DOI 10.31489/2022M3/4-16

UDC 517.949.8, 533.7

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Global solvability of a nonlinear Boltzmann equation

In this paper, based on the splitting method scheme, the existence and uniqueness theorem on the whole time interval $t \in [0, T), T \leq \infty$ for the full nonlinear Boltzmann equation in the nonequilibrium case is proved where the intermolecular interactions are hard-sphere molecule and central forces. Considering the existence of a bounded solution in the space $\mathbf{C}$, the strict positivity of the solution to the full nonlinear Boltzmann equation is proved when the initial function is positive. On the basis of this some mathematical justification of the $H-$theorem of Boltzmann is shown.

Keywords: full nonlinear Boltzmann equation, splitting method, existence and uniqueness theorem on the whole time for the nonlinear Boltzmann equation, positivity of the solution to the nonlinear Boltzmann equation, Boltzmann's $H-$theorem.

Introduction

The Boltzmann equation [1] is a complex nonlinear integro-differential equation and refers to difficult-to-study mathematical objects. Proof of the existence and uniqueness theorem for a solution of the Cauchy problem for a spatially homogeneous Boltzmann equation begins with the work of T. Carleman [2].

H. Grad [3] proved the first existence theorem in the "small" for spatially nonhomogeneous Boltzmann equation in the case of Maxwellian molecules when the initial function tends to Maxwellian distribution function in a special norm.

The world's leading experts on kinetic equations provided a review monograph [4], on the current state of mathematical theory of the Boltzmann equation starting with its derivation, theorem existence and uniqueness and methods of solution. They wrote: «... For over 110 years this equation attracts the attention of researchers, but only in recent years it has proved *global solvability* spatially $-$ nonhomogeneous problem in the case of a small deviation of the gas state from equilibrium positions - more general results are not obtained to this day ...» [4].

T. Carleman in [2] pointed out that solving the full Boltzmann equation for practical problems can be only done through approximate mathematical methods. In this connection, we have chosen the splitting method to solve the full nonlinear Boltzmann equation. Splitting methods for solving a class of various applied problems were developed by G. I. Marchuk [5].

In Kazakhstan, the study of the nonlinear equation and its corresponding discrete models began in S.K. Godunov and U.M. Sultangazin works [6].

In this connection, to solve the full nonlinear Boltzmann equation in the class of positive initial functions, the splitting method was applied [7], [8]. First, based on this method boundedness of positive solutions in the space continuous functions was got. With the help of the boundedness of the solution and of the established a priori estimates, the convergence of the scheme splitting method and uniqueness of the limiting element were proved. The found limiting element satisfies the equivalent integral Boltzmann equation. Thus, a weak solvability of the nonlinear Boltzmann equation as a whole in time.

From modern bibliographic sources it follows that there are no the existence and uniqueness theorems as a whole in time for the nonlinear Boltzmann equation in a nonequilibrium case when intermolecular interactions are hard-sphere molecules or central forces.

1 Statement of the problem for a nonlinear Boltzmann equation

Cauchy problem for the full nonlinear Boltzmann equation for molecules − hard spheres of radius χ in the domain

$$Q = \Big(t \in [0,T),\ T \le \infty;\ \mathbf{x} = (x_1, x_2, x_3) \in G \equiv \{\, 0 \le x_\alpha \le 1,\ \alpha = \overline{1,3}\,\};$$
$$\mathbf{v} = (\xi_1, \xi_2, \xi_3) \in V_3 \equiv \{-\infty \le \xi_\alpha \le \infty,\ \alpha = \overline{1,3}\,\}\Big)$$

with respect to the distribution function $f = f(t, \mathbf{x}, \mathbf{v})$ is written as [1], [2]:

$$\frac{\partial f}{\partial t} + (\mathbf{v}, \nabla) f = \mathbf{J}(f) - f\mathbf{S}(f) \equiv \mathbf{B}(f, f), \tag{1}$$

with an initial

$$f(t, \mathbf{x}, \mathbf{v}) \mid_{t=0} = \varphi(\mathbf{x}, \mathbf{v}) \tag{2}$$

and periodic boundary conditions*

$$f(t, \mathbf{x}, \mathbf{v})\big|_{\Gamma_{0x_\alpha}} = f(t, \mathbf{x}, \mathbf{v})\big|_{\Gamma_{1x_\alpha}},\ \ \alpha = \overline{1,3}, \tag{3}$$

where

$$\mathbf{S}(f) = \iiint\limits_{-\infty}^{\infty} \int\limits_0^{\pi/2} \int\limits_0^{2\pi} f(t, \mathbf{x}, \mathbf{v}_1) K(\theta, \mathbf{w}) d\varepsilon d\theta d\mathbf{v}_1 \equiv \int\limits_{V_3 \times \Sigma} f(t, \mathbf{x}, \mathbf{v}_1) K(\theta, \mathbf{w}) d\sigma d\mathbf{v}_1;$$

$$\mathbf{J}(f) = \int\limits_{V_3 \times \Sigma} f(t, \mathbf{x}, \mathbf{v}') f(t, \mathbf{x}, \mathbf{v}_1') K(\theta, \mathbf{w}) d\sigma d\mathbf{v}_1,\ \ K(\theta, \mathbf{w}) = 0.25\chi^2 \mid \mathbf{w} \mid \sin(2\theta),$$

$\mathbf{v}$, $\mathbf{v}_1$ are the velocity vectors of two colliding molecules, $\mathbf{w} = \mathbf{v} - \mathbf{v}_1$ is relative velocity vector; velocity of molecules after collisions $\mathbf{v}', \mathbf{v}_1'$, are related to $\mathbf{v}, \mathbf{v}_1$ by the dynamic relation: $\mathbf{v}' = \mathbf{v} + \mathbf{g}(\mathbf{g}, \mathbf{w})$, $\mathbf{v}_1' = \mathbf{v}_1 - \mathbf{g}(\mathbf{g}, \mathbf{w})$; $\mathbf{g}$ *is* unit vector in the direction of scattering of molecules:
$\mathbf{g} = \big(\sin\theta\cos\varepsilon,\ \sin\theta\sin\varepsilon,\ \cos\theta\big)$; $(\theta, \varepsilon) \in \mathbf{\Sigma} \equiv \big\{0 \le \theta \le \pi;\ 0 \le \varepsilon \le 2\pi\big\}$; $\Gamma_{\rho x_\alpha}-$ edge cube G perpendicular to the axis x_α, passing through $x_\alpha = \rho$, ρ takes the value either 0 or 1.

