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Inverse problems of determining coefficients of time type in a degenerate parabolic equation

The paper is devoted to the study of the solvability of inverse coefficient problems for degenerate parabolic equations of the second order. We study both linear inverse problems – the problems of determining an unknown right-hand side (external influence), and nonlinear problems of determining an unknown coefficient of the equation itself. The peculiarity of the studied work is that its unknown coefficients are functions of a time variable only. The work aims to prove the existence and uniqueness of regular solutions to the studied problems (having all the generalized in the sense of S.L. Sobolev derivatives entering the equation).

Keywords: degenerate parabolic equations, linear inverse problems, non-linear inverse problems, regular solutions, existence.

Introduction

The paper studies the solvability of some inverse problems of finding the solution to a degenerate parabolic equation and a certain coefficient of the equation itself. If the unknown coefficient determines the free term (external influence) in the equation, then such an inverse problem will be linear, but if the unknown coefficient is a multiplier for one or another derivative of the solution, then it will be nonlinear. In this paper, both linear and nonlinear inverse problems will be studied.

The problems studied in the work will have two features.The first of them is that inverse coefficient problems for time-variable degenerate parabolic equations will be studied. The second feature is that the unknown coefficient in our problems will also be a function of the time variable only.

Inverse problems for parabolic equations without degeneracy and with unknown coefficients depending only on the time variable seem to be thoroughly studied (see [1–12]. As for similar problems for time-variable degenerate parabolic equations, there are few works here – only works [13–15] can be named, and in these works either the nature of degeneracy is different, or the problem itself is completely distinct.

Note the following. The presence of degeneracy in parabolic equations means that the well-posed boundary value problems for them may differ significantly from the classical initial boundary value problems for non-degenerate equations (see [16–19]). This is the situation that will be studied in this paper – a situation in which the boundary conditions in linear problems will be different than in natural initial boundary value problems.

All constructions and reasoning in the work will be conducted based on the Lebesgue L_p and SobolevW_p^l spaces. The necessary definitions and description of the properties of functions from these spaces can be found in monographs [20–22].

The purpose of this work is to prove the existence and uniqueness of regular solutions to the problem, i.e., solutions having all the generalized in the sense of S.L. Sobolev derivatives, included in the corresponding equation.

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The presence of additional unknown functions in inverse problems requires that, in addition to the boundary conditions natural for a particular class of differential equations, some additional conditions are also set – overdetermination conditions. In this paper, overdetermination conditions, called integral overdetermination conditions in the literature, will be used. Inverse coefficient problems, linear and nonlinear, with integral overdetermination conditions are well-studied for both classical (elliptic, parabolic and hyperbolic) and non-classical differential equations. However, for time-variable degenerate parabolic equations, inverse coefficient problems with integral overdetermination have not been studied before.

Overall, the content of the work consists of four parts. The first part provides studied linear and nonlinear problems statement. The second part investigates the solvability of linear inverse problems for degenerate parabolic equations of the second order. The third part studies the solvability of some nonlinear inverse coefficient problem for degenerate parabolic equations of the second order. Finally, the fourth part describes some generalizations and amplifications of the results obtained in the second and third parts of the work, discusses their possible development.

Problem statement

Let Ω be a bounded domain from the space R^n with a smooth (for simplicity – infinitely differentiable) boundary Γ, Q be a cylinder $\Omega \times (0,T)$ variables (x,t) of finite height T, $S = \Gamma \times (0,T)$ be the lateral boundary of Q.

Next, let $\varphi(t), c(x,t), f(x,t), N(x), h(x,t), \mu(t)$ and $u_0(x)$ be the given functions defined at $x \in \bar{\Omega}$, $t \in [0,T]$, respectively. L is a differential operator whose action on a given function $v(x,t)$ is determined by the equality

$$Lv = \varphi(t)v_t - \Delta v + c(x,t)v$$

(Δ is the Laplace operator for variables $x_1, x_2, ..., x_n$).

