$

Then the functions $u(x,t)$ and $f(x)$ satisfy (1) and the homogeneous conditions (2) and (5).

Let

$$u_0(t) = \frac{1}{\sqrt{\pi}} \int_0^{\pi} u(x,t)\,dx; \tag{16}$$

$$u_{2k}(t) = \sqrt{\frac{2}{\pi}} \int_0^{\pi} u(x,t)\cos 2kx dx, k \in \mathbb{N}; \tag{17}$$

$$u_{2k+1}(t) = \sqrt{\frac{2}{\pi}} \int_0^{\pi} u(x,t)\cos(2k+1)x dx, k \in \mathbb{N}_0; \tag{18}$$

$$f_0 = \frac{1}{\sqrt{\pi}} \int_0^{\pi} f(x)dx; \tag{19}$$

$$f_{2k} = \sqrt{\frac{2}{\pi}} \int_0^{\pi} f(x)\cos 2kx dx, k \in \mathbb{N}; \tag{20}$$

$$f_{2k+1} = \sqrt{\frac{2}{\pi}} \int_0^{\pi} f(x)\cos(2k+1)x dx, k \in \mathbb{N}. \tag{21}$$

Applying the operator $\frac{\partial^2}{\partial t^2}$ to the equation (16) we have

$$u_0''(t) = \frac{1}{\sqrt{\pi}} \int_0^{\pi} \mathcal{D}_t^{\alpha} u(x,t)\,dx = \frac{1}{\sqrt{\pi}} \int_0^{\pi} \left(u_{xx}(x,t) - \varepsilon u_{xx}(\pi - x,t)\right)dx + f_0.$$

Integrating by parts and taking into account the homogeneous conditions (2) and (6) , we obtain

$$u_0''(t) = f_0,$$

$$u(0) = 0\ ,\ u(T) = 0,\ u'(0) = 0.$$

Hence it is easy to get $f_0 = 0, u_0(t) \equiv 0$.

In a similar way for the functions (17)–(21) it is easy to prove that

$$f_{2k} = 0, f_{2k+1} = 0, u_{2k}(t) \equiv 0, u_{2k+1}(t) \equiv 0.$$

Further, by the completeness of the system (10) in $L^2(0,\pi)$ we obtain

$$f(x) \equiv 0, u(x,t) \equiv 0, 0 \le t \le T, 0 \le x \le \pi.$$

The uniqueness of the solution of the problem N is proved.

The uniqueness of the solution of the problem D can be proved similarly.

6 Proof of the existence of the solution

We give the full proof for the problem D. The existence of the solution of the problem N is proved analogously.

As the eigenfunctions system (10) of the problem D forms an orthonormal basis in $L^2(0,\pi)$ (this follows from the self-adjoint problem (7), (8)), the functions $u(x,t)$ and $f(x)$ can be expanded as follows

$$u(x,t) = \sum_{k=0}^{\infty} u_{2k+1}(t)\sin(2k+1)x + \sum_{k=1}^{\infty} u_{2k}(t)\sin 2kx; \tag{22}$$

$$f(x) = \sum_{k=0}^{\infty} f_{2k+1}\sin(2k+1)x + \sum_{k=1}^{\infty} f_{2k}\sin 2kx, \tag{23}$$

where $f_{2k+1}, f_{2k}, u_{2k+1}(t), u_{2k}(t)$ are unknown. Substituting (22) and (23) into (1), we obtain the following equation for the functions $u_{2k+1}(t)$, $u_{2k}(t)$ and the constantsf_{2k+1}, f_{2k} :

$$u''_{2k+1}(t) + (1-\varepsilon)(2k+1)^2 u_{2k+1}(t) = f_{2k+1};$$

$$u''_{2k}(t) + (1+\varepsilon) 4k^2 u_{2k}(t) = f_{2k}.$$

Solving these equations [13], we obtain

$$u_{2k+1}(t) = \frac{f_{2k+1}}{(1-\varepsilon)(2k+1)^2} + C_{1k}\cos\sqrt{1-\varepsilon}(2k+1)t + C_{2k}\sin\sqrt{1-\varepsilon}(2k+1)t;$$

$$u_{2k}(t) = \frac{f_{2k}}{(1+\varepsilon)4k^2} + D_{1k}\cos\sqrt{1+\varepsilon}2kt + D_{2k}\sin\sqrt{1+\varepsilon}2kt,$$

where the constants $C_{1k}, C_{2k}, D_{1k}, D_{2k}, f_{2k+1}, f_{2k}$ are unknown. To find these constants, we use the conditions (2). Let ψ_{2k}, ψ_{2k+1} be the coefficients of the expansions of $\psi(x)$

$$\psi_{2k+1} = \sqrt{\frac{2}{\pi}}\int_0^\pi \psi(x)\sin(2k+1)x dx;$$

$$\psi_{2k} = \sqrt{\frac{2}{\pi}}\int_0^\pi \psi(x)\sin 2kx dx.$$

We first find C_{1k}, C_{2k} :

$$u_{2k+1}(0) = \frac{f_{2k+1}}{(1-\varepsilon)(2k+1)^2} + C_{1k} = 0;$$

$$u'_{2k+1}(0) = C_{2k} = 0;$$

$$u_{2k+1}(T) = \frac{f_{2k+1}}{(1-\varepsilon)(2k+1)^2}\left(1-\cos\sqrt{1-\varepsilon}(2k+1)T\right) = \psi_{2k+1}.$$

The constant f_{2k+1} is represented as

$$f_{2k+1} = \frac{(1-\varepsilon)(2k+1)^2\psi_{2k+1}}{1-\cos\sqrt{1-\varepsilon}(2k+1)T}.$$

Now we find D_{1k}, D_{2k}:

$$u_{2k}(0) = \frac{f_{2k}}{(1+\varepsilon)4k^2} + D_{1k} = 0;$$

$$u'_{2k}(0) = D_{2k} = 0;$$

$$u_{2k}(T) = \frac{f_{2k}}{(1+\varepsilon)4k^2}\left(1-\cos\sqrt{1+\varepsilon}2kT\right) = \psi_{2k}.$$

For the constant f_{2k}, we find:

$$f_{2k} = \frac{(1+\varepsilon)4k^2\psi_{2k}}{1-\cos\sqrt{1+\varepsilon}2kT}.$$

Substituting $u_{2k}(t)$, $u_{2k+1}(t)$, f_{2k}, f_{2k+1} into (22) and (23), we find

$$u(x,t) = \sum_{k=0}^{\infty}\frac{\left(1-\cos\sqrt{1-\varepsilon}(2k+1)t\right)\sin(2k+1)x}{\left(1-\cos\sqrt{1-\varepsilon}(2k+1)T\right)}\psi_{2k+1} +$$
$$+\sum_{k=1}^{\infty}\frac{\left(1-\cos\sqrt{1+\varepsilon}2kt\right)\sin 2kx}{\left(1-\cos\sqrt{1+\varepsilon}2kT\right)}\psi_{2k}.$$

Suppose that

$$\psi^{(i)}(0)=0\ ,\ \psi^{(i)}(\pi)=0\ ,\ i=0,1,2,3,4,$$

then

$$\psi_{2k+1}=\frac{1}{(2k+1)^4}\psi_{2k+1}^{(4)},\quad \psi_{2k}=\frac{1}{16k^4}\psi_{2k}^{(4)}.$$

Then we have (12).

Similarly,

$$f(x)=\sum_{k=0}^{\infty}\frac{(1-\varepsilon)\,\psi_{2k+1}^{4}}{\left(1-\cos\sqrt{1-\varepsilon}\,(2k+1)\,T\right)(2k+1)^2}\sin(2k+1)\,x+$$

$$+\sum_{k=1}^{\infty}\frac{(1+\varepsilon)\,\psi_{2k}^{4}}{\left(1-\cos\sqrt{1+\varepsilon}2kT\right)4k^2}\sin 2kx.$$

Now for the convergence of the series, we have the following estimate

$$|u(x,t)|\leq\sum_{k=0}^{\infty}\frac{\left(1-\cos\sqrt{1-\varepsilon}\,(2k+1)\,t\right)}{\left(1-\cos\sqrt{1-\varepsilon}\,(2k+1)\,T\right)(2k+1)^4}|\psi_{2k+1}^{(4)}|+$$

$$+\sum_{k=1}^{\infty}\frac{\left(1-\cos\sqrt{1+\varepsilon}2kt\right)}{\left(1-\cos\sqrt{1+\varepsilon}2kT\right)16k^4}|\psi_{2k}^{(4)}|\leq$$

$$\leq C\sum_{k=0}^{\infty}\frac{1}{(2k+1)^4}|\psi_{2k+1}^{(4)}|+C\sum_{k=1}^{\infty}\frac{1}{16k^4}|\psi_{2k}^{(4)}|<\infty. \tag{24}$$

Similarly for $f(x)$ we obtain the estimate

$$|f(x)|\leq\sum_{k=0}^{\infty}\frac{(1-\varepsilon)\,|\psi_{2k+1}^{(4)}|}{\left(1-\cos\sqrt{1-\varepsilon}\,(2k+1)\,T\right)(2k+1)^2}+\sum_{k=1}^{\infty}\frac{(1+\varepsilon)\,|\psi_{2k}^{(4)}|}{\left(1-\cos\sqrt{1+\varepsilon}2kT\right)4k^2}\leq$$

$$\leq C\sum_{k=0}^{\infty}\frac{|\psi_{2k+1}^{(4)}|}{(2k+1)^2}+C\sum_{k=1}^{\infty}\frac{|\psi_{2k}^{(4)}|}{4k^2}. \tag{25}$$

Since by hypotheses of Theorem 1, the function $\psi^{(4)}$ is continuous on $[0,\pi]$, then by the Bessel inequality for the trigonometric series the following series converge:

$$\sum_{k=1}^{\infty}\left|\psi_{2k}^{(4)}\right|^2\leq C\left\|\psi^{(4)}(x)\right\|_{L_2(0,\pi)}^2; \tag{26}$$

$$\sum_{k=0}^{\infty}\left|\psi_{2k+1}^{(4)}\right|^2\leq C\left\|\psi^{(4)}(x)\right\|_{L_2(0,\pi)}^2, \tag{27}$$

which implies the boundedness of the set

$$\left\{\psi_{2k}^{(4)}\right\}_{k=1}^{\infty},\left\{\psi_{2k+1}^{(4)}\right\}_{k=0}^{\infty}.$$

Therefore, by the Weierstrass M-test (see [14]), the series (24) and (25) converge absolutely and uniformly in the domain $\bar{\Omega}$.

Now we show the possibility of termwise differentiation of the series (24) twice in the variable x and twice in the variable t. For this purpose, we prove that the series obtained by means of term by term differentiation converge absolutely and uniformly on $\bar{\Omega}$. Given the estimates (26) and (27) we have

$$|u_{xx}(x,t)|\leq\sum_{k=0}^{\infty}\frac{\left(1-\cos\sqrt{1-\varepsilon}\,(2k+1)\,t\right)}{\left(1-\cos\sqrt{1-\varepsilon}\,(2k+1)\,T\right)(2k+1)^2}|\psi_{2k+1}^{(4)}|+$$

$$+\sum_{k=1}^{\infty}\frac{\left(1-\cos\sqrt{1+\varepsilon}2kt\right)}{\left(1-\cos\sqrt{1+\varepsilon}2kT\right)4k^2}|\psi_{2k}^{(4)}|\leq$$

$$\leq C\sum_{k=0}^{\infty}\frac{1}{(2k+1)^2}|\psi_{2k+1}^{(4)}|+C\sum_{k=1}^{\infty}\frac{1}{4k^2}|\psi_{2k}^{(4)}|<\infty;$$

$$|u_{tt}(x,t)|\leq\sqrt{1-\varepsilon}\sum_{k=0}^{\infty}\frac{\left(|\sin\sqrt{1-\varepsilon}\,(2k+1)\,t|\right)}{\left(1-\cos\sqrt{1-\varepsilon}\,(2k+1)\,T\right)(2k+1)^2}\left|\psi_{2k+1}^{(4)}\right|+$$

$$+\sqrt{1+\varepsilon}\sum_{k=1}^{\infty}\frac{\left|\sin\sqrt{1+\varepsilon}2kt\right|}{\left(1-\cos\sqrt{1+\varepsilon}2kT\right)4k^2}|\psi_{2k}^{(4)}|\leq$$

$$\leq C\sum_{k=0}^{\infty}\frac{1}{(2k+1)^2}|\psi_{2k+1}^{(4)}|+C\sum_{k=1}^{\infty}\frac{1}{4k^2}|\psi_{2k}^{(4)}|<\infty.$$

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Р. Тапдигоглу, Б.Т. Төребек

Инволюциялы толқын теңдеуі үшін дереккөзді кері есептер

Мақалада екі түрлі шекаралық шартпен, атап айтқанда, Дирихле және Нейман шекаралық шарттарымен берілген инволюциясы толқын теңдеуі үшін кері есептер класы қарастырылды. Осы есептердің шешімінің бар болуы мен жалғыздығы дәлелденді. Шешім әрбір есептің сәйкес ортогоналды базистері арқылы жіктелген қатар арқылы алынды. Ол шешімдердің жинақтылығы дәлелденді.

Кілт сөздер: кері есеп, инволюция, бейлокал толқын теңдеуі, Штурм-Лиувилл есебі, шешімнің бар болуы, шешімнің жалғыздығы.

Р. Тапдигоглу, Б.Т. Торебек

Обратные задачи источника для волнового уравнения с инволюцией

В статье рассмотрен класс обратных задач для волнового уравнения с инволюцией для случаев двух разных граничных условий, а именно граничных условий Дирихле и Неймана. Доказаны существование и единственность решений этих задач. Решения получены в виде разложения рядов с использованием набора подходящих ортогональных базисов для каждой задачи. Также доказана сходимость полученных решений.

Ключевые слова: обратная задача, инволюция, нелокальное волновое уравнение, задача Штурма-Лиувилля, существование решения, единственность решения.

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Dynamics of HIV-1 infected population acquired via different sexual contacts route: a case study of Turkey

This paper aims to study the transmission dynamics of HIV/AIDS in heterosexual, men having sex with men (MSM)/bisexuals and others in Turkey. Four equilibrium points were obtained which include disease free and endemic equilibrium points. The global stability analysis of the equilibria was carried out using the Lyapunov function which happens to depend on the basic reproduction number R_0. If $R_0 < 1$ the disease free equilibrium point is globally asymptotically stable and the disease dies out, and if $R_0 > 1$, the endemic equilibrium point is stable and epidemics will occur. We use raw data obtained from Kocaeli University, PCR Unit, Turkey to analyze and predict the trend of HIV/AIDS among heterosexuals, MSM/bisexual, and others. The basic reproduction number for heterosexuals, MSM/bisexuals, and others was found to be 1.08, 0.6719, and 0.050, respectively. This shows that the threat posed by HIV/AIDS among heterosexuals is greater than followed by MSM/bisexuals, and than the others. So, the relevant authorities should put priorities in containing the disease in order of their threat.

Keywords: HIV/AIDS, Basic reproduction number, Equilibrium point, Stability, Lyapunov function, MSM, Heterosexual.

Introduction

One of the major factors leading to the prevalence and epidemics of HIV/AIDS is the increase in the population of men having sex with men (MSM). Most of the countries affected are the United States, Canada, Australia, and New Zealand. However, in some of the under-developed and developing countries, heterosexual sex, injection drug use, and/or transfusion of contaminated blood remain the main mode of transmission of the disease [1] and [2]. A sudden rise of HIV cases in MSM was detected in the continent of Africa, Asia, South America, and Eastern Europe [3]. According to a report in China, the rates of HIV infection in MSM is increasing dramatically, while other means of transmissions are either decreasing or remaining under control. The rate among MSM increased from 12.2% in 2007 to 35.5% in 2009. As a result, China is declared as one of the countries experiencing the rise of HIV epidemics in MSM [4].

In a report by the European surveillance network, Euro HIV, the number of new HIV cases in MSM rises from 2538 to 5016 during 1999-2006 across 13 Western European countries. This signifies almost 100% increase in new cases among MSM [5]. Central European countries experienced low prevalence of new HIV cases among MSM, but a sudden increase started in the year 1999 with 130 cases. This number increases by more than 100% in 2006, where 295 new cases were recorded. The countries in this region contributed at least 50% of the total number of HIV cases in MSM recorded across Europe. In Eastern Europe, not more than 1% of newly reported cases were in MSM, and no increase was discovered over time [6]. The annual HIV incidence rate in the Netherlands was 1.2% for a period of 6 years (1999-2005). However, a relative increase was noticed among MSM [7].

In 2008, the US Centers for Disease Control and Prevention (CDC) reported that, around 1.1 million people were infected with HIV/AIDS in 2006 in the United States. Almost half (48.1 percent) of this reported figure were MSM. In the same year, CDC also stated that the number of new cases of HIV/AIDS in MSM increased to 8.6% during 2001-2006. In black MSM, an increase in the number of newly HIV/AIDS infected individuals was reported from 2001-2006. Most of the victims were young adults aged 13-24 years; which recorded a 93.1% increase. Despite the fact that the blacks forms just 13% of the US population, the number of HIV/AIDS cases diagnosed in black MSM aged 13-24 years (7658) was at least twice the number diagnosed in whites (3221) [8].

According to a report in 2006, 56,300 new adult HIV infections were recorded, of which 53% were in MSM. Among the new HIV infections in men, 72% were in MSM, and of new infections in MSM, 46% were in whites, 35% were in blacks and 19% were in Hispanics. Therefore, the estimated HIV incidence among black MSM was 5.9 times bigger than among white MSM [9].

Mathematical models help to study the dynamics, spread and control of infectious diseases. It highlights and explains the possible outcome of an epidemic and aids in suggesting the various control measures. Kermarck and McKendrick in 1927 formulated an SIR model which studied the dynamics of infectious diseases [10]. The most important parameter that determines the outcome of an epidemic is the basic reproduction ratio (R_0). It is the number of secondary infection caused by a single infective individual in a completely susceptible population. If the basic reproduction ratio is less than one, then there is not going to be an epidemic, which means that the disease will die out. Otherwise, an epidemic will occur [11] and [12]. Many mathematical models in literature have studied the dynamics of HIV/AIDS [13].

Our aim in this research is to study the dynamics of HIV/AIDS among heterosexuals, MSM/bisexuals, and others in Turkey. We use data obtained from Kocaeli University, PCR Unit, Turkey to analyze and predict the trend of HIV/AIDS among these groups. This is because Kocaeli University has the largest HIV data collection center in Turkey.

Model formulation

The model is given by the system of differential equations as follows, where the meaning of parameters/variables is given in Table 1.

$$\begin{cases} \frac{dS}{dt} = \Lambda - \beta_1 SH_1 - \beta_2 SH_2 - \beta_3 SH_3 - \mu S; \\ \frac{dH_1}{dt} = \beta_1 SH_1 - (\mu + v)S; \\ \frac{dH_2}{dt} = \beta_2 SH_2 - (\mu + v)S; \\ \frac{dH_3}{dt} = \beta_3 SH_3 - (\mu + v)S, \end{cases} \tag{1}$$

$S(0) > 0, H_1(0) \geq 0, H_2(0) \geq 0, H_3(0) \geq 0.$

Table 1

Meaning of parameters and variables of model (1)

Parameters/Variables	**Meaning**
S	Population of susceptible
H_1	HIV positive of heterosexual population
H_2	HIV positive of MSM population
H_3	HIV positive of other population
Λ	Recruitment rate
$\frac{1}{v}$	Duration spent in the HIV stage
$\frac{1}{\mu}$	Life expectancy
β_1	Incidence rate between heterosexuals
β_2	Incidence rate between MSM
β_3	Incidence rate between others

In our model, S(0) is considered to be the whole population in a specific year, H_2 consists of both MSM and bisexuals HIV positive population, and β_2 is the transmission rate of HIV through MSM or bisexuals. Moreover, we refer to HIV positive of other population as those that acquire the disease through contaminated blood transfusion, contact with infected sharp objects and so on.

