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On a boundary problem for the fourth order equation with the third derivative with respect to time

In this paper, we consider a boundary value problem in a rectangular domain for a fourth-order homogeneous partial differential equation containing the third derivative with respect to time. The uniqueness of the solution of the stated problem is proved by the method of energy integrals. Using the method of separation of variables, the solution of the considered problem is sought as a multiplication of two functions $X(x)$ and $Y(y)$. To determine $X(x)$, we obtain a fourth-order ordinary differential equation with four boundary conditions at the segment boundary $[0, p]$, and for a $Y(y)$ – third-order ordinary differential equation with three boundary conditions at the boundary of the segment $[0, q]$. Imposing conditions on the given functions, we prove the existence theorem for a regular solution of the problem. The solution of the problem is constructed in the form of an infinite series, and the possibility of term-by-term differentiation of the series with respect to all variables is substantiated. When substantiating the uniform convergence, it is shown that the "small denominator" is different from zero.

Keywords: Initial boundary problem, Fourier method, uniqueness, existence, eigenvalue, eigenfunction, functional series, absolute and uniform convergence.

Introduction

Problems about the vibrations of rods, beams and plates, which are of great importance in structural mechanics, lead to differential equations with a higher order than the string equation.

The study of many problems of gas dynamics, the theory of elasticity, the theory of plates and shells comes to the consideration of differential equations with higher order partial derivatives. From the point of view of physical applications, the fourth order differential equations are also of great interest (see [1–6]).

In the field of modern science and technology, initial-boundary value problems for fourth-order equations are of great importance. For example, aircraft wings, bridge slabs, floor systems, and window panes are modeled as plates with various types of end supports, which are successfully described in terms of fourth-order equations [7–9].

The monograph by T.D. Dzhuraev, A. Sopuev [10] is devoted to the classification of differential equations with partial derivatives of the fourth order, the formulation and solution of boundary value problems for such equations.

In the paper [11], a problem with boundary conditions for a non-homogeneous fourth-order equation with multiple characteristics and one lower term was considered.

In [12], a boundary value problem for a fourth-order equation of the form

$$u_{xxxx} - u_{tt} = f(x, t)$$

was investigated.

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In [13], a problem was solved with initial and boundary conditions for the beam oscillation equation of the form

$$a^2 u_{xxxx} + u_{tt} = 0,$$

in which a beam of length l is clamped with ends in a massive vise.

In [14], a boundary value problem for a degenerate higher order equation with lower terms was studied.

In [15–19], the boundary value problems for a third-order equation with multiple characteristics containing second derivatives with respect to time were discussed.

The boundary value problems for fourth-order equations with the third derivative in time have been little studied [20, 21].

1 Formulation of the problem

In the domain $D = \{(x, y) : 0 < x < p, 0 < y < q\}$ we consider the equation

$$\mathrm{L}\left[u\right] \equiv \frac{\partial^4 u}{\partial x^4} - \frac{\partial^3 u}{\partial y^3} = 0, \tag{1}$$

where $p, q \in R$.

Problem A. Find a solution to equation (1) in the domain D from the class $u\left(x, y\right) \in C_{x,y}^{4,3}\left(D\right) \cap C_{x,y}^{3,2}\left(\overline{D}\right)$ such that satisfies the following boundary conditions:

$$u\left(0, y\right) = u\left(p, y\right) = u_{xx}\left(0, y\right) = u_{xx}\left(p, y\right) = 0, \quad 0 \le y \le q, \tag{2}$$

$$u_y\left(x, 0\right) = \psi_1\left(x\right), \quad u_{yy}\left(x, 0\right) = \psi_2\left(x\right), \quad u_{yy}\left(x, q\right) = \psi_3\left(x\right), \quad 0 \le x \le p, \tag{3}$$

where $\psi_i\left(x\right)$, $i = \overline{1, 3}$ are the given sufficiently smooth functions, and

$$\psi_1\left(0\right) = \psi_1\left(p\right) = \psi_1^{''}\left(0\right) = \psi_1^{''}\left(p\right) = 0, \qquad \psi_i\left(0\right) = \psi_i\left(p\right) = 0, \quad i = 2, 3. \tag{4}$$

2 The uniqueness of the solution to the problem A

Theorem 1. If the problem A has a solution, then it is unique.

