DOI 10.31489/2022M2/59-71

UDC 517.956

Zh.A. Balkizov[1,*], Z.Kh. Guchaeva[2], A.Kh. Kodzokov[2]

[1] *Institute of Applied Mathematics and Automation, KBSC RAS, Nalchik, Russia;*
[2] *Kabardino-Balkarian State University named after H.M. Berbekov, Nalchik, Russia*
(E-mail: Giraslan@yandex.ru, Proporwiz@yandex.ru, Kodzoko@mail.ru)

Inner boundary value problem with displacement for a second order mixed parabolic-hyperbolic equation

This paper investigates inner boundary value problems with a shift for a second-order mixed-hyperbolic equation consisting of a wave operator in one part of the domain and a degenerate hyperbolic operator of the first kind in the other part. We find sufficient conditions for the given functions to ensure the existence of a unique regular solution to the problems under study. In some special cases, solutions are obtained explicitly.

Keywords: wave equation, degenerate hyperbolic equation of the first kind, Volterra integral equation, Fredholm integral equation, Tricomi method, method of integral equations, methods of fractional calculus theory.

Introduction. Notation. Formulation of the problem

In the Euclidean plane with independent variables x and y consider the equation

$$0 = \begin{cases} (-y)^m u_{xx} - u_{yy} + \lambda (-y)^{\frac{m-2}{2}} u_x, & y < 0, \\ u_{xx} - u_{yy} + f, & y > 0, \end{cases} \tag{1}$$

where λ, m are given numbers; $m > 0$, $|\lambda| \leq \frac{m}{2}$; $f = f(x, y)$ is a given function; $u = u(x, y)$ is an unknown function.

When $y < 0$ equation (1) is a degenerate hyperbolic equation of the first kind [1]

$$(-y)^m u_{xx} - u_{yy} + \lambda (-y)^{\frac{m-2}{2}} u_x = 0, \tag{2}$$

but when $y < 0$ coincides with the inhomogeneous wave equation

$$u_{xx} - u_{yy} + f(x, y) = 0. \tag{3}$$

Equation (2) belongs to the class of the first kind degenerate hyperbolic equations [1; 21], that is, at no point of the degenerate line $y = 0$ the tangent line does coincide with the characteristic direction of the equation (2). An important property of equation (2) is the fact that when $|\lambda| \leq \frac{m}{2}$ the Cauchy problem is correct for it in the usual formulation with data on the parabolic degeneracy line $y = 0$ despite that the Protter condition [2] is violated. When $m = 2$ equation (2) turns into the Bitsadze-Lykov equation [3; 37], [4], [5; 234], and for $\lambda = 0$ from equation (2) we come to the Gellerstedt equation, which, as shown in the monograph [6; 234], finds application in the problem of determining the shape of the dam slot. Apart from that as well the particular case for equation (2) is the Tricomi equation, which presents the theoretical basis for transonic gas dynamics [7; 38], [8; 280].

Equation (1) is considered in the domain $\Omega = \Omega_1 \cup \Omega_2 \cup I$, where Ω_1 is the domain restricted by characteristics $\sigma_1 = AC : x - \frac{2}{m+2} (-y)^{\frac{m+2}{2}} = 0$ and $\sigma_2 = CB : x + \frac{2}{m+2} (-y)^{\frac{m+2}{2}} = r$ of equation

*Corresponding author.
E-mail: Giraslan@yandex.ru

(2) for $y<0$, outgoing from the point $C=(r/2,y_c)$, $y_c=-\left[\frac{(m+2)r}{4}\right]^{\frac{2}{m+2}}$, passing through the points $A=(0,0)$ and $B=(r,0)$, and the segment $I=AB$ of the strait line $y=0$; Ω_2 is the domain restricted by characteristics $\sigma_3=AD:\ x-y=0$, $\sigma_4=BD:\ x+y=r$ of equation (3), outgoing from the points A and B intersecting at the point $D=\left(\frac{r}{2},\,\frac{r}{2}\right)$ and the segment $I=AB$.