Inverse problem I: Find the functions $u(x,t)$ and $q(t)$ connected in the cylinder Q by the equation

$$Lu = f(x,t) + q(t)h(x,t), \tag{1}$$

when the conditions for the function $u(x,t)$ are met

$$u(x,t)|_S = 0, \tag{2}$$

$$\int_\Omega N(x)u(x,t)dx = 0, t \in (0,T). \tag{3}$$

Inverse problem II: Find the functions $u(x,t)$ and $q(t)$ connected in the cylinder Q by the equation (1), when the conditions (2) and (3) are met for the function $u(x,t)$, as well as the conditions

$$u(x,0) = u(x,T) = 0, x \in \Omega. \tag{4}$$

The inverse problems I and II are linear inverse problems for the parabolic equation $Lu = F$. Note that in the problem I there are no boundary conditions for the variable t, in the problem II, on the contrary, two boundary conditions are set for the variable t. Both of these situations do not seem to be characteristic of first-order differential equations (with respect to a time variable), nevertheless, sufficient conditions will be specified for each of them to ensure the existence and uniqueness of regular solutions to the corresponding inverse problems.

Along with the inverse problems I and II, is easy to study linear inverse problems for equation (1) with the setting of one boundary condition for the variable t at $t = 0$ or at $t = T$. The sufficient conditions for the existence and uniqueness of regular solutions to such problems is presented in the fourth part of the work.

Let us consider the problem statement of a nonlinear inverse problem for degenerate parabolic equations.

Inverse problem III: Find the functions $u(x,t)$ and $q(t)$ connected in the cylinder Q by the equation

$$Lu + q(t)u = f(x,t), \tag{5}$$

when the function $u(x,t)$ fulfills the condition (2), as well as the conditions

$$u(x,0) = u_0(x), x \in \Omega, \tag{6}$$

$$\int_\Omega N(x)u(x,t)dx = \mu(t), t \in (0,T). \tag{7}$$

The inverse problem III corresponds to the usual first initial-boundary value problem for parabolic equations of the second order, the inhomogeneity of conditions (2) and (7) is explained by the nonlinearity of the problem.

Solvability of inverse problems I and II

Let us put

$$h_0(t) = \int_\Omega N(x)h(x,t)dx,$$

$$h_1(x,t) = \frac{h(x,t)}{h_0(t)},$$

$$f_0(t) = \int_\Omega N(x)f(x,t)dx,$$

$$f_1(x,t) = f(x,t) - h_1(x,t)f_0(t).$$

Next, by the given function $v(x,t)$, we define the functions $A_1(t;v)$ and $A_2(t;v)$:

$$A_1(t;v) = \int_\Omega N(x)\Delta v(x,t)dx,$$

$$A_2(t;v) = \int_\Omega c(x,t)N(x)v(x,t)dx.$$

For the function $\omega(x)$ from the space $W_2^2(\Omega) \cap \overset{0}{W}{}_2^1(\Omega)$ there are inequalities

$$\int_\Omega \omega^2(x)dx \le d_0 \sum_{i=1}^n \int_\Omega \omega_{x_i}^2(x)dx \le d_0^2 \int_\Omega [\Delta\omega(x)]^2 dx \tag{8}$$

with the number d_0 defined only by the domain Ω (see [20–22]). We will need these inequalities and the actual number d_0 below.

In addition to the number d_0, we will also need the following numbers:

$$\overline{h}_1 = \max_{\overline{Q}} |h_1(x,t)|,$$

$$N_1 = \overline{h}_1 ||N||_{L_2(\Omega)} mes^{1/2}\Omega,$$

$$N_2 = d_0\overline{h}_1 \max_{0\le t\le T}[\int_\Omega c^2(x,t)N^2(x)dx]^{1/2} mes^{1/2}\Omega.$$

Theorem 1. Let the conditions be satisfied

$$\varphi(t) \in C^1([0;T]), \varphi(0) \leq 0, \varphi(T) \geq 0; \tag{9}$$

$$\begin{gathered} c(x,t) = c_1(x,t) + c_0,\ c_1(x,t) \in C^2(\overline{Q}) \\ c_1(x,t) \geq 0,\ \ \Delta c_1(x,t) \leq 0 \ \text{at} \ (x,t) \in \overline{Q}, c_0 = const > 0; \end{gathered} \tag{10}$$

$$2c_0 - \varphi^{'}(t) \geq \overline{c}_0 > 0,\ \ 2c_0 + \varphi^{'}(t) \geq \overline{c}_1 > 0 \ \text{at} \ (x,t) \in \overline{Q}; \tag{11}$$

$$N(x) \in W^1_\infty(\Omega), h(x,t) \in L_\infty(Q), h_t(x,t) \in L_2(Q); \tag{12}$$

$$|h_0(t)| \geq \overline{h}_0 > 0 \ \text{at} \ t \in [0,T]; \tag{13}$$

$$N_1 + N_2 < 1. \tag{14}$$

Then for any function $f(x,t)$ such that $f(x,t) \in L_2(Q), f_t(x,t) \in L_2(Q)$, the inverse problem has the solution $(u(x,t),\ q(t))$ such that $u(x,t) \in W_2^{2,1}(Q)$, $q(t) \in L_2(Q)$.

