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A problem with shift for a mixed-type model equation of the second kind in an unbounded domain

This article studies a problem with shift in the characteristics of different families in an unbounded domain for a mixed-type model equation of the second kind. The elliptic part of this problem is the vertical half-strip; the hyperbolic part is the characteristic triangle bounded by the characteristics of the equation. Using the extremum principle we prove the uniqueness of the solution. With the integral equations method we prove the existence of the solution.

Keywords: mixed-type equation of the second kind, problem with a shift, uniqueness and existence of a solution, extremum principle, method of integral equations.

1 Statement of the problem

Consider the following equation

$$u_{xx} + \operatorname{sign} y|y|^m u_{yy} = 0, \quad 0 < m < 1 \tag{1}$$

in unbounded mixed domain $\Omega = \Omega_1 \cup J \cup, \Omega_2$, where $\Omega_1 = \{(x,y) : 0 < x < 1,\ 0 < y < +\infty\}$ $J = \{(x,y) : 0 < x < 1,\ y = 0\}$ and Ω_2 is the domain of half-plane $y < 0$, bounded by the characteristics of equation (1)

$$AC: \ x - [2/(2-m)]\,(-y)^{(2-m)/2} = 0, \quad BC: \ x + [2/(2-m)]\,(-y)^{(2-m)/2} = 1,$$

going out of points $A(0,0)$ and $B(1,0)$ and intersecting at point $C(\frac{1}{2}, -\left(\frac{2-m}{4}\right)^{\frac{2}{2-m}})$, and by the AB segment of the abscissa axis, we assume the following notation:

$$\beta = m/\left(2m-4\right), \ J_1 = \{(x,y) : 0 < y < +\infty,\ x = 0\}, \ J_2 = \{(x,y) : 0 < y < +\infty,\ x = 1\},$$

$$\theta_0(x) = \left(\frac{x}{2}, -\left[\frac{2-m}{2}\cdot\frac{x}{2}\right]^{\frac{2}{2-m}}\right), \quad \theta_1(x) = \left(\frac{1+x}{2}, -\left[\frac{2-m}{2}\cdot\frac{1-x}{2}\right]^{\frac{2}{2-m}}\right).$$

Problem S^∞. Find function $u\,(x,y)$ that satisfies the following conditions:
$u(x,y) \in C(\Omega \cup \overline{J_1} \cup \overline{J_2} \cup AC \cup BC) \cap C^1(\Omega_1 \cup J) \cap C^1(\Omega_2 \cup J) \cap C^2(\Omega_1 \cup \Omega_2)$, it satisfies equation (1) in domains Ω_1 and Ω_2, and has the following property $u_y(x,+0) = \nu\,(x) \in C^1(J)$ and at the ends of the interval it can turn to infinity of order -2β for $x = 0$ and of order $\frac{1}{2} - \beta$ for $x = 1$ with the following boundary conditions:

$$u(0,y) = \varphi_1\,(y)\,, \ u(1,y) = \varphi_2\,(y)\,, \ 0 \le y < +\infty, \tag{2}$$

$$\lim_{y\to+\infty} u(x,y) = 0, \ \text{uniformly in } x \in [0,1]\,, \tag{3}$$

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$$a\left(x\right)D_{0x}^{1-\beta}u\left[\theta_0\left(x\right)\right]+b(x)D_{x1}^{1-\beta}u\left[\theta_1\left(x\right)\right]=c(x),\ \forall x\in J, \tag{4}$$

$$u\left(x,-0\right)=u\left(x,+0\right), \quad \frac{\partial u\left(x,-0\right)}{\partial y}=-\frac{\partial u\left(x,+0\right)}{\partial y}. \tag{5}$$