Existence of Equilibria

Equating the equations in (1) to zero and solving simultaneously we find four equilibrium points. They are as follows. Disease free equilibrium point $E_0 = (\frac{\Lambda}{\mu}, 0, 0, 0)$, which always exists and endemic equilibria.

$E_1 = \left(\frac{\mu+v}{\beta_1}, \frac{\Lambda\beta_1-\mu(\mu+v)}{\beta_1(\mu+v)}, 0, 0\right), E_2 = \left(\frac{\mu+v}{\beta_2}, 0, \frac{\Lambda\beta_2-\mu(\mu+v)}{\beta_2(\mu+v)}, 0\right),\ and\ E_3 = \left(\frac{\mu+v}{\beta_3}, 0, 0, \frac{\Lambda\beta_3-\mu(\mu+v)}{\beta_3(\mu+v)}\right).$
The endemic equilibria are biologically meaningful (exists) when
$\frac{\beta_1\Lambda}{\mu(\mu+v)} > 1, \frac{\beta_2\Lambda}{\mu(\mu+v)} > 1,\ and\ \frac{\beta_3\Lambda}{\mu(\mu+v)} > 1$, respectively.

Basic Reproduction Ratio

Basic reproduction ratio is the number of secondary infections caused by a single infective individual in a completely susceptible population. It is denoted by R_0. We use the next generation of matrix (NGM) method to compute the basic reproduction ratio and it is given by

$$R_0 = max[R_1, R_2, R_3] \tag{2}$$

where $R_1 = \frac{\beta_1\Lambda}{\mu(\mu+v)},\ R_2 = \frac{\beta_2\Lambda}{\mu(\mu+v)},\ and\ R_3 = \frac{\beta_3\Lambda}{\mu(\mu+v)}$.

Global Stability Analysis of the Equilibria

Theorem 1. E_0 is globally asymptotically stable when $R_0 < 1$.

Proof. We consider the following Lyapunov function;
$V = (S - S_0 lnS) + H_1 + H_2 + H_3 + C$, where $C = S_0 lnS_0 - S_0$. Hence,

$$\frac{dV}{dt} = (1-\frac{S_0}{S})\frac{dS}{dt} + \frac{dH_1}{dt} + \frac{dH_2}{dt} + \frac{dH_3}{dt} =$$
$$= (1-\frac{S_0}{S})(\Lambda-\beta_1SH_1-\beta_2SH_2-\beta_3SH_3-\mu S)+(\beta_1SH_1-(\mu+v)H_1)+(\beta_2SH_2-(\mu+v)H_2)+(\beta_3SH_3-(\mu+v)H_3) =$$
$$= \mu S_0\left(2-\frac{S_0}{S}-\frac{S}{S_0}\right) + (\beta_1S_0-(\mu+v))H_1 + (\beta_2S_0-(\mu+v))H_2 + (\beta_3S_0-(\mu+v))H_3.$$

Therefore $\frac{dV}{dt} < 0$ if $\beta_1S_0-(\mu+v) < 0$, $\beta_2S_0-(\mu+v) < 0$ and $\beta_3S_0-(\mu+v) < 0$ which means $R_0 < 1$.

Theorem 2. The endemic equilibrium point E_1 is globally asymptotically stable when $R_1 > 1, R_2 < 1$and $R_3 < 1$

Proof. The proof is similar to Theorem 1 by considering the following Lyapunov function;

$$V = (S - S^*lnS) + (H_1 - H_1^*lnH_1) + H_2 + H_3 + C, \text{ where } C = S^*lnS^* - S^* - H_1^* + H_1^*lnH_1^*.$$

Theorem 3. The endemic equilibrium point E_2 is globally asymptotically stable when $R_1 > 1, R_2 > 1$ and $R_3 < 1$.

Proof. The proof is similar to Theorem 1 by considering the following Lyapunov function;

$$V = (S - S^*lnS) + H_1 + H_2 - H_2^*lnH_2 + H_3 + C, \text{ where } C = S^*lnS - S^* - H_2^* + H_2^*lnH_2^*.$$

Theorem 4. The endemic equilibrium point E_3 is globally asymptotically stable when $R_1 < 1, R_2 < 1$ and $R_3 > 1$.

Proof. The result can be achieved using the following Lyapunov function;

$$V = (S - S^*lnS) + H_1 + H_2 + (H_3 - H_3^*lnH_3 + C, \text{ where } C = S^*lnS^* - S^* - H_3^* + H_3^*lnH_3^*.$$

Numerical Simulations

We use the raw data obtained from Kocaeli University, PCR Unit, Turkey from January 2016 to March 2017. This unit is a collection center for HIV in Turkey. Table 2, 3, Pictures 1 and 2 show the outcomes of the analysis of the model.

The dynamics of the heterosexuals, MSM/bisexuals and others is given by Picture 1.

Table 2

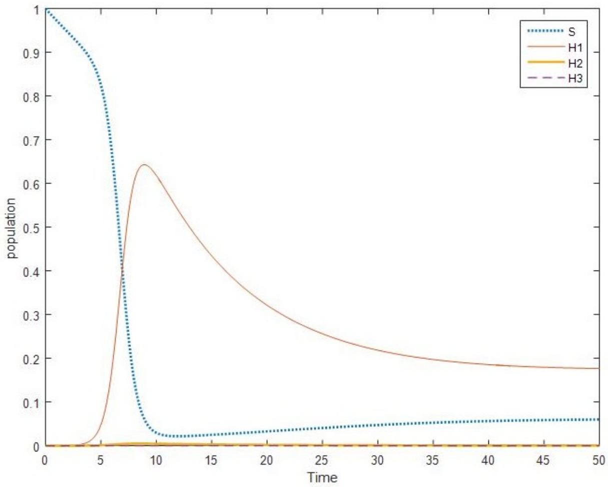
Analysis of the model with incidence rates

H_1	H_2	H_2
β_1	β_2	β_3
1.863	1.155	0.086

Table 3

Analysis of the model with Basic reproduction ratios

H_1	H_2	H_2
R_1	R_2	R_3
1.08	0.6719	0.050



Picture 1. Parameter values are $\beta_1 = 1.863$, $\beta_2 = 1.155$, $\beta_3 = 0.086$, $\mu = 0.055$, $\Lambda = 0.023$ and $v = 0.063$

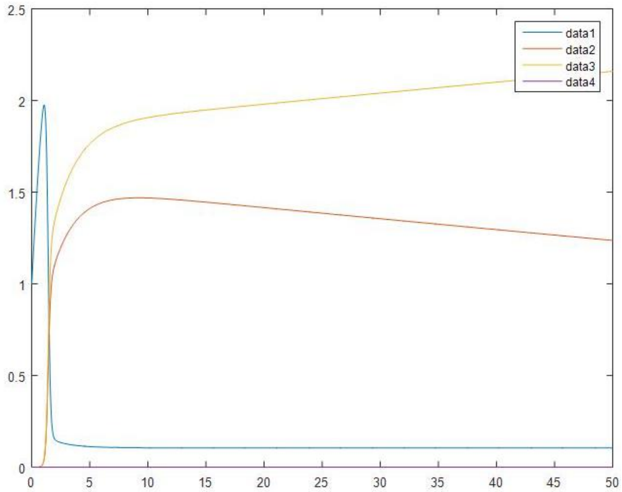
Discussion and Conclusion

We constructed a mathematical model that studies the transmission dynamics of HIV/AIDS in heterosexuals, MSM/bisexuals and others. The basic reproduction ratio was computed using the next generation matrix method. Four equilibrium points were obtained which include disease free and endemic equilibrium points. The global stability analysis of each of the equilibria was conducted using the Lyapunov function, and the stability of the equilibrium points depends on the basic reproduction ratio R_0. If the basic reproduction ratio is less than one, there will be no epidemic, which means the disease will die out, otherwise, an epidemic will occur.

From table 3, the basic reproduction ratios of heterosexuals, MSM/bisexuals, and others are 1.08, 0.6719, and 0.050, respectively. Based on the table and Picture 1, the epidemics will occur in the heterosexuals, while no epidemics will occur in the MSM/bisexuals and others. Although no epidemics will occur in MSM, but the basic reproduction ratio is close to 1, so if care is not taken in MSM/bisexuals epidemic may occur as time goes on. The threat posed by HIV/AIDS through heterosexuals is greater, than followed by MSM, and than the others. However, this is subject to the initial values which can change the nature of the graph, because, normally the data is usually collected at the early stages of the infection (trauma stage), so patients may hide their sexual status or refuse to give accurate information. We therefore recommend the use of IVD drug data in any community with similar settings as that of Turkey like the Asian countries, African countries, and Islamic countries for data collection.

The relevant authorities should put priorities in containing the disease in order of their threat. It is also evident that the incidence rate plays a vital role in posing this threat as can be seen from Table 2, therefore to contain the disease it is advisable to employ the appropriate measures in decreasing the incidence rates. Finally, Picture 1 shows the prediction of these epidemics for the three cases in 50 years.

Istanbul is the most populous city and center for tourism in Turkey. It is one of the largest cities in Europe and considered to be the home of immigrants. Hence, the population in Istanbul has more freedom, is well mixed, versatile, and tolerant. These are among the major factors that give rise to the increase in MSM population in Istanbul because people are not afraid to say their sexual status. Therefore, tourism, immigration, and increase in MSM population make Istanbul the center for the spread of HIV infection in Turkey.



Picture 2. Parameter values are $\beta_1 = 4.44$, $\beta_2 = 4.51$, $\beta_3 = 0.19$, $\mu = 0.41$, $\Lambda = 1.65$ and $v = 0.063$

The aforementioned reasons and facts served as our motivation to consider the data collected for Istanbul only and put it in our model in order to analyze the dynamics of the transmission route of HIV. Picture 2 shows this dynamics and predicts what will possibly happen in the next 50 years. It can be observed that the transmission of HIV in MSM is increasing while the transmission via heterosexuals and others is decreasing.

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Әрқилы жыныстық қатынастар арқылы ВИЧ-1 инфекциясын жұқтырған халықтың динамикасы: Түркияда жүргізілген зерттеу

Мақала Түркиядағы гетеросексуалдар, ер адамдар мен ер адамдар арасындағы сексуалдық қатынас (ЕСЕ)/бисексуалдар және басқа да адамдардың арасында ВИЧ/СПИД инфекциясының таралу динамикасын зерттеуге арналған. Аурулардан таза және эндемиялық тепе-теңдік нүктелерін қамтитын төрт тепе-теңдік нүктесі табылған. Негізгі көбею санынан R_0 тәуелді Ляпунов функциясын пайдалану арқылы тепе-теңдік орнықтылығына ауқымды талдау жасалды. Егер $R_0 < 1$ болса, ауруы жоқ тепе-теңдік нүктесі асимптотикалық тұрғыдан ауқымды орнықты болады да, аурулар жойылады. Ал $R_0 > 1$ болса, эндемиялық тепе-теңдік нүктесі тұрақты кейіпке ие болады да, эпидемиялар болып тұрады. Гетеросексуалдар, ЕСЕ/бисексуалдар және басқа да адамдардардың арасында ВИЧ/СПИД инфекциясының таралу тенденциясына талдау және болжам жасау үшін біз Коджаэли университетінен, Түркиядағы орталығынан алынған өңделмеген мәліметтерді пайдаланылды. Гетеросексуалдар, ЕСЕ/бисексуалдар және басқалардың көбеюінің базалық саны тиісінше 1.086, 0.6719, 0.050 болатыны анықталды. Бұл мәліметтерден, басқа топтарға қарағанда, ВИЧ/СПИД қатері гетеросексуалдар арасында жоғары екені байқалды. Сондықтан тиісті мекемелер ауру қоздырғыштарын тежеу мәселесінде басымдылықтар белгілеуі тиіс.

Кілт сөздер: ВИЧ/СПИД, көбеюдің базалық саны, тепе-теңдік нүктесі, тұрақтылық, Ляпунов функциясы, ЕСЕ, гетеросексуалды.

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Динамика населения, инфицированного ВИЧ-1, приобретенного через разные половые контакты: исследование в Турции

Статья направлена на изучение динамики передачи ВИЧ/СПИДа среди гетеросексуалов, мужчин, имеющих секс с мужчинами (МСМ)/бисексуалов и других лиц в Турции. Получены четыре точки равновесия, которые включают свободные от болезней и эндемичные точки равновесия. Глобальный анализ устойчивости равновесий проводился с использованием функции Ляпунова, которая зависит от основного числа воспроизведения R_0. Если $R_0 < 1$, точка равновесия без болезней является глобально асимптотически устойчивой, болезнь вымирает, а если $R_0 > 1$, точка эндемического равновесия стабильна — будут происходить эпидемии. Авторами использованы необработанные данные, полученные из Университета Коджаэли, ПЦР, в Турции для анализа и прогнозирования тенденции к ВИЧ/СПИДу среди гетеросексуалов, МСМ/бисексуалов и других. Было установлено, что базовое число воспроизведения гетеросексуалов, МСМ/бисексуалов и других было 1.08, 0.6719 и 0.050 соответственно. Это свидетельствует о том, что угроза, которую представляет ВИЧ/СПИД среди гетеросексуалов, выше, чем МСМ/бисексуалы. Таким образом, соответствующие органы должны поставить приоритеты в сдерживании болезни в порядке их угрозы.

Ключевые слова: ВИЧ/СПИД, базовый номер воспроизводства, точка равновесия, стабильность, функция Ляпунова, МСМ, гетеросексуальный.

MSC 35L20, 35R30, 65N21

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Identification hyperbolic problems with the Neumann boundary condition

In the present study, an identification problem with the Neumann boundary condition for a one-dimensional hyperbolic equation is investigated. Stability estimates for the solution of the identification problem are established. Furthermore, a first order of accuracy difference scheme for the numerical solution of the identification problems for hyperbolic equations with the Neumann boundary condition is presented. Stability estimates for the solution of the difference scheme are established. This difference scheme is tested on an example and some numerical results are presented.

Keywords: source identification problem, hyperbolic differential equations, difference schemes.

Introduction

Identification problems take an important place in applied sciences and engineering, and have been studied by many authors (see, e.g., [1–4] and the references given therein). The theory and applications of source identification problems for partial differential equations have been given in various papers (see, e.g., [3, 5–8] and the references given therein).

The well-posedness of the unknown source identification problem for parabolic and delay parabolic equations have been well-investigated (see, e.g., [9–14], and the references given therein).

The solvability of the inverse problems in various formulations with various overdetermination conditions for telegraph and hyperbolic equations were studied in many works (see, e.g., [15–18] and the references given therein). Some new representations were given for the solutions and coefficients of the equations of mathematical physics in [5, 19–23]. They gave such formulas for evolution equations of first and second-order in time, in particular for parabolic and hyperbolic equations in the linear and nonlinear cases.

In this study, we consider the time-dependent source identification problem for a one-dimensional hyperbolic equation with the Neumann boundary condition

$$\begin{cases} \frac{\partial^2 u(t,x)}{\partial t^2} - \frac{\partial}{\partial x}\left(a\left(x\right)\frac{\partial u(t,x)}{\partial x}\right) = p\left(t\right)q\left(x\right) + f\left(t,x\right), x \in \left(0,l\right), t \in \left(0,T\right), \\ u\left(0,x\right) = \varphi\left(x\right), u_t\left(0,x\right) = \psi\left(x\right), x \in \left[0,l\right], \\ u_x\left(t,0\right) = u_x\left(t,l\right) = 0, \int_0^l u\left(t,x\right)dx = \zeta\left(t\right), t \in \left[0,T\right], \end{cases} \tag{1}$$

where $u\left(t,x\right)$ and $p\left(t\right)$ are unknown functions, $a\left(x\right) \geq a > 0$, $f\left(t,x\right), \zeta\left(t\right), \varphi\left(x\right)$ and $\psi\left(x\right)$ are sufficiently smooth functions and $q\left(x\right)$ is a sufficiently smooth function assuming $q'\left(0\right) = q'\left(l\right) = 0$ and $\int_0^l q\left(x\right)dx \neq 0$.

Our interest in this study is to investigate the stability of differential and difference identification problems. The stability estimate for the solution of problem (1) is established. A first order of accuracy difference scheme for the numerical solution of identification hyperbolic problems with the Neumann boundary condition (1) is presented. The theoretical statements for solution of this difference scheme are supported by result of the numerical experiments.

Stability of differential problem (1)

To formulate our results, we introduce the Banach space $C(H) = C([0,T], H)$ of all abstract continuous functions $\phi(t)$ defined on $[0,T]$ with values in H equipped with the norm

$$\|\phi\|_{C(H)} = \max_{0 \leqslant t \leqslant T} \|\phi(t)\|_H .$$

Let $L_2[0,l]$ be a space of all square integrable functions $\gamma(x)$ defined on $[0,l]$ equipped with the norm

$$\|\gamma\|_{L_2[0,l]} = \left(\int_0^l |\gamma(x)|^2 \, dx \right)^{\frac{1}{2}},$$

and let $W_2^1[0,l], W_2^2[0,l]$ be Sobolev spaces with norms

$$\|\gamma\|_{W_2^1[0,l]} = \left(\int_0^l \left[\gamma^2(x) + \gamma_x^2(x) \right] dx \right)^{\frac{1}{2}};$$

$$\|\gamma\|_{W_2^2[0,l]} = \left(\int_0^l \left[\gamma^2(x) + \gamma_{xx}^2(x) \right] dx \right)^{\frac{1}{2}},$$

respectively. We introduce the positive definite self-adjoint operator A defined by the formula

$$Au = -\frac{d}{dx}\left(a(x) \frac{du(x)}{dx} \right) \tag{2}$$

with the domain

$$D(A) = \{ u : u, u'' \in L_2[0,l], u'(0) = u'(l) = 0 \}.$$

Throughout the present paper, M denotes positive constants, which may differ in time and thus is not a subject of precision. However, we will use the notation $M(\alpha, \beta, \gamma, ...)$ to stress the fact that the constant depends only on $\alpha, \beta, \gamma, ...$

We have the following theorem on the stability of problem (1).