The initial function $\varphi(\mathbf{x}, \mathbf{v})$ satisfies condition (3) and it is such that

$$\begin{cases} 0 < \varphi(\mathbf{x}, \mathbf{v}) \in C(G \times V_3) \wedge \Big(\|\varphi(\mathbf{v})\|_{L_\infty(G)} \le \frac{const}{(1+|\mathbf{v}|^2)^{\frac{\gamma}{2}}},\ \gamma > 6\Big); \\ \mathbf{J}(\varphi) \le \int\limits_{V_3\times\Sigma} \|\varphi(\mathbf{v}')\|_{L_\infty(G)} \cdot \|\varphi(\mathbf{v}_1')_{L_\infty(G)}\| K(\theta, \mathbf{w}) d\sigma d\mathbf{v}_1 = q_1(\mathbf{v}) < \infty; \\ \mathbf{S}(\varphi) \le \int\limits_{V_3\times\Sigma} \|\varphi(\mathbf{v}_1)\|_{L_\infty(G)} K(\theta, \mathbf{w}) d\sigma d\mathbf{v}_1 = q_2(\mathbf{v}) < \infty, \end{cases} \tag{4}$$

where $\|\varphi(\mathbf{v})\|_{L_\infty(G)} = \sup\limits_{\mathbf{x}\in G} \mid \varphi(\mathbf{x}, \mathbf{v}) \mid$ at every $\mathbf{v} \in V_3$, $\int\limits_{V_3} q_k(\mathbf{v}) d\mathbf{v} = const$, $k = 1, 2$. Following [2], requirements (4) for the initial function were taken into account, that improper integrals were convergent in the velocity space.

Lemma 1. For periodic functions, the following integration-by-parts formula over the cube G_3 is valid

$$\int\limits_{G_3} V \Delta U \, \mathbf{dx} = -\int\limits_{G_3} \nabla V \, \nabla U \, \mathbf{dx}. \tag{5}$$

*or mirror reflections of molecules from the boundary of the G domain.

Proof. Let us write down the integration-by-parts formula

$$\int\limits_{\Omega} V\Delta U\,\mathbf{dx} = -\int\limits_{\Omega} \nabla V\,\nabla U\,\mathbf{dx} + \int\limits_{\partial\Omega} V\frac{\partial U}{\partial \mathbf{n}}\,\mathbf{dx}. \tag{6}$$

From the properties of the cube surface, it follows

$$\int\limits_{\partial G_3} V\frac{\partial U}{\partial \mathbf{n}}\,\mathbf{dx} = \sum_{\kappa=1}^{3}\Bigg(\int\limits_{\Gamma_{0,x_\kappa}} V\frac{\partial U}{\partial x_\kappa} n_\kappa\,dx_\beta dx_\gamma + \\ + \int\limits_{\Gamma_{1,x_\kappa}} V\frac{\partial U}{\partial x_\kappa} n_\kappa\,dx_\beta dx_\gamma\Bigg),\ (\beta,\gamma)\in\{1,2,3\}\wedge(\beta,\gamma)\neq\kappa, \tag{7}$$

where $\mathbf{n}$ is the outward normal vector to the cube surface, then considering the value of the normal component in formula (7), $n_\kappa = \begin{cases} -1, & \rho = 0, \\ +1, & \rho = 1, \end{cases}$ we have

$$\int\limits_{\partial G_3} V\frac{\partial U}{\partial \mathbf{n}}\,\mathbf{dx} = \sum_{\kappa=1}^{3}\Bigg(\int\limits_{\Gamma_{0,x_\kappa}} V\frac{\partial U}{\partial x_\kappa} n_\kappa\,dx_\beta dx_\gamma + \int\limits_{\Gamma_{1,x_\kappa}} V\frac{\partial U}{\partial x_\kappa} n_\kappa\,dx_\beta dx_\gamma\Bigg) = 0.$$

Taking into account this relation, from (6) we get (5).

2 Existence and uniqueness theorems

To solve problem (1)−(3) we use the of splitting method [5]. On $[0,T)$ we introduce the time grid $\omega^\tau = \{t_n = n\tau \leq \infty,\ \tau > 0,\ n = 0,1,\cdots;\}$, and*

$$\tau < 1/\int\limits_{V_3}(q_1+q_2)d\mathbf{v}. \tag{8}$$

Suppose an approximation is known $f^n(\mathbf{x},\mathbf{v})$, at time $n\tau$. Then the schemes of the splitting method corresponding to the problem (1)–(3) are written as follows:

$$\frac{f^{n+1/2}-f^n}{\tau} = \mathbf{B}(f^n,f^n), \tag{9}$$

$$\frac{f^{n+1}-f^{n+1/2}}{\tau} + (\mathbf{v},\nabla)f^{n+1} = 0, \tag{10}$$

with initial and periodic boundary conditions

$$f^0(\mathbf{x},\mathbf{v}) = \varphi(\mathbf{x},\mathbf{v}), \quad f^{n+1}\big|_{\Gamma_{0x_\alpha}} = f^{n+1}\big|_{\Gamma_{1x_\alpha}}. \tag{11}$$

Let the known approximation $f^n(\mathbf{x},\mathbf{v})$ has all the properties (4) of the initial function.

Introduce a shift operator $\mathbf{T}^{-1/2}$ such that $\mathbf{T}^{-1/2}f^n = f^{n-1/2}$, that is, the operator $\mathbf{T}^{-1/2}-$ acting on the function f^n returns its value obtained by the previous fractional step of the splitting method. Acting this operator on scheme (10), we find the difference-differential analog of the continuity equation

*Condition (8) on the step τ is necessary for the solution positivity of the splitting method schemes.

(or mass conservation equation) at each $\mathbf{v} \in V_3$ corresponding to the first fractional step of the splitting method, that is

$$\frac{f^{n+1/2} - f^n}{\tau} + (\mathbf{v}, \nabla) f^{n+1/2} = 0, \quad f^{n+1/2}\big|_{\Gamma_{0x_\alpha}} = f^{n+1/2}\big|_{\Gamma_{1x_\alpha}}. \tag{12}$$

The boundary condition was obtained from (9), since the function f^n has this property.

It is easy to see that there is the maximum principle on spatial variable $\mathbf{x} \in G$ for problems (10)-(11) and (12).