Proof. Consider the boundary value problem: Find the function $u(x,t)$, which is the solution to the equation in the cylinder Q.

$$Lu = f_1(x,t) - h_1(x,t)[A_1(t;u) - A_2(t;u)] \tag{15}$$

and such that the condition (2) is satisfied for it. In this problem, equation (15) is a degenerate parabolic integro-differential equation (similar equations are also called "loaded" [23], [24]). We will prove its solvability in the space $W_2^{2,1}(Q)$ using the regularization method and the continuation method by parameter.

Let ε be a positive number. Consider the boundary value problem: Find the function $u(x,t)$, which in the cylinder Q is the solution to the equation

$$-\varepsilon u_{tt} + Lu = f_1(x,t) - h_1(x,t)[A_1(t;u) - A_2(t;u)] \tag{16}$$

and such that condition (2) is met for it, as well as the condition

$$u_t(x,0) = u_t(x,T) = 0,\ \ x \in \Omega. \tag{17}$$

This problem is a mixed boundary value problem for an elliptic (non-degenerate) "loaded" equation (16); its solvability in the space $W_2^2(Q)$ is not difficult to show using the continuation method by parameter [25].

Let λ be a number from the segment [0;1]. Consider a family of problems: Find the function $u(x,t)$, which in the cylinder Q is the solution to the equation

$$-\varepsilon u_{tt} + Lu = f_1(x,t) - \lambda h_1(x,t)[A_1(t;u) - A_2(t;u)] \tag{18}$$

and such that conditions (2) and (17) are met for it.

Boundary value problem (18), (2), (17) in the case of $\lambda = 0$ with a fixed ε and if the conditions of the theorem are met, it is solvable in the space $W_2^2(Q)$ for any function $f(x,t)$ belonging to the space $L_2(Q)$ (see [21]). Further, integrating by parts in equality (19)

$$\varepsilon \int_Q u_{tt} \Delta u dx dt - \int_Q Lu \Delta u dx dt = - \int_Q \{f_1(x,t) - \lambda h_1(x,t)[A_1(t;u) - A_2(t;u)]\} \Delta u dx dt \tag{19}$$

(which is a consequence of equation (18)), using conditions (9)–(14) and applying the Helder and Young inequalities, it is easy to obtain that for all possible solutions $u(x,t)$ to the boundary value problem (18), (2), (17) we take an estimate

$$\varepsilon \sum_{i=1}^{n} \int_Q u_{x_i t}^2 dxdt + \int_Q (\Delta u)^2 dxdt \leq M_1 \int_Q f^2 dxdt \tag{20}$$

with a constant M_1, defined only by the functions $\varphi(t), c(x,t), h(x,t), N(x)$, as well as the domain Ω.

Consider now the equality

$$\varepsilon \int_Q u_{tt}^2 dxdt - \int_Q Luu_{tt} dxdt = -\int_Q \{f_1(x,t) - \lambda h_1(x,t)[A_1(t;u) - A_2(t;u)]\} u_{tt} dxdt. \tag{21}$$

Integrating again by parts, using conditions (9)–(14) and applying the Helder and Young inequalities, we obtain that for all possible solutions $u(x,t)$ to the boundary value problem (18), (2), (17) a priori estimate is performed

$$\varepsilon \int_Q u_{tt}^2 dxdt + \sum_{i=1}^{n} \int_Q u_{x_i t}^2 dxdt \leq M_2 \int_Q f^2 dxdt, \tag{22}$$

where the constant M_2 is defined by the functions $\varphi(t), c(x,t), h(x,t)$ and $N(x)$, the domain Ω, and the number ε.

From estimates (20) and (22), as well as from the second basic inequality for elliptic operators [21], it follows that for solutions $u(x,t)$ to the boundary value problem (18), (2), (17) the next estimate is true

$$||u||_{W_2^2(Q)} \leq M_3 ||f||_{L_2(Q)}, \tag{23}$$

where the constant M_3 is defined by the functions $\varphi(t), c(x,t), h(x,t)$ and $N(x)$, the domain Ω, and the number ε.