Here $a^2\left(x\right)+b^2(x)\neq 0$, $\forall x\in\overline{J}$; $a\left(x\right)x^{-\beta}+b(x)(1-x)^{-\beta}\neq 0$, $\forall x\in\overline{J}$; the functions $\varphi_i\left(y\right)\in C(J_i)$ are such that $\varphi_1\left(0\right)=0$, $\varphi_2\left(0\right)=0$, and the integrals

$$\int\limits_0^\infty s^{-\frac{m}{2(2-m)}}\left|\varphi_i\left[\left(\frac{2-m}{2}s\right)^{\frac{2}{2-m}}\right]\right|ds, \quad \int\limits_0^\infty s^{-\frac{m}{(2-m)}}\left|\varphi_i\left[\left(\frac{2-m}{2}s\right)^{\frac{2}{2-m}}\right]\right|ds\ \ (i=1,2)$$

converge; $-1\leq\frac{b(x)(1-x)^{-\beta}}{a(x)x^{-\beta}+b(x)(1-x)^{-\beta}}\leq 0$; $a(x)=a_1(x)x^p,\ p<\beta,\ a_1\left(x\right),b\left(x\right),c\left(x\right)\in C(\overline{J})\cap C^3(J)$; here $D_{sx}^{\alpha}[f(x)]$ is the operator of fractional integro-differentiation in the sense of Riemann-Liouville [1].

2 Uniqueness of the solution

The solution to the Cauchy problem has the following form [2]:

$$u(x,y)=\int\limits_0^1 T\left\{\left[x-(1-2\beta)(-y)^{\frac{1}{1-2\beta}}\right]t\right\}\left[x+(1-2\beta)(-y)^{\frac{1}{1-2\beta}}-xt+(1-2\beta)(-y)^{\frac{1}{1-2\beta}}t\right]^{-\beta}\times$$

$$\times\left[x-(1-2\beta)(-y)^{\frac{1}{1-2\beta}}\right]^{1-\beta}(1-t)^{-\beta}dt+\frac{1}{2\cos\pi\beta}[2(1-2\beta)]^{1-2\beta}\times$$

$$\times\int\limits_0^1 T\left[x-(1-2\beta)(-y)^{\frac{1}{1-2\beta}}(2t-1)\right](-y)t^{-\beta}\left(1-t\right)^{-\beta}dt-\frac{\Gamma(2-2\beta)}{\Gamma^2(1-\beta)}\times$$

$$\times\int\limits_0^1 \nu_1\left[x-(1-2\beta)(-y)^{\frac{1}{1-2\beta}}(2t-1)\right](-y)t^{-\beta}(1-t)^{-\beta}dt, \tag{6}$$

where

$$\nu_1(x)=u_y(x,\ -0), \quad u(x,0)=\tau(x)=\Gamma(1-2\beta)D_{0x}^{2\beta-1}T(x). \tag{7}$$

Considering the definitions and properties of operators of fractional integro-differentiation in the sense of Riemann-Liouville from (6), we have (8)

$$U\left[\theta_0(x)\right]=\frac{\Gamma\left(1-\beta\right)}{2\cos\left(\pi\beta\right)}D_{0x}^{\beta-1}T(x)x^{-\beta}-\frac{\Gamma(2-2\beta)}{\Gamma(1-\beta)[2(1-2\beta)]^{1-2\beta}}D_{x1}^{\beta-1}\nu_1(x)(1-x)^{-\beta}. \tag{8}$$

$$U\left[\theta_1(x)\right]=\Gamma(1-\beta)D_{0x}^{\beta-1}T(x)(1-x)^{-\beta}+\frac{\Gamma(1-\beta)}{2\cos\pi\beta}D_{0x}^{\beta-1}T(x)(1-x)^{-\beta}-$$

$$-\frac{\Gamma(2-2\beta)}{\Gamma(1-\beta)[2(1-2\beta)]^{1-2\beta}}D_{0x}^{\beta-1}\nu_1(x)x^{-\beta}. \tag{9}$$