Let us first consider problem (12) in the form

$$f^{n+1/2} + \tau(\mathbf{v}, \nabla) f^{n+1/2} = f^n, \;\; f^{n+1/2}\big|_{\Gamma_{0x_\alpha}} = f^{n+1/2}\big|_{\Gamma_{1x_\alpha}}.$$

Applying the maximum principle to this problem, we find an estimate for the solution $f^{n+1/2}(\mathbf{x}, \mathbf{v})$ in the space $C(G)$

$$\sup_{\mathbf{x}\in G} |f^{n+1/2}(\mathbf{v})| \le \sup_{\mathbf{x}\in G} |f^n(\mathbf{v})|, \;\; \forall \mathbf{v} \in V_3.$$

Then in the same way from problem (10), (11) we obtain an estimate

$$\sup_{\mathbf{x}\in G} |f^{n+1}(\mathbf{v})| \le \sup_{\mathbf{x}\in G} |f^{n+1/2}(\mathbf{v})|, \;\; \forall \mathbf{v} \in V_3.$$

Combining the found estimates, we have

$$\sup_{\mathbf{x}\in G} |f^{n+1}(\mathbf{v})| \le \sup_{\mathbf{x}\in G} |f^{n+1/2}(\mathbf{v})| \le \sup_{\mathbf{x}\in G} |f^n(\mathbf{v})|, \;\; \forall \mathbf{v} \in V_3.$$

From here, summing over n, we find the main estimate

$$\sup_{\mathbf{x}\in G} |f^{n+1}(\mathbf{v})| \le \sup_{\mathbf{x}\in G} |f^{n+1/2}(\mathbf{v})| \le \|\varphi(\mathbf{v})\|_{L_\infty(G)} = q_0(\mathbf{v}), \;\; \forall \mathbf{v} \in V_3 \tag{13}$$

that allows us to obtain estimates for the nonlinear terms of the equation (9).

Consider first

$$\mathbf{J}(f^n) = \int\limits_{V_3\times\Sigma} f^n(\mathbf{x}, \mathbf{v}') \cdot f^n(\mathbf{x}, \mathbf{v}_1') K(\theta, \mathbf{w}) d\sigma d\mathbf{v}_1.$$

From here

$$\begin{aligned} 1.\ |\mathbf{J}(f^n)| \le \int\limits_{V_3\times\Sigma} |f^n(\mathbf{x}, \mathbf{v}')| \cdot |f^n(\mathbf{x}, \mathbf{v}_1')| K(\theta, \mathbf{w}) d\sigma d\mathbf{v}_1 \le \\ \le \int\limits_{V_3\times\Sigma} \sup_{\mathbf{x}\in G} |f^n(\mathbf{x}, \mathbf{v}')| \cdot \sup_{\mathbf{x}\in G} |f^n(\mathbf{x}, \mathbf{v}_1')| K(\theta, \mathbf{w}) d\sigma d\mathbf{v}_1 = \\ = \int\limits_{V_3\times\Sigma} \|f^n(\mathbf{v}')\|_{L_\infty(G)} \cdot \|f^n(\mathbf{v}_1')\|_{L_\infty(G)} K(\theta, \mathbf{w}) d\sigma d\mathbf{v}_1 \le \\ \le \int\limits_{V_3\times\Sigma} \|\varphi(\mathbf{v}')\|_{L_\infty(G)} \cdot \|\varphi(\mathbf{v}_1')\|_{L_\infty(G)} K(\theta, \mathbf{w}) d\sigma d\mathbf{v}_1 = q_1(\mathbf{v}) < \infty. \end{aligned} \tag{14}$$

2. $f^n(\mathbf{x},\mathbf{v}) \mid \mathbf{S}(f^n(\mathbf{x},\mathbf{v}_1)) \mid \le \sup_{\mathbf{x}\in G} |f^n(\mathbf{v})| \int\limits_{V_3\times\Sigma} \mid f^n(\mathbf{x},\mathbf{v}_1) \mid K(\theta,\mathbf{w})d\sigma d\mathbf{v}_1 \le$

$$\le q_0(\mathbf{v}) \int\limits_{V_3\times\Sigma} \|f^n(\mathbf{v}_1\|_{L_\infty(G)} K(\theta,\mathbf{w})d\sigma d\mathbf{v}_1 \le$$

$$\le q_0(\mathbf{v}) \int\limits_{V_3\times\Sigma} \|\varphi(\mathbf{v}_1)\|_{L_\infty(G)} K(\theta,\mathbf{w})d\sigma d\mathbf{v}_1 = q_2(\mathbf{v}) < \infty. \quad (15)$$

It is now easy to obtain an estimate for the difference derivative $f_{\bar{t}}^{n+1/2}$ using (14), (15), based on the equation (9):

$$\sup_{\mathbf{x}\in G} \mid (f^{n+1/2} - f^n)/\tau \mid \le \mid \mathbf{B}(f^n, f^n) \mid \le \mid \mathbf{J}(f^n)| + |f^n(\mathbf{v})| \mid \mathbf{S}(f^n) \mid \le q_1(\mathbf{v}) + q_2(\mathbf{v}) = q_3(\mathbf{v}). \quad (16)$$

Adding equations (9), (10), on the integer step we obtain the difference-differential Boltzmann equation

$$(f^{n+1} - f^n)/\tau + (\mathbf{v},\nabla)f^{n+1} = \mathbf{B}(f^n, f^n). \quad (17)$$

with initial and boundary conditions

$$f^0(\mathbf{x},\mathbf{v}) = \varphi(\mathbf{x},\mathbf{v}), \quad f^{n+1}\big|_{\Gamma_{0x_\alpha}} = f^{n+1}\big|_{\Gamma_{1x_\alpha}}. \quad (18)$$

From here

$$\mid (f^{n+1} - f^n)/\tau \mid \le \mid \mathbf{B}(f^n, f^n) \mid + \mid (\mathbf{v},\nabla)f^{n+1} \mid .$$

When the function $f^{n+1}(\mathbf{x},\mathbf{v})$ reaches its maximum value at extremum points reaches $\mathbf{x}^e$ in G for every $\mathbf{v}$ in V_3 by virtue of the maximum principle, we have

$$\mid (f^{n+1} - f^n)/\tau \mid (\mathbf{x}) \le \mid (f^{n+1} - f^n)/\tau \mid (\mathbf{x}^e) \le \mid \mathbf{B}(f^n, f^n) \mid (\mathbf{x}^e).$$

From here

$$\sup_{\mathbf{x}\in G} \left|(f^{n+1} - f^n)/\tau\right| \le \left|\mathbf{B}(f^n, f^n)\right| = q_3(\mathbf{v}). \quad (19)$$

Now from (17) we find

$$\sup_{\mathbf{x}\in G} \left|(\mathbf{v},\nabla)f^{n+1}\right| \le 2q_3(\mathbf{v}). \quad (20)$$

Remark 1. The functions $q_k(\mathbf{v}) \in C(V_3)$. $k = \overline{0,3}$, i.e., they are positive continuous summable functions and continuous depending on the integral of norm for the initial function $\varphi(\mathbf{v})$ over the domain V_3.

