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Existential prime convex Jonsson theories and their models

This article is devoted to the study of the model-theoretic properties of special subclasses of Jonsson theories and their classes of models. Namely, introduce a new class of Jonsson theories is existentially prime convex Jonsson theories. The notion of convexity of theory was previously known, it was introduced by A.Robinson. The notion of existential simplicity introduced by the author of this article. Both conditions form a broad natural classes of theories. Thus two natural restrictions on Jonsson define a new class of theories with rich model-theoretic properties. Besides defined formulation of the problem, which define a new direction in the study Jonsson theories.

Key words: Jonsson sets, the model-theoretic properties, the existentially-prime closed model.

In the study of complete theories was obtained classical result of R.Vought [1] about the description of counting atomic and prime models. Recall that the model is called prime if it is elementary embedded in any model of this theory. And the model is called atomic if any of its finite sequence implements some a main type. So theorem of R. Vought says that the model is prime if and only if it is countable and atomic.

Let L is countable language of first-order. We will consider theory of this language, which satisfy the natural requirements as a certain kind of axiomatizability and completeness, property of joint embeding and amalgam. Such theories are not generally complete of theories.

In the study of Jonsson theories [2] occur naturally notions of algebraic of prime models and various types of atomic models. This problem is related with the general problem of describing countable models or as they say, small models. In the case where the theory is not necessarily complete usually considered isomorphic embeddings instead elementary. Recall that the model of the theory is called of algebraically prime if it is isomorphic to be embedded in any model of the theory.

In a famous study [3] the authors introduce various types of atomic models and try to get the results are similar to R.Vought for algebraic simplicity. The main conclusion of this study was to understand the fact that the theorem is similar in the case of a complete theory can not be obtained with such a formulation of the problem. Considered theories were usually inductive. Later it became clear that for the positive progress of this problem, you must naturally narrow circle of the theories. The following results were known.

Definition 1. T satisfies joint embedding property (JEP), if for any two models of T there is a third model of T in which the first two are isomorphic invested.

Lemma 1. The theory T satisfies the JEP if and only if it has a universal model.

Definition 2. T is complete for existential sentences, if for any existential sentence σ or $T \models \sigma$ or $T \models \neg\sigma$.

Complete for existential sentences, the theory satisfies the JEP, but the reverse is not true.

A natural choice among inductive theories with such conditions is Jonsson theory. Let us recall its definition.

The theory T is call Jonsson, if:

1) the theory T has infinite models;

2) the theory T is inductive;

3) the theory T has the joint embedding property (JEP)

4) the theory T has the amalgam property(AP).

A. Robinson [4] introduced the notion of convex theories.

Definition 3. Theory T is convex if for any model $\overline{A}$ of T and any family $\{\overline{B}_i : i \in I\}$ of substructures models $\overline{A}$ which are models of the theory $\overline{A}$, of their intersection $\cap B_i$ is a model of T, provided that if the intersection is not empty. If, moreover, the intersection is never empty, then T to be strongly convex.

The convex theories are theories with the following important algebraic property that every is non-empty subset model of T generates only a substructure, which is a model of T (i.e. the intersection of all models of T is contained in the model that contains the set). If T is strongly convex, then the intersection of all models of T is contained in the model of T, which is also a model of T (since every model of T contains the core model and the core model is strictly not contain any model of T).

If T satisfies the JEP and it is strongly convex, then core model of the theory T is unique up to isomorphism. This model is isomorphic to exactly one substructure of each model of T, and it is uniquely determined as the higher structure with this property. When we talk about the notion of structure, keep in mind the language model.

Note that the core model is a prime algebraic model of the theory.

Definition 4. $\overline{C}$ is a core structure for theory T if $\overline{C}$ is isomorphic only one substructure to each model of T.

Note that the core structure of T will not necessarily be a model of T and the same theory (even complete) can have different not isomorphic to the core structures. Core structure for T has a structure that can be selected only inside each model of T. The following result [5] characterizes the core structure.

Theorem 1. For any theory T the following conditions are equivalent:

1) $\overline{C}$ is a core structure of theory T;

2) $\overline{C}$ is a model of each universal joint proposal with T and there is existential formula $\varphi_i(x)$ and $k_i \in \omega$ such $i \in I$ that:

$$\overline{C}, T \models \exists^{=k_1} \text{ for all } i \in I \text{ and } \overline{C} \models \forall x \bigvee_{i \in I} \varphi_i.$$

Note that if T is strongly convex and has exactly one core model (in particular, if T satisfies JEP), then this core model is the core structure for T. Therefore, we can apply Theorem 1 for core models of strongly convex theories and get the next Robinson result.

Corollary 1. Let $\overline{C}$ is the core model of strongly convex of T. Then there are existential formulas $\varphi_i(x), i \in I$ such that: $T \models \exists^{<\omega} x \varphi_i$ for all $i \in I$ and $\overline{C} \models \forall x \bigvee_{i \in I} \varphi_i$.