From estimate (23), from the solvability in the space $W_2^2(Q)$ of the problem (18), (2), (17) in the case of $\lambda = 0$, as well as from the theorem on the continuation method by parameter [25], it follows that for a fixed ε, for an arbitrary λ from the segment [0,1] and if conditions (9)–(14) are met, the boundary value problem (18), (2), (17) will be solvable in the space $W_2^2(Q)$ for any function $f(x,t)$ from the space $L_2(Q)$.

Let $\{\varepsilon_m\}_{m=1}$ be a sequence of positive numbers converging to zero. According to the above, the boundary value problem (18), (2), (17) in the case of $\varepsilon = \varepsilon_m$ and $\lambda = 1$, there is a solution $u_m(x,t)$ belonging to the space $W_2^2(Q)$. For the family $\{u_m(x,t)\}_{m=1}^{\infty}$, there is a priori estimate (20) which is uniform by ε. Next, on the right side of the equality (21) with $\varepsilon = \varepsilon_m$, we will integrate by parts with respect to the variable t. Further, using the conditions of the theorem and applying the Helder and Young inequalities, we obtain that for the functions $u_m(x,t)$ there is a true estimate

$$\varepsilon_m \int_Q u_{mtt}^2 dxdt + \sum_{i=1}^{n} \int_Q u_{mx_i t}^2 dxdt \leq M_4 \int_Q (f^2 + f_t^2) dxdt, \tag{24}$$

the constant M_4 where is defined only by the functions $\varphi(t), c(x,t), h(x,t)$ and $N(x)$, as well as the domain Ω.

Estimates (20) and (24) for functions $u_m(x,t)$, the reflexivity property of the Hilbert space [25], as well as embedding theorems [20–22] mean that there are functions $u_{m_k}(x,t), k = 1,2,...,$ and $u(x,t)$ such that for $k \to \infty$ there are convergences

$$u_{m_k}(x,t) \to u(x,t) \text{ weakly in } W_2^{2,1}(Q),$$
$$u_{m_k x_i}(x,t) \to u_{x_i}(x,t) \text{ strongly in } L_2(Q) \text{ for } \text{ß} = 1, ..., n,$$
$$u_{m_k x_i}(x,t) \to u_{x_i}(x,t) \text{ strongly in } L_2(S) \text{ for } \text{ß} = 1, ..., n,$$
$$\varepsilon_{m_k} u_{m_k tt}(x,t) \to 0 \text{ weakly in } L_2(Q).$$

From these convergences, as well as from the representation

$$A_1(t; u_{m_k}) = -\sum_{i=1}^{n} \int_\Omega N_{x_i}(x) u_{m_k x_i}(x,t)dx - \int_\Gamma N(x) \tfrac{\partial u_{m_k}}{\partial \nu} ds$$

it follows that for the limit function $u(x,t)$, equation (15) will be fulfilled. The function $u(x,t)$ belongs to the space $W_2^{2,1}(Q)$.

Let us put

$$q(t) = \tfrac{1}{h_0(t)}[A_2(t;u) - A_1(t;u) - f_0(t)].$$

The functions $u(x,t)$ and $q(t)$ are connected in the cylinder Q by equation (1). We show that the condition (3) holds for the function $u(x,t)$.

We multiply equation (1) with the function $q(t)$ defined above by the function $N(x)$ and integrate over the domain Ω. Considering the form of the functions $h_0(t), f_0(t), h_1(x,t), A_1(t;u)$ and $A_2(t;u)$, we obtain that for the function $\omega(t)$ which is defined by equality

$$\omega(t) = \int_\Omega N(x) u(x,t) dx,$$

the equation is performed

$$\varphi(t)\omega_t + c_0\omega = 0.$$

Multiplying this equation by the function ω and integrating over the segment $[0,T]$, we get

$$\omega(t) \equiv 0 \quad \text{at} \quad t \in [0,T].$$

Hence, it follows that for the function $u(x,t)$, which is the solution to the boundary value problem (15), (2) the overdetermination condition (3) is satisfied.

All of the above means that the found functions $u(x,t)$ and $q(t)$ give the desired solution to the inverse problem I.

The theorem is proved.

There is a similar result to the above for the inverse problem II.