Now, substituting (8) and (9) into the boundary condition (4) considering (7), we obtain

$$\frac{\Gamma(2-2\beta)}{\Gamma(1-\beta)[2(1-2\beta)]^{1-2\beta}}\left[a(x)x^{-\beta}+b(x)(1-x)^{-\beta}\right]\nu_1(x)=$$

$$= c(x) + \frac{\Gamma(1-\beta)}{2\cos\pi\beta}\left[a(x)x^{-\beta} + b(x)(1-x)^{-\beta}\right]T(x) + \tag{10}$$

$$+b(x)\Gamma(1-\beta)D_{x1}^{1-\beta}D_{0x}^{\beta-1}T(x)(1-x)^{-\beta}.$$

Next, consider the superposition of two operators

$$D_{x1}^{1-\beta}D_{0x}^{\beta-1}T(x)(1-x)^{-\beta},$$

where function $T(x)$ is continuous in the interval $(0,1)$ and integrable on the segment $[0,1]$. The following equality holds

$$D_{x1}^{1-\beta}D_{0x}^{\beta-1}T(x)(1-x)^{-\beta} = T(x)(1-x)^{-\beta}\cos\pi\beta + \frac{\sin\pi\beta}{\pi}\int\limits_0^1 T(t)\frac{(1-t)^{1-2\beta}}{(1-x)^{1-\beta}(t-x)} \tag{11}$$

(the integral here is understood in the sense of the Cauchy's principal value). From (10), considering the properties mentioned above, we conclude that $\nu_1(x)$ belongs to the class of functions integrable on the segment $[0,1]$ and continuous in the interval $(0,1)$.

Theorem. Problem S^∞ cannot have more than one solution.

Proof. Let $u(x,y)$ be the solution to homogeneous problem S^∞. At that $c(x)\equiv 0$. We can prove that $u(x,y)\equiv 0$ in $\Omega\cup J_1\cup J_2\cup\overline{AC}\cup\overline{BC}$.

First, we prove that $u(x,y)\equiv 0$ in $\Omega_1\cup J_1\cup J_2\cup\overline{AB}$. Let us assume the opposite. Then there is domain $\Omega_{1\rho}=\{(x,y):\ 0<x<1,\ 0<y<\rho\}$, in which $u(x,y)\not\equiv 0$. Therefore, $\sup\limits_{\overline{\Omega}_{1\rho}}|u(x,y)|>0$ and this value is reached at some point $(\xi,\eta)\in\overline{\Omega}_{1\rho}$.

We introduce the notation $\partial\Omega_{1\rho}=AB\cup BD\cup DP\cup PA$, where