Theorem 1. Assume that $\varphi \in W_2^2[0,l], \psi \in W_2^1[0,l]$ and $f(t,x)$ is a continuously differentiable function in t and square integrable in x, $\zeta(t)$ is a twice continuously differentiable function. Suppose that $q(x)$ is a sufficiently smooth function assuming $q'(0) = q'(l) = 0$ and $\int_0^l q(x)\,dx \neq 0$. Then for the solution of problem (1) the following stability estimates hold

$$\left\| \frac{\partial^2 u}{\partial t^2} \right\|_{C(L_2[0,l])} + \|u\|_{C\left(W_2^2[0,l]\right)} \leqslant M_1(q) \left[\|\varphi\|_{W_2^2[0,l]} + \|\psi\|_{W_2^1[0,l]} + \right. \tag{3}$$

$$\left. + \|f(0,\cdot)\|_{L_2[0,l]} + \left\| \frac{\partial f}{\partial t} \right\|_{C(L_2[0,l])} + \|\zeta''\|_{C[0,T]} \right];$$

$$\|p\|_{C[0,T]} \leqslant M_2(q) \left[\|\varphi\|_{W_2^2[0,l]} + \|\psi\|_{W_2^1[0,l]} + \|\zeta''\|_{C[0,T]} + \right. \tag{4}$$

$$\left. + \|f(0,\cdot)\|_{L_2[0,l]} + \left\| \frac{\partial f}{\partial t} \right\|_{C(L_2[0,l])} \right].$$

Proof. We will use the following substitution

$$u(t,x) = w(t,x) + \eta(t) q(x), \tag{5}$$

where $\eta(t)$ is the function defined by the formula

$$\eta(t) = \int_0^t (t-s) p(s)\, ds, \eta(0) = \eta'(0) = 0. \tag{6}$$

It is easy to see that $w\left(t,x\right)$ is the solution of problem

$$\begin{cases} \frac{\partial^2 w(t,x)}{\partial t^2}-\frac{\partial}{\partial x}\left(a\left(x\right)\frac{\partial w(t,x)}{\partial x}\right)=f\left(t,x\right)+\eta\left(t\right)\left[\frac{d}{dx}\left(a\left(x\right)q'\left(x\right)\right)\right], \\ x\in\left(0,l\right), t\in\left(0,T\right), \\ w\left(0,x\right)=\varphi\left(x\right), w_t\left(0,x\right)=\psi\left(x\right), x\in\left[0,l\right], \\ u_x\left(t,0\right)=u_x\left(t,l\right)=0,, t\in\left[0,T\right]. \end{cases} \tag{7}$$

Applying the integral overdetermined condition $\int_0^l u\left(t,x\right)dx=\zeta\left(t\right)$ and substitution (5), we get

$$\eta\left(t\right)=\frac{\zeta\left(t\right)-\int_0^l w\left(t,x\right)dx}{\int_0^l q\left(x\right)dx}.$$

From that and $p\left(t\right)=\eta''\left(t\right)$ it follows

$$p\left(t\right)=\frac{\zeta''\left(t\right)-\int_0^l \frac{\partial^2}{\partial t^2}w\left(t,x\right)dx}{\int_0^l q\left(x\right)dx}.$$

Applying $\int_0^l q\left(x\right)dx\neq 0$, we get the estimate

$$\left|p\left(t\right)\right|\leq M_3\left(q\right)\left[\left|\zeta''\left(t\right)\right|+\left\|\frac{\partial^2 w\left(t,.\right)}{\partial t^2}\right\|_{L_2[0,l]}\right] \tag{8}$$

for all $t\in\left[0,T\right]$. From that it follows

$$\left\|p\right\|_{C[0,T]}\leqslant M_3\left(q\right)\left[\left\|\zeta''\right\|_{C[0,T]}+\left\|\frac{\partial^2 w}{\partial t^2}\right\|_{C(L_2[0,l])}\right]. \tag{9}$$

Now, using substitution (5), we get

$$\frac{\partial^2 u\left(t,x\right)}{\partial t^2}=\frac{\partial^2 w\left(t,x\right)}{\partial t^2}+p\left(t\right)q\left(x\right).$$

Applying the triangle inequality, we obtain

$$\left\|\frac{\partial^2 u}{\partial t^2}\right\|_{C(L_2[0,l])}\leqslant\left\|\frac{\partial^2 w}{\partial t^2}\right\|_{C(L_2[0,l])}+\left\|p\right\|_{C[0,T]}\left\|q\right\|_{L_2[0,l]}. \tag{10}$$

Therefore, the proof of estimates (3) and (4) is based on equation (1), the triangle inequality, estimates (9), (10) and on the following stability estimate

$$\left\|\frac{\partial^2 w}{\partial t^2}\right\|_{C(L_2[0,l])}\leqslant M_4(q,a)\left[\left\|\varphi\right\|_{W_2^2[0,l]}+\left\|\psi\right\|_{W_2^1[0,l]}+\right. \tag{11}$$

$$\left.+\left\|f\left(0,\cdot\right)\right\|_{L_2[0,l]}+\left\|\frac{\partial f}{\partial t}\right\|_{C(L_2[0,l])}+\left\|\zeta''\right\|_{C[0,T]}\right]$$

for the solution of problem (7). It was proved in [16] for the identification hyperbolic problem with the Dirichlet boundary condition. The proof of (11) is carried out according to the same approach. This completes the proof of Theorem 1.

Stability of the difference scheme

To formulate our results on difference problem, we introduce the Banach space $C_\tau(H)=C([0,T]_\tau,H)$ of all abstract grid functions $\phi^\tau=\{\phi(t_k)\}_{k=0}^N$ defined on

$$[0,T]_\tau=\{t_k=k\tau,0\leqslant k\leqslant N,N\tau=T\},$$

with values in H equipped with the norm

$$\|\phi^\tau\|_{C_\tau(H)}=\max_{0\leqslant k\leqslant N}\|\phi(t_k)\|_H .$$

Moreover, $L_{2h}=L_2[0,l]_h$ is the Hilbert space of all grid functions $\gamma^h(x)=\{\gamma_n\}_{n=0}^M$ defined on

$$[0,l]_h=\{x_n=nh,0\leqslant n\leqslant M,Mh=l\},$$

equipped with the norm

$$\left\|\gamma^h\right\|_{L_{2h}}=\left\{\sum_{i=0}^M|\gamma_i|^2h\right\}^{\frac12},$$

and $W_{2h}^1=W_2^1[0,l]_h, W_{2h}^2=W_2^2[0,l]_h$ are the discrete analogues of Sobolev spaces of all grid functions $\gamma^h(x)=\{\gamma_n\}_{n=0}^M$ defined on $[0,l]_h$ with norms

$$\left\|\gamma^h\right\|_{W_{2h}^1}=\left\{\sum_{i=0}^M|\gamma_i|^2h+\sum_{i=1}^M\left|\frac{\gamma_i-\gamma_{i-1}}{h}\right|^2h\right\}^{\frac12};$$

$$\left\|\gamma^h\right\|_{W_{2h}^2}=\left\{\sum_{i=0}^M|\gamma_i|^2h+\sum_{i=1}^{M-1}\left|\frac{\gamma_{i+1}-2\gamma_i+\gamma_{i-1}}{h^2}\right|^2h\right\}^{\frac12},$$

respectively. For the differential operator A defined by (2), we introduce the self-adjoint positive definite difference operator A_h defined by the formula

$$A_h\varphi^h(x)=\left\{-\frac1h\left(a(x_{n+1})\frac{\varphi_{n+1}-\varphi_n}{h}-a(x_n)\frac{\varphi_n-\varphi_{n-1}}{h}\right)\right\}_{n=1}^{M-1}, \tag{12}$$

acting in the space of grid functions $\varphi^h(x)=\{\varphi_n\}_{n=0}^M$ defined on $[0,l]_h$ satisfying the conditions $\varphi_1-\varphi_0=\varphi_M-\varphi_{M-1}=0$.

For the numerical solution $\left\{\{u_n^k\}_{k=0}^N\right\}_{n=0}^M$ of problem (1), we consider the first order of accuracy difference scheme

$$\begin{cases}\frac{u_n^{k+1}-2u_n^k+u_n^{k-1}}{\tau^2}-\left(a(x_{n+1})\frac{u_{n+1}^{k+1}-u_n^{k+1}}{h^2}-a(x_n)\frac{u_n^{k+1}-u_{n-1}^{k+1}}{h^2}\right)=p_kq(x_n)+f(t_k,x_n);\\ t_k=k\tau,x_n=nh,1\leqslant k\leqslant N-1,1\leqslant n\leqslant M-1,N\tau=T;\\ u_n^0=\varphi(x_n),\frac{u_n^1-u_n^0}{\tau}=\psi(x_n),0\leqslant n\leqslant M,Mh=l;\\ u_1^{k+1}-u_0^{k+1}=u_M^{k+1}-u_{M-1}^{k+1}=0;\\ \sum_{i=1}^{M-1}u_i^{k+1}h=\zeta(t_{k+1}),-1\leqslant k\leqslant N-1.\end{cases} \tag{13}$$

Here, it is assumed that $q_1-q_0=q_M-q_{M-1}=0$ and $\sum_{i=1}^{M-1}q_i\neq0$. We have the following theorem on the stability of difference scheme (13).

Theorem 2. For the solution of difference scheme (13), the following stability estimates hold

$$\left\|\left\{\frac{u_{k+1}^h-2u_k^h+u_{k-1}^h}{\tau^2}\right\}_{k=1}^{N-1}\right\|_{C_\tau(L_{2h})}+\left\|\left\{u_{k+1}^h\right\}_{k=1}^{N-1}\right\|_{C_\tau\left(W_{2h}^2\right)}\leqslant \tag{14}$$

$$\leqslant M_5(q)\left[\left\|\varphi^h\right\|_{W_{2h}^2}+\left\|\psi^h\right\|_{W_{2h}^1}+\left\|f_1^h\right\|_{L_{2h}}+\left\|\left\{\frac{f_k^h-f_{k-1}^h}{\tau}\right\}_{k=2}^{N-1}\right\|_{C_\tau(L_{2h})}+\right.$$

$$\left.+\left\|\left\{\frac{\zeta_{k+1}-2\zeta_k+\zeta_{k-1}}{\tau^2}\right\}_{k=1}^{N-1}\right\|_{C[0,T]_\tau}\right],$$

$$\left\|\{p_k\}_{k=1}^{N-1}\right\|_{C[0,T]_\tau}\leqslant M_6(q)\left[\left\|\varphi^h\right\|_{W_{2h}^2}+\left\|\psi^h\right\|_{W_{2h}^2}+\left\|f_1^h\right\|_{L_{2h}}+\right. \tag{15}$$

$$\left.+\left\|\left\{\frac{f_k^h-f_{k-1}^h}{\tau}\right\}_{k=2}^{N-1}\right\|_{C_\tau(L_{2h})}+\left\|\left\{\frac{\zeta_{k+1}-2\zeta_k+\zeta_{k-1}}{\tau^2}\right\}_{k=1}^{N-1}\right\|_{C[0,T]_\tau}\right].$$

Here and throughout this subsection $f_k^h(x)=\{f(t_k,x_n)\}_{n=0}^M, 1\le k\le N-1$.

Proof. We will use the following substitution

$$u_n^k=w_n^k+\eta_k q_n, \tag{16}$$

where

$$q_n=q(x_n)$$

and

$$\eta_{k+1}=\sum_{i=1}^{k}(k+1-i)\,p_i\tau^2, 1\leqslant k\leqslant N-1, \eta_0=\eta_1=0. \tag{17}$$

It is easy to see that $\left\{\left\{w_n^k\right\}_{k=0}^N\right\}_{n=0}^M$ is the solution of difference problem

$$\begin{cases} \frac{w_n^{k+1}-2w_n^k+w_n^{k-1}}{\tau^2}-\frac{1}{h}\left(a(x_{n+1})\frac{w_{n+1}^{k+1}-w_n^{k+1}}{h}-a(x_n)\frac{w_n^{k+1}-w_{n-1}^{k+1}}{h}\right)= \\ =f(t_k,x_n)+\frac{1}{h}\left[a(x_{n+1})\frac{q_{n+1}-q_n}{h}-a(x_n)\frac{q_n-q_{n-1}}{h}\right]\eta_{k+1}; \\ 1\leqslant k\leqslant N-1, 1\leqslant n\leqslant M-1; \\ w_n^0=\varphi(x_n), \frac{w_n^1-w_n^0}{\tau}=\psi(x_n), 0\leqslant n\leqslant M, \\ u_1^{k+1}-u_0^{k+1}=u_M^{k+1}-u_{M-1}^{k+1}=0, -1\leqslant k\leqslant N-1. \end{cases} \tag{18}$$

Applying the overdetermined condition $\sum_{i=1}^{M-1}u_i^{k+1}h=\zeta(t_{k+1})$ and substitution (16), one can obtain that

$$\eta_{k+1}=\frac{\zeta_{k+1}-\sum_{i=1}^{M-1}w_i^{k+1}h}{\sum_{i=1}^{M-1}q_ih}. \tag{19}$$

Then, using formulas $p_k=\frac{\eta_{k+1}-2\eta_k+\eta_{k-1}}{\tau^2}$ and (19), we get

$$p_k=\frac{\zeta_{k+1}-2\zeta_k+\zeta_{k-1}-\sum_{i=1}^{M-1}\left(w_i^{k+1}-2w_i^k+w_i^{k-1}\right)h}{\tau^2\sum_{i=1}^{M-1}q_ih},$$

$$|p_k| \leqslant M_7(q)\left[\left|\frac{\zeta_{k+1}-2\zeta_k+\zeta_{k-1}}{\tau^2}\right| + \left\|\frac{w_{k+1}^h-2w_k^h+w_{k-1}^h}{\tau^2}\right\|_{L_{2h}}\right] \tag{20}$$

for all $1 \leqslant k \leqslant N-1$. From that it follows

$$\left\|\{p_k\}_{k=1}^{N-1}\right\|_{C[0,T]_\tau} \leqslant M_7(q)\left[\left\|\left\{\frac{\zeta_{k+1}-2\zeta_k+\zeta_{k-1}}{\tau^2}\right\}_{k=1}^{N-1}\right\|_{C[0,T]_\tau} + \right. \tag{21}$$

$$\left. + \left\|\left\{\frac{w_{k+1}^h-2w_k^h+w_{k-1}^h}{\tau^2}\right\}_{k=1}^{N-1}\right\|_{C_\tau(L_{2h})}\right].$$

Now, using substitution (16), we get

$$\frac{u_n^{k+1}-2u_n^k+u_n^{k-1}}{\tau^2} = \frac{w_n^{k+1}-2w_n^k+w_n^{k-1}}{\tau^2} + p_k q(x_n).$$

Applying the triangle inequality, we obtain

$$\left\|\left\{\frac{u_{k+1}^h-2u_k^h+u_{k-1}^h}{\tau^2}\right\}_{k=1}^{N-1}\right\|_{C_\tau(L_{2h})} \leqslant \tag{22}$$

$$\leqslant \left\|\left\{\frac{w_{k+1}^h-2w_k^h+w_{k-1}^h}{\tau^2}\right\}_{k=1}^{N-1}\right\|_{C_\tau(L_{2h})} +$$

$$+ \left\|\{p_k\}_{k=1}^{N-1}\right\|_{C[0,T]_\tau} \left\|\{q(x_n)\}_{n=0}^{M}\right\|_{L_{2h}}.$$

Therefore, the proof of estimates (14) and (15) are based on equation (13), the triangle inequality, estimates (21), (22) and on the following stability estimate

$$\left\|\left\{\frac{w_{k+1}^h-2w_k^h+w_{k-1}^h}{\tau^2}\right\}_{k=1}^{N-1}\right\|_{C_\tau(L_{2h})} \leqslant M_8(q)\times \tag{23}$$

$$\times\left[\left\|\varphi^h\right\|_{W_{2h}^2} + \left\|\psi^h\right\|_{W_{2h}^1} + \left\|f_1^h\right\|_{L_{2h}} + \left\|\left\{\frac{f_k^h-f_{k-1}^h}{\tau}\right\}_{k=2}^{N-1}\right\|_{C_\tau(L_{2h})} + \right.$$

$$\left. + \left\|\left\{\frac{\zeta_{k+1}-2\zeta_k+\zeta_{k-1}}{\tau^2}\right\}_{k=1}^{N-1}\right\|_{C[0,T]_\tau}\right]$$

for the solution of difference problem (18). It was proved in ([9]) for the identification hyperbolic problem with the Dirichlet boundary condition. The proof of (23) is carried out according to the same approach. This completes the proof of Theorem 2.

Numerical Experiments

In this section, we study the numerical solution of the identification problem

$$\begin{cases} \frac{\partial^2 u(t,x)}{\partial t^2} - \frac{\partial^2 u(t,x)}{\partial x^2} = p(t)(1+\cos x) + e^{-t}\cos x, x \in (0,\pi), t \in (0,1); \\ u(0,x) = 1+\cos x, u_t(0,x) = -(1+\cos x), x \in [0,\pi]; \\ u_x(t,0) = u_x(t,\pi) = 0, t \in [0,1]; \\ \int_0^\pi u(t,x)\,dx = \pi e^{-t}, t \in [0,1] \end{cases} \tag{24}$$

for a hyperbolic differential equation. The exact solution pair of this problem is $(u(t,x), p(t)) = (e^{-t}(1+\cos x), e^{-t})$.

For the numerical solution of problem (24), we present the following first order of accuracy difference scheme for the approximate solution for the problem (24)

$$\begin{cases} \frac{u_n^{k+1}-2u_n^k+u_n^{k-1}}{\tau^2} - \frac{u_{n+1}^{k+1}-2u_n^{k+1}+u_{n-1}^{k+1}}{h^2} = p_k\left(1+\cos x_n\right) + e^{-t_{k+1}}\cos x_n; \\ t_k = k\tau, x_n = nh, 1 \leqslant k \leqslant N-1, 1 \leqslant n \leqslant M-1; \\ u_n^0 = 1+\cos x_n, \frac{u_n^1-u_n^0}{\tau} = -\left(1+\cos x_n\right), 0 \leqslant n \leqslant M, Mh=\pi, N\tau=1; \\ u_0^{k+1}-u_1^{k+1} = u_M^{k+1}-u_{M-1}^{k+1} = 0; \\ \sum_{i=1}^{M-1} u_i^{k+1}h = \pi e^{-t_{k+1}}, -1 \leqslant k \leqslant N-1. \end{cases} \tag{25}$$

Algorithm for obtaining the solution of identification problem (25) contains three stages. Actually, let us define

$$u_n^k = w_n^k + \eta_k\left(1+\cos x_n\right), 0 \le k \le N, 0 \le n \le M, \tag{26}$$

Applying difference scheme (25) and formula (26), we will obtain the formula

$$\eta_{k+1} = \frac{\pi e^{-t_{k+1}} - \sum_{i=1}^{M-1} w_i^{k+1}h}{\sum_{i=1}^{M-1}\left(1+\cos x_i\right)h}, -1 \le k \le N-1 \tag{27}$$

and the difference scheme

$$\begin{cases} \frac{w_n^{k+1}-2w_n^k+w_n^{k-1}}{\tau^2} - \frac{w_{n+1}^{k+1}-2w_n^{k+1}+w_{n-1}^{k+1}}{h^2} + \frac{2(\cos h-1)}{h^2}\cos x_n \frac{\sum_{i=1}^{M-1} w_i^{k+1}h}{\sum_{i=1}^{M-1}(1+\cos x_i)h} = \\ = \left[1+\frac{2\pi(\cos h-1)}{h^2\sum_{i=1}^{M-1}(1+\cos x_i)h}\right] e^{-t_{k+1}}\cos x_n,\ 1 \leqslant k \leqslant N-1, 1 \leqslant n \leqslant M-1; \\ w_n^0 = 1+\cos x_n, \frac{w_n^1-w_n^0}{\tau} = -\left(1+\cos x_n\right), , 0 \leqslant n \leqslant M; \\ w_0^{k+1}-w_1^{k+1} = w_M^{k+1}-w_{M-1}^{k+1} = 0, -1 \leqslant k \leqslant N-1. \end{cases} \tag{28}$$

In the first stage, we find numerical solution $\left\{\left\{w_n^k\right\}_{k=0}^N\right\}_{n=0}^M$ of corresponding first order of accuracy auxiliary difference scheme (28). For obtaining the solution of difference scheme (28), we will write it in the matrix form as

$$\begin{cases} Aw^{k+1} + Bw^k + Cw^{k-1} = \varphi^k, 1 \leqslant k \leqslant N-1; \\ w^0 = \left\{1+\cos x_n\right\}_{n=0}^M, w^1 = \left\{(1-\tau)\left(1+\cos x_n\right)\right\}_{n=0}^M, \end{cases} \tag{29}$$

where A, B, C are $(M+1)\times(M+1)$ square matrices, $w^s, s = k, k\pm 1, f^k$ are $(M+1)\times 1$ column matrices and

$$A = \begin{bmatrix} 1 & -1 & 0 & \cdot & 0 & 0 & -1 \\ b & a+c_1 & b+c_1 & \cdot & c_1 & c_1 & 0 \\ 0 & b+c_2 & a+c_2 & \cdot & c_2 & c_2 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & c_{M-2} & c_{M-2} & \cdot & a+c_{M-2} & b+c_{M-2} & 0 \\ 0 & c_{M-1} & c_{M-1} & \cdot & b+c_{M-1} & a+c_{M-1} & b \\ 0 & 0 & 0 & \cdot & 0 & -1 & 1 \end{bmatrix}_{(M+1)\times(M+1)},$$

$$B = \begin{bmatrix} 0 & 0 & 0 & \cdot & 0 & 0 & 0 \\ 0 & e & 0 & \cdot & 0 & 0 & 0 \\ 0 & 0 & e & \cdot & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & e & 0 & 0 \\ 0 & 0 & 0 & \cdot & 0 & e & 0 \\ 0 & 0 & 0 & \cdot & 0 & 0 & 0 \end{bmatrix}_{(M+1)\times(M+1)} \quad C = \begin{bmatrix} 0 & 0 & 0 & \cdot & 0 & 0 & 0 \\ 0 & g & 0 & \cdot & 0 & 0 & 0 \\ 0 & 0 & g & \cdot & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & g & 0 & 0 \\ 0 & 0 & 0 & \cdot & 0 & g & 0 \\ 0 & 0 & 0 & \cdot & 0 & 0 & 0 \end{bmatrix}_{(M+1)\times(M+1)},$$

$$\varphi^k = \begin{bmatrix} 0 \\ \varphi_1^k \\ \cdot \\ \varphi_{M-1}^k \\ 0 \end{bmatrix}_{(M+1)\times 1} \quad w^s = \begin{bmatrix} 0 \\ w_1^s \\ \cdot \\ w_{M-1}^s \\ 0 \end{bmatrix}_{(M+1)\times 1}, \text{ for } s = k, k \pm 1.$$

Here,

$$a = \frac{1}{\tau^2} + \frac{2}{h^2}, b = -\frac{1}{h^2}, e = -\frac{2}{\tau^2}, g = \frac{1}{\tau^2};$$

$$d = \sum_{i=1}^{M-1} (1 + \cos x_i)\, h, c_n = \frac{2(\cos h - 1)}{hd} \cos x_n, 1 \leqslant n \leqslant M - 1,$$

$$\varphi_n^k = \left[1 + \frac{2\pi(\cos h - 1)}{h^2 \sum_{i=1}^{M-1} (1 + \cos x_i h)}\right] e^{-t_{k+1}} \cos x_n, 1 \leqslant k \leqslant N - 1, 1 \leqslant n \leqslant M - 1.$$

So, we have the initial value problem for the second order difference equation (29) with respect to k with matrix coefficients A, B and C. Since $w^0$and $w^1$are given, we can obtain $\left\{ \{w_n^k\}_{k=0}^N \right\}_{n=0}^M$ by (29).