Proof. Let the problem A have two solutions $u_1\left(x, y\right)$ and $u_2\left(x, y\right)$. Then the function $u\left(x, y\right) = u_1\left(x, y\right) - u_2\left(x, y\right)$ satisfies equation (1) and the uniform boundary conditions. Let us prove that $u\left(x, y\right) \equiv 0$ in $\bar{D}$.

In the domain D the following identity is valid:

$$uL\left[u\right] \equiv \frac{\partial}{\partial x}\left(uu_{xxx} - u_x u_{xx}\right) - \frac{\partial}{\partial y}\left(uu_{yy} - \frac{1}{2}u_y^2\right) + u_{xx}^2 = 0.$$

Integrating the identity over the domain D, we have

$$\int\limits_0^q \left[u\left(p, y\right) u_{xxx}\left(p, y\right) - u\left(0, y\right) u_{xxx}\left(0, y\right)\right]dy -$$

$$- \int\limits_0^q \left[u_x\left(p, y\right) u_{xx}\left(p, y\right) - u_x\left(0, y\right) u_{xx}\left(0, y\right)\right]dy -$$

$$- \int\limits_0^p \left[u\left(x, q\right) u_{yy}\left(x, q\right) - u\left(x, 0\right) u_{yy}\left(x, 0\right)\right]dx +$$

$$+ \frac{1}{2} \int\limits_0^p \left[u_y^2\left(x, q\right) - u_y^2\left(x, 0\right)\right]dx + \int\limits_0^p \int\limits_0^q u_{xx}^2 dxdy = 0.$$

Taking the homogeneous boundary conditions into consideration we obtain

$$\frac{1}{2}\int\limits_0^p u_y^2\left(x,q\right)dx + \int\limits_0^p\int\limits_0^q u_{xx}^2 dxdy = 0.$$

From the second term we obtain

$$u_{xx} = 0 \quad \Rightarrow \quad u\left(x,y\right) = x\cdot f_1\left(y\right) + f_2\left(y\right), \quad \left(x,y\right)\in D.$$

Assuming $x = 0$ we get
$$u\left(0,y\right) = f_2\left(y\right) = 0 \quad \Rightarrow \quad f_2\left(y\right) = 0,$$

and supposing $x = p$ we attain

$$u\left(p,y\right) = p\cdot f_1\left(y\right) = 0 \quad \Rightarrow \quad f_1\left(y\right) = 0.$$

Hence, $u\left(x,y\right) \equiv 0$, $\left(x,y\right)\in\overline{D}$.

Theorem 1 is proved.

3 Existence of a solution to the problem A

In order to prove the existence of the solution of the problem A, we will first consider the following auxiliary problem: find a nontrivial solution of equation (1) such that satisfies conditions (2) and can be represented as
$$u\left(x,y\right) = X\left(x\right)Y\left(y\right). \tag{5}$$

Substituting (5) into equation (1) and separating the variables, we find the following ordinary differential equations with respect to the functions $X\left(x\right)$ and $Y\left(y\right)$:

$$X^{(4)}\left(x\right) - \lambda^4 X\left(x\right) = 0, \tag{6}$$

$$Y^{(3)}\left(y\right) - \lambda^4 Y\left(y\right) = 0, \tag{7}$$

where λ^4 is the split parameter.