$$AB=\{(x,y):\ 0<x<1,\ y=0\},\ BD=\{(x,y):\ x=1,\ 0<y<\rho\},$$

$$DP=\{(x,y):\ 0<x<1,\ y=\rho\},\ PA=\{(x,y):\ x=0,\ 0<y<\rho\}.$$

According to the extremum principle for elliptic equations [3], it follows that $(\xi,\eta)\notin\Omega_{1\rho}$. Due to homogeneous conditions (2) $(\xi,\eta)\notin\overline{BD}\cup\overline{PA}$. Then $(\xi,\eta)\in AB\cup\overline{DP}$. Let $(\xi,\eta)\in AB$, i.e., $\sup\limits_{\overline{\Omega}_{1\rho}}|u(x,y)| = \sup\limits_{\overline{AB}}|u(x,y)| = |u(\xi,0)|>0,\ 0<\xi<1$. Then if $u(\xi,0)>0$ (<0), i.e., $(\xi,0)$ is a point of positive maximum (negative minimum) of function $u(x,y)$, then according to the sign lemma proved in [4], and due to the Zaremba-Giraud principle [3], it follows that $(\xi,\eta)\notin AB$. Therefore, $(\xi,\eta)\in\overline{DP}$, i.e. $\sup\limits_{\overline{\Omega}_{1\rho}}|u(x,y)| = \sup\limits_{0\le x\le 1}|u(x,\rho)|>0$. Taking arbitrary number $\rho_1>\rho$, we obtain by the same method $\sup\limits_{\overline{\Omega}_{1\rho_1}}|u(x,y)| = \sup\limits_{0\le x\le 1}|u(x,\rho_1)|>0$. Since $\Omega_{1\rho}\subset\Omega_{1\rho_1}$, then $\sup\limits_{\overline{\Omega}_{1\rho_1}}|u(x,y)|\ge$ $\ge\sup\limits_{\overline{\Omega}_{1\rho}}|u(x,y)|>0$, i.e. $\sup\limits_{0\le x\le 1}|u(x,\rho_1)|\ge\sup\limits_{0\le x\le 1}|u(x,\rho)|>0$. This implies that $\lim\limits_{y\to+\infty}u(x,y)\neq 0$, which contradicts condition (3). Therefore, $u(x,y)\equiv 0$, $(x,y)\in\Omega_1\cup l_1\cup l_2\cup\overline{AB}$. Hence, from (6) and (10), it follows that $u(x,y)\equiv 0$ in $\overline{\Omega}_2$. Therefore, $u(x,y)\equiv 0$, $(x,y)\in\Omega\cup l_1\cup l_2\cup\overline{AC}\cup\overline{BC}$, whence follows the assertion of the theorem.

3 Existence of the solution

Solving the problem N in the area of ellipticity of equation (1) according to the S.V. Falkovich method [5], we obtain the solution in the following form:

$$u(x,y) = k\sqrt{y}\int\limits_0^1 \nu(t)\sum_{n=1}^{\infty}\frac{\sin n\pi x \sin n\pi t}{n^{\alpha}}K_{\alpha}\left(2n\pi\alpha y^{\frac{1}{2\alpha}}\right)dt+$$

$$+\sqrt{y}\int\limits_0^{\infty}\left[\frac{2\alpha}{s}\right]^{\alpha}\varphi_1\left[\left(\frac{s}{2\alpha}\right)^{2\alpha}\right]sds\int\limits_0^{\infty}\lambda\frac{sh\,(1-x)\,\lambda}{sh\lambda}J_{-\alpha}\left(2\lambda\alpha y^{\frac{1}{2\alpha}}\right)J_{-\alpha}(\lambda s)\,d\lambda+$$

$$+\sqrt{y}\int\limits_0^{\infty}\left[\frac{2\alpha}{s}\right]^{\alpha}\varphi_2\left[\left(\frac{s}{2\alpha}\right)^{2\alpha}\right]sds\int\limits_0^{\infty}\lambda\frac{shx\lambda}{sh\lambda}J_{-\alpha}\left(2\lambda\alpha y^{\frac{1}{2\alpha}}\right)J_{-\alpha}(\lambda s)\,d\lambda, \qquad (12)$$

where $\alpha = \frac{1}{2-m}$, $k = -\frac{4\sin\pi\alpha\Gamma(1+\alpha)}{\pi(\pi\alpha)^{\alpha}}$, $\Gamma(z)$ is the gamma function [1], $K_{\alpha}(z)$ and $J_{\alpha}(z)$ are the Macdonald and Bessel functions, respectively [6]. Passing to the limit as $y \to 0$ in formula (12), we obtain the main functional relation between $\tau(x)$ and $\nu(x)$ brought from the area of ellipticity of equation (1):

$$\tau(x) = -\frac{2\Gamma(1+\alpha)}{\Gamma(1-\alpha)}\left(\frac{1}{\pi\alpha}\right)^{2\alpha}\int\limits_0^1 \nu(t)\sum_{n=1}^{\infty}\frac{\sin n\pi x \sin n\pi t}{n^{2\alpha}}dt + F_1(x). \qquad (13)$$