Theorem 2. Let the condition (25) be satisfied

$$\varphi(t) \in C^1([0;T]), \varphi(0) > 0, \varphi(T) < 0; \tag{25}$$

as well as conditions (10)–(14). Then for any function $f(x,t)$ such that $f(x,t) \in L_2(Q)$, $f_t(x,t) \in L_2(Q)$, $f(x,0) = f(x,T) = 0$ for $x \in \Omega$ the inverse problem II has a solution $(u(x,t), q(t))$ such that $u(x,t) \in W_2^{2,1}(Q), q(t) \in L_2([0,T])$.

The proof of this theorem is carried out in general analogous to the proof of Theorem 1, the only difference is that in the boundary value problem for equation (16), conditions are not (2) and (17), they are (2) and (4).

Solvability of the inverse problem III

The study of the solvability of the nonlinear inverse problem III will also be carried out by using the transition to integro-differential (loaded) equations. For simplicity, we will limit ourselves to analyzing the case of $c(x,t) \equiv 0$; the general case will differ from the one considered only by the greater cumbersomeness of conditions and calculations.

Let us put

$$M_1 = d_0 \int_Q \frac{f^2(x,t)}{\varphi(t)} dxdt + \int_\Omega u_0^2(x)dx.$$

Theorem 3. Let the conditions be satisfied

$$\varphi(t) \in C([0;T]), [\varphi(t)]^{-1} \in L_2([0;T]), \varphi(t) \geq 0, \text{ when } t \in [0;T]; \tag{26}$$

$$c(x,t) \equiv 0 \text{ at } (x,t) \in \overline{Q}; \tag{27}$$

$$\begin{gathered} N(x) \in W_\infty^2(\Omega) \cap \overset{0}{W}{}_2^1(\Omega),\ \mu(t) \in W_\infty^1([0,T]),\ u_0(x) \in W_2^2(\Omega) \cap \overset{0}{W}{}_2^1(\Omega), \\ f(x,t) \in L_\infty(0,T; \overset{0}{W}{}_2^1(\Omega)); \end{gathered} \tag{28}$$

$$\mu(t) \geq \mu_0 > 0,\ f_0(t) - \varphi(t)\mu'(t) \geq \mu_1 > 0 \text{ at } t \in [0,T]; \tag{29}$$

$$M_1^{1/2} ||\Delta N||_{L_2(\Omega)} \leq \mu_1; \tag{30}$$

$$\int_\Omega N(x)u_0(x)dx = \mu(0), \tag{31}$$

Then the inverse problem III has a solution $(u(x,t), q(t))$ such that $u(x,t) \in L_\infty(0,T; W_2^2(\Omega) \cap \overset{0}{W}{}_2^1(\Omega))$, $u_t(x,t) \in L_2(Q)$, $q(t) \in L_\infty([0,T])$, $q(t) \geq 0$ at $t \in [0,T]$.

Proof. Let $\{\varepsilon_m\}_{m=1}^\infty$ be a sequence of positive numbers converging to 0. Denote $\varphi_m(t) = \varphi(t) + \varepsilon_m$. Next, we define the cutting function $G_M(\xi), \xi \in R:$

$$G_M(\xi) = \begin{cases} \xi, & \text{if } \|\xi| < M, \\ M, & \text{if } \xi \geq M, \\ -M, & \text{if } \xi \leq -M. \end{cases}$$

Let M_0 be a number from the interval $(0, \mu_1]$. Consider the boundary value problem: Find the function $u(x,t)$, which is the solution to the equation in the cylinder Q

$$\varphi_m(t)u_t - \Delta u + \varepsilon_m \varphi_m(t)\Delta^2 u + \frac{1}{\mu(t)}[f_0(t) - \varphi(t)\mu'(t) + G_{M_0}(A_1(t;u))]u = f(x,t) \tag{32}$$

and such that conditions (2) and (6) are met for it, as well as the condition

$$\Delta u(x,t)|_S = 0. \tag{33}$$

In this problem, equation (32) for a fixed m is a non-degenerate parabolic equation of the fourth order with bounded nonlinearity in the lower term. Using standard energy estimates for parabolic equations [26], the Galerkin method or the fixed-point method, it is easy to establish that the problem (32), (2), (6), (33) has a solution $u_m(x,t)$ belonging to the space $W_2^{4,1}$. We show that using the functions $u_m(x,t)$ it is possible to find a solution to the inverse problem III.