Considering the boundary conditions (2), we generate the following problem for equation (6):

$$\begin{cases} X^{(4)} - \lambda^4 X = 0, \\ X\left(0\right) = X\left(p\right) = X''\left(0\right) = X''\left(p\right) = 0. \end{cases} \tag{8}$$

A nontrivial solution to problem (8) exists if and only if

$$\lambda_n^4 = \left(\frac{\pi n}{p}\right)^4, \quad n = 1, 2, 3, \dots.$$

These numbers are the eigenvalues of problem (8), and their corresponding eigenfunctions have the following form:
$$X_n\left(x\right) = \sqrt{\frac{2}{p}}\sin\frac{\pi n}{p}x. \tag{9}$$

A general solution (7) has the form

$$Y_n\left(y\right) = C_1 e^{k_n y} + e^{-\frac{1}{2}k_n y}\left(C_2\cos\left(\frac{\sqrt{3}}{2}k_n y\right) + C_3\sin\left(\frac{\sqrt{3}}{2}k_n y\right)\right), \tag{10}$$

where $k_n = \sqrt[3]{\lambda_n^4} = \left(\frac{\pi n}{p}\right)^{4/3}$, $n \in N$ and C_i, $i = \overline{1,3}$ are unknown constants for now.

According to (9) and (10), it follows from equation (5) that the functions

$$u_n(x,y) = \sqrt{\frac{2}{p}}\left(C_1 e^{k_n y} + e^{-\frac{1}{2}k_n y}\left(C_2 \cos\left(\frac{\sqrt{3}}{2}k_n y\right) + C_3 \sin\left(\frac{\sqrt{3}}{2}k_n y\right)\right)\right)\sin\frac{\pi n}{p}x$$

are the particular solutions of equation (1), which satisfy homogeneous conditions (2).

Due to the linearity and homogeneity of (1), the sum of the particular solutions can also be the solution of equation (1). Taking this into account we will seek the solution of problem A in the form

$$u(x,y) = \sqrt{\frac{2}{p}}\sum_{n=1}^{\infty}\left(C_1 e^{k_n y} + e^{-\frac{1}{2}k_n y}\left(C_2 \cos\left(\frac{\sqrt{3}}{2}k_n y\right) + C_3 \sin\left(\frac{\sqrt{3}}{2}k_n y\right)\right)\right)\sin\frac{\pi n}{p}x. \qquad (11)$$

Assuming temporarily that the series in (11) and its derivatives converge uniformly and requiring the function defined by the series (11) to satisfy the boundary conditions (3) we obtain

$$u_y(x,0) = \psi_1(x) = \sqrt{\frac{2}{p}}\sum_{n=1}^{\infty}\psi_{1n}\sin\frac{\pi n}{p}x,$$

$$u_{yy}(x,0) = \psi_2(x) = \sqrt{\frac{2}{p}}\sum_{n=1}^{\infty}\psi_{2n}\sin\frac{\pi n}{p}x,$$

$$u_{yy}(x,q) = \psi_3(x) = \sqrt{\frac{2}{p}}\sum_{n=1}^{\infty}\psi_{3n}\sin\frac{\pi n}{p}x,$$

where

$$\begin{cases} k_n C_1 - \dfrac{1}{2}k_n C_2 + \dfrac{\sqrt{3}}{2}k_n C_3 = \psi_{1n}, \\[2mm] k_n^2 C_1 - \dfrac{1}{2}k_n^2 C_2 - \dfrac{\sqrt{3}}{2}k_n^2 C_3 = \psi_{2n}, \\[2mm] k_n^2 e^{k_n q}C_1 + k_n^2 e^{-\frac{1}{2}k_n q}\cos\left(\dfrac{\sqrt{3}}{2}k_n q - \dfrac{2\pi}{3}\right)C_2 + k_n^2 e^{-\frac{1}{2}k_n q}\sin\left(\dfrac{\sqrt{3}}{2}k_n q - \dfrac{2\pi}{3}\right)C_3 = \psi_{3n}. \end{cases} \qquad (12)$$