Other examples the core structure are given by models rigidly embedded in all models of the theory T. $\overline{C}$ is firmly embedded in $\overline{A}$, if there is exactly one isomorphism is mapping $\overline{C}$ in $\overline{A}$. Then, $\overline{C}$ is firmly embedded in every model of T if and only if $\overline{C}$ is the core structure of T and has not its own automorphism. Therefore, we get the following result Kreysela [6].

Corollary 2. $\overline{C}$ is firmly embedded in every model of T if and only if the condition 2) of theorem 1 is true $k_i = 1$ for all $i \in I$.

It should be noted that if there is any core structure for T, then there is a unique maximal core structure of T (i.e. core structure in which every other core structure for T can be embedded), namely the union of all core structures contained in a model of T. If T is strongly convex and satisfies the JEP, the core model of the theory T has the maximum core structure.

Definition 5. $\overline{A}$ is a core model of T if the $\overline{A}$ model of T and $\overline{A}$ is isomorphic to only one sub-model of each model of T.

From [5] can be learned that is true.

Theorem 2. Let T is $\forall\exists$-theory. Then the following conditions are equivalent:

1) T is the core model;

2) as the only $\psi(x)$ is existential and $T \models \exists x\psi$ then there is a certain existential $\varphi(x)$ and integer k such that $T \models \exists^{=k} x\varphi \wedge \exists x(\varphi \wedge \psi)$ and (#) if $T \models (\sigma_1 \vee \sigma_2)$, where σ_1, σ_2 are existential sentence, $T \models \sigma_1$ or $T \models \sigma_2$. We recall some necessary definitions.

Let L is the signature and K is class models of L (L-structures). We say that K is an inductive class if K closed under unions of chains.

The theory T is called $\forall\exists$-axiomatizable if T it can be axiomatized $\forall\exists$-sentences.

Definition 6. The model A of theory T is called existentially closed if whenever $A \subseteq B$ and $B \models T$, we have $A_A \models \sigma$ for each existential formula σ of theory $Th(B_A)$.

Remark. If the model A is existentially closed, and $A' \cong A$ the model A' also is existentially closed.

Theorem 3 [1]. Let K is inductive class of L-structures, A is structure of K. Then there is existentially closed structure B in K such that $A \subseteq B$.

We introduce the following definition of a new class of theories.

Definition 7. The inductive theory T is called the existential-prime if:

1) it has a prime algebraically (AP) model, a class of its AP denote ;

2) $T_{AP} \cap E_T \oslash$, where E_T is class of existentially closed of T.

So how Jonsson theories are inductive, we can consider the Jonsson theory that existentially prime and then among them to consider convex. The most glaring example showing that many of these theories is an example of the theory of groups. This example is characterized in that it is an example of imperfect Jonsson theory. In the case of the theory of Abelian groups, we have a example of a perfect convex Jonsson theory. But even in the case of divisible Abelian groups [7], there is the question of the existence of their class AP that says nontriviality review Booking existential prime convex Jonsson theories. Thus, all of the above indicates the relevance of the themes in line with the study of incomplete inductive theories.

It is clear that the study of a new class theory should ask certain research programs, namely to formulate objectives. It is clear that we must begin with the perfect event, ie consider the theory to be perfect. In this case, we know that this theory has a model companion and it is exactly its center [2]. A further refinement is that we consider universally axiomatized Jonsson theory. Such theories are called Robinson's. Further we will work with the special subsets of the semantic model of the above theories.

Let T is complete Jonsson theory for the existential sentences in L and its semantic model has C. Suppose X is a subset of the semantic model of T. We say that the set X is Σ-definable if it is definable some existential formula. The set X is called Jonsson in T if it satisfies the following properties:

X is Σ-definable subset of C;

$dcl(X)$ is the bearer of a some existential closed submodel of C where $dcl(X)$ is the set of all X-definable elements $a \in C$ such that for some formula $\varphi(x) \in L(X)$, it follows that $\varphi(C) = \{a\}$. Let X be a Jonsson set and M is some existentially closed submodel of semantic model of C, with $dcl(X) = M$.

Consider $Th_{\forall\exists}(M) = T_M$. We call TM Jonsson fragment of Jonsson plurality X. We say that all $\forall\exists'$-corollary arbitrary theory create Jonsson fragment of this theory, if the deductive closure of these $\forall\exists$-corollary there are Jonsson theory.

As part of the above definitions and considerations we have the following results. The assumption of completeness of some of this theory is necessary due to the following fact.

Lemma 2. In the case of theory of Robinson from existential completes follows «JEP» , the reverse is not true.

Theorem 4. Let theory T is existentially complete perfect existentially-prime strongly convex theory of Robinson. Let X be Jonsson set in T and M such existentially closed submodel semantic model of C theory T, that $dcl(X) = M$.