Now, applying formula (17), we can obtain

$$p_k = \frac{\eta_{k+1} - 2\eta_k + \eta_{k-1}}{\tau^2}, 1 \leq k \leq N - 1. \tag{30}$$

In the second stage, we obtain $\{p_k\}_{k=1}^{N-1}$ by formulas (27) and (30). Finally, in the third stage, we obtain $\left\{ \{u_n^k\}_{k=0}^N \right\}_{n=0}^M$ by formulas (26) and (27). The errors are computed by

$$E_u = \max_{0 \leqslant k \leqslant N} \left(\sum_{n=0}^{M} \left| u(t_k, x_n) - u_n^k \right|^2 h \right)^{\frac{1}{2}}; \tag{31}$$

$$E_p = \max_{1 \leqslant k \leqslant N-1} \left| p(t_k) - p_k \right|,$$

where $u(t,x), p(t)$ represent the exact solution, u_n^k represent the numerical solutions at (t_k, x_n) and p_k represent the numerical solutions at t_k. The numerical results are given in the following Table.

Table

Error analysis

Error	$N = M = 20$	$N = M = 40$	$N = M = 80$	N=M=160
E_u	0.0501	0.0250	0.0124	0.0062
E_p	0.0472	0.0244	0.0124	0.0063

As it is seen in Table, we get some numerical results. If N and M are doubled, the value of errors decrease by a factor of approximately 1/2 for first order difference scheme (25).

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А. Ашыралыев, Ф. Эммараб

Шеттік шарты Нейман түрінде болатын идентификациялық гиперболалық есептер

Мақала шеттік шарты Нейман түріндегі бір өлшемді гиперболалық түрдегі теңдеу үшін идентификациялау есебін зерттеуге арналған. Идентификациялау есебінің шешімі үшін орнықтылық бағалаулары алынған. Шеттік шарты Нейман түріндегі гиперболалық теңдеулер үшін идентификациялау есебін сандық шешу үшін дәлдігі бірінші ретті айырымдық схема ұсынылған. Айырымдық схеманың шешімі үшін орнықтылық бағалаулары келтірілген. Бұл айырымдық схема қарапайым есеп үшін тексеріліп, сандық есептеулер нәтижесі келтірілген.

Кілт сөздер: көздерді идентификациялау есебі, гиперболалық дифференциалды теңдеу, айырымдық схемалары.

А. Ашыралыев, Ф. Эммараб

Идентификационные гиперболические задачи с граничным условием Неймана

В статье изучаена задача идентификации с граничным условием Неймана для одномерного гиперболического уравнения. Установлены оценки устойчивости решения задачи идентификации. Кроме того, представлена разностная схема первого порядка точности для численного решения задач идентификации для гиперболических уравнений с граничным условием Неймана. Установлены оценки устойчивости решения разностной схемы. Эта разностная схема проверена на примере и представлены некоторые численные результаты.

Ключевые слова: задача идентификации источника, гиперболические дифференциальные уравнения, разностные схемы.

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Numerical solution of the nonlocal boundary value problem for elliptic equations

In the present paper a second order of accuracy two-step difference scheme for an approximate solution of the nonlocal boundary value problem for the elliptic differential equation $-v''(t) + Av(t) = f(t)$ $(0 \le t \le T)$, $v(0) = v(T) + \varphi$, $\int\limits_0^T v(s)ds = \psi$ in an arbitrary Banach space E with the strongly positive operator A is presented. The stability of this difference scheme is established. In application, the stability estimates for the solution of the difference scheme for the elliptic differential problem with the Neumann boundary condition are obtained. Additionally, the illustrative numerical result is provided.

Keywords: stability; positive operators; elliptic equation; numerical results, two-step difference scheme.

Introduction

The well-posedness in various Banach spaces of the local boundary value problem for the elliptic equation

$$-v''(t) + Av(t) = f(t)\ (0 \le t \le T), v(0) = v_0, v(T) = v_T \tag{1}$$

in an arbitrary Banach space E with the positive operator A and its related applications have been investigated by many researchers (see, for example, [1–3] and the references given therein).

In mathematical modeling, elliptic equations are used together with local boundary conditions specifying the solution on the boundary of the domain. In some cases, classical boundary conditions cannot describe process or phenomenon precisely. Therefore, mathematical models of various physical, chemical, biological or environmental processes often involve nonclassical conditions. The well-posedness of various nonlocal boundary value problems for partial differential and difference equations has been studied extensively by many researchers (see, e.g., [4–21] and the references given therein).

In the present paper the abstract nonlocal boundary value problem for differential equation of elliptic type

$$-v''(t) + Av(t) = f(t) \quad (0 \le t \le T), v(0) = v(T) + \varphi, \int\limits_0^T v(s)ds = \psi \tag{2}$$

in the arbitrary Banach space E with the positive operator A is considered. The second order of approximation two-step difference scheme

$$\begin{cases} -\frac{u_{k+1} - 2u_k + u_{k-1}}{\tau^2} + Au_k = f_k, f_k = f(t_k), t_k = k\tau, 1 \le k \le N-1, N\tau = T; \\ u_0 = u_N + \varphi, \sum\limits_{i=1}^{N} u_i \tau = \psi \end{cases} \tag{3}$$

for the approximate solution of problem (2) is presented. The stability of this difference scheme is established. In application, the stability estimates for the solution of the difference scheme for the elliptic differential problem with the Neumann boundary condition are obtained. Additionally, the illustrative numerical result is provided.

Auxiliary results

In this section, we give some auxiliary statements from [1] which will be useful in the sequel. We consider the second order of accuracy difference scheme

$$-\frac{u_{k+1}-2u_k+u_{k-1}}{\tau^2}+Au_k=f_k, f_k=f(t_k), t_k=k\tau, 1\leq k\leq N-1, N\tau=T, \quad (4)$$

$$u_0=v_0, u_N=v_T.$$

of the approximate solution of the boundary value problem (1). This problem is uniquely solvable, and the following formula holds

$$u_k=(I-R^{2N})^{-1}\{(R^k-R^{2N-k})u_0+ \quad (5)$$
$$+(R^{N-k}-R^{N+k})u_N-(R^{N-k}-R^{N+k})(I+\tau B)\times$$
$$\times(2I+\tau B)^{-1}B^{-1}\sum_{i=1}^{N-1}(R^{N-i}-R^{N+i})f_i\tau+$$
$$+(I+\tau B)(2I+\tau B)^{-1}B^{-1}\sum_{i=1}^{N-1}(R^{|k-i|}-R^{k+i})f_i\tau, 1\leq k\leq N-1,$$

where

$$B=B(\tau,A)=\frac{\tau A}{2}+\sqrt{\left(\frac{\tau A}{2}\right)^2+A}, R=R(\tau B)=(I+\tau B)^{-1}.$$

Note that $B(\tau,A)\neq A^{\frac{1}{2}}$ but then $B(\tau,A)\to A^{\frac{1}{2}}$ as $\tau\to 0$ and it has same spectral properties of $A^{\frac{1}{2}}$ under some assumptions for A.

Let us denote by $C_\tau(E)=C([0,T]_\tau,E)$ the normed space of grid functions $\varphi^\tau=\{\varphi_k\}_{k=0}^N$ for fixed $\tau=\frac{T}{N}$ with the norm

$$\|\varphi^\tau\|_{C_\tau(E)}=\max_{0\leq k\leq N}\|\varphi_k\|_E.$$

From the formula (5) it follows that the investigation of the stability and well-posedness of difference scheme (4) relies in an essential manner on a number of properties of the powers of the operator $(I+\tau B)^{-1}$. We were not able to obtain the estimates for powers of the operator $(I+\tau B)^{-1}$ in the general cases of operator A. We begin by deriving some estimates for powers of the operator $(I+\tau B)^{-1}$ with a strongly positive operator A in a Banach space E.

Lemma 1. Let A be a strongly positive operator in a Banach space E. Then $-A$ is a generator of the analitic semigroup $\exp\{-tA\}$ $(t\geq 0)$ with exponentially decreasing norm, when $t\longrightarrow+\infty$, i. e. we have the following estimates

$$\|\exp\{-tA\}\|_{E\longrightarrow E}\leq M\ e^{-t\delta}\ (t>0); \quad (6)$$

$$\|tA\exp\{-tA\}\|_{E\longrightarrow E}\leq M\ e^{-t\delta}(t>0) \quad (7)$$

for $1\leq M<+\infty$, $0<\delta<+\infty$. Here M does not depend on τ.

Lemma 2. Let $-A$ be a generator of the analytic semigroup $\exp\{-tA\}$ $(t\geq 0)$ with exponentially decreasing norm, when $t\longrightarrow+\infty$. Then the following estimates hold for any $k\geq 1$:

$$\left\|(\lambda I+\tau B)^{-k}\right\|_{E\to E}\leq M[\lambda+\tau^2a(A)]^{-k}; \quad (8)$$

$$||k\tau B(I+\tau B)^{-k}||_{E\to E}\leq M, \quad (9)$$

where M does not depend on τ.

We have the following results.

Theorem 3. Let A be a strongly positive operator in a Banach space E. Then the difference problem (4) is stable in $C_\tau(E)$. For the solution of the difference problem (4) the following stability inequality is satisfied:

$$\|u^\tau\|_{C_\tau(E)}\leq M[\|f^\tau\|_{C_\tau(E)}+\|u_0\|_E+||u_N\|_E],$$

where M does not depend on f^τ, u_0, u_N and τ.

Stability of difference problem (3)

We consider the difference problem (3). Using formula (5) and the nonlocal conditions

$$u_0 = u_N + \varphi, \sum_{i=1}^{N} u_i \tau = \psi,$$

we get

$$u_0 = (2I + \tau B)^{-1}(I - R^N)^{-1}(I + R^N)\left\{B\psi - (I + \tau B)B^{-1}\sum_{i=1}^{N-1} f_i\tau\right\} - \tag{10}$$
$$-(I - R^N)^{-1}(I + \tau B)(I - R^{N+1})(2I + \tau B)^{-1}\varphi +$$
$$+(I - R^N)^{-1}(I + \tau B)(2I + \tau B)^{-1}B^{-1}\sum_{i=1}^{N-1}\left(R^{N-i} + R^{N+i}\right) f_i\tau.$$

$$u_N = (2I + \tau B)^{-1}(I - R^N)^{-1}(I + R^N)\left\{B\psi - (I + \tau B)B^{-1}\sum_{i=1}^{N-1} f_i\tau\right\} - \tag{11}$$
$$-(I - R^N)^{-1}(I - R^{N-1})(2I + \tau B)^{-1}\varphi +$$
$$+(I - R^N)^{-1}(I + \tau B)(2I + \tau B)^{-1}B^{-1}\sum_{i=1}^{N-1}\left(R^{N-i} + R^{N+i}\right) f_i\tau.$$

Actually, applying formula (5), we get

$$\psi = u_N\tau + \sum_{k=1}^{N-1} u_k\tau = (I - R^{2N})^{-1}\left\{\sum_{k=1}^{N}(R^k - R^{2N-k})\,(u_N + \varphi)\,\tau + \right.$$
$$+\sum_{k=1}^{N}(R^{N-k} - R^{N+k})u_N\tau - \sum_{k=1}^{N-1}(R^{N-k} - R^{N+k})(I + \tau B)\times$$
$$\left.\times(2I + \tau B)^{-1}B^{-1}\sum_{i=1}^{N-1}(R^{N-i} - R^{N+i})f_i\tau^2\right\} +$$
$$+(I + \tau B)(2I + \tau B)^{-1}B^{-1}\sum_{i=1}^{N-1}\sum_{k=1}^{N-1}(R^{|k-i|} - R^{k+i})f_i\tau^2.$$

By computing and interchanging the order of summation, we obtain

$$\psi = (I - R^{2N})^{-1}\,(I - R)^{-1}\left\{\left(R - R^{N+1} - R^N + R^{2N}\right)(u_N + \varphi)\,\tau + \left(I - R^N - R^{N+1} + R^{2N+1}\right)u_N\tau\right\} -$$
$$-(I - R^{2N})^{-1}\,(I - R)^{-1}\,(I + \tau B)(2I + \tau B)^{-1}B^{-1}\left(R - R^N - R^{N+1} + R^{2N}\right)\sum_{i=1}^{N-1}(R^{N-i} - R^{N+i})f_i\tau^2 +$$
$$+\,(I - R)^{-1}\,(I + \tau B)(2I + \tau B)^{-1}B^{-1}\sum_{i=1}^{N-1}\left(I - R^i + R - R^{N-i} - R^{i+1} + R^{N+i}\right) f_i\tau^2.$$

It follows that

$$\psi - (I + R^N)^{-1}(I - R^{N-1})B^{-1}\varphi = (I + R^N)^{-1}\left(I - R^N\right)(2I + \tau B)B^{-1}u_N -$$
$$-(I + R^N)^{-1}(I + \tau B)(2I + \tau B)^{-1}B^{-2}\left(I - R^{N-1}\right)\sum_{i=1}^{N-1}(R^{N-i} - R^{N+i})f_i\tau +$$
$$+(I + \tau B)^2(2I + \tau B)^{-1}B^{-2}\sum_{i=1}^{N-1}\left(\left(I + R\right)(I - R^i) - R^{N-i} + R^{N+i}\right) f_i\tau.$$

Thus

$$u_N=(2I+\tau B)^{-1}(I-R^N)^{-1}(I+R^N)B\left\{\psi-(I+R^N)^{-1}(I-R^{N-1})B^{-1}\varphi+\right.$$

$$+(I+R^N)^{-1}(I+\tau B)(2I+\tau B)^{-1}B^{-2}\left(I-R^{N-1}\right)\sum_{i=1}^{N-1}(R^{N-i}-R^{N+i})f_i\tau-$$

$$\left.-(I+\tau B)^2(2I+\tau B)^{-1}B^{-2}\sum_{i=1}^{N-1}\left((I+R)\left(I-R^i\right)-R^{N-i}+R^{N+i}\right)f_i\tau\right\}=$$

$$=(2I+\tau B)^{-1}(I-R^N)^{-1}(I+R^N)B\psi-(I-R^N)^{-1}(I-R^{N-1})(2I+\tau B)^{-1}\varphi+$$

$$+(I-R^N)^{-1}(I+\tau B)^2(2I+\tau B)^{-2}B^{-1}\left(R-R^N\right)\sum_{i=1}^{N-1}(R^{N-i}-R^{N+i})f_i\tau-$$

$$\left.-(I-R^N)^{-1}(I+\tau B)(2I+\tau B)^{-1}(I+R^N)B^{-1}\sum_{i=1}^{N-1}f_i\tau\right\}-$$

$$\left.-(I-R^N)^{-1}(I+\tau B)^2(2I+\tau B)^{-2}(I+R^N)B^{-1}\sum_{i=1}^{N-1}\left(-(I+R)R^i-R^{N-i}+R^{N+i}\right)f_i\tau\right\}=$$

$$=(2I+\tau B)^{-1}(I-R^N)^{-1}(I+R^N)\left\{B\psi-(I+\tau B)B^{-1}\sum_{i=1}^{N-1}f_i\tau\right\}-$$

$$-(I-R^N)^{-1}(I-R^{N-1})(2I+\tau B)^{-1}\varphi+$$

$$+(I-R^N)^{-1}(I+\tau B)(2I+\tau B)^{-1}B^{-1}\sum_{i=1}^{N-1}\left(R^{N-i}+R^{N+i}\right)f_i\tau.$$

From that there follow formulas (10) and (11).

Theorem 4. Let A be a strongly positive operator in a Banach space E and $\psi=A^{-1}\sum\limits_{i=1}^{N-1}f_i\tau,\varphi=0$. Then difference problem (13) is stable in $C_\tau(E)$. For a solution of the difference problem (13) the following stability inequalities holds

$$\| u^\tau \|_{C_\tau(E)}\leq M_1\| f^\tau \|_{C_\tau(E)},$$

where M_1 does not depend on f^τ and τ.

Proof. By Theorem 3 we have the following estimate

$$\| u^\tau \|_{C_\tau(E)}\leq M[\| f^\tau \|_{C_\tau(E)}+\| u_0\|_E+||u_N\|_E] \tag{12}$$

for solution of problem (4). Therefore, to prove the theorem it is sufficient to establish estimates for $\| u_0\|_E$ and $||u_N\|_E$. Applying formulas (10) and (11), the triangle inequality and estimates (8), (9), we get

$$\| u_0\|_E\leq M_1\| f^\tau \|_{C_\tau(E)};$$

$$||u_N\|_E\leq M_1\| f^\tau \|_{C_\tau(E)}.$$

Theorem 4 is proved.