From the hyperbolic area we have relation (10) between $\nu(x)$ and $T(x)$ which, considering (11), has the following form:

$$\frac{\Gamma(2-2\beta)}{\Gamma(1-\beta)\left[2(1-2\beta)\right]^{1-2\beta}}\nu_1(x) = -\frac{c(x)}{a(x)x^{-\beta}+b(x)(1-x)^{-\beta}} + \frac{\Gamma(1-\beta)}{2\cos\pi\beta}T(x) + \qquad (14)$$

$$+\frac{\Gamma(1-\beta)\,b(x)}{a(x)x^{-\beta}+b(x)(1-x)^{-\beta}}\left[T(x)(1-x)^{-\beta}\cos\pi\beta + \frac{\sin\pi\beta}{\pi(1-x)^{1-\beta}}\int\limits_0^1\frac{T(t)\,dt}{(1-t)^{2\beta-1}(t-x)}\right].$$

Taking into account the gluing conditions (5), we eliminate $T(x)$ from (13) and (14). After some transformations, we obtain a singular integral equation:

$$A(x)\rho(x) + \frac{B(x)}{\pi i}\int\limits_0^1\rho(t)\left(\frac{1}{t-x}+\frac{1-2t}{t+x-2xt}\right)dt+$$

$$+\cos\pi\beta\mu(x)\int\limits_0^1\rho(t)K_1(x,t)dt = F(x), \qquad (15)$$

where $A(x) = 1-\sin\pi\beta$, $B(x) = -i\cos\pi\beta\left[1+2\mu(x)\right]$, $\rho(x) = \nu(x)x^{-2\beta}$,

$$\mu(x) = \frac{b(x)(1-x)^{-\beta}}{a(x)x^{-\beta}+b(x)(1-x)^{-\beta}},$$

$$F(x) = \frac{\Gamma(1-\beta)\left[2(1-2\beta)\right]^{1-2\beta}x^{-2\beta}}{\Gamma(2-2\beta)}\left[\frac{\Gamma(1+2\beta)}{\Gamma(1-2\beta)}D_{0x}^{1-2\beta}F_1(x) - \frac{\Gamma(1+2\beta)\,c(x)}{a(x)x^{-\beta}+b(x)(1-x)^{-\beta}}\right],$$

$$K_1(x,t) = \left(\frac{t}{x}\right)^{1-2\beta}\left[\frac{1}{t+x-2xt} - \frac{1}{t+x} + \sum_{n=1}^{\infty}\left[\left(\frac{2n+t}{t}\right)^{1-2\beta}\frac{1}{2n-x+t}+\right.\right.$$

$$+\left(\frac{2n-t}{t}\right)^{1-2\beta}\frac{1}{2n-t+x}-\left(\frac{2n-t}{t}\right)^{1-2\beta}\frac{1}{2n-x-t}-\left(\frac{2n+t}{t}\right)\frac{1}{2n+x+t}\Bigg]$$

is the weakly-singular kernel. Since $A^2(x)-B^2(x)\neq 0$, therefore, the singular integral equation (15) is of the normal type. Now, changing the variables

$$z=\frac{t^2}{1-2t+t^2}\quad\text{and}\quad w=\frac{x^2}{1-2x+x^2}.$$

equation (15) is reduced to a singular integral equation with the Cauchy kernel. Then, applying the Carleman-Vekua regularization method [7, 8], we obtain an equivalent Fredholm equation of the second kind, the unconditional solvability of which follows from the uniqueness of the problem solution.

References

1 Смирнов М.М. Уравнения смешанного типа / М.М. Смирнов. — М.: Высш. шк., 1985. — 304 с.

2 Кароль А.И. Дзета-функция вырождающихся эллиптических операторов: автореф. дис. ... канд. физ.-мат. наук / А.И. Кароль. — Л.: ЛГУ, 1981. — 13 с.

3 Бицадзе А.В. Краевые задачи для эллиптических уравнений второго порядка / А.В Бицадзе. — М.: Наука, 1966. — 203 с.