We can see from (12) that the numbers ψ_{in} are the Fourier coefficients of the function $\psi_i(x)$ when they are expanded into the Fourier series in terms of sines on the interval $(0, p)$, i.e.

$$\psi_{in} = \sqrt{\frac{2}{p}}\int_0^p \psi_i(\xi)\sin\frac{\pi n}{p}\xi d\xi, \quad i = \overline{1,3}.$$

Let us calculate the determinant of the system (12), i.e.

$$\Delta = \begin{vmatrix} k_n & -\dfrac{1}{2}k_n & \dfrac{\sqrt{3}}{2}k_n \\[2mm] k_n^2 & -\dfrac{1}{2}k_n^2 & -\dfrac{\sqrt{3}}{2}k_n^2 \\[2mm] k_n^2 e^{k_n q} & k_n^2 e^{-\frac{1}{2}k_n q}\cos\left(\dfrac{\sqrt{3}}{2}k_n q - \dfrac{2\pi}{3}\right) & k_n^2 e^{-\frac{1}{2}k_n q}\sin\left(\dfrac{\sqrt{3}}{2}k_n q - \dfrac{2\pi}{3}\right) \end{vmatrix} =$$

$$= \sqrt{3}k_n^5 e^{k_n q}\bar{\Delta}, \quad \bar{\Delta} = \frac{1}{2} + e^{-\frac{3}{2}k_n q}\sin\left(\frac{\sqrt{3}}{2}k_n q - \frac{\pi}{6}\right).$$

Lemma. For an arbitrary positive q, the inequality $\Delta > 0$ holds .

Proof. Write the determinant in the form

$$\Delta = \sqrt{3} k_n^5 e^{k_n q} \bar{\Delta}(x_n), \quad \bar{\Delta}(x_n) = \frac{1}{2} + e^{-\sqrt{3} x_n} \sin\left(x_n - \frac{\pi}{6}\right),$$

where, $x_n = \dfrac{\sqrt{3}}{2} k_n q = \dfrac{\sqrt{3}}{2} \sqrt[3]{\lambda_n^4} q > 0$, $n \in N$.

Find the minimum value of $\bar{\Delta}(x_n)$. To do this, calculate the first order derivative

$$\frac{d\bar{\Delta}(x_n)}{dx_n} = 2 e^{-\sqrt{3} x_n} \sin\left(\frac{\pi}{3} - x_n\right).$$

1) When $0 < x_n < \frac{\pi}{3}$, we have $\bar{\Delta}'(x_n) > 0$ .This means that $\bar{\Delta}(x_n)$ increases at finite discrete values of x_n , but it does not reach its maximum value. Then the function $\bar{\Delta}(x_n)$ takes its minimum value at $n = 1$ and we have the estimation

$$\bar{\Delta}(x_n) \geq \frac{1}{2} + e^{-\sqrt{3} x_1} \sin\left(x_1 - \frac{\pi}{6}\right) = \delta_1 > 0,$$

where $x_1 = \dfrac{\sqrt{3}}{2} \sqrt[3]{\lambda_1^4} q$.

2) If $x_n \geq \dfrac{\pi}{3}$, then the function $\bar{\Delta}(x_n)$ takes its first minimum when the argument is $\dfrac{4\pi}{3}$ and we achieve

$$\overline{\Delta}(x_n) \geq \frac{1}{2}\left(1 - e^{-4/\sqrt{3}\pi}\right) = \delta_2 > 0.$$

3) For sufficiently large values of x , it is obvious that the function $\bar{\Delta}(x)$ tends to $\frac{1}{2}$. From here we find

$$\bar{\Delta} \geq \delta = \min\{\delta_1; \delta_2\} > 0.$$

Considering the above considerations we conclude that $\Delta_n > 0$. The lemma is proved.

Hence, the system of equations (12) has a unique solution.