Proposition 1. Each problem (12) and (10)–(11) has a unique positive continuous solution that is bounded in Q, and it is periodic function over x_α, i.e., it possesses properties (4) of the initial function, since the approximation $f^n(\mathbf{x},\mathbf{v})$ is such. The periodicity is shown in the same ways as in [7], and the rest of the properties have already been proven.

3 *Compact solutions and existence*

We denote the set of found approximate solutions to problems (9), (10)–(11) by $\{f^\tau\}$, and the the set of interpolated values on the interval $[0,T)$ by $\tilde{f}^\tau$.

In the velocity space V_3, we introduce a ball V_{R^τ} with the center at the origin of coordinates and with the radius $R^\tau = O(1/\tau^k) < \infty$, where $1 \ll k = const \wedge k \in N$ resulting in a finite bounded domain $Q_{R^\tau} = [0,T) \times G \times V_{R^\tau} \subset Q$.

Since all the estimates are established in the domain Q then they are valid in Q_{R^τ}. The validity of the estimates are not violated when the radius of the ball R^τ increases arbitrarily large as τ tends to zero.

Moreover, from estimates of (13), (14), (15), (16) and (19), (20) it follows the uniform boundedness of the norms for the interpolated functions

$$\tilde{f}^\tau, \quad \mathbf{J}(\tilde{f}^\tau), \quad \mathbf{S}(\tilde{f}^\tau), \quad \tilde{f}^\tau_{\bar{t}}, \ (\mathbf{v}, \nabla)\tilde{f}^\tau$$

in $C(Q)$ at $\tau \to 0$. From here it follows the equicontinuity of $\{\tilde{f}^\tau\}$ in $C(Q)$. Hence, the set $\tilde{f}^\tau$ is compact in the space $C(Q)$. A convergent subsequence can be distinguished from this set. It converges in $C(Q)$ to some element $f(t, \mathbf{x}, \mathbf{v}) \in C(Q)$. Due to compactness, the following limit transitions take place at $\tau \to 0$:

$$\tilde{f}^\tau \to f, \ \tilde{f}^\tau_{\bar{t}} \to f_t, \ \mathbf{J}(\tilde{f}^\tau) \to \mathbf{J}(f), \ \tilde{f}^\tau \mathbf{S}(\tilde{f}^\tau) \to f\mathbf{S}(f),$$
$$\tilde{f}^\tau(t, \mathbf{x}, \mathbf{v}) \mid_{t=0} \to f(t, \mathbf{x}, \mathbf{v}) \mid_{t=0} = \varphi(\mathbf{x}, \mathbf{v}), \ (\mathbf{v}, \nabla)\tilde{f}^\tau \to (\mathbf{v}, \nabla)f,$$
$$\tilde{f}^\tau(t, \mathbf{x}, \mathbf{v})\Big|_{\Gamma_{0x_\alpha}} = \tilde{f}^\tau(t, \mathbf{x}, \mathbf{v})\Big|_{\Gamma_{1\mathbf{x}_\alpha}} \to f(t, \mathbf{x}, \mathbf{v})\Big|_{\Gamma_{0x_\alpha}} = f(t, \mathbf{x}, \mathbf{v})\Big|_{\Gamma_{1x_\alpha}}, \alpha = \overline{1, 3},$$
$$Q_{R^\tau} \to Q.$$

Thus, going to the limit in the difference-differential problem (17)–(18) we make sure that the limit element $f(t, \mathbf{x}, \mathbf{v})$ uniformly satisfies problem (1)–(3) for the nonlinear Boltzmann equation.

4 Uniqueness

Let there be two solutions $f(t, \mathbf{x}, \mathbf{v})$ and $F(t, \mathbf{x}, \mathbf{v})$ of problem (1)$-$(3). Let us write down the equations for their difference $U = f - F$:

$$\frac{\partial U}{\partial t} + (\mathbf{v}, \nabla)U = \mathbf{B}(f, f) - \mathbf{B}(F, F), \tag{21}$$

in the domain $Q = [0, T) \times G \times V_3$ with zero initial $U \mid_{t=0} = 0$ and periodic boundary condition

$$U(t, \mathbf{x}, \mathbf{v})\Big|_{\Gamma_{0x_\alpha}} = U(t, \mathbf{x}, \mathbf{v})\Big|_{\Gamma_{1x_\alpha}}, \ \alpha = \overline{1, 3}. \tag{22}$$

Note that all improper integrals in the calculations make sense, i.e. they are converging integrals. Multiply equation (21) by $2U$ and integrate by domain V_3:

$$\frac{\partial}{\partial t}\int\limits_{V_3} U^2 d\mathbf{v} + \int\limits_{V_3} (\mathbf{v}, \nabla)U^2 d\mathbf{v} = 2\int\limits_{V_3} U\Big(\mathbf{B}(f, f) - \mathbf{B}(F, F)\Big)d\mathbf{v}. \tag{23}$$

Remark 2. In ([2], p. 13), there are formulas (8), (9) of the involutive transformation. For transformation (8), properties are briefly written as

$$U' = \mathbf{P}(U),$$

a) $\mathbf{P}$ is an involutive transformation, i.e. $\mathbf{P}\big(\mathbf{P}(U) = U\big)$,
b) Transformation $\mathbf{P}$ preserves the volume element $d\sigma d\mathbf{v_1} d\mathbf{v}$.

Definition 1. We call two single-valued functions sign equivalent, i.e., $U \sim W$, in the domain Q for $\forall t \in [0, T)$ such that

$$U(t, \mathbf{x}, \mathbf{v}), W(t, \mathbf{x}, \mathbf{v}) \in C(G \times V_3) \cap L_1(V_3), \forall t \in [0, T),$$

and properties
a) $s\mathrm{ignU} = \mathrm{signW}$ in Q,
b) $U(M^j) = W(M^j) = 0$, where $M^j, j = 0, 1, \cdots$, are zeros of these functions in Q.