Application

Now, we will give the application of Theorem 4 to elliptic equations. Let Ω be the unit open cube in the $n-$dimensional Euclidean space $\mathbb{R}^n$ $(0<x_k<1, 1\leq k\leq n)$ with boundary S, $\overline{\Omega}=\Omega\cup S$. In $[0,T]\times\Omega$ we consider the nonlocal boundary value problem for the multidimensional elliptic equation

$$\begin{cases} -\frac{\partial^2 u(y,x)}{\partial y^2} - \sum\limits_{r=1}^{n} \alpha_r(x)\frac{\partial^2 u(y,x)}{\partial x_r^2} + \delta u(y,x) = f(y,x); \\ x = (x_1, \dots, x_n) \in \Omega, 0 < y < T; \\ u(0,x) = u(T,x), \int\limits_0^T u(s,x)ds = 0,\ x \in \overline{\Omega}; \\ \frac{\partial u(y,x)}{\partial \overrightarrow{m}} = 0,\ x \in S, \end{cases} \tag{13}$$

where $\alpha_r(x)$ $(x \in \Omega)$ and $f(y,x)$ $(y \in (0,T),\ x \in \Omega)$ are given smooth functions and $\alpha_r(x) > 0,\ \delta > 0$ is a sufficiently large number. Here, $\overrightarrow{m}$ is a normal vector to S. The discretization of problem (13) is also carried out in two steps. In the first step, let us define the grid sets

$$\overline{\Omega}_h = \{x = x_j = (h_1 j_1, ..., h_m j_m),\ j = (j_1, ..., j_m);$$

$$0 \le j_r \le M_r,\ h_r M_r = 1,\ r = 1, ..., m, \};$$

$$\Omega_h = \overline{\Omega}_h \cap \Omega,\ S_h = \overline{\Omega}_h \cap S.$$

We introduce the Banach spaces $L_{2h} = L_2(\overline{\Omega}_h)$ and $C_h = C(\overline{\Omega}_h)$ of the grid functions $\varphi^h(x) = \{\varphi(h_1 j_1, ..., h_m j_m)\}$ defined on $\overline{\Omega}_h$, equipped with the norms

$$\left\|\varphi^h\right\|_{L_{2h}} = \left(\sum_{x\in\overline{\Omega}_h} \left|\varphi^h(x)\right|^2 h_1 \cdots h_m\right)^{1/2} \tag{14}$$

and

$$\left\|\varphi^h\right\|_{C_h} = \sup_{x\in\overline{\Omega}_h} \left|\varphi^h(x)\right|, \tag{15}$$

respectively. To the differential operator A generated by problem (13), we assign the difference operator A_h^x by the formula

$$A_h^x u^h(x) = -\sum_{r=1}^{m} \left(a_r(x) u_{\overline{x}_r}^h\right)_{x_r, j_r} \tag{16}$$

acting in the space of grid functions $u^h(x)$, satisfying the condition $\frac{\partial u^h(x)}{\partial \overrightarrow{m}}$ $(\forall\ x \in S_h)$. It is known that A_h^x is a self-adjoint positive definite operator in $L_2(\overline{\Omega}_h)$ and $C(\overline{\Omega}_h)$. With the help of A_h^x, we arrive at the nonlocal boundary value problem

$$\begin{cases} -\frac{d^2 u^h(y,x)}{dy^2} + A_h^x u^h(t,x) = f^h(t,x); \\ x \in \Omega_h, 0 < y < T; \\ u^h(0,x) = u^h(T,x), \int\limits_0^T u^h(s,x)ds = 0,\ x \in \overline{\Omega}_h; \\ \frac{\partial u^h(x)}{\partial \overrightarrow{m}} = 0,\ x \in S_h. \end{cases} \tag{17}$$

In the second step, we replace problem (17) by the second order of accuracy difference scheme (3)

$$\begin{cases} -\frac{u_{k+1}^h(x) - 2u_k^h(x) + u_{k-1}^h(x)}{\tau^2} + A_h^x u_k^h(x) = f_k^h(x), f_k^h(x) = f^h(y_k, x); \\ y_k = k\tau, 1 \le k \le N-1, N\tau = T, x \in \Omega_h; \\ u_0^h(x) = u_N^h(x), \sum\limits_{i=1}^{N} u_i^h(x)\tau = 0, x \in \overline{\Omega}_h. \end{cases} \tag{18}$$

Using the results of Theorem 4, we can obtain the following theorem.

Theorem 5. Let τ and h be sufficiently small numbers and $\sum\limits_{i=1}^{N-1} f^h(y_i,x)=0$. Then, solutions of difference scheme (18) satisfy the following estimates

$$\max_{0\le k\le N}\left\|u_k^h\right\|_{L_{2h}}\le M_1\max_{1\le k\le N-1}\left\|f_k^h\right\|_{L_{2h}};$$

$$\max_{0\le k\le N}\left\|u_k^h\right\|_{C_h}\le M_1\max_{1\le k\le N-1}\left\|f_k^h\right\|_{C_h},$$

here M_1 does not depend on τ, h and $f_k^h, 1\le k\le N-1$.

The illustrative numerical result

When the analytical methods do not work properly, the numerical methods for obtaining approximate solutions of partial differential equations play an important role in applied mathematics.

For numerical analysis, we consider the nonlocal boundary problem for the two dimensional elliptic equation

$$\begin{cases} -\frac{\partial^2 u}{\partial t^2}-\frac{\partial^2 u}{\partial x^2}+u=3\cos t\cos x, \qquad 0<t<2\pi,\ 0<x<2\pi; \\ u\left(0,x\right)=u\left(2\pi,x\right); \qquad \int_0^{2\pi}u\left(s,x\right)ds=0,\ 0\le x\le 2\pi; \\ u_x\left(t,0\right)=u_x\left(t,2\pi\right)=0, \qquad 0\le x\le 2\pi. \end{cases} \tag{19}$$

The exact solution of this problem is

$$u\left(t,x\right)=\cos t\cos x.$$

For an approximate solution of the nonlocal boundary problem (19), we consider the set $[0,2\pi]_\tau\times[0,2\pi]_h$ of a family of grid points depending on the small parameters τ and h

$$[0,2\pi]_\tau\times[0,2\pi]_h=\left\{(t_k,x_n): t_k=k\tau, 0\le k\le N, N\tau=2\pi,\ \ x_n=nh, 0\le n\le M, Mh=2\pi\right\}.$$

For a numerical solution, we consider the difference scheme of the second order of accuracy in t and the first order of accuracy in x.

$$\begin{cases} -\frac{u_n^{k+1}+u_n^k+u_n^{k-1}}{\tau^2}-\frac{u_{n+1}^k+u_n^k+u_{n-1}^k}{h^2}+u_n^k=3\cos t_k\cos x_n, 1\le k\le N-1,\ 1\le n\le M-1; \\ u_n^0=u_n^N,\ \sum_{i=0}^{N-1}u_n^i=0,\ 0\le n\le M; \\ u_1^k-u_0^k=u_M^k-u_{M-1}^k=0,\ 0\le k\le N. \end{cases} \tag{20}$$

It is the system of algebraic equations and it can be written in the matrix form

$$\begin{cases} Au_{n+1}+Bu_n+Cu_{n-1}=D\varphi_n,\ 1\le n\le M-1, \\ u_0=u_1, u_{M-1}=u_M. \end{cases} \tag{21}$$

Here,

$$A=C=\begin{bmatrix} 0 & 0 & 0 & 0 & . & 0 & 0 & 0 & 0 \\ 0 & a & 0 & 0 & . & 0 & 0 & 0 & 0 \\ 0 & 0 & a & 0 & . & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a & . & 0 & 0 & 0 & 0 \\ . & . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & . & a & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & . & 0 & a & 0 & 0 \\ 0 & 0 & 0 & 0 & . & 0 & 0 & a & 0 \\ 0 & 0 & 0 & 0 & . & 0 & 0 & 0 & 0 \end{bmatrix}_{(N+1)\times(N+1)}, B=\begin{bmatrix} 1 & 0 & 0 & 0 & . & 0 & 0 & 0 & -1 \\ c & b & c & 0 & . & 0 & 0 & 0 & 0 \\ 0 & c & b & c & . & 0 & 0 & 0 & 0 \\ 0 & 0 & c & b & . & 0 & 0 & 0 & 0 \\ . & . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & . & b & c & 0 & 0 \\ 0 & 0 & 0 & 0 & . & c & b & c & 0 \\ 0 & 0 & 0 & 0 & . & 0 & c & b & c \\ 0 & 1 & 1 & 1 & . & 1 & 1 & 1 & 1 \end{bmatrix}_{(N+1)\times(N+1),}$$

where $a=-\frac{1}{h^2},\ b=\frac{2}{\tau^2}+\frac{2}{h^2}+1,\ c=-\frac{1}{\tau^2},$

$$\varphi_n = \begin{bmatrix} c\varphi_n^0 \\ \varphi_n^1 \\ . \\ \varphi_n^{N-1} \\ \varphi_n^N \end{bmatrix}_{(N+1)\times 1} = \begin{bmatrix} 0 \\ 3\cos t_1 \cos x_n \\ . \\ 3\cos t_{N-1}\cos x_n \\ 0 \end{bmatrix}_{(N+1)\times 1},$$

and $D = I_{N+1}$ is the identity matrix,

$$u_s = \begin{bmatrix} cu_s^0 \\ u_s^1 \\ . \\ u_s^{N-1} \\ u_s^N \end{bmatrix}_{(N+1)\times 1}, \quad s = n-1,\ n,\ n+1.$$

Therefore, to solve the matrix equation (21), we will use the modified Gauss elimination method. We seek the solution of the matrix equation by the following form:

$$u_n = \alpha_{n+1}u_{n+1} + \beta_{n+1}, \quad n = M-1, ..., 1, 0, \tag{22}$$

where $u_M = (I - \alpha_M)^{-1}\beta_M$, α_j $(j = 1, ..., M-1)$ are $(N+1)\times(N+1)$ square matrices, β_j $(j = 1, ..., M-1)$ are $(N+1)\times 1$ column matrices, α_1 is the identity and β_1 are zero matrices and

$$\alpha_{n+1} = -(B + C\alpha_n)^{-1}A;$$

$$\beta_{n+1} = (B + C\alpha_n)^{-1}(D\varphi_n - C\beta_n), n = 1, ..., M-1.$$

Now, we give the error analysis between exact solutions $u(t_k, x_n)$ and the approximate solutions u_n^k for the different values of N and M. The errors are computed by the formula

$$E_M^N = \max_{0\le k\le N, 0\le n\le M} \left| u(t_k, x_n) - u_n^k \right|. \tag{23}$$

The numerical results for the difference scheme (20) are given in the following tables 1, 2.

Table 1

Two dimensional	$N, M = 20, 20$	$N, M = 40, 40$	$N, M = 80, 80$
Difference scheme	0.1329	0.0607	0.0290

Table 2

Two dimensional	$N, M = 20, 400$	$N, M = 40, 1600$	$N, M = 80, 6400$
Difference scheme	0.0029	$7.1859e-04$	$1.7955e-04$

As it is seen in Table 1, if N and M are doubled, the value of errors decrease by a factor of approximately 1/2. Moreover, as it is seen in Table 2, if N is doubled and $M \geq N^2$, the value of errors decrease by a factor of approximately 1/4 difference scheme as the second order of accuracy.

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А. Ашыралыев, А. Хамад

Эллипстік теңдеулер үшін локалдық емес шеттік есептерді сандық шешу

Мақалада қатаң оң A операторы бар эллипстік теңдеу үшін мына түрдегі $-v''(t)+Av(t)=f(t)$, $(0\leq t\leq T)$, $v(0)=v(T)+\varphi$, $\int\limits_0^T v(s)\,ds=\psi$ локалдық емес шеттік есепті жуықтап шешуге арналған екінші ретті дәлдігі бар екі адымды айрымдық схема келтірілген. Есеп қандай да бір E Банах кеңістігінде қарастырылды. Айрымдық схеманың орнықты болатыны көрсетілген. Қосымшада шеттік шарттары Нейман түріндегі дифференциалды есеп үшін айырымдық схема шешімінің орнықтылығын бағалаулар көрсетілген. Сондай-ақ сандық есептеулердің нәтижелері берілген.

Кілт сөздер: орнықтылық, оң операторлар, эллипстік теңдеулер, сандық нәтиже, екінші ретті дәлдігі.

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Численное решение нелокальной краевой задачи для эллиптических уравнений

В статье представлена двухшаговая разностная схема второго порядка точности для приближенного решения нелокальной краевой задачи для эллиптического дифференциального уравнения $-v''(t) + Av(t) = f(t) \quad (0 \leq t \leq T),\ v(0) = v(T) + \varphi,\ \int\limits_0^T v(s)ds = \psi$ в произвольном банаховом пространстве E с сильно положительным оператором A. Установлена устойчивость этой разностной схемы. В приложении получены оценки устойчивости решения разностной схемы для эллиптической дифференциальной задачи с граничным условием Неймана. Кроме того, приведен демонстрационный численный результат.

Ключевые слова: устойчивость, положительные операторы, эллиптическое уравнение, численные результаты, двухшаговая разностная схема.

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A note on the second order of accuracy difference scheme for elliptic-parabolic equations in Hölder spaces

The present paper is devoted to the study of a second order of accuracy difference scheme for a solution of the elliptic-parabolic equation with nonlocal boundary condition. The well-posedness of the second order of accuracy difference scheme in Hölder spaces is established. Coercivity estimates in Hölder norms for an approximate solution of a nonlocal boundary value problem for elliptic-parabolic differential equation are obtained. Results of numerical experiments are presented in order to support the aforementioned theoretical statements.

Keywords: difference scheme, elliptic-parabolic equation, Hölder spaces, well-posedness, coercivity inequalities.

Introduction

In the last decades, boundary value problems with nonlocal boundary conditions have been an important research topic in many natural phenomena. Methods and theories of solutions of the nonlocal boundary value problems for elliptic, parabolic, and mixed type differential equations have been studied extensively in a large cycle of papers (see, for example, [1–20] and the references given therein).

In paper [1], the well-posedness of the nonlocal boundary value problem

$$\begin{cases} \frac{-d^2u(t)}{dt^2} + Au(t) = g(t), 0 \le t \le 1, \\ \frac{du(t)}{dt} - Au(t) = f(t), -1 \le t \le 0, \\ u(0+) = u(0-), u^{'}(0+) = u^{'}(0-), u(1) = u(-1) + \mu \end{cases} \tag{1}$$

in Hölder spaces was determined. Furthermore, the coercivity inequalities for solutions of the nonlocal boundary value problem for elliptic-parabolic equations were obtained.

In article [2], the first order of accuracy difference scheme

$$\begin{cases} -\frac{u_{k+1}-2u_k+u_{k-1}}{\tau^2} + Au_k = g_k, g_k = g(t_k), t_k = k\tau, 1 \le k \le N-1, \\ \frac{u_k-u_{k-1}}{\tau} - Au_{k-1} = f_k, f_k = f(t_{k-1}), t_k = (k-1)\tau, -(N-1) \le k \le -1, \\ u_1 - u_0 = u_0 - u_{-1}, u_N = u_{-N} + \mu \end{cases}$$

for an approximate solution of problem (1) was constructed. Also the well-posedness of the difference scheme in Hölder spaces was proven. Moreover, coercivity estimates in Hölder norms for the solutions of difference scheme for elliptic-parabolic equations were derived.

In this study, the well-posedness of the following second order of accuracy difference scheme

$$\begin{cases} -\frac{u_{k+1}-2u_k+u_{k-1}}{\tau^2}+Au_k=g_k, g_k=g(t_k), \\ t_k=k\tau, 1\leq k\leq N-1, N\tau=1; \\ \frac{u_k-u_{k-1}}{\tau}-\frac{1}{2}(Au_k+Au_{k-1})=f_k, f_k=f(t_{k-\frac{1}{2}}); \\ t_{k-\frac{1}{2}}=(k-\frac{1}{2})\tau, -(N-1)\leq k\leq 0; \\ u_2-4u_1+3u_0=-3u_0+4u_{-1}-u_{-2},\ u_N=u_{-N}+\mu \end{cases} \tag{2}$$

for the approximate solution of nonlocal boundary value problem (1) in Hölder spaces is presented. In addition coercivity inequalities for solutions of difference schemes are obtained.

The rest of this paper is organized as follows. In section 2, the main theorem on well-posedness of the difference scheme (2) will be presented. In section 3, an application of the main theorem will be given. In section 4, the numerical results are presented. Finally, in section 5, the conclusion will be given.

Well-posedness of the difference scheme (2)

Throughout this paper, we have adopted the following symbols. H denotes a Hilbert space and $A=\delta I$, where $\delta>\delta_0>0$, is a self-adjoint positive definite operator. I is an identity operator, $B=\frac{1}{\tau}(\tau A+\sqrt{A(4+\tau^2A)}$ is a given self-adjoint positive definite operator and $B\geq\delta^{\frac{1}{2}}I$. In addition, $R=(I+\tau B)^{-1}$ is a bounded operator defined on the whole space H. The following operators

$$P=(I-\frac{\tau A}{2}), G=(I+\frac{\tau A}{2})^{-1}, K=(I+2\tau A+\frac{5}{4}(\tau A)^2)^{-1}$$

exist and are bounded for the self-adjoint positive operator A.

Lemma 1. The following necessary estimates for P^k, R^k and T_τ are satisfied in [3] and [4]:

$$||P^k||_{H\to H}\leq 1,\ \ ||G||_{H\to H}\leq 1,\ \ k\tau||AP^kG^2||_{H\to H}\leq M,\ k\geq 1,\ \delta>0; \tag{3}$$

$$||R^K||_{H\to H}\leq M(1+\tau B)^{-k},\ \ k\tau||BR^K||_{H\to H}\leq M,\ k\geq 1,\ \delta>0; \tag{4}$$

where A is a self-adjoint positive operator and M is independent of τ.

From these estimates it follows that

$$||(I+B^{-1}A(I+\tau A+\frac{\tau}{2}G^{-2})K(I-R^{2N-1})+K(I-\frac{\tau^2A}{2})G^{-2}R^{2N-1}- \tag{5}$$
$$-K(I-\frac{\tau^2A}{2})G^{-2}(2I+\tau B)R^NP^N)^{-1})^{-1}||_{H\to H}\leq M.$$

Here, we will study well-posedness of (2) in Hölder space. Consider $F_\tau(H)=F([a,b]_\tau,H)$ as the linear space of mesh functions $\varphi^\tau=\{\varphi_k\}_{N_a}^{N_b}$ defined on $[a,b]_\tau=\{t_k=kh, N_a\leq k\leq N_b, N_a\tau=a, N_b\tau=b\}$ with values in Hilbert space H.