Let us introduce the following notation:

$$\varepsilon_1=\frac{m-2\lambda}{2\,(m+2)},\ \varepsilon_2=\frac{m+2\lambda}{2\,(m+2)},\ \varepsilon=\varepsilon_1+\varepsilon_2=\frac{m}{m+2},$$

$$\gamma_1=\frac{\Gamma\,(\varepsilon)}{\Gamma\,(\varepsilon_1)},\ \gamma_2=\frac{\Gamma\,(2-\varepsilon)\,(2-2\varepsilon)^{\varepsilon-1}}{\Gamma\,(1-\varepsilon_2)};$$

$$a(x)=\frac{\alpha_2(x)+\gamma_1\alpha_1(x)}{\alpha_3(x)-\gamma_1\alpha_1(x)},\ b(x)=\frac{\gamma_1\beta_1(x)+\beta_3(x)}{\gamma_2\beta_1(x)-\beta_2(x)};$$

$$K(x,t)=\frac{a(x)}{(t-x)^{1-\varepsilon}}+\int\limits_x^t\frac{a'\,(s)}{(t-s)^{1-\varepsilon}}ds,\ L(x,t)=\begin{cases}K\,(r,\,t)\,, & 0\le x<t,\\ K\,(r,t)-K(x,t), & t<x\le r,\end{cases}$$

$$F_1(x)=2\psi_1(x)-\psi_1\,(r)+\int\limits_x^r\frac{\psi_2(t)}{\alpha_3(t)-\gamma_2\alpha_1(t)}dt-\int\limits_x^r\int\limits_0^{\frac{r-t}{2}}f\,(t+s,s)\,dsdt,$$

$$F_2(x)=b(x)\left[2\,\varphi_1(x)-\varphi_1\,(0)-\int\limits_0^{x/2}\int\limits_t^{x-t}f\,(s,t)\,dsdt\right]-\frac{\varphi_2(x)}{\gamma_2\beta_1(x)-\beta_2(x)};$$

$$\theta_{00}(x)=\left(\frac{x}{2},-\left(\frac{m+2}{4}\right)^{2/(m+2)}x^{2/(m+2)}\right)=\left(\frac{x}{2},-(2-2\varepsilon)^{\varepsilon-1}\,x^{1-\varepsilon}\right),$$

$$\theta_{r0}(x)=\left(\frac{r+x}{2},-\left(\frac{m+2}{4}\right)^{2/(m+2)}(r-x)^{2/(m+2)}\right)=\left(\frac{r+x}{2},-(2-2\varepsilon)^{\varepsilon-1}\,(r-x)^{1-\varepsilon}\right)$$

– affixes of characteristics intersection points that leave the point $(x,0)$ with characteristics of AC and BC of equation (3), correspondingly;

$$\theta_{01}(x)=\left(\frac{x}{2},\,\frac{x}{2}\right),\ \theta_{r1}(x)=\left(\frac{r+x}{2},\,\frac{r-x}{2}\right)$$

– affixes of characteristics intersection points that leave the point $(x,0)$ with characteristics AD and BD of equation (3), correspondingly;

$$B\,(p,\,q)=\int_0^1t^{p-1}\,(1-t)^{q-1}\,dt,\ \Gamma(p)=\int_0^\infty\exp\,(-t)\ t^{p-1}dt,\ B\,(p,q)=\frac{\Gamma(p)\,\Gamma\,(q)}{\Gamma\,(p+q)}$$

– Euler integrals of the first and second kind and their relationship;

$$D_{cx}^{\alpha}g(t)=\begin{cases}\frac{sgn(x-c)}{\Gamma(-\alpha)}\int\limits_c^x\frac{g(t)}{|x-t|^{1+\alpha}}dt, & \alpha<0,\\ sgn^{[\alpha]+1}\,(x-c)\ \frac{d^{[\alpha]+1}}{dx^{[\alpha]+1}}D_{cx}^{\alpha-[\alpha]-1}g(t), & \alpha>0\end{cases}$$