Lemma 2. There is the inequality

$$\int\limits_{V_3} U\Big(\mathbf{B}(f,f)-\mathbf{B}(F,F)\Big)d\mathbf{v} \le 0. \tag{24}$$

Proof. Consider the expression system

$$\begin{cases} \int\limits_{V_3} \Phi\mathbf{B}(f,f)d\mathbf{v} = \int\limits_{V_3^2\times\Sigma} \Phi[f,f]d\sigma d\mathbf{v}_1 d\mathbf{v}, \\ \\ \int\limits_{V_3} \Phi\mathbf{B}(F,F)d\mathbf{v} = \int\limits_{V_3^2\times\Sigma} \Phi[F,F]d\sigma d\mathbf{v}_1 d\mathbf{v}, \end{cases} \tag{25}$$

where

$$[f,f] = \big(f'f_1' - ff_1\big)K(\theta,\mathbf{w}), \tag{26}$$

$V_3^2 = V_3\times V_3$, $\Phi = \Phi(t,\mathbf{x},\mathbf{v})$ is an arbitrary continuous in Q and summable in V_3 function.

From the first expression of system (25), subtracting the second expression, respectively, we get

$$\int\limits_{V_3} \Phi\big(\mathbf{B}(f,f)-\mathbf{B}(F,F)\big)d\mathbf{v} = \int\limits_{V_3^2\times\Sigma} \Phi\big([f,f]-[F,F]\big)d\sigma d\mathbf{v}_1 d\mathbf{v}.$$

Here we use the well-known involutive transformation $\mathbf{P}$ (see Remark 2).

Applying $\mathbf{P}$ to the integrand on the right parts, we have

$$\int\limits_{V_3} \Phi\big(\mathbf{B}(f,f)-\mathbf{B}(F,F)\big)d\mathbf{v} = -\int\limits_{V_3^2\times\Sigma} \Phi'\big([f,f]-[F,F]\big)d\sigma d\mathbf{v}_1 d\mathbf{v}.$$

Adding the latter with the previous expression, we find

$$\int\limits_{V_3} \Phi\big(\mathbf{B}(f,f)-\mathbf{B}(F,F)\big)d\mathbf{v} = \frac{1}{2}\int\limits_{V_3^2\times\Sigma} (\Phi-\Phi')\big([f,f]-[F,F]\big)d\sigma d\mathbf{v}_1 d\mathbf{v}.$$

In this formula, we make the change of variables $\mathbf{v}_1 \rightleftarrows \mathbf{v}$ and find

$$\int\limits_{V_3} \Phi\big(\mathbf{B}(f,f)-\mathbf{B}(F,F)\big)d\mathbf{v} = \frac{1}{4}\int\limits_{V_3^2\times\Sigma} (\Phi+\Phi_1-\Phi'-\Phi_1')\big([f,f]-[F,F]\big)d\sigma d\mathbf{v}_1 d\mathbf{v}.$$

Hence the square brackets on the right side, replacing the expressions according to the formula (26), we find

$$\int\limits_{V_3} \Phi\big(\mathbf{B}(f,f)-\mathbf{B}(F,F)\big)d\mathbf{v} = \frac{1}{4}\int\limits_{V_3^2\times\Sigma} (\Phi+\Phi_1-\Phi'-\Phi_1')\times$$
$$\times\Big(\big(f'f_1'+FF_1\big)-\big(ff_1+F'F_1'\big)\Big)d\sigma d\mathbf{v}_1 d\mathbf{v}. \tag{27}$$

If we put $\Phi = \ln(f/F)$, then from (27) we arrive at formula

$$\int\limits_{V_3} \ln(f/F)\big(\mathbf{B}(f,f)-\mathbf{B}(F,F)\big)d\mathbf{v} =$$
$$= \frac{1}{4}\int\limits_{V_3^2\times\Sigma} \ln\Big(\frac{ff_1F'F_1'}{f'f_1'FF_1}\Big)\Big(\big(f'f_1'+FF_1\big)-\big(ff_1+F'F_1'\big)\Big)d\sigma d\mathbf{v}_1 d\mathbf{v}. \tag{28}$$

We must define the sign definiteness of the complex integral (28). In this case, it is difficult to check the sign of the second the integrand in the domain Q. Since we are interested in only integral is definite in sign, then using Definition 1 we write the sign equivalence functions for the terms of the second integrand

$$\big(f'f_1' - ff_1\big) \sim ln\frac{f'f_1'}{ff_1},\ \big(FF_1 - F'F_1'\big) \sim ln\frac{FF_1}{F'F_1'}.$$

Using these relations, we rewrite the integral (28) sign equivalent form

$$\int\limits_{V_3} \ln(f/F)\big(\mathbf{B}(f,f) - \mathbf{B}(F,F)\big)d\mathbf{v} \sim \frac{1}{4}\int\limits_{V_3^2\times\Sigma} \ln\Big(\frac{ff_1F'F_1'}{f'f_1'FF_1}\Big)\Big(ln\frac{f'f_1'}{ff_1}+$$

$$+ ln\frac{FF_1}{F'F_1'}\Big)K(\theta,\mathbf{w})d\sigma d\mathbf{v}_1 d\mathbf{v} = \frac{1}{4}\int\limits_{V_3^2\times\Sigma} \ln\Big(\frac{ff_1F'F_1'}{f'f_1'FF_1}\Big)ln\Big(\frac{f'f_1'FF_1}{ff_1F'F_1'}\Big)K(\theta,\mathbf{w})d\sigma d\mathbf{v}_1 d\mathbf{v} =$$

$$= -\frac{1}{4}\int\limits_{V_3^2\times\Sigma} \ln^2\Big(\frac{ff_1F'F_1'}{f'f_1'FF_1}\Big)K(\theta,\mathbf{w})d\sigma d\mathbf{v}_1 d\mathbf{v} \le 0.\ \forall t\in[0,T). \quad (29)$$

According to Definition 1, the functions $U = f - F$ and $\Phi = \ln(f/F)$ are also sign equivalent that is, $U \sim \Phi$, since $signU = sign\Phi$ in Q, Thus, (29) implies inequality (24), since $U = f - F \wedge \Phi = \ln(f/F)$, then we will see that $signU = sign\Phi$ in Q, As a result, we arrive at the inequality (24).

$$\int\limits_{V_3} \ln(f/F)\big(\mathbf{B}(f,f) - \mathbf{B}(F,F)\big)d\mathbf{v} \le 0, \Longrightarrow \int\limits_{V_3} U\big(\mathbf{B}(f,f) - \mathbf{B}(F,F)\big)d\mathbf{v} \le 0.$$

Lemma 2 is proved.