Multiply equation (32) by the function $[\varphi(t)]^{-1}u_m(x,t)$ and integrate over the domain Ω and over the time variable from 0 to the current point. After integrating by parts and reassigning variables, we get equality

$$\begin{gathered}\int_\Omega u_m^2(x,t)dx+\sum_{i=1}^n\int_0^t\int_\Omega\frac{u_{mx_i}^2(x,\tau)}{\varphi_m(\tau)}dxd\tau+\varepsilon_m\int_0^t\int_\Omega[\Delta u_m(x,\tau)]^2dxd\tau+\\+\int_0^t\int_\Omega\frac{u_m^2(x,\tau)}{\mu(\tau)}[f_0(\tau)-\varphi(\tau)\mu'(\tau)+G_{M_0}(A_1(\tau;u_m))]dxd\tau=\\=\int_0^t\int_\Omega\frac{f(x,\tau)}{\varphi_m(\tau)}u_m(x,\tau)dxd\tau+\frac{1}{2}\int_\Omega u_0^2(x)dx.\end{gathered}\tag{34}$$

Due to conditions (26) and (29), all the terms of the left side of this equality are non-negative. Applying the Young's inequality and inequality (8) to the first term of the right part (34), taking into account also condition (28), we obtain that for functions $u_m(x,t)$ for $t\in[0,T]$, the evaluation is performed

$$\int_\Omega u_m^2(x,t)dx\le M_1.\tag{35}$$

Analyzing the equalities obtained after multiplying equation (32) by the functions $-[\varphi_m(t)]^{-1}\Delta u_m(x,t)$, $[\varphi_m(t)]^{-1}\Delta^2u_m(x,t)$ with subsequent integration over the domain Ω and over the time variable from 0 to the current point, we obtain by using conditions (26), (28) and (29), and the Helder inequality and the inequality (35) that for the functions $u_m(x,t)$ the evaluation is performed

$$\begin{gathered}\sum_{i=1}^n\int_\Omega u_{mx_i}^2(x,t)dx+\int_\Omega[\Delta u_m(x,t)]^2dx+\varepsilon_m\sum_{i=1}^n\int_0^t\int_\Omega(\Delta u_{mx_i}(x,t))^2dxd\tau+\\+\varepsilon_m\int_0^t\int_\Omega(\Delta^2u_m(x,t))^2dxd\tau\le M_2\end{gathered}\tag{36}$$

with a constant M_2 defined only by the functions $\varphi(t),\mu(t),N(x),f(x,t)$ and $u_0(x)$, as well as the domain Ω and the number T. To obtain the last necessary estimate, multiply equation (32) by the function $[\varphi_m(t)]^{-1}u_{mt}(x,t)$ and integrate over the cylinder Q. After simple transformations using the conditions of the theorem, the Gelder inequality and estimates (35) and (36), we obtain that for the function $u_m(x,t)$ the inequality holds

$$\int_Q u_{mt}^2(x,t)dxdt\le M_3\tag{37}$$

with a constant M_3 defined only by the functions $\varphi(t),\mu(t),N(x),f(x,t)$ and $u_0(x)$, as well as the domain Ω and the number T. Let's clarify the value of the number M_0: $M_0=\mu_1$. With this choice of the number M_0, it follows from the estimate (34) and condition (30) that $G_{M_0}(A_1(t;u_m))=A_1(t;u_m)$ is satisfied in equations (32). Further, from the estimates (34)–(37) and from the reflexivity property of the Hilbert space, as well as from the embedding theorems, it follows that there exists a subsequence $\{u_{mk}(x,t)\}_{m=1}^\infty$ from the sequence of solutions to the boundary value problem (32), (2), (6), (33), and the function $u(x,t)$ such that for $k\to\infty$ there are convergences:

$$u_{m_k}(x,t)\to u(x,t)\ \text{ weak in }\ W_2^{2,1}(Q)\ \text{and strong in }\ L_2(Q),$$

$$\varepsilon_{m_k}\Delta^2u_{m_k}(x,t)\to0\ \text{ weak in }\ L_2(Q).$$

Let us put

$$q(t)=\frac{1}{\mu(t)}[f_0(t)-\varphi(t)\mu'(t)+A_1(t;u)],\tag{38}$$

$$\omega(t)=\int_\Omega N(x)u(x,t)dx-\mu(t).\tag{39}$$

For the function $u(x,t)$ and for the function $q(t)$ defined by equality (38) in the cylinder Q, equation