4 Ивашкина Г.А. Краевая задача со смещением для уравнения второго рода / Г.А. Ивашкина, Л.М. Невоструев // Диффренц. уравнения. — 1978. — 14. — № 2. — С. 281–290.

5 Фалькович С.В. Об одном случае решения задачи Трикоми / С.В. Фалькович // Трансзвуковые течения газа: межвуз. сб. — Вып. 1. — Саратов, 1964. — С. 3–8.

6 Кузнецов Д.С. Специальные функции / Д.С. Кузнецов. — М.: Высш. шк., 1962. — 424 с.

7 Мусхелишвили Н.И. Сингулярные интегральные уравнения / Н.И. Мусхелишвили. — М.: Физматлит, 1962. — 600 с.

8 Гахов Ф.Д. Краевые задачи / Ф.Д. Гахов. — М.: ГИФМЛ, 1963. — 639 с.

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Шексіз облыста екінші текті аралас типті модельдік теңдеу үшін ығысу есебі

Мақалада екінші текті аралас типті модельдік теңдеу үшін шектелмеген облыстағы әртүрлі сипаттамаларының ығысу есебі зерттелген. Облыстың эллипстік бөлігі – тік жарты жолақ, ал гиперболалық бөлігі – теңдеу сипаттамаларымен шектелген сипаттамалық үшбұрыш. Шешімнің бірегейлігі экстремум принципі арқылы, ал шешімнің бар екендігі интегралдық теңдеулер әдісімен дәлелденген.

Кілт сөздер: екінші текті аралас типті теңдеу, ығысуы бар есеп, шешімнің жалғыздығы және бар болуы, экстремум принципі, интегралдық теңдеулер әдісі.

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Задача со смещением для модельного уравнения смешанного типа второго рода в неограниченной области

В статье в неограниченной области для модельного уравнения смешанного типа второго рода исследована задача со смещением на характеристиках различных семейств. Эллиптическая часть области представляет собой вертикальную полуполосу, а гиперболическая часть — характеристический треугольник, ограниченный характеристиками уравнения. Единственность решения доказана с помощью принципа экстремума, а существование решения — методом интегральных уравнений.

Ключевые слова: уравнение смешанного типа второго рода, задача со смещением, единственность и существование решения, принцип экстремума, метод интегральных уравнений.

References

1 Smirnov, M.M. (1985). *Uravneniia smeshannogo tipa [Mixed-type equations]*. Moscow: Vysshaia shkola [in Russian].

2 Karol, A.I. (1981). Dzeta-funktsiia vyrozhdaiushchikhsia ellipticheskikh operatorov [Zeta function of degenerate elliptic operators]. *Extended abstract of candidate's thesis.* Leningrad: Leningradkii gosudarstvennyi universitet [in Russian].

3 Bitsadze, A.V. (1966). *Kraevye zadachi dlia ellipticheskikh uravnenii vtorogo poriadka [Boundary value problems for second order elliptic equations]*. Moscow: Nauka [in Russian].

4 Ivashkina, G.A., & Nevostruev, L.M. (1978). Kraevaia zadacha so smeshcheniem dlia uravneniia vtorogo roda [Boundary value problem with a shift for an equation of the second kind]. *Differentsialnye uravneniia — Differential Equations, 14*, 2, 281–290 [in Russian].

5 Falkovich, S.V. (1964). Ob odnom sluchae resheniia zadachi Trikomi [On a case of solving the Tricomi problem] *Okolozvukovye potoki gaza — Transonic gas flows.* Saratov: Saratov University, 3–6 [in Russian].

6 Kuznetsov, D.S. (1962). *Spetsialnye funktsii [Special functions]*. Moscow: Vysshaia shkola [in Russian].

7 Muskhelishvili, N.I. (1962). *Singuliarnye integralnye uravneniia [Singular integral equations]*. Moscow: GIFML [in Russian].

8 Gakhov, F.D. (1963). *Kraevye zadachi [Boundary value problems]*. Moscow: Fizmatgiz [in Russian].