Then the following conditions are equivalent:

1) theory T_M has an core structure;

2) theory T has a core model; where T is the center of the theory T_M;

3) when $ever\varphi(x)$ there is an existential formula and deducible in T, then there is some existential formula $\psi(x)$ and n is an integer, such that in derivable $\exists^n x\varphi \wedge \exists x(\varphi \wedge \psi)$, and if $\models (\sigma_1 \vee \sigma_2)$, where $\sigma 1, \sigma_2$ are some existential sentences, then $\models \sigma_1$ or $\models \sigma_2$.

Proof. Should be using Theorems 1 and 2 under the conditions of the theorem and the fact that a lot Jonsson set inherits from the theory that the properties diagram then properties diagram of existential closed model M and semantic model is same.

Theorem 5. Let theory T is existentially complete perfect existentially-prime of strongly convex theory of Robinson. Let X is Jonsson set in T and M such existentially closed submodel semantic model C theory T, that $dcl(X) = M$.

Then M is core structure of T if and only if M is core model center of T, where T is center of theory T_M.

Proof follows from the application of Theorem 5.

We associate the notion of algebraic ease with the notion of the core model in connection with the bulge theory in the existentially complete perfect existential easy strongly convex Robinson's theory.

The following definition is taken from [3]:

Definition 8. Formula $\varphi(\overline{x})$ is Δ-formula, if there are exist formulas $\psi_1(\overline{x})$ and $\psi_2(\overline{x})$ such that $T \models (\varphi \leftrightarrow \psi_1)$ and $T \models (\varphi \leftrightarrow \psi_2)$.

Thus, Δ-formula are invariant formula with respect to investments of formula between theory models T. Together with the existential formulas (Σ-formula) they constitute the main classes of formulas, that used to define relationships in a algebraic prime models.

(a) $\overline{A}$ is Σ-nice algebraically prime model of T, if $\overline{A}$ is countable model of theory T and for each model $\overline{B}$ of T, each $n \in \omega$ and all $a_0, \ldots, a_{n-1} \in A, b_0, \ldots, b_{n-1} \in B$ if $(\overline{A}, a_0, \ldots, a_{n-1}) \Rightarrow_\exists (\overline{B}, b_0, \ldots, b_{n-1},)$ then for each $a_n \in A$ exist $b_n \in B$ than $(\overline{A}, a_0, \ldots, a_n) \Rightarrow_\exists (\overline{B}, b_0, \ldots, b_n)$;

(b) $\overline{A}$ is Σ-nice algebraically prime model of T, if a condition in which (a) is true when replacing $\Rightarrow_\exists$ on $\equiv_\exists$;

(c) $\overline{A}$ is Δ-nice algebraically prime model of T, if a condition in which (a) is true when replacing $\Rightarrow_\exists$ on $\equiv_\Delta$.

Under these definitions the following result.

Theorem 6. Suppose that the theory T is existentially complete perfect existentially-prime strongly convex Robinson's theory.

Let X of Jonsson set is in T and M such that existentially closed submodel semantic model C of T, that $dcl(X) = M$.

Then the following conditions are equivalent: 1) M is Σ-nice; 2) M is existentially closed and Σ-nice.

All undefined here definitions and notions related to the theory the Jonsson can be found in [2].

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А.Р.Ешкеев

Дөңес экзистенциалды жай йонсондық теориялар

Мақала арнайы ішкі кластардың йонсондық жиындарының теориялық-модельдік қасиеттерін және олардың модельдер класын зерттеуге арналған. Әсіресе йонсондық теориялардың жаңа класы — экзистенционалды жай дөңес йонсондық теориялар ұғымы енгізілді. Дөңес теориялар ұғымын А.Робинсон енгізген. Экзистенционалды жай ұғымы мақаланың авторына тиесілі. Екі шарт та теориялардың кең табиғи кластарын құрайды. Осыдан йонсондықтың екі табиғи кластарының шектері байытылған теориялық-модельдік қасиеттермен жаңа теориялар класын анықтайды. Осыдан басқа, мәселенің тұжырымы анықталды, мұнда йонсондық теорияны оқу аясында жаңа бағытты айқындалды.

А.Р.Ешкеев

Выпуклые экзистенциально простые йонсоновские теории

Данная статья посвящена исследованию теоретико-модельных свойств специальных подклассов йонсоновских теорий и их классов моделей. А именно вводится новый класс йонсоновских теорий — экзистенциально простые выпуклые йосоновские теории. Понятие выпуклости теории было известно ранее, оно было введено А.Робинсоном. Понятие экзистенциальной простоты вводится автором данной статьи. Оба условия образуют широкие естественные классы теорий. Таким образом, два естественных ограничения на йонсоновость определяют новый класс теорий с богатыми теоретико-модельными свойствами. Кроме этого, определяется постановка задачи, которая определяет новое направление в рамках изучения йонсоновских теорий.

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