(5) is fulfilled. Further, for the function $u(x,t)$, conditions (2) and (6) are fulfilled. We show that the overdetermination condition (7) is satisfied for the function $u(x,t)$. Multiply equation (5) by the function $N(x)$ and integrate over the domain Ω. Comparing the obtained equality with equality (38), we come to the equation for the function $\omega(t)$:

$$\varphi(t)\omega'(t) + q(t)\omega(t) = 0. \tag{40}$$

Since the function $\omega(t)$ is bounded on the segment $[0,T]$, the function $[\varphi(t)]^{-1}$ belongs to the space $L_2([0,T])$, then (40) can be written as

$$\omega'(t) + \tfrac{q(t)}{\varphi(t)}\omega(t) = 0.$$

Multiplying the last equality by the function $\omega(t)$ and integrating, we come to equality

$$\frac{1}{2}\omega^2(t) + \int_0^t \frac{q(\tau)}{\varphi(\tau)}\omega^2(\tau)d\tau = \frac{1}{2}\omega^2(0). \tag{41}$$

Since the function $q(t)$ is non-negative on the segment $[0,T]$ and $\omega(0) = 0$ (due to condition (31)), then from (41) it follows that $\omega(t)$ is an identically zero function on the segment $[0,T]$.

The equality to zero of the function $\omega(t)$ and the formula (39) mean that the overdetermination condition (7) is satisfied for the found function $u(x,t)$.

So, for the functions $u(x,t)$ and $q(t)$ defined above, equation (5) is fulfilled, boundary conditions (2) and (6) are fulfilled, as well as the overdetermination condition (7). Belonging of the functions $u(x,t)$ and $q(t)$ to the required classes follows from a priori estimates (34)–(37). Consequently, these functions will give the desired solution to the inverse problem III.

The theorem is proved.

Comments and additions

1. Throughout the work, it is assumed that certain inequalities or conditions for functions from the Lebesgue or Sobolev spaces (conditions (13), (14), etc.) are fulfilled in the sense of their truth almost everywhere on the corresponding set, that is, truth everywhere except, perhaps, for some set of zero Lebesgue measure.

2. The approaches to proving the solvability of the corresponding inverse problems in clause 3 and clause 4 are significantly different. First of all, we note that the statement of problem III does not imply, despite the possible reversal of the function $\varphi(t)$ to zero at $t = 0$, the liberation of the set $\{x \in \Omega,\ t = 0\}$ from carrying the initial condition. Further, the conditions of Theorem 3 do not imply differentiability of the function $\varphi(t)$, which is required in Theorems 1 and 2. All this is explained by the fact that the conditions of Theorem 3 allow only weak degeneracy at $t = 0$, with weak degeneracy and the nondifferentiability of the function $\varphi(t)$ at the points of its vanishing, the liberation of the initial manifold of the initial data does not occur.

Note also that the conditions on the right side of $f(x,t)$ in Theorems 1, 2, and 3 differ significantly.

3. The paper studies the solvability of some inverse problems for model parabolic equations. Similar results (with minor changes) can be obtained for more general equations, for example, with the replacement of the Laplace operator by an elliptic operator

$$\sum_{i=1}^{n} \tfrac{\partial}{\partial x_i}(a^{ij}(x)u_{x_j}),$$

or for equations with first derivatives in variables $x_1, ..., x_n$,, etc.

4. Let one of the conditions be satisfied in equation (1)

$$\varphi(0) > 0, \ \ \varphi(T) \geq 0,$$

or

$$\varphi(0) \leq 0, \ \ \varphi(T) < 0.$$

In these cases, the inverse problems of finding the functions $u(x,t)$ and $q(t)$ connected in the cylinder Q by equation (1) will be correct when conditions (2) and (3) are set, as well as the initial or final conditions:

$$u(x,0) = 0, \ x \in \Omega,$$

or

$$u(x,T) = 0, \ x \in \Omega.$$

The proof of the corresponding existence theorems is carried out analogously to the proof of Theorem 1, the only difference is that in regularized problems either such conditions are given

$$u(x,0) = u_t(x,T) = 0, \ x \in \Omega,$$

or else such

$$u_t(x,0) = u(x,T) = 0, \ x \in \Omega,$$

and for the function $f(x,t)$, $f(x,0) = 0$ or $f(x,T) = 0$ must be executed for $x \in \Omega$.