Now for functional equation (23), integrating over the domain G taking into account the boundary condition (22) and Lemmas 1 and 2, we obtain the main the inequality for the uniqueness of the solution

$$\frac{d}{dt}\int\limits_{G\times V_3} U^2(t,\mathbf{x},\mathbf{v})d\mathbf{v}d\mathbf{x} \le 2\int\limits_{G\times V_3} U\Big(\mathbf{B}(f,f) - \mathbf{B}(F,F)\Big)d\mathbf{v}d\mathbf{x} \le 0.$$

The latter we will rewrite

$$\frac{d}{dt}\int\limits_{G\times V_3} U^2(t,\mathbf{x},\mathbf{v})d\mathbf{v}d\mathbf{x} - 2\int\limits_{G\times V_3} U\Big(\mathbf{B}(f,f) - \mathbf{B}(F,F)\Big)d\mathbf{v}d\mathbf{x} \le 0,$$

from the left side, discarding the non-negative bounded integral justified in estimates (13)–(16) and, integrating over t, we obtain

$$\int\limits_{G\times V_3} U^2(t,\mathbf{x},\mathbf{v})d\mathbf{v}d\mathbf{x} \le \int\limits_{G\times V_3} U^2(0,\mathbf{x},\mathbf{v})d\mathbf{v}d\mathbf{x},\ \forall t\in[0,T).$$

From here $\int\limits_{G\times V_3} U^2(t,\mathbf{x},\mathbf{v})d\mathbf{v}d\mathbf{x} \le 0,\ \Longrightarrow U(t,\mathbf{v},\mathbf{x}) \equiv 0.\ \forall(t,\mathbf{x},\mathbf{v}) \in Q.$

As a result, we show the existence and uniqueness of the positive solution to the full nonlinear Boltzmann equation from the space

$$f(t,\mathbf{x},\mathbf{v}) \in C^1\big(0,T;C(G\times V_3)\cap L_1(V_3)\big) \wedge \big((\mathbf{v},\nabla f);\mathbf{B}(f,f)\big) \in C(Q)\cap L_1(V_3), \quad (30)$$

it consists of the union of some functional spaces, as the space of continuously differentiable functions $f(t,\mathbf{x},\mathbf{v})$ by $t \in [0,T)$ and at each t continuous in $(\mathbf{x},\mathbf{v})$ in the domain $G \times V_3$ and summable over $\mathbf{v}$ in V_3, and the functions $(\mathbf{v},\nabla)f$; $\mathbf{B}(f,f)$ at each t continuous over all variables in Q and summable over $\mathbf{v}$ in V_3.

Definition 2. The solution $f(t,\mathbf{x},\mathbf{v})$ with properties (30) uniformly satisfying the Boltzmann equation (1) with initial boundary conditions (2), (3) in the domain Q will be called strong.

As a result, it was proved next main theorems

Theorem 1. If the initial function satisfies conditions (3), (4), then there is a unique strong positive solution to (1)–(3) for the whole time interval $t \in [0,T), T \leq \infty$ satisfying uniformly the Boltzmann equation (1) everywhere in Q.

When intermolecular interactions are determined by central forces, then $K(\theta,\mathbf{w})$ is determined by the formula (see [2], p. 15)

$$K_c(\theta,\mathbf{w}) =\mid \mathbf{w} \mid \rho; \quad \mathbf{w} = \mathbf{v} - \mathbf{v}_1; \quad \mathbf{\Sigma} \equiv \{0 \leq \rho \leq \rho_0;\ 0 \leq \theta \leq 2\pi\}, \quad d\sigma = d\rho d\theta,$$

where ρ is the target distance of the colliding molecules, ρ_0 is the radius of action of the molecule. Initial function $\varphi(\mathbf{x},\mathbf{v})$ satisfies condition (3) and such that

$$\begin{cases} 0 < \varphi(\mathbf{x},\mathbf{v}) \in C(G \times V_3) \wedge \Big(\|\varphi(\mathbf{v})\|_{C(G)} \leq \frac{const}{(1+|\mathbf{v}|^2)^{\frac{\gamma}{2}}},\ \gamma > 6\Big); \\ \mathbf{J}(\varphi) \leq \int\limits_{V_3\times\Sigma} \|\varphi(\mathbf{v}')\| \cdot \|\varphi(\mathbf{v}_1')\| K_c(\theta,\mathbf{w}) d\sigma d\mathbf{v}_1 = h_1(\mathbf{v}) < \infty; \\ \mathbf{S}(\varphi) \leq \int\limits_{V_3\times\Sigma} \|\varphi(\mathbf{v}_1)\| K_c(\theta,\mathbf{w}) d\sigma d\mathbf{v}_1 = h_2(\mathbf{v}) < \infty; \end{cases} \tag{31}$$

where $\int\limits_{V_3} h_k(\mathbf{v})d\mathbf{v} = const,\ k = 1,2.$

The existence and uniqueness theorem of the Cauchy problem for the Boltzmann equation with intermolecular interaction $K_c(\theta,\mathbf{w})$ is also proved as Theorem 1, by a literal repetition, the formulation will be:

Theorem 2. If the initial function satisfies conditions (3) (31), then there exists a unique strong positive solution of problem (1)-(3) on the whole time interval $t \in [0,T), T \leq \infty$ satisfying uniformly the Boltzmann equation (1) everywhere in Q.

Corollary 1. The existence and uniqueness theorems 1 for the nonlinear Boltzmann equation (1) are trivial for the Boltzmann equation in the case of Maxwellian molecules with corresponding relaxations of the requirement from the initial function.

5 *Positivity of the solution to the Boltzmann equation*

Lemma 3. Since there exists a bounded solution of the Boltzmann equation (1) with positive initial condition (2), then the value $\mathbf{B}(f,f)$ of the collisions integral makes sense and the solution $f(t,\mathbf{x},\mathbf{v}) \in Q$ is positive.