Below, we determine all unkown numbers C_i, $i = \overline{1,3}$:

$$C_1 = \frac{1}{\Delta}\left[\psi_{1n} k_n^4 e^{-\frac{1}{2} k_n q} \sin\left(\frac{\sqrt{3}}{2} k_n q\right) - \psi_{2n} k_n^3 e^{-\frac{1}{2} k_n q} \cos\left(\frac{\sqrt{3}}{2} k_n q + \frac{\pi}{6}\right) + \frac{\sqrt{3}}{2} \psi_{3n} k_n^3\right],$$

$$C_2 = \frac{1}{\Delta}\left[\psi_{1n} k_n^4 \left(e^{-\frac{1}{2} k_n q} \sin\left(\frac{\sqrt{3}}{2} k_n q + \frac{\pi}{3}\right) - \frac{\sqrt{3}}{2} e^{k_n q}\right) - \right.$$

$$\left. - \psi_{2n} k_n^3 \left(e^{-\frac{1}{2} k_n q} \sin\left(\frac{\sqrt{3}}{2} k_n q + \frac{\pi}{3}\right) + \frac{\sqrt{3}}{2} e^{k_n q}\right) + \sqrt{3} \psi_{3n} k_n^3\right],$$

$$C_3 = \frac{1}{\Delta}\left[\psi_{1n} k_n^4 e^{k_n q} \left(\frac{1}{2} - e^{-\frac{3}{2} k_n q} \cos\left(\frac{\sqrt{3}}{2} k_n q + \frac{\pi}{3}\right)\right) + \psi_{2n} k_n^3 e^{k_n q} \left(e^{-\frac{3}{2} k_n q} \cos\left(\frac{\sqrt{3}}{2} k_n q + \frac{\pi}{3}\right) - \frac{1}{2}\right)\right].$$

In what follows, the maximum value of all found positive known numbers in estimates will be denoted by M .

Taking account condition (4), we integrate $\psi_1(x)$ by parts four times, and $\psi_i(x)$, $i = 2, 3$ by parts two times we get the following estimates:

$$|\psi_1| \leq M \frac{|\Psi_{1n}|}{n^4}, \quad |\psi_i| \leq M \frac{|\Psi_{in}|}{n^2}, \quad i = 2, 3, \tag{13}$$

where

$$\Psi_{1n} = \int\limits_0^p \psi_1^{(4)}(\xi) X_n(\xi) d\xi, \qquad \Psi_{in} = \int\limits_0^p \psi_i^{''}(\xi) X_n(\xi) d\xi, \quad i = 2,3.$$

For C_i, $i = \overline{1,3}$ we can write the following estimations:

$$|C_1|\, e^{k_n q} \leq M\left(\frac{|\psi_{1n}|}{k_n} e^{-\frac{1}{2}k_n q} + \frac{|\psi_{2n}|}{k_n^2} e^{-\frac{1}{2}k_n q} + \frac{|\psi_{3n}|}{k_n^2}\right) \leq M\left(\frac{|\Psi_{1n}|}{n^{\frac{16}{3}}} + \frac{|\Psi_{2n}|}{n^{\frac{14}{3}}} + \frac{|\Psi_{3n}|}{n^{\frac{14}{3}}}\right),$$

$$|C_2| \leq M\left(\left(\frac{|\psi_{1n}|}{k_n} + \frac{|\psi_{2n}|}{k_n^2}\right)\left(e^{-\frac{3}{2}k_n q} + \frac{\sqrt{3}}{2}\right) + \frac{|\psi_{3n}|}{k_n^2} e^{-k_n q}\right) \leq M\left(\frac{|\Psi_{1n}|}{n^{\frac{16}{3}}} + \frac{|\Psi_{2n}|}{n^{\frac{14}{3}}} + \frac{|\Psi_{3n}|}{n^{\frac{14}{3}}}\right),$$

$$|C_3| \leq M\left(\frac{|\psi_{1n}|}{k_n}\left(\frac{1}{2} + e^{-\frac{3}{2}k_n q}\right) + \frac{|\psi_{2n}|}{k_n^2}\left(\frac{1}{2} + e^{-\frac{3}{2}k_n q}\right)\right) \leq M\left(\frac{|\Psi_{1n}|}{n^{\frac{16}{3}}} + \frac{|\Psi_{2n}|}{n^{\frac{14}{3}}}\right).$$