5. The condition of turning the function $N(x)$ to zero at $x \in \Gamma$ (see condition (28)) can be abandoned. Let the condition be true for the function $f(x,t)$

$$f(x,t) \in L_\infty(0,T; W_2^2(\Omega) \cap \overset{0}{W}{}_2^1(\Omega)).$$

Consider the problem: Find the function $\overline{u}(x,t)$, which is the solution to the equation in the cylinder Q

$$\varphi(t)\overline{u}_t + \tfrac{1}{\mu(t)}[f_0(t) - \varphi(t)\mu'(t) + \textstyle\int_\Omega N(y)\overline{u}(y,t)dy]\overline{u} = \Delta f(x,t)$$

and such that the condition (2) and the condition are fulfilled for it

$$\overline{u}(x,0) = \Delta u_0(x), \ \ x \in \Omega.$$

The existence of regular solutions to this problem (under conditions similar to conditions (26)–(31)) is easy to prove by the method by which Theorem 3 was proved. Finding the function $\overline{u}(x,t)$, it will not be difficult to further find the desired solution $(u(x,t), q(t))$ to the inverse problem III.

6. On the contrary, if in the inverse problems I and II the function $N(x)$ vanishes at $x \in \Gamma$ and belongs to the space $W_2^2(\Omega)$, then using the representation

$$A_1(t;u) = \textstyle\int_\Omega \Delta N(y)u(y,t)dy$$

it is not difficult to obtain a condition other than (14) for the solvability of inverse problems I and II.

7. The conditions of theorems 1 and 2 are satisfied if the measure of the domain Ω is small, the functions $c(x,t)$ and $\varphi(t)$ are small.

We show that in the inverse problem III, the set of initial data for which all the conditions of Theorem 3 are satisfied is not empty.

Let $n = 1$, $\Omega = (0;1)$, $N(x) = x(1-x)$, $u_0(x)$ and $\widetilde{f}(x)$ be functions from the space $W_2^2(\Omega) \cap \overset{0}{W}{}_2^1(\Omega)$ positive in Ω. Next, let γ be a positive number, a number for which the inequality holds

$$2(\int_\Omega u_0^2(x)dx)^{\frac{1}{2}} < \gamma \int_\Omega \widetilde{f}(x)N(x)dx.$$

If now $\mu(t)$ is an arbitrary decreasing on the segment $[0,T]$ continuously differentiable function such that

$$\mu(0) = \int_\Omega N(x)u_0(x)dx,$$

$f(x,t)$ and $\varphi(t)$ are functions of $\gamma\widetilde{f}(x)$ and t^α, $0 < \alpha < \frac{1}{2}$, then all the conditions of the Theorem 3 will be executed for sufficiently small numbers T.

Other examples can be given for the inverse problems I, II, and III.

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Өзгешеленген псевдопараболалық теңдеуге қойылған коэффициентті кері есептер

Мақала өзгешеленген екінші ретті параболалық теңдеулер үшін коэффициентті кері есептердің шешілімділігін зерттеуге арналған. Сызықтық кері есептер ретінде теңдеудің белгісіз оң жағын (сыртқы әсер) анықтау есептері және белгісіз бір коэффициентті анықтаудың сызықтық емес есебі қарастырылған. Зерттелетін жұмыстардың ерекшелігі – олардағы белгісіз коэффициенттер тек уақыт айнымалысының функциялары болып табылады. Жұмыстың мақсаты – зерттелетін есептердің тұрақты шешімдерінің (теңдеуге қатысатын функциялардың С.Л. Соболев мағынасында барлық жалпылама туындылары бар) бар және жалғыздығын дәлелдеу.

Кілт сөздер: өзгешеленген параболалық теңдеулер, сызықтық кері есептер, сызықты емес кері есептер, регуляр шешімдер, шешімнің бар болуы.

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Обратные задачи определения коэффициентов временного типа в вырождающемся параболическом уравнении

Статья посвящена исследованию разрешимости обратных коэффициентных задач для вырождающихся параболических уравнений второго порядка. Изучены как линейные обратные задачи — задачи определения неизвестной правой части (внешнего воздействия), так и нелинейные задачи определения некоторого коэффициента самого уравнения. Особенностью изучаемых работ является то, что неизвестные коэффициенты в них являются функциями лишь от временной переменной. Цель работы — доказательство существования и единственности регулярных решений изучаемых задач (решений, имеющих все обобщенные, по С.Л. Соболеву, производные, входящие в уравнение).

Ключевые слова: вырождающиеся параболические уравнения, линейные обратные задачи, нелинейные обратные задачи, регулярные решения, существование.

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