Proof. The Boltzmann equation (1) is written along the trajectory

$$\frac{d}{d\tau} f(t,\mathbf{x}-\mathbf{v}(t-\tau),\mathbf{v}) = \frac{\partial f}{\partial t} + (\mathbf{v},\nabla) f(t,\mathbf{x}-\mathbf{v}(t-\tau),\mathbf{v}) = \mathbf{B}(f,f)(t,\mathbf{x}-\mathbf{v}(t-\tau),\mathbf{v}). \tag{32}$$

We put $f = U^{-1}$, since f exists and it is a bounded solution of the Boltzmann equation, then (32) can be rewritten as

$$\frac{d}{d\tau} U(t,\mathbf{x}-\mathbf{v}(t-\tau),\mathbf{v}) = -U\mathbf{B}(U,U)(t,\mathbf{x}-\mathbf{v}(t-\tau),\mathbf{v}).$$

From here, integrating we find

$$U(t,\mathbf{x},\mathbf{v}) = \varphi(\mathbf{x},\mathbf{v})\exp(-\mathbf{B}(U,U)t) > 0, \forall(\mathbf{x},\mathbf{v}) \in G\times V_3,$$

it was required to prove.

6 *H−Boltzmann theorem*

Let us multiply the Boltzmann equation (1) by the function $1+\ln f(t,\mathbf{x},\mathbf{v})$. Then integrate over the domain $G\times V_3$ and, considering mass conservation [2], we find

$$\frac{d}{dt}\int\limits_{G\times V_3} f\ln f d\mathbf{v}d\mathbf{x} + \int\limits_{G\times V_3}(\mathbf{v},\nabla)f\ln f d\mathbf{v}d\mathbf{x} =$$

$$= \int\limits_{G\times V_3}\int\limits_{V_3\times\Sigma}\ln f\big(f'f_1' - ff_1\big)K(\theta,\mathbf{w})d\sigma d\mathbf{v}_1 d\mathbf{v}d\mathbf{x}. \quad (33)$$

Hence, the second summand of the left part, integrating over the parts, taking into account the boundary condition (3) and using the lemma 1, we have:

$$\int\limits_{G\times V_3}(\mathbf{v},\nabla)f\ln f d\mathbf{v}d\mathbf{x} = 0. \quad (34)$$

Using the involutive transformation $\mathbf{P}$ (see note 2), the integral in the right-hand side (33) can be written as

$$\int\limits_{G\times V_3}\int\limits_{V_3\times\Sigma}\ln f\big(f'f_1' - ff_1\big)K(\theta,\mathbf{w})d\sigma d\mathbf{v}_1 d\mathbf{v}d\mathbf{x} =$$

$$-\frac{1}{4}\int\limits_{G\times V_3^2\times\Sigma}\Big(\ln f' + \ln f_1' - \ln f - \ln f_1\Big)\big(f'f_1' - ff_1\big)K(\theta,\mathbf{w})d\sigma d\mathbf{v}_1 d\mathbf{v}d\mathbf{x} =$$

$$= -\frac{1}{4}\int\limits_{G\times V_3^2\times\Sigma}\ln\frac{f'f_1'}{ff_1}\big(f'f_1' - ff_1\big)K(\theta,\mathbf{w})d\sigma d\mathbf{v}_1 d\mathbf{v}d\mathbf{x}. \quad (35)$$

Wherefore, using the signequivalence of the function $ln\frac{f'f_1'}{ff_1} \sim \big(f'f_1' - ff_1\big)$ and denoting

$$H(t) = \int\limits_{G\times V_3} f(t,\mathbf{x},\mathbf{v})\ln f(t,\mathbf{x},\mathbf{v})d\mathbf{v}d\mathbf{x},$$

considering (34), (35) from (33), we find

First case:

$$\frac{d}{dt}H \sim -\frac{1}{4}\int\limits_{G\times V_3}\int\limits_{V_3\times\Sigma} ln^2\frac{f'f_1'}{ff_1}K(\theta,\mathbf{w})d\sigma d\mathbf{v}_1 d\mathbf{v}d\mathbf{x} \le 0.$$

Second case:

$$\frac{d}{dt}H \sim -\frac{1}{4}\int\limits_{G\times V_3}\int\limits_{V_3\times\Sigma}\big(f'f_1' - ff_1\big)^2 K(\theta,\mathbf{w})d\sigma d\mathbf{v}_1 d\mathbf{v}d\mathbf{x} \le 0.$$

From these cases it follows that

$$\frac{d}{dt}H \le 0. \quad (36)$$

Lemma 4. If the positive initial function $\varphi(\mathbf{x},\mathbf{v})$ is an additionally function to the requirements (4) and satisfies the condition

$$\int\limits_{G\times V_3} \varphi(\mathbf{x},\mathbf{v})|\ln\varphi(\mathbf{x},\mathbf{v})|d\mathbf{v}d\mathbf{x}<\infty,$$

then the following inequality holds

$$|\int\limits_{G\times V_3} f(t,\mathbf{x},\mathbf{v})\ln f(t,\mathbf{x},\mathbf{v})d\mathbf{v}d\mathbf{x}|<\int\limits_{G\times V_3} \varphi(\mathbf{x},\mathbf{v})|\ln\varphi(\mathbf{x},\mathbf{v})|d\mathbf{v}d\mathbf{x}<\infty,\ \forall t. \tag{37}$$

Integrating inequality (36) over t in the range from 0 to t, we obtain (37).

Above we proved the strict positivity of the solution $f(t,\mathbf{x},\mathbf{v})$ to problem (1)-(3), when the initial function $\varphi(\mathbf{x},\mathbf{v})$ is positive, thus the logarithm function of the distribution is lawful and, moreover, it follows from (30), (37) that $\exists \ln f(t,\mathbf{x},\mathbf{v})$ for all $(t,\mathbf{x},\mathbf{v})\in Q$.