– fractional integro-differentiation operator (in the Riemann-Liouville sense) of order $|\alpha|$ with starting point c [5], [6], [9]; the regularized fractional derivative (Caputo derivative) is defined using the equality [10]

$$\partial_{cx}^{\alpha} g(t) = sgn^n\,(x-c)\; D_{cx}^{\alpha-n} g^{(n)}(t),\;\; n-1<\alpha\le n,\;\; n\in N;$$

and it is related to the Riemann-Liouville derivative by the relation [10]

$$\partial_{cx}^{\alpha} g(t) = D_{cx}^{\alpha} g(t) - \sum_{k=0}^{n-1} \frac{g^{(k)}\,(c)}{\Gamma\,(k-\alpha+1)},$$

where $n-1<\alpha\le n$, $n\in N$;

$$E_\rho\,(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma\,(1+n\rho)},\;\; E_\rho\,(z,\,\mu) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma\,(\mu+n\rho^{-1})},\;\; E_\rho\,(z,\,1) = E_{1/\rho}\,(z)$$

– the Mittag-Leffler function and the function of Mittag-Leffler type [11].

Assume the function $u=u(x,y)$ of class $C\left(\bar{\Omega}\right)\cap C^1\left(\Omega\right)\cap C^2\left(\Omega_1\cup\Omega_2\right)$, $u_x,\,u_y\,\in L_1\left(I\right)$ in the domain Ω is a regular solution, which by substitution transforms equation (1) into identity.

Problem 1. Find a regular solution of equation (1) in the domain Ω satisfying the conditions

$$u\left[\theta_{r1}(x)\right] = \psi_1(x), \quad 0\le x\le r, \tag{4}$$

$$\alpha_1(x)\;(r-x)^{\varepsilon_2}\,D_{rx}^{1-\varepsilon_1}\,\left\{u\left[\theta_{r0}(t)\right]\right\} + \alpha_2(x)\,D_{rx}^{1-\varepsilon} u\,(t,\,0) + \alpha_3(x)\,u_y(x,0) = \psi_2(x), \quad 0<x<r, \tag{5}$$

where $\alpha_1(x)$, $\alpha_2(x)$, $\alpha_3(x)$, $\psi_1(x)$, $\psi_2(x)$ are given functions on the line segment $0\le x\le r$, moreover $\alpha_1^2(x)+\alpha_2^2(x)+\alpha_3^2(x)\ne 0 \quad \forall\, x\in[0,r]$.

Problem 2. Find a regular solution of equation (1) in the domain Ω satisfying the conditions

$$u\left[\theta_{01}(x)\right] = \varphi_1(x), \quad 0\le x\le r, \tag{6}$$

$$\beta_1(x)\;(r-x)^{1-\varepsilon_1}\,D_{rx}^{\varepsilon_2}\left\{(r-t)^{\varepsilon-1}\,u\left[\theta_{r0}(t)\right]\right\} +$$

$$+\beta_2(x)\,D_{rx}^{\varepsilon-1} u_y\,(t,\,0) + \beta_3(x)\,u(x,0) = \varphi_2(x), \quad 0<x<r, \tag{7}$$

where $\beta_1(x)$, $\beta_2(x)$, $\beta_3(x)$, $\varphi_1(x)$, $\varphi_2(x)$ are given functions on the line segment $0\le x\le r$, moreover $\beta_1^2(x)+\beta_2^2(x)+\beta_3^2(x)\ne 0 \quad \forall\, x\in[0,r]$.

Earlier, the Goursat problem for the first kind degenerate hyperbolic equations was investigated in [12], [13]. The criterion of continuity for the Goursat problem for equation (2) is investigated in [12], and the solution of the Goursat problem for a model equation degenerating inside the domain is written in explicit form in [13]. The first boundary value problem for the hyperbolic equation degenerating inside the domain is considered in [14]. Boundary value problems for degenerate hyperbolic equations in a characteristic quadrangle with data on opposite characteristics were investigated in [15–17].

Inner boundary value problems 1 and 2 considered in this paper belong to the class of boundary value problems with a displacement of the Zhegalov-Nakhushev [18–20] and are generalization of the Goursat problem and problems with data on opposite characteristic lines for an equation of the type (1). The displacement problems for the first kind hyperbolic equations degenerating inside the domain were previously studied in [21–24]. The displacement problems for the first kind degenerate hyperbolic equation of the type (2) were investigated in [25], presented as generalization of the first and second Darboux problems. A rather complete bibliography of works devoted to the formulation