Let $C([a,b]_\tau,H), C^\alpha([-1,1]_\tau,H), \tilde{C}^\alpha([-1,1]_\tau,H), \tilde{C}^{\frac{\alpha}{2}}([-1,1]_\tau,H), \tilde{C}^\alpha([0,1]_\tau,H)$ be Banach spaces with the norms

$$||\varphi^\tau||_{C([a,b]_\tau,H)}=\max_{N_a\leq k\leq N_b}||\varphi_k||_H,$$

$$||\varphi^\tau||_{C^\alpha([-1,1]_\tau,H)}=||\varphi^\tau||_{C([-1,1]_\tau,H)}+\sup_{-N\leq k<k+r\leq 0}||\varphi_{k+r}-\varphi_k||_H(r\tau)^{-\frac{\alpha}{2}}+$$
$$+\sup_{1\leq k<k+r\leq N-1}||\varphi_{k+r}-\varphi_k||_H(r\tau)^{-\alpha},$$

$$||\varphi^\tau||_{\tilde{C}^\alpha([-1,1]_\tau,H)}=||\varphi^\tau||_{C([-1,1]_\tau,H)}+\sup_{-N\leq k<k+2r\leq 0}||\varphi_{k+2r}-\varphi_k||_H(2r\tau)^{-\frac{\alpha}{2}}+$$
$$+\sup_{1\leq k<k+r\leq N-1}||\varphi_{k+r}-\varphi_k||_H(r\tau)^{-\alpha},$$

$$||\varphi^\tau||_{\tilde{C}^{\frac{\alpha}{2}}([-1,0]_\tau,H)} = ||\varphi^\tau||_{C([-1,0]_\tau,H)} + \sup_{-N\le k<k+2r\le 0} ||\varphi_{k+2r}-\varphi_k||_H (2r\tau)^{-\frac{\alpha}{2}},$$

$$||\varphi^\tau||_{\tilde{C}^{\alpha}([0,1]_\tau,H)} = ||\varphi^\tau||_{C([0,1]_\tau,H)} + \sup_{1\le k<k+r\le N-1} ||\varphi_{k+r}-\varphi_k||_H (r\tau)^{-\alpha}.$$

Recall that the Banach space $E_\alpha = E_\alpha(B,H)$, where $0<\alpha<1$ consists of $v\in H$, for which the following norm is finite [5]

$$||v||_{E_\alpha} = \sup_{z>0} z^\alpha ||B(z+B)^{-1}v||.$$

The following holds for all $\beta<\alpha$:

$$D(B)\subset E_\alpha(B,H)\subset E_\beta(B,H)\subset H.$$

Theorem 1. Assume that $(I+\tau B)(f_{-N+1}+g_{N-1})\in E_\alpha$, $(I+\tau B)(f_0+g_1)\in E_{\frac{\alpha}{2}}$, and $A\mu\in E_\alpha$. Then, the solution of difference problem (2) obeys the coercivity inequalities

$$\left\|\{\tau^{-2}(u_{k+1}-2u_k+u_{k-1})\}_1^{N-1}\right\|_{C^\alpha([0,1]_\tau,H)} + \left\|\{\tau^{-1}(u_k-u_{k-1})\}_{-N+1}^0\right\|_{\tilde{C}^{\frac{\alpha}{2}}([-1,0]_\tau,H)} +$$

$$+\left\|\{Au_k\}_1^{N-1}\right\|_{C^\alpha([0,1]_\tau,H)} + \left\|\left\{\frac{1}{2}(Au_k+Au_{k-1})\right\}_{-N+1}^0\right\|_{\tilde{C}^{\frac{\alpha}{2}}([-1,0]_\tau,H)} \le$$

$$\le M_1\left\{||A\mu||_{E_\alpha} + ||(I+\tau B)(f_0+g_1)||_{E_{\frac{\alpha}{2}}} + ||(I+\tau B)(f_{-N+1}+g_{N-1})||_{E_\alpha} +\right.$$

$$\left.+\frac{1}{\alpha(1-\alpha)}\left[||f^\tau||_{\tilde{C}^{\frac{\alpha}{2}}([-1,0]_\tau,H)} + ||g^\tau||_{C^\alpha([0,1]_\tau,H)}\right]\right\}$$

and

$$\left\|\{\tau^{-2}(u_{k+1}-2u_k+u_{k-1})\}_1^{N-1}\right\|_{C^\alpha([0,1]_\tau,H)} + \left\|\{\tau^{-1}(u_k-u_{k-1})\}_{-N+1}^0\right\|_{C^{\frac{\alpha}{2}}([-1,0]_\tau,H)} +$$

$$+\left\|\{Au_k\}_1^{N-1}\right\|_{C^\alpha([0,1]_\tau,H)} + \left\|\left\{\frac{1}{2}(Au_k+Au_{k-1})\right\}_{-N+1}^0\right\|_{C^{\frac{\alpha}{2}}([-1,0]_\tau,H)} \le$$

$$\le M_2\left\{||A\mu||_{E_\alpha} + ||(I+\tau B)(f_0+g_1)||_{E_{\frac{\alpha}{2}}} + \|(I+\tau B)(f_{-N+1}+g_{N-1})\|_{E_\alpha} +\right.$$

$$\left.+\frac{1}{\alpha(1-\alpha)}\left[\left\|\left(I+\frac{\tau A}{2}\right)f^\tau\right\|_{C^{\frac{\alpha}{2}}([-1,0]_\tau,H)} + \|g^\tau\|_{C^\alpha([0,1]_\tau,H)}\right]\right\}.$$

Here M_1 and M_2 do not depend on f^τ, g^τ, μ, τ, and α.

Proof. By [6], we obtain

$$\left\|\left\{\tau^{-1}(u_k-u_{k-1})\right\}_{-N+1}^0\right\|_{\tilde{C}^{\frac{\alpha}{2}}([-1,0]_\tau,H)} + \left\|\left\{\frac{1}{2}(Au_k+Au_{k-1})\right\}_{-N+1}^0\right\|_{\tilde{C}^{\frac{\alpha}{2}}([-1,0]_\tau,H)} \le \tag{6}$$

$$\le M_1\left[\frac{1}{\alpha(1-\frac{\alpha}{2})}||f^\tau||_{C^{\frac{\alpha}{2}}([-1,0]_\tau,H)} + ||Au_0||_{E_{\frac{\alpha}{2}}}\right]$$

and

$$\left\|\left\{\tau^{-1}(u_k-u_{k-1})\right\}_{-N+1}^0\right\|_{C^{\frac{\alpha}{2}}([-1,0]_\tau,H)} + \left\|\left\{\frac{1}{2}(Au_k+Au_{k-1})\right\}_{-N+1}^0\right\|_{C^{\frac{\alpha}{2}\le}([-1,0]_\tau,H)} \le \tag{7}$$

$$\le M_2\left[\frac{1}{\alpha(1-\frac{\alpha}{2})}\left\|\left(I+\frac{\tau A}{2}\right)f^\tau\right\|_{C^{\frac{\alpha}{2}}([-1,0]_\tau,H)} + ||Au_0||_{E_{\frac{\alpha}{2}}}\right]$$

for the solution of an inverse Cauchy difference problem

$$\begin{cases} \tau^{-1}(u_k-u_{k-1}) - \frac{A}{2}(u_k+u_{k-1}) = f_k, \\ -(N-1)\le k\le 0, \quad u_0 \text{ is given.} \end{cases}$$

By [3] and [7], we get

$$\left\|\{\tau^{-2}(u_{k+1}-2u_k+u_{k-1})\}_1^{N-1}\right\|_{C^\alpha([0,1]_\tau,H)}+||\{Au_k\}_1^{N-1}||_{C^\alpha([0,1]_\tau,H)}\leq$$

$$\leq M\left[\frac{1}{\alpha(1-\alpha)}||g^\tau||_{C^\alpha([0,1]_\tau,H)}+||Au_0||_{E_\alpha}+||Au_N||_{E_\alpha}\right] \tag{8}$$

for the solution of boundary value problem

$$\begin{cases} \tau^{-2}(u_{k+1}-2u_k+u_{k-1})+Au_k=g_k, \\ 1\leq k\leq N-1,\ u_0,\ u_N \text{ are given.}\end{cases}$$

Then, the proof of Theorem 1 is based on coercivity inequalities (6)–(8) and estimates

$$||Au_0||_{E_\alpha}\leq M\left\{\frac{1}{\alpha(1-\frac{\alpha}{2})}\left[||f^\tau||_{C^{\frac{\alpha}{2}}([-1,0]_\tau,H)}+|||g^\tau||_{C^\alpha([0,1]_\tau,H)}\right]+\right. \tag{9}$$

$$\left.+||A\mu||_{E_\alpha}+||(I+\tau B)(f_0+g_1)||_{E_\alpha}+||(I+\tau B)f_{-1}||_{E_\alpha}\right\}$$

and

$$||Au_N||_{E_\alpha}\leq M\left\{\frac{1}{\alpha(1-\frac{\alpha}{2})}\left[||f^\tau||_{C^\alpha([-1,0]_\tau,H)}+|||g^\tau||_{C^\alpha([0,1]_\tau,H)}\right]+\right. \tag{10}$$

$$\left.+||A\mu||_{E_\alpha}+||(I+\tau B)(f_0+g_1)||_{E_\alpha}\right\}$$

for the solution of the boundary value problem (2). Estimates (9) and (10) follow from the formulae

$$Au_0=\frac{1}{2}T\tau KG^{-2}\times\left\{(2I-\tau^2A)\left\{(2+\tau B)R^N\left[-\tau\sum_{s=-N+1}^{0}AP^{s+N-1}G(f_s-f_{-N+1})+A\mu\right]-\right.\right.$$

$$-R^{N-1}AB^{-1}\sum_{s=1}^{N-1}R^{N-s}(g_s-g_{N-1})\tau+R^{N-1}AB^{-1}\sum_{s=1}^{N-1}R^{N+s}(g_s-g_1)\tau+$$

$$\left.+(I-R^{2N})AB^{-1}\sum_{s=1}^{N-1}R^{s-1}(g_s-g_1)\tau\}\right\}+$$

$$+(I-R^{2N})(I+\tau B)(\tau B^{-1}Ag_1-4GB^{-1}Af_0+PGB^{-1}Af_0+GB^{-1}Af_{-1})+$$

$$+(2I-\tau^2A)(2+\tau B)R^N(P^N-I)f_{-N+1}+$$

$$+AB^2(R^{N-1}-I)\left[R^{N-1}g_{N-1}+(R^{2N}-R^{2N-1}-I)g_1\right]\}$$

and

$$Au_N=\frac{1}{2}P^NT\tau KG^{-2}\times\left\{(2I-\tau^2A)\left\{(2+\tau B)R^N\left[-\tau\sum_{s=-N+1}^{0}AP^{s+N-1}G(f_s-f_{-N+1})+A\mu\right]-\right.\right.$$

$$\left.-R^{N-1}AB^{-1}\sum_{s=1}^{N-1}R^{N-s}(g_s-g_{N-1})\tau+R^{N-1}AB^{-1}\sum_{s=1}^{N-1}R^{N+s}(g_s-g_1)\tau\right\}+$$

$$+(I-R^{2N})AB^{-1}\sum_{s=1}^{N-1}BR^{s-1}(g_s-g_1)\tau\}+$$

$$+(I-R^{2N})(I+\tau B)(\tau B^{-1}Ag_1-4GB^{-1}Af_0+PGB^{-1}Af_0+GB^{-1}Af_{-1})+$$

$$+(2I-\tau^2A)(2+\tau B)R^N(P^N-I)f_{-N+1}+$$

$$+AB^2(R^{N-1}-I)\{R^{N-1}g_{N-1}+(R^{2N}-R^{2N-1}-I)g_1\}\}-$$

$$-\tau\sum_{s=-N+1}^{0}AP^{s+N-1}G(f_s-f_{-N+1})+A\mu+(P^N-I)_{f_{-N+1}},$$

for the solution of problem (2) and estimates (3)–(5). The proof of Theorem 1 is complete.

An application of main theorem

In this section, an application of Theorem 1 is presented. Let Ω be a unit cube in the n-dimensional Euclidean space R^n $(0 < x_k < 1, 1 \leq k \leq n)$ with boundary S, $\bar{\Omega} = \Omega \cup S$, in $[-1,1] \times \Omega$. A nonlocal boundary value problem

$$\begin{cases} -u_{tt} - \sum\limits_{r=1}^{n} (a_r(x) u_{x_r})_{x_r} = g(t,x), 0 < t < 1, x \in \Omega, \\ u_t + \sum\limits_{r=1}^{n} (a_r(x) u_{x_r})_{x_r} = f(t,x), -1 < t < 0, x \in \Omega, \\ u(0+,x) = u(0-,x), u_t(0+,x) = u_t(0-,x), x \in \bar{\Omega}, \\ u(t,x) = 0, x \in S, -1 \leq t \leq 1, u(1,x) = u(-1,x), x \in \bar{\Omega} \end{cases} \tag{11}$$

is considered, where $a_r(x)$ $(x \in \Omega)$, $g(t,x)$ $(t \in (0,1), x \in \bar{\Omega})$, $f(t,x)$ $(t \in (-1,0), x \in \bar{\Omega})$ are given smooth functions and $a_r(x) \geq a \geq 0$ is a sufficiently large number.

The discretization of problem (11) is carried out in two steps. In the first step, the grid sets

$$\bar{\Omega}_h = \{x = x_m = (h_1 m_1, ..., h_n m_n), m = (m_1, m_2, ..., m_n),$$

$$0 \leq m_r \leq N, h_r N_r = 1, r = 1, ..., n\}, \Omega_h = \bar{\Omega}_h \cap \Omega, S_h = \bar{\Omega}_h \cap S$$

are defined.

We introduce the Hilbert space $L_{2h} = L_{2h}(\bar{\Omega})$ of the grid functions $\varphi^h(x) = \{\varphi(h_1 m_1, h_2 m_2, \cdots, h_n m_n)\}$ defined on $\bar{\Omega}_h$, equipped with the norm

$$||\varphi^h||_{L_{2h}} = \left(\sum_{x \in \bar{\Omega}_h} |\varphi^h(x)|^2 h_1 \cdots h_n \right)^{\frac{1}{2}},$$

and the Hilbert spaces $W_{2h}^1 = W_2^1(\bar{\Omega}_h)$, $W_{2h}^2 = W_2^2(\bar{\Omega}_h)$ defined on $\bar{\Omega}_h$, equipped with the norms

$$||\varphi^h||_{W_{2h}^1} = \left(\sum_{x \in \bar{\Omega}_h} \sum_{r=1}^{n} |(\varphi^h)_{x_r}|^2 h_1 \cdots h_n \right)^{\frac{1}{2}},$$

$$||\varphi^h||_{W_{2h}^2} = ||\varphi^h||_{L_{2h}} + \left(\sum_{x \in \bar{\Omega}_h} \sum_{r=1}^{n} |(\varphi^h)_{x_r \bar{x}_r, m_r}|^2 h_1 \cdots h_n \right)^{\frac{1}{2}}.$$

It is known that the differential expression

$$A_h^x u^h = -\sum_{r=1}^{n} (a_r(x) u_{\bar{x}_r}^h)_{x_{r,m_r}} \tag{12}$$

defines a positive operator A_h^x acting in the space of grid functions $u^h(x)$, satisfying the condition $u^h(x) = 0$, for all $x \in S_h$. With the help of A_h^x, we arrive at the nonlocal boundary value problem

$$\begin{cases} -\frac{d^2 u^h(t,x)}{dt^2} + A_h^x u^h(t,x) = g^h(t,x), 0 < t < 1, x \in \Omega_h, \\ \frac{du^h(t,x)}{dt} - A_h^x u^h(t,x) = f^h(t,x), -1 < t < 0, x \in \Omega_h, \\ u^h(0+,x) = u^h(0-,x), \frac{du^h(0+,x)}{dt} = \frac{du(0-,x)}{dt}, x \in \bar{\Omega}_h, \\ u^h(1,x) = u^h(-1,x), x \in \bar{\Omega}_h \end{cases} \tag{13}$$

for an infinite system of ordinary differential equations. In the second step problem (13) is replaced by the difference scheme (2) (see [7]).

$$\begin{cases} -\frac{u_{k+1}^h(x)-2u_k^h(x)+u_{k-1}^h(x)}{\tau^2}+A_h^x u_k^h(x)=g_k^h(x), \\ g_k^h(x)=g^h(t_k,x), t_k=k\tau, 1\le k\le N-1, N\tau=1, x\in\Omega_h, \\ \frac{u_k^h(x)-u_{k-1}^h(x)}{\tau^2}-\frac{A_h^x}{2}(u_k^h(x)+u_{k-1}^h(x))=f_k^h(x), \\ f_k^h(x)=f^h(t_{k-\frac{1}{2}},x_n), t_{k-\frac{1}{2}}=(k-\frac{1}{2})\tau, -N+1\le k\le 0, x\in\Omega_h, \\ -u_2^h(x)+4u_1^h(x)-3u_0^h(x)=3u_0^h(x)-4u_{-1}^h(x)+u_{-2}^h(x), x\in\bar\Omega_h, \\ u_N^h(x)=u_{-N}^h(x), x\in\bar\Omega_h. \end{cases}$$

Theorem 2. Let τ and $|h|$ be sufficiently small numbers. Then, the solution of difference scheme (11) obeys the coercivity stability estimates

$$||\{\tau^{-2}(u_{k+1}^h-2u_k^h+u_{k-1}^h)\}_1^{N-1}||_{C^\alpha([0,1]_\tau,L_{2h})}+||\{\tau^{-1}(u_k^h-u_{k-1}^h)\}_{-N+1}^0||_{\tilde C^{\frac{\alpha}{2}}([-1,0]_\tau,L_{2h})}+$$
$$+||\{u_k^h\}_1^{N-1}||_{C^\alpha([0,1]_\tau,W_{2h}^2)}+||\left\{\frac{1}{2}(u_k^h+u_{k-1}^h)\right\}_{-N+1}^0||_{\tilde C^{\frac{\alpha}{2}}([-1,0]_\tau,W_{2h}^2)}\le$$
$$\le M_3\left\{||f_0^h+g_1^h||_{W_{2h}^1}+\tau||f_0^h+g_1^h||_{W_{2h}^2}+||f_{-N+1}^h+g_{N-1}^h||_{W_{2h}^1}+\tau||f_{-N+1}^h+g_{N-1}^h||_{W_{2h}^2}+\right.$$
$$\left.+\frac{1}{\alpha(1-\alpha)}\left[||\{f_k^h\}_{-N+1}^0||_{\tilde C^{\frac{\alpha}{2}}([-1,0]_\tau,L_{2h})}+||\{g_k^h\}_1^{N-1}||_{C^\alpha([0,1]_\tau,L_{2h})}\right]\right\},$$
$$||\{\tau^{-2}(u_{k+1}^h-2u_k^h+u_{k-1}^h)\}_1^{N-1}||_{C^\alpha([0,1]_\tau,L_{2h})}+||\{\tau^{-1}(u_k^h-u_{k-1}^h)\}_{-N+1}^0||_{C^{\frac{\alpha}{2}}([-1,0]_\tau,L_{2h})}+$$
$$+||\{u_k^h\}_1^{N-1}||_{C^\alpha([0,1]_\tau,W_{2h}^2)}+||\{\frac{1}{2}(u_k^h+u_{k-1}^h\}_{-N+1}^0||_{C^{\frac{\alpha}{2}}([-1,0]_\tau,W_{2h})}\le$$
$$\le M_4\left\{||f_0^h+g_1^h||_{W_{2h}^1}+\tau||f_0^h+g_1^h||_{W_{2h}^2}+||f_{-N+1}^h+g_{N-1}^h||_{W_{2h}^1}+\tau||f_{-N+1}^h+g_{N-1}^h||_{W_{2h}^2}+\right.$$
$$\left.+\frac{1}{\alpha(1-\alpha)}\left[||\{f_k^h\}_{-N+1}^0||_{C^{\frac{\alpha}{2}}([-1,0]_\tau,L_{2h})}+\tau||\{f_k^h\}_{-N+1}^0||_{C^{\frac{\alpha}{2}}([-1,0]_\tau,W_{2h}^2)}+||\{g_k^h\}_1^{N-1}||_{C^\alpha([0,1]_\tau,L_{2h})}\right]\right\},$$

where M_3 and M_4 do not depend on τ, h, α, f_k^h, $-N+1\le k\le 0$, and $g_k^h(x)$, $1\le k\le N-1$.

Applying the symmetry properties of the difference operator A_h^x acting in the space of grid functions $u^h(x)$, Theorem 1, and the theorem on coercivity of elliptic difference problem [8] conclude the proof of Theorem 2.

Numerical results

We have not been able to obtain a sharp estimate for the constants figuring in the inequalities in order to support theoretical statements. So, we will give the following results of numerical experiments of the following nonlocal boundary value problem

$$\begin{cases} \frac{\partial^2 u}{\partial t^2}+\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=(1-2\pi^2)e^t\sin\pi x\sin\pi y, 0<t<1,\ 0<x,y<1; \\ \frac{\partial u}{\partial t}+\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=(1-2\pi^2)e^t\sin\pi x\sin\pi y,\ \ -1<t\le 0,\ 0<x,y<1; \\ u\left(1,x,y\right)-u\left(-1,x,y\right)=(e-e^{-1})\sin\pi x\sin\pi y,\ x,y\in[0,1] \end{cases} \tag{14}$$

for a two dimensional elliptic-parabolic equation with the following Dirichlet conditions

$$\begin{cases} u\left(0-,x,y\right)=u\left(0+,x,y\right),u_t\left(0-,x,y\right)=u_t\left(0+,x,y\right); \\ u\left(t,0,y\right)=u\left(t,1,y\right)=0,\ y\in[0,1],\ t\in[0,1]; \\ u(t,x,0)=u(t,x,1)=0,\ x\in[0,1],\ t\in[0,1]. \end{cases} .$$

The exact solution of problem (14) is $u(t,x,y) = e^t \sin \pi x \sin \pi y$.

Now, we give the results of the numerical analysis in order to compare and conclude the accuracy of solutions for the first and second order of accuracy difference schemes. The numerical solutions are recorded for different values of N and M and $u^k_{n,m}$ represents the numerical solutions of these difference schemes at $u(t_k, x_n, y_m)$.