Theorem 2. If $\psi_1(x) \in C^4[0,p]$, $\psi_i(x) \in C^2[0,p]$, $i = 2,3$ and the corresponding conditions (4) are satisfied, then the solution of the problem A exists and it is represented by the series (11).

Proof. If the series (11) and its derivatives u_{xxxx}, u_{yyy} converge uniformly in the region $\bar{D}$, then the function $u(x,y)$ defined by this series will be the solution of the problem A.

From (11) we have

$$|u(x,y)| \leq \sum_{n=1}^\infty \left(|C_1|\, e^{k_n q} + |C_2| + |C_3|\right) |X_n(x)|. \tag{14}$$

Then, taking account of (13) it is obtained from (14) that

$$|u(x,y)| \leq M\left(\sum_{n=1}^\infty \frac{|\Psi_{1n}|}{n^{\frac{16}{3}}} + \sum_{n=1}^\infty \frac{|\Psi_{2n}|}{n^{\frac{14}{3}}} + \sum_{n=1}^\infty \frac{|\Psi_{3n}|}{n^{\frac{14}{3}}}\right) < \infty.$$

This implies that the series (11) converges absolutely and uniformly.

Now let us prove that the partial derivatives of the series (11) with respect to both variables included in the equation also converge absolutely and uniformly in the region $\bar{D}$. Calculating the derivatives with respect to y, it follows from (11) we obtain

$$\frac{\partial^3 u}{\partial y^3} = \sum_{n=1}^\infty k_n^3 \left[C_1 e^{k_n y} + e^{-\frac{1}{2}k_n y}\left(C_2 \cos\left(\frac{\sqrt{3}}{2}k_n y\right) + C_3 \sin\left(\frac{\sqrt{3}}{2}k_n y\right)\right)\right] X_n(x). \tag{15}$$

From (15) we determine the estimation

$$\left|\frac{\partial^3 u}{\partial y^3}\right| \leq \sum_{n=1}^\infty k_n^3 \left(|C_1|\, e^{k_n q} + |C_2| + |C_3|\right) |X_n(x)| \leq M\left(\sum_{n=1}^\infty \frac{|\Psi_{1n}|}{n^{\frac{4}{3}}} + \sum_{n=1}^\infty \frac{|\Psi_{2n}|}{n^{\frac{2}{3}}} + \sum_{n=1}^\infty \frac{|\Psi_{3n}|}{n^{\frac{2}{3}}}\right). \tag{16}$$

Using the Cauchy-Bunyakovsky and Bessel inequality we attain

$$\left|\frac{\partial^3 u}{\partial y^3}\right| \leq M\left(\sum_{n=1}^\infty \frac{|\Psi_{1n}|}{n^{\frac{4}{3}}} + \sqrt{\sum_{n=1}^\infty |\Psi_{2n}|^2}\sqrt{\sum_{n=1}^\infty \frac{1}{n^{\frac{4}{3}}}} + \sqrt{\sum_{n=1}^\infty |\Psi_{3n}|^2}\sqrt{\sum_{n=1}^\infty \frac{1}{n^{\frac{4}{3}}}}\right) \leq$$

$$\leq M\left(\sum_{n=1}^\infty \frac{|\Psi_{1n}|}{n^{\frac{4}{3}}} + \|\Psi_{2n}\|\sqrt{\sum_{n=1}^\infty \frac{1}{n^{\frac{4}{3}}}} + \|\Psi_{3n}\|\sqrt{\sum_{n=1}^\infty \frac{1}{n^{\frac{4}{3}}}}\right) < \infty,$$

where

$$\sum_{n=1}^{\infty} |\Psi_{1n}|^2 \leq \left\| \psi_1^{(4)}(x) \right\|_{L_2[0;p]}^2, \quad \sum_{n=1}^{\infty} |\Psi_{in}|^2 \leq \left\| \psi_i^{(2)}(x) \right\|_{L_2[0,p]}^2, \quad i = 2, 3.$$