It follows from (36) that the $H(t)$ function never increases in time and is constant if and only if the distribution function is locally-Maxwellian. Indeed, the equality in (36) is achieved, if and only if: in the first case $\ln^2\frac{f'f_1'}{ff_1}=0,\Longrightarrow\ \ln\frac{f'f_1'}{ff_1}=0$, from here

$$\ln f(\mathbf{v}')+\ln f(\mathbf{v}_1')-\ln f(\mathbf{v})-\ln f(\mathbf{v}_1)=0, \tag{38}$$

and in the second case $f'f_1'=ff_1$, By logarithmizing both parts of the latter, we have the ratio

$$\ln f(\mathbf{v})+\ln f(\mathbf{v}_1)-\ln f(\mathbf{v}')-\ln f(\mathbf{v}_1')=0. \tag{39}$$

Eventually, we see that equations (38), (39) coincide and it follows from them that $\ln f(t,\mathbf{x},\mathbf{v})$ is a summator invariant, i.e.,

$$\ln f(t,\mathbf{x},\mathbf{v})=a+\mathbf{b}\mathbf{v}+c|\mathbf{v}|^2,\ \forall(t,\mathbf{x})\in[0,T)\times G,$$

where a,c are scalar function and $\mathbf{b}$ is a vector constant. Hence, following [2], we obtain

$$f\equiv f_0=C\exp(\big[-\alpha(|v|-|v_0|)^2\big]),$$

where f_0 is the local-Maxwell distribution.

$$C>0\quad\text{and}\quad\alpha>0,\text{and}\alpha=1/(2kT),$$

k is the Boltzmann constant, T is the temperature, v_0 is the average velocity, $C=\rho(2\pi kT)^{-\frac{3}{2}}$, ρ is the density. ρ,T,v_0 can depend on $(t,\mathbf{x})$.

References

1 Больцман Л.Э. Лекции по теории газов / Л.Э. Больцман. — М.: Гостехиздат, 1953. — 554 с.

2 Карлеман Т. Математические задачи кинетической теории газов / Т. Карлеман. — М.: Изд-во ИЛ, 1960. — 150 с.

3 Град Г. Кинетическая теория газов / Г. Град // Сб. ст.: Термодинамика газов. — М.: Машиностроение, 1970. — С. 5–109.

4 Либовиц Дж.Л. Неравновесные явления. Уравнение Больцмана / Дж.Л. Либовиц, Е.У. Монтролл (ред.). — М.: Мир, 1986. — 272 с.

5 Марчук Г.И. Методы расщепления / Г.И. Марчук. — М.: Наука, 1988. — 263 с.

6 Годунов С.К. О дискретных моделях кинетического уравнения Больцмана / С.К. Годунов, У.М. Султангазин // УМН. — 1971. — Т. 26. — Вып. 3(159). — С. 3–51.

7 Akysh A.Sh. Convergence of a splitting method for the nonlinear Boltzmann equation / A.Sh. Akysh // Numerical Analysis and Applications. — 2013. — 6. — No. 2. — P. 111–118.

8 Акыш А.Ш. О разрешимости нелинейного уравнения Больцмана / А.Ш. Акыш // В книге: Неклассические уравнения математической физики. — Новосибирск: Ин-т математики им. С.Л. Соболева СО РАН, 2007. — С. 15–23.

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Алматы, Қазақстан

Бейсызықты Больцман теңдеуінің барлық уақытта шешілетіндігі

Жұмыста ыдырату әдісінің негізінде толық бейсызықты Болцман теңдеуінің барлық уақыт аралығында $t \in [0, T),\ T \leq \infty$ молекулалардың тепетеңдіксіз күйі ортасында және олардың әсерлесуі қатты сфералы молекулалар болса немесе қақтығысуы орталық күш арқылы орындалса жалқы шешуінің болатындығы теоремасы дәлелденген. Үзіліссіз функциялар кеңістігінде тұйық шешуі болғандықтан бейсызықты Болцман теңдеуінің бастапқы шарты оң болғанда, шешудің әрқашанда оң болатыны дәлелденді. Соңғының негізінде Болцман $H-$теоремасының кейбір математикалық негіздеуі көрсетілген.

Кілт сөздер: толық бейсызықты Болцман теңдеуі, ыдырату әдісі, барлық уақыт аралығында бейсызықты Болцман теңдеуінің жалқы шешуінің болатындығы теоремасы, бейсызықты Болцман теңдеуінің оң шешуі, Болцман $H-$теоремасының математикалық негіздемесі.

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Алматы, Казахстан

Глобальная разрешимость нелинейного уравнения Больцмана

В статье с помощью схемы метода расщепления доказана теорема существования и единственности на всем промежутке времени $t \in [0, T),\ T \leq \infty$, для полного нелинейного уравнения Больцмана в неравновесном случае, когда межмолекулярные взаимодействия являются молекул-твердыми сферами и центральными силами. На основе существования ограниченного решения в пространстве $\mathbf{C}$ подтверждена строгая положительность решений полного нелинейного уравнения Больцмана, когда начальная функция положительна. На основании этого показано некоторое математическое обоснование $H-$теоремы Больцмана.

Ключевые слова: полное нелинейное уравнение Больцмана, метод расщепления, теорема существования и единственности на всем промежутке времени для нелинейного уравнения Больцмана, положительность решений нелинейного уравнения Больцмана, H-теорема Больцмана.

References

1 Boltzmann, L. (1953). *Lektsii po teorii gazov [Lectures on gas theory].* Moscow: Gostekhizdat [in Russian].

2 Carleman, T. (1960). *Matematicheskie zadachi kineticheskoi teorii gazov [Mathematical problems of kinetic theory of gases].* Moscow: Izdatelstvo inostrannoi literatury [in Russian].

3 Grad, H. (1970). *Kineticheskaia teoriia gazov [Kinetic theory of gases].* V sbornike statei: Termodinamika gazov — In collection articles: Thermodynamics of gases. Moscow: Mashinostroenie, 5–109 [in Russian].

4 Liebovitz, J.L., & Montroll, E.U. (1986). *Neravnovesnye yavleniia. Uravnenie Boltsmana [Nonequilibrium phenomena. Boltzmann equation].* Moscow: Mir [in Russian].

5 Marchuk, G.I. (1988). *Metody rasshchepleniia [Splitting methods].* Moscow: Nauka [in Russian].

6 Godunov, S.K., & Sultangazin, U.M. (1971). O diskretnykh modeliakh kineticheskogo uravneniia Boltsmana [About discrete models of the kinetic Boltzmann equation]. *Uspekhi matematicheskikh nauk — Russian Mathematical Surveys, 26,* 3(159), 3–51 [in Russian].

7 Akysh, A.Sh. (2013). Convergence of a splitting method for the nonlinear Boltzmann equation. *Numerical Analysis and Applications, 6*(2), 111–118.

8 Akysh, A.Sh. (2007). O razreshimosti nelineinogo uravneniia Boltsmana [On the solvability of a nonlinear Boltzmann equation]. *V knige: Neklassicheskie uravneniia matematicheskoi fiziki — In the book: Nonclassical equations of mathematical Physics.* Novosibirsk: Institut matematiki imeni S.L. Soboleva SO RAN [in Russian].