Table is constructed for $N = M = 10, 20, 30$, respectively and the error is computed by the following formula

$$E = \max_{-N \le k \le N, 1 \le n,m \le M-1} |u(t_k, x_n, y_m) - u^k_{n,m}|.$$

The results of the exact and numerical solutions are given in the following Table.

Table

Error analysis

Method	N=M=10	N=M=20	N=M=30
1^{st} order of accuracy d. s.	0.0938	0.0459	0.0237
2^{nd} order of accuracy d. s.	0.0122	0.0031	0.0014

Therefore, the results confirm that the second order of accuracy difference scheme is more accurate comparing with the first order of accuracy difference scheme.

Conclusion

In the present work, the second order of accuracy difference scheme for the approximate solution of problem (1) has been presented. Also, the theorem on well-posedness of this problem in Hölder spaces has been established and the coercivity estimates for the solution of the second order difference schemes for the approximate solution of the nonlocal boundary value elliptic-parabolic problem have been constructed. Furthermore, the numerical experiments have been given. Some of results of the present article were presented in the conference proceedings [20] and [29] as extended abstracts without proofs and without numerical results of error analysis, respectively.

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Гёльдер кеңістіктеріндегі эллипстік-параболалық түрдегі теңдеулер үшін дәлдігі екінші ретті айырымдық схемалар жөнінде ескертпе

Мақала шеттік шарттары локалдық емес эллипстік-параболалық түрдегі теңдеулерді шешу үшін дәлдігі екінші ретті айырымдық схемаларды зерттеуге арналған. Дәлдігі екінші ретті айырымдық схеманың Гельдер кеңістіктерінде орнықты болатындығы көрсетілген. Шеттік шарттары локалдық емес эллипстік-параболалық түрдегі теңдеудің жуық шешімі үшін Гельдер нормасында коэрцитивті бағалаулар алынған. Теориялық тұжырымдар жұмыста келтірілген сандық есептеулермен расталды.

Кілт сөздер: айырымдық схема, эллипстік-параболалық түрдегі теңдеу, Гельдер кеңістіктері, коэрцитивті теңсіздіктер.

А. Ашыралыев, О. Герсек, Е. Зус

Замечание о разностной схеме второго порядка точности для эллиптико-параболических уравнений в пространствах Гёльдера

Статья посвящена изучению разностной схемы второго порядка точности для решения эллиптико-параболического уравнения с нелокальным граничным условием. Установлена корректность разностной схемы второго порядка точности в пространствах Гёльдера. Получены оценки коэрцитивности в нормах Гёльдера для приближенного решения нелокальной краевой задачи для эллиптико-параболического дифференциального уравнения. Результаты численных экспериментов представлены для поддержки упомянутых выше теоретических утверждений.

Ключевые слова: разностная схема, эллиптико-параболическое уравнение, пространства Гёльдера, коэрцитивные неравенства.

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Unconditional basicity of eigenfunctions' system of Sturm-Liouville operator with an involutional perturbation

In this paper the question on unconditional basicity of the system of eigenfunctions of the involutive perturbed Sturm-Liouville operator is investigated. The Green's function of the operator under consideration in the case of constant coefficients is constructed. The estimates of the Green's functions are obtained. The existence of the Green's function is shown in the case when the operator under consideration has a variable coefficient. The theorem on the equiconvergence of expansions with respect to the eigenfunctions of the indicated operators is proved with the help of the Green's function. The basicity of the eigenfunctions of the operator under consideration in the class $L_2(-1,1)$ is proved. It is established that the basis from the eigenfunctions of the involutive perturbed Sturm-Liouville operator is the unconditional basis.

Keywords: Involution, eigenfunction, eigenvalue, basis, Green's function.

Introduction

In the present paper we study a spectral problem of the form

$$Lu = -u''(x) + \alpha u''(-x) + q(x)u(x) = \lambda u(x), \quad -1 < x < 1, \quad u(-1) = 0, \quad u(1) = 0, \tag{1}$$

where $q(x) \in C[-1,1]$ — is complex-valued function. The parameter α belongs to the interval $(-1,1)$. If $q(x) \equiv 0$, then the spectral problem (1)

$$-u''(x) + \alpha u''(-x) = \lambda u(x), \quad u(-1) = 0, \ \ u(1) = 0 \tag{2}$$

is well-known [1], it has eigenvalues

$$\lambda_{k1} = (1-\alpha)\left(k+\frac{1}{2}\right)^2 \pi^2, \quad \lambda_{k2} = (1+\alpha)k^2\pi^2, \quad k = 0, \pm 1, \pm 2, ... \tag{3}$$

and eigenfunctions

$$\left\{u_{k1}(x) = \cos\left(l+\frac{1}{2}\right)\pi x, \quad k_1 = 0,1,2,...; \quad u_{k2}(x) = \sin k\pi x, \quad k_2 = 1,2,...\right\}, \tag{4}$$

which form a Riesz basis in $L_2(-1,1)$.

We show that the eigenfunctions' systems of the spectral problem (1) forms a basis in $L_2(-1,1)$.

Results on the spectral properties of one-dimensional differential operators with involution (we use the simplest one, that is, with reflection $v(x) = -x$ on $[-1,1]$) are actively applied in research of PDE. The recent papers by Aleorov, Kirane, and Malik [2], Kirane and Al-Sati [3] give natural examples. Various applications of differential operators with involutions can be found in [4].

Spectral theory of differential operators with involution forms a specific niche in the study of ODE. Eigenfunction expansions for the first-order differential operators with involution are considered in [5–7]. An example of second-order differential operators with involution are discussed in [8–10]. A specific example of a boundary-value problem for the second-order differential operator with involution that produces an infinite number of associated functions is given in [11, 12]. We also note valuable results on the Green's function for the boundary value problems related to functional-differential operators with involution (see Cabada and Tojo [13, 14]) and new types of non-classical Sturm-Liouville problems (see Aidemir, Mukhtarov et al. [15, 16]).

Green's function of the boundary value problem with involution

Along with the boundary value problem (2), we consider the non-homogeneous boundary value problem

$$-u''(x)+\alpha u''(-x)=\lambda u(x)+f(x),\quad -1<x<1;$$
$$u(-1)=0,\quad u(1)=0, \tag{5}$$

where $-1<\alpha<1$, with the arbitrary continuous function $f(x)$. We note that equation in (5) contains an involution and corresponds to the homogeneous boundary value problem (2). It is clear that the functions

$$u_1(x)=\cos(\alpha_0\rho x),\quad u_2(x)=\sin(\alpha_1\rho x),\quad \rho=\sqrt{\lambda};$$
$$\alpha_0=\sqrt{\frac{1}{1-\alpha}},\quad \alpha_1=\sqrt{\frac{1}{1+\alpha}},$$

give linearly independent solutions to the homogeneous equation (2).

As usual, the Green's function $G(x,t,\lambda)$ of the boundary value problem (2) is the kernel of the integral

$$u(x)=\int_{-1}^{1}G(x,t,\lambda)f(t)\,dt$$

that provides a solution to the problem (5).

Theorem 1. If λ is not an eigenvalue of the problem (2), then the non-homogeneous boundary value problem (5) is solvable for any continuous function $f(x)$ and its solution can be represented in the form

$$u(x)=\frac{1}{2}\frac{\alpha_0}{\rho}\frac{\sin\alpha_0\rho}{\cos\alpha_0\rho}\cos(\alpha_0\rho x)\int_{-1}^{1}\cos(\alpha_0\rho t)f(t)\,dt-$$
$$-\frac{1}{2}\frac{\alpha_1}{\rho}\frac{\cos\alpha_1\rho}{\sin\alpha_1\rho}\sin(\alpha_1\rho x)\int_{-1}^{1}\sin(\alpha_1\rho t)f(t)\,dt+$$
$$+\frac{1}{2\rho}\left\{\int_{-1}^{-x}[\alpha_0\cos(\alpha_0\rho x)\sin(\alpha_0\rho t)-\alpha_1\sin(\alpha_1\rho x)\cos(\alpha_1\rho t)]f(t)\,dt-\right.$$
$$-\int_{-x}^{x}[\alpha_0\cos(\alpha_0\rho t)\sin(\alpha_0\rho x)-\alpha_1\sin(\alpha_1\rho t)\cos(\alpha_1\rho x)]f(t)\,dt-$$
$$\left.-\int_{x}^{1}[\alpha_0\cos(\alpha_0\rho x)\sin(\alpha_0\rho t)-\alpha_1\sin(\alpha_1\rho x)\cos(\alpha_1\rho t)]f(t)\,dt\right\}.$$

Proof. Since the functions $\cos(\alpha_0\rho x)$, $\sin(\alpha_1\rho x)$ are solutions to the homogeneous equation in (2), it is sufficient to show that the function

$$g_0(x)\equiv\frac{1}{2}\int_{-1}^{1}g(x,t,\lambda)f(t)\,dt=$$
$$=\frac{1}{2\rho}\int_{-1}^{-x}[\alpha_0\cos(\alpha_0\rho x)\sin(\alpha_0\rho t)-\alpha_1\sin(\alpha_1\rho x)\cos(\alpha_1\rho t)]f(t)\,dt+$$
$$+\frac{1}{2\rho}\int_{-x}^{x}[-\alpha_0\cos(\alpha_0\rho t)\sin(\alpha_0\rho x)+\alpha_1\sin(\alpha_1\rho t)\cos(\alpha_1\rho x)]f(t)\,dt+$$

$$+\frac{1}{2\rho}\int_{x}^{1}\left[-\alpha_0\cos\left(\alpha_0\rho x\right)\sin\left(\alpha_0\rho t\right)+\alpha_1\sin\left(\alpha_1\rho x\right)\cos\left(\alpha_1\rho t\right)\right]f\left(t\right)dt$$

satisfies the equation in (1). The direct calculation of its first derivative

$$g'_0\left(x\right)=\frac{1}{2\rho}\int_{-1}^{-x}\left[\alpha_0(\cos\left(\alpha_0\rho x\right))'\sin\left(\alpha_0\rho t\right)-\alpha_1(\sin\left(\alpha_1\rho x\right))'\cos\left(\alpha_1\rho t\right)\right]f\left(t\right)dt+$$

$$+\frac{1}{2\rho}\int_{-x}^{x}\left[-\alpha_0\cos\left(\alpha_0\rho t\right)\left(\sin\left(\alpha_0\rho x\right)\right)'+\alpha_1\sin\left(\alpha_1\rho t\right)\left(\cos\left(\alpha_1\rho x\right)\right)'\right]f\left(t\right)dt+$$

$$+\frac{1}{2\rho}\int_{x}^{1}\left[-\alpha_0(\cos\left(\alpha_0\rho x\right))'\sin\left(\alpha_0\rho t\right)+\alpha_1(\sin\left(\alpha_1\rho x\right))'\cos\left(\alpha_1\rho t\right)\right]f\left(t\right)dt.$$

and its second derivative

$$g''_0\left(x\right)=-\frac{f\left(x\right)+\alpha f\left(-x\right)}{1-\alpha^2}+$$

$$+\frac{1}{2\rho}\int_{-1}^{-x}\left[\alpha_0(\cos\left(\alpha_0\rho x\right))''\sin\left(\alpha_0\rho t\right)-\alpha_1(\sin\left(\alpha_1\rho x\right))''\cos\left(\alpha_1\rho t\right)\right]f\left(t\right)dt+$$

$$+\frac{1}{2\rho}\int_{-x}^{x}\left[-\alpha_0\cos\left(\alpha_0\rho t\right)\left(\sin\left(\alpha_0\rho x\right)\right)''+\alpha_1\sin\left(\alpha_1\rho t\right)\left(\cos\left(\alpha_1\rho x\right)\right)''\right]f\left(t\right)dt+$$

$$+\frac{1}{2\rho}\int_{x}^{1}\left[-\alpha_0(\cos\left(\alpha_0\rho x\right))''\sin\left(\alpha_0\rho t\right)+\alpha_1(\sin\left(\alpha_1\rho x\right))''\cos\left(\alpha_1\rho t\right)\right]f\left(t\right)dt$$

verify the equality in (5). The boundary conditions in (5) can be checked directly. The theorem is proved.

The theorem implies the following corollary.

Corollary 1. The Green's function of the boundary value problem (2) has the form

$$G\left(x,t,\lambda\right)=\frac{1}{2}\frac{\alpha_0}{2\rho}\frac{\sin\alpha_0\rho}{\cos\alpha_0\rho}\left(\cos\alpha_0\rho x\right)\left(\cos\alpha_0\rho t\right)-$$

$$-\frac{1}{2}\frac{\alpha_1}{2\rho}\frac{\cos\alpha_1\rho}{\sin\alpha_1\rho}\left(\sin\alpha_1\rho x\right)\left(\sin\alpha_1\rho t\right)+$$

$$+\frac{1}{2\rho}\begin{cases}\alpha_0\left(\cos\alpha_0\rho x\right)\left(\sin\alpha_0\rho t\right)-\alpha_1\left(\sin\alpha_1\rho x\right)\left(\cos\alpha_1\rho t\right), & t\leq -x;\\ -\alpha_0\left(\cos\alpha_0\rho t\right)\left(\sin\alpha_0\rho x\right)+\alpha_1\left(\sin\alpha_1\rho t\right)\left(\cos\alpha_1\rho x\right), & -x\leq t\leq x;\\ -\alpha_0\left(\cos\alpha_0\rho x\right)\left(\sin\alpha_0\rho t\right)+\alpha_1\left(\sin\alpha_1\rho x\right)\left(\cos\alpha_1\rho t\right), & t\geq x.\end{cases}$$

Using the explicit form of the Green function one can write down the expansion of an arbitrary function $f\left(x\right)$ from $L_1(-1,1)$ in the eigenfunctions of the spectral problem (2). The poles of the Green's function are the zeros of the functions $\cos\alpha_0\rho,\ \sin\alpha_1\rho$:

$$\rho_{k1}=\sqrt{\lambda_{k1}}=\sqrt{(1-\alpha)}\left(k+\frac{1}{2}\right)\pi,\quad k=0,1,2,\ ...;$$

$$\rho_{k2}=\sqrt{\lambda_{k2}}=\sqrt{(1+\alpha)}k\pi,\quad k=1,2,...$$

If the number $\sqrt{\frac{1-\alpha}{1+\alpha}}$ is not even, then all eigenvalues are single. On the complex ρ-plane we consider the circles $P_{k1},\quad k=0,1,2,...;\ P_{k2},\ k=1,2,\ ...$, with a common center at the origin and respective radius:

$$P_{k1}:|\rho|=\sqrt{1-\alpha}\left(k+\frac{1}{2}\right)\pi+\frac{1}{8};\quad P_{k2}:|\rho|=\sqrt{1+\alpha}\,k\pi+\frac{1}{8}.$$

These circles do not overlap and do not pass through the points ρ_{k1} and $\rho_{k2.}$ When $\lambda = \rho^2$ the circles P_{k1}, P_{k2} turn into the circles

$$\widetilde{P}_{k1} : |\lambda| = \left(\sqrt{1-\alpha}\left(k+\frac{1}{2}\right)\pi + \frac{1}{8}\right)^2; \quad \widetilde{P}_{k2} : |\lambda| = \left(\sqrt{1+\alpha}k\pi + \frac{1}{8}\right)^2$$

in the λ-plane, respectively. For any function $f(x) \in L_1(-1.1)$, the partial sums of the eigenfunctions's expansions for the spectral problem (2) can be written as [17]

$$\sigma_m(f) = -\frac{1}{2\pi i}\int\limits_{\widetilde{P}_m}\left(\int\limits_{-1}^{1} G(x,t,\lambda) f(t)\, dt\right) d\lambda = -\frac{1}{2\pi i}\int\limits_{P_m}\left(\int\limits_{-1}^{1} G\left(x,t,\rho^2\right) f(t)\, dt\right) 2\rho d\rho,$$

where P_m — is the circle with the radius $\rho_m = \max\left\{\rho_{k1} + \frac{1}{8}, \rho_{k2} + \frac{1}{8}\right\}$.

Further, changing the order of integration and using the residue theorem, we calculate the integral over the circle P_m

$$\sigma_m(f) = -\frac{1}{2\pi i}\int\limits_{-1}^{1}\left[\int\limits_{P_m} G(x,t,\lambda)\, 2\rho d\rho\right] f(t)\, dt =$$

$$= \int\limits_{-1}^{1}\sum_{k=0}^{m}\cos\left(k+\frac{1}{2}\right)\pi x \cos\left(k+\frac{1}{2}\right)\pi t\, f(t)\, dt +$$

$$+\int\limits_{-1}^{1}\sum_{k=1}^{m}\sin k\pi x\, \sin k\pi t\, f(t)\, dt = \sum_{k=0}^{m}\left(\int\limits_{-1}^{1} f(t)\cos\left(k+\frac{1}{2}\right)\pi t dt\right)\cos\left(k+\frac{1}{2}\right)\pi x +$$

$$+\sum_{k=1}^{m}\left(\int\limits_{-1}^{1} f(t)\sin k\pi t dt\right)\sin k\pi x.$$

Thus, the partial sums of the eigenfunction expansions for the spectral problem (2) of the arbitrary integrable function $f(x)$ has the form

$$\sigma_m(f) = \sum_{k=0}^{m} a_k \cos\left(k+\frac{1}{2}\right)\pi x + \sum_{k=1}^{m} b_k \sin k\pi x, \qquad (6)$$

where

$$a_k = \int\limits_{-1}^{1} f(t)\cos\left(k+\frac{1}{2}\right)\pi t dt, \quad b_k = \int\limits_{-1}^{1} f(t)\sin k\pi t dt.$$

Note that the system $\left\{\sin k\pi x,\, \cos\left(n+\frac{1}{2}\right)\pi x\right\}$, $k = 1, 2, ...$, $n = 0, 1, 2...$, is a complete orthogonal system in $L_2(-1,1)$. Therefore, for all $f(x) \in L_2(-1,1)$ the partial sums $\sigma_m(f)$ of the form (6) converge to the function $f(x)$ with respect to the norm of the space $L_2(-1,1)$.

Further we need an estimate of the Green's function.

Let $\rho = \Re\rho + i\,\Im\rho$ and denote $\rho_0 = \Im\rho$.

Let $O_\varepsilon(\rho_k) = \{\rho : |\rho - \rho_{ki}| < \varepsilon, i = 1, 2\}$ be a circle of sufficiently small radius ε.