Therefore, the series (16) converges absolutely and uniformly. The absolute and uniform convergence of the partial derivative of the fourth order in x series (11) follows from the equality $\dfrac{\partial^4 u}{\partial x^4} = \dfrac{\partial^3 u}{\partial y^3}$.

Theorem 2 is proved.

Conclusion

The article considers an initial boundary value problem for a fourth-order equation containing the third time derivative with multiple characteristics. Uniqueness theorems are proved using the method of energy integrals. The existence of a solution is shown with the help of conditions imposed on given functions constructed as a series by the Fourier method and a regular solution of this series.

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Уақыт бойынша үшінші туындысы бар төртінші ретті теңдеу үшін шекаралық есеп жайында

Мақалада уақыт бойынша үшінші туындысы бар біртекті төртінші ретті дербес туындылы дифференциалдық теңдеу үшін тікбұрышты облыстағы шекаралық есеп қарастырылды. Қойылған есептің шешімінің жалғыздығы энергия интегралдары әдісімен дәлелденген. Айнымалыларды бөлу әдісін қолданып, есептің шешімі $X(x)$ және $Y(y)$ екі функцияның көбейтіндісі түрінде іздестіріледі. $X(x)$ анықтау үшін $[0, p]$ кесіндісінің шекарасында төрт шекаралық шарты бар төртінші ретті қарапайым дифференциалдық теңдеуін, ал $Y(y)$ анықтау үшін $[0, q]$ кесіндісінің шекарасында үш шекаралық шарты бар үшінші ретті қарапайым дифференциалдық теңдеуді аламыз. Берілген функцияларға шарттарды қою арқылы есептің тұрақты шешімінің бар екендігі туралы теорема дәлелденеді. Қойылған есептің шешімі шексіз қатар түрінде құрылып, қатардың барлық айнымалыларға қатысты

қатарды мүшелеп дифференциалдау мүмкіндігі анықталған. Бірқалыпты жинақталуды табу кезінде «кіші бөлгіш» нөлге тең емес екені анықталды.

Кілт сөздер: бастапқы-шеттік есеп, Фурье әдісі, шешімнің жалғыздығы, бар болуы, меншікті мән, меншікті функция, функционалдық қатар, абсолютті және бірқалыпты жинақталу.

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О граничной задаче для уравнения четвёртого порядка, содержащего третью производную по времени

В статье рассмотрена краевая задача в прямоугольной области для однородного дифференциального уравнения в частных производных четвёртого порядка, содержащего третью производную по времени. Единственность решения поставленной задачи доказана методом интегралов энергии. Используя метод разделения переменных, решение задачи ищется в виде произведения двух функций $X(x)$ и $Y(y)$. Для определения $X(x)$ получаем обыкновенное дифференциальное уравнение четвёртого порядка с четырьмя граничными условиями на границе сегмента $[0, p]$, а для $Y(y)$ – обыкновенное дифференциальное уравнение третьего порядка с тремя граничными условиями на границе сегмента $[0, q]$. Налагая определенные условия на заданные функции, доказана теорема существования регулярного решения задачи. Решение поставленной задачи построено в виде бесконечного ряда, обоснована возможность почленного дифференцирования ряда по всем переменным. При доказательстве равномерной сходимости установлена отличность от нуля «малого знаменателя».

Ключевые слова: начально-краевая задача, метод Фурье, единственность, существование, собственное значение, собственная функция, функциональный ряд, абсолютная и равномерная сходимость.

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