Lemma 2. If $\rho \notin O_\varepsilon(\rho_k)$, then the Green's function $G(x,t,\lambda)$ of the boundary value problem (2) satisfies the following uniform estimate

$$|G(x,t,\lambda)| \le C|\rho|^{-1} r(x,t,\rho)$$

with $-1 \le x$, $t \le 1$, where

$$r(x,t,\rho) = \left(e^{-\alpha_2|\rho_0|(2-|x|-|t|)} + e^{-\alpha_2|\rho_0|(||x|-|t||)}\right), \ \alpha_2 = \min\{\alpha_1, \alpha_0\}.$$

Proof. In the case when $t \geq x$ the Green's function can be rewritten in the form

$$G(x,t,\lambda) = \frac{\alpha_0}{4i\rho}\left\{-\frac{e^{-i\alpha_0\rho}}{e^{i\alpha_0\rho}+e^{-i\alpha_0\rho}}\left[e^{i\alpha_0\rho(x+t)}+e^{i\alpha_0\rho(t-x)}\right]+\right.$$

$$\left.+\frac{e^{i\alpha_0\rho}}{e^{i\alpha_0\rho}+e^{-i\alpha_0\rho}}\left[e^{i\alpha_0\rho(x-t)}+e^{i\alpha_0\rho(-x-t)}\right]\right\}+$$

$$+\frac{\alpha_1}{4i\rho}\left\{-\frac{e^{-i\alpha_1\rho}}{e^{i\alpha_1\rho}-e^{-i\alpha_1\rho}}\left[e^{i\alpha_1\rho(x+t)}-e^{i\alpha_1\rho(t-x)}\right]-\right.$$

$$\left.-\frac{e^{i\alpha_1\rho}}{e^{i\alpha_1\rho}-e^{-i\alpha_1\rho}}\left[-e^{i\alpha_1\rho(x-t)}+e^{i\alpha_1\rho(-x-t)}\right]\right\}.$$

For sufficiently large $|\rho|$ the Green's function satisfies the following inequality

$$|G(x,t,\lambda)| \leq \frac{\alpha_0}{4|\rho|}\left\{\frac{-e^{\alpha_0\rho_0}}{|e^{-\alpha_0\rho_0}-e^{\alpha_0\rho_0}|}\left[e^{-\alpha_0\rho_0(x+t)}+e^{-\alpha_0\rho_0(t-x)}\right]+\right.$$

$$\left.+\frac{e^{-\alpha_0\rho_0}}{|e^{-\alpha_0\rho_0}-e^{\alpha_0\rho_0}|}\left[e^{-\alpha_0\rho_0(x-t)}+e^{-\alpha_0\rho_0(-x-t)}\right]\right\}+$$

$$+\frac{\alpha_1}{4|\rho|}\left\{\frac{e^{\alpha_1\rho_0}}{|e^{-\alpha_1\rho_0}-e^{\alpha_1\rho_0}|}\left[e^{-\alpha_1\rho_0(x+t)}+e^{-\alpha_1\rho_0(t-x)}\right]+\right.$$

$$\left.+\frac{e^{-\alpha_1\rho_0}}{|e^{-\alpha_1\rho_0}-e^{\alpha_1\rho_0}|}\left[e^{-\alpha_1\rho_0(x-t)}+e^{-\alpha_1\rho_0(-x-t)}\right]\right\}.$$

Since $t > x > 0$, one has $t + x > t - x$, $x - t > -x - t$. Therefore,

$$|G(x,t,\lambda)| \leq \frac{\alpha_0}{4|\rho|}\left[e^{-\alpha_0\rho_0(2-x-t)}+e^{-\alpha_0\rho_0(t-x)}\right]+\frac{\alpha_1}{4|\rho|}\left[e^{-\alpha_1\rho_0(2-x-t)}+e^{-\alpha_1\rho_0(t-x)}\right]$$

if $\rho_0 > 0$ and

$$|G(x,t,\lambda)| \leq \frac{\alpha_0}{4|\rho|}\left[e^{\alpha_0\rho_0(2-x-t)}+e^{\alpha_0\rho_0(t-x)}\right]+\frac{\alpha_1}{4|\rho|}\left[e^{\alpha_1\rho_0(2-x-t)}+e^{\alpha_1\rho_0(t-x)}\right]$$

if $\rho_0 < 0$.

Thus, for $t > x > 0$ the Green's function satisfies the following estimate

$$|G(x,t,\lambda)| \leq \frac{c_1}{|\rho|}\left(e^{-\alpha_2|\rho_0|(2-x-t)}+e^{-\alpha_2|\rho_0|(t-x)}\right), \quad \alpha_2 = \min\{\alpha_0, \alpha_1\}.$$

In the case of $-x < t < x$ the proof of lemma is similar to the previous case while the estimate of the Green's functions takes the form

$$|G(x,t,\lambda)| \leq \frac{c_2}{|\rho|}\left[e^{-\alpha_2|\rho_0|(2-x-|t|)}+e^{-\alpha_2|\rho_0|(x-|t|)}\right].$$

In the case of $t < -x$ the estimate transforms into the following inequality

$$|G(x,t,\lambda)| \leq \frac{c_3}{|\rho|}\left[e^{-\alpha_2|\rho_0|(2-|t|-x)}+e^{-\alpha_2|\rho_0|(|t|-x)}\right].$$

The last three inequalities provide the desired estimate

$$|G(x,t,\lambda)| \leq \frac{C}{|\rho|}\left[e^{-\alpha_2|\rho_0|(2-|x|-|t|)}+e^{-\alpha_2|\rho_0|||x|-|t||}\right].$$

Lemma is proved.

Theorems on the basis property of the eigenfunctions of the spectral problem (1)

We are interested in the possibility of expanding the arbitrary function $f(x) \in L_2(-1,1)$ in converging series related to the spectral problem (1) in the case when the complex-valued coefficient $q(x)$ is continuous over the interval $(-1,1)$.

We assume that there exists the Green's function $G_q(x,t,\lambda)$ of the boundary value problem (1). Let $G(x,t,\lambda)$ be the Green's function of the problem (2). Since almost everywhere on the interval $(-1,1)$ we have the relations:

$$-\frac{\partial^2 G(x,t,\lambda)}{\partial x^2} + \alpha \frac{\partial^2 G(-x,t,\lambda)}{\partial x^2} = \lambda G(x,t,\lambda);$$
$$-\frac{\partial^2 G_q(x,t,\lambda)}{\partial x^2} + \alpha \frac{\partial^2 G_q(-x,t,\lambda)}{\partial x^2} + q(x) G_q(x,t,\lambda) = \lambda G_q(x,t,\lambda)$$

then

$$-\frac{\partial^2 (G_q(x,t,\lambda) - G(x,t,\lambda))}{\partial x^2} + \alpha \frac{\partial^2 (G_q(x,t,\lambda) - G(x,t,\lambda))_{x=-x}}{\partial x^2} -$$
$$-\lambda (G_q(x,t,\lambda) - G(x,t,\lambda)) = -q(x) G_q(x,t,\lambda).$$

The difference $G_q(x,t,\lambda) - G(x,t,\lambda)$ clearly satisfies the boundary condition (1). Therefore outside the poles of the function $G(x,t,\lambda)$ the Green's function $G_q(x,t,\lambda)$ satisfies the equality

$$G_q(x,t,\lambda) - G(x,t,\lambda) = -\int_{-1}^{1} G(x,s,\lambda) q(s) G_q(s,t,\lambda) ds. \tag{7}$$

Existence of the Green's function for the boundary value problem (1) is equivalent to the existence of a solution to the integral equation (7). We come to the following theorem.

Theorem 3. If the number $\sqrt{\frac{1-\alpha}{1+\alpha}}$ is not even, then for all sufficiently large ρ, $\rho \notin O_\varepsilon(\rho_k)$, then there exists a solution to the integral equation (7).

Proof. Let $G_{q0}(x,t,\lambda) \equiv 0$ and

$$G_{q,p+1}(x,t,\lambda) = G(x,t,\lambda) - \int_{-1}^{1} G(x,s,\lambda) q(s) G_q(s,t,\lambda) ds \tag{8}$$

for all sufficiently large $|\rho|$.

For the Green's function $G(x,t,\lambda)$ of the problem (2) the estimate holds

$$|G(x,t,\lambda)| \le \frac{C}{|\rho|} r(x,t),$$

where

$$r(x,t) = e^{-\alpha_2 |\rho_0| ||x|-|t||} + e^{-\alpha_2 |\rho_0| (2-||x|-|t||)}.$$

Relation (8) with $p=0$ yields the estimate

$$|G_{q1}(x,t,\lambda)| = |G(x,t,\lambda)| \le \frac{C}{|\rho|} r(x,t).$$

For brevity, we introduce the notation

$$\max |G_{q1}(x,t,\lambda)| |\rho| r^{-1}(x,t) = C_0;$$
$$\max |G_{qp+1}(x,t,\lambda) - G_{qp}(x,t,\lambda)| |\rho| r^{-1}(x,t) = C_p, \tag{9}$$

where the maximum is taken with respect to $x \in [-1,1]$, for fixed t and sufficiently large $|\rho|$ laying outside the poles of the function $G(x,t,\lambda)$.

Let us show that

$$C_j \le \frac{C}{2^j}, \quad j = 0, 1, 2, ...p. \tag{10}$$

For $j = 0$ estimate (10) follows from the first estimate (9). Let us assume that estimate (10) holds in the case $j = 1, 2, ...p$ and prove it for $j = p + 1$. Taking into account the notation in (9), we get the inequality

$$C_{p+1} \le C \cdot C_p |\rho|^{-1} \max \int_{-1}^{1} r(x,s)\, r(s,t)\, r^{-1}(x,t)\, |q(s)|\, ds. \tag{11}$$

Here we have the relation

$$r(x,s) \cdot r(s,t) = \left(e^{-\alpha_0|\rho_0|(2-|x|-|s|)} + e^{-\alpha_0|\rho_0|||x|-|s||}\right) \times$$

$$\times \left(e^{-\alpha_0|\rho_0|(2-|s|-|t|)} + e^{-\alpha_0|\rho_0|||s|-|t||}\right) =$$

$$= e^{-\alpha_0|\rho_0|(4-|x|-2|s|-|t|)} + e^{-\alpha_0|\rho_0|(2-|x|-|s|+||s|-|t||)} +$$

$$+ e^{-\alpha_0|\rho_0|(2-|s|-|t|+||x|-|s||)} + e^{-\alpha_0|\rho_0|(||x|-|s||+||s|-|t||)}.$$

The triangle inequality yields

$$||x| - |t|| \le ||x| - |s|| + ||s| - |t||\,.$$

The inequality

$$|t| = |t| + |s| - |s| \ge |s| - ||t| - |s||$$

implies

$$|x| + |t| \ge |x| + |s| - ||t| - |s||\,;$$

and the inequality

$$|x| \ge |s| - ||x| - |s||\,;$$

implies

$$|x| + |t| \ge |t| + |s| - ||x| - |s||\,.$$

Therefore

$$||x| - |t|| = ||x| - 1 + 1 - |t|| < 1 - |x| + 1 - |t| < 1 - |x| + 1 - |t| + 2 - 2\,|s| = 4 - |x| - |t| - 2\,|s|\,.$$

Hence

$$r(x,s) \cdot r(s,t) \le 2r(x,t).$$

This inequality and the inequality (11) imply

$$C_{p+1} \le 2CC_p|\rho|^{-1} \int_{-1}^{1} |q(s)|\, ds.$$

For sufficiently large $|\rho|$, the inequality

$$2C|\rho|^{-1} \int_{-1}^{1} |q(s)|\, ds \le \frac{1}{2}$$

holds true.

Consequently, $C_{p+1} \le \frac{C_p}{2}$ for any p and hence the desired inequality (10) is verified.

It follows from the inequality (10) that the series

$$\sum_{1}^{\infty} \left(G_{q,p+1}(x,t,\lambda) - G_{qp}(x,t,\lambda)\right)$$

uniformly converges and hence its partial sum

$$S_n(x) = G_{q,p+n}(x,t,\lambda) - G_{q1}(x,t,\lambda)$$

converges also.

Therefore the sequence $G_{qp}(x,t,\lambda)$ uniformly converges to its limit $G_q(x,t,\lambda)$ which satisfies the equation (2). The theorem is proved.

Let

$$\sigma_m(f) = -\frac{1}{2\pi i}\int_{-1}^{1}\left[\int_{P_m} G(x,t,\lambda)2\rho d\rho\right] f(t)\,dt$$

be the partial sum of eigenfunction expansions related to the spectral problem (2), where $f(x) \in L_1(-1,1)$. Denote by

$$S_m(f) = -\frac{1}{2\pi i}\int_{-1}^{1}\left[\int_{P_m} G_q(x,t,\lambda)2\rho d\rho\right] f(t)\,dt$$

the partial sum of the eigenfunction expansion related to the spectral problem (1).

The sequence $S_m(f)$ is said to be *equiconvergent* with the sequence $\sigma_m(f)$ on an interval $-1 \le x \le 1$ if the difference $S_m - \sigma_m$ vanishes uniformly on the interval as $m \to \infty$.

Theorem 4. If the number $\sqrt{\frac{1-\alpha}{1+\alpha}}$ *is not even, then for any function* $f(x) \in L_1(-1,1)$ *sequence* $S_m(f)$ *equiconverges with the sequence* $\sigma_m(f)$ *on the interval* $-1 \le x \le 1$.

Proof. Consider the relation

$$S_m(f) - \sigma_m(f) = -\frac{1}{2\pi i}\int_{P_m}\left\{\int_{-1}^{1}\left[G_q(x,t,\lambda) - G(x,t,\lambda)\right] f(t)\,dt\right\} 2\rho d\rho. \tag{12}$$

It follows from the proof of Theorem 2 that

$$|G_q(x,t,\lambda)| \le \frac{2C}{|\rho|} r(x,t).$$

This estimate and the equality (7) yield that

$$|G_q(x,t,\lambda) - G(x,t,\lambda)| \le 4^2 |\rho|^{-2} r(x,t) \int_{-1}^{1} |q(s)|\,ds.$$

Then the equality (12) gives the estimate

$$|S_m(f) - \sigma_m(f)| \le \frac{2C^2}{\pi}\int_{P_m}\left[\int_{-1}^{1} r(x,t)|f(t)|\,dt\right]\frac{2|\rho|}{|\rho|^2}d\rho \cdot \int_{-1}^{1}|q(s)|\,ds =$$

$$= \frac{4C^2}{\pi}\int_{-1}^{1}|q(s)|\,ds \int_{P_m}\left[\int_{-1}^{1} r(x,t)|f(t)|\,dt\right]\left|\frac{d\rho}{\rho}\right|.$$

If we use the notation

$$C_1 = \frac{4C^2}{\pi}\int_{-1}^{1}|q(s)|\,ds,$$

then

$$|S_m(f) - \sigma_m(f)| \le C_1 \int_{P_m}\left[\int_{-1}^{1} r(x,t)|f(t)|\,dt\right]\left|\frac{d\rho}{\rho}\right|.$$

Let us divide the interval $(0,1)=\Delta_1+\Delta_2$ into two parts:

$$\Delta_1=(-1+\delta,-x-\delta)\cup(-x+\delta,x-\delta)\cup(x+\delta,1-\delta);$$

$$\Delta_2=(-1,-1+\delta)\cup(-x-\delta,-x+\delta)\cup(x-\delta,x+\delta)\cup(1-\delta,1)$$

with a sufficiently small positive value of $\delta>0$. Then

$$|S_m(f)-\sigma_m(f)|\le C_1\int\limits_{P_m}\int\limits_{\Delta_1}\left(e^{-\alpha_0|\rho_0|||x|-|t||}+e^{-\alpha_0|\rho_0|(2-||x|-|t||)}\right)\times$$

$$\times|f(t)|\,dt\left|\frac{d\rho}{\rho}\right|+2C_1\pi\int\limits_{\Delta_2}|f(t)|\,dt. \tag{13}$$

Since

$$\int\limits_{\Delta_2}|f(t)|\,dt=\int\limits_{-1}^{-1+\delta}|f(t)|\,dt+\int\limits_{-x-\delta}^{-x+\delta}|f(t)|\,dt+\int\limits_{x-\delta}^{x+\delta}|f(t)|\,dt+\int\limits_{1-\delta}^{1}|f(t)|\,dt,$$

the choice of δ can make the second term in (13) less than $\frac{\varepsilon}{2}$.

If ρ_m is the radius of the circle P_m, then the partition of the integral

$$\int\limits_{P_m}e^{-\alpha_0|\rho_0|\delta}\left|\frac{d\rho}{\rho}\right|=\int\limits_0^{\frac{\pi}{4}}e^{-\alpha_0\delta\rho_m|\sin t|}dt+\int\limits_{\frac{\pi}{4}}^{\frac{3\pi}{4}}e^{-\alpha_0\delta\rho_m|\cos t|}dt+$$

$$+\int\limits_{\frac{3\pi}{4}}^{\frac{5\pi}{4}}e^{-\alpha_0\delta\rho_m|\sin t|}dt+\int\limits_{\frac{5\pi}{4}}^{\frac{7\pi}{4}}e^{-\alpha_0\delta\rho_m|\cos t|}dt+\int\limits_{\frac{7\pi}{4}}^{2\pi}e^{-\alpha_0\delta\rho_m|\sin t|}dt$$

provides the estimate

$$\int\limits_{P_m}e^{-\alpha_0|\rho_0|\delta}\left|\frac{d\rho}{\rho}\right|<\frac{C_2}{|\rho_m|\,\delta}.$$

With sufficiently large value of m, the first term in (13) can be made less than $\frac{\varepsilon}{2}$.

The theorem is proved.

Remark. In [18, 19] the boundary value problem

$$-u''(-x)+q(x)\,u(x)=\lambda u(x);$$
$$u(-1)=u(1),\ \ u'(-1)=u'(1)$$

is considered and the theorems similar to Theorems 2 and 3 are obtained.

Theorem 5. If the number $\sqrt{\frac{1-\alpha}{1+\alpha}}$ is not even, then the system of eigenfunctions of the spectral problem (1) forms the basis in $L_2(-1,1)$.

Proof. Let $\|\cdot\|_2$ denote the norm in $L_2(-1,1)$. Then for any function $f(x)\in L_2(-1,1)$, one obtains the estimate

$$\|f-S_m\|_2\le\|f-\sigma_m\|_2+\|\sigma_m-S_m\|_2<\varepsilon$$

as the first term is less than $\frac{\varepsilon}{2}$ by virtue of the basis property of the eigenfunctions of the spectral problem (2), and the second term is less than $\frac{\varepsilon}{2}$ by virtue of the equiconvergence Theorem 3. Theorem 4 is proved.

Unconditional basicity of the system of eigenfunctions of the spectral problem (1) does not follow from Theorem 4. By Theorem 4 the system of eigenfunctions of the spectral problem (1) forms a basis in $L_2(-1,1)$. It is well-known that for any basis u_k in a Hilbert space $L_2(-1,1)$ the estimate

$$\|u_k\|_{L_2(-1,1)}\|v_k\|_{L_2(-1,1)}\le C$$

holds [20], where v_k is biortogonally adjoint system to u_k. Since the system of eigenfunctions of the spectral problem (1) forms a basis in $L_2(-1,1)$, then by Theorems of L.V. Kritskov and A.M. Sarsenbi [21] this basis is an unconditional basis in the same space. Thus, we get the following result

Theorem 6. Let all the conditions of Theorem 4 be satisfied. Then the system of eigenfunctions of the spectral problem (1) forms an unconditional basis in $L_2(-1,1)$.

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Ә.Ә. Сәрсенбі

Инволютивті толқытылған Штурм-Лиувилл операторының меншікті функцияларының шартсыз базис болуы

Мақалада шеттік шарттары Дирихле түрінде болатын инволюциясы бар Штурм-Лиувилл операторы меншікті функциялар жүйесінің базис болуы туралы мәселе зерттелген. Коэффициенттері тұрақты инволюциясы бар Штурм-Лиувилл операторының Грин функциясы құрылып, Грин функциясы үшін бағалаулар алынған. Коэффициенттері айнымалы инволюциясы бар Штурм-Лиувилл операторының да Грин функциясының бар болуы туралы теорема дәлелденген. Осы нәтижелердің көмегімен айтылып отырған екі оператордың меншікті функциялары бойынша жіктеулері бірқалыпты қабаттаса жинақталатындығы көрсетілген. Коэффициенті айнымалы инволюциясы бар Штурм-Лиувилл операторы меншікті функцияларының жүйесі $L_2(-1,1)$ кеңістігінде базис болатындығы көрсетілген. Және мұндай базистің шартсыз базис болатындығы дәлелденген.

Кілт сөздер: инволюция, меншікті функциялар, меншікті мәндер, базис, Грин функциясы.

А.А. Сарсенби

Безусловная базисность собственных функций инволютивно возмущенного оператора Штурма-Лиувилля

В статье исследован вопрос о безусловной базисности системы собственных функций инволютивно возмущенного оператора Штурма-Лиувилля. Построена функция Грина изучаемого оператора в случае постоянных коэффициентов. Получены оценки функций Грина. При наличии переменного коэффициента у изучаемого оператора показано существование функции Грина. Доказаны теорема о равносходимости разложений по собственным функциям указанных операторов с помощью функции Грина, а также базисность собственных функций в классе $L_2(-1,1)$ изучаемого оператора. Установлено, что базис из собственных функций инволютивно возмущенного оператора Штурма-Лиувилля является безусловным базисом.

Ключевые слова: инволюция, собственная функция, собственные значения, базис, функция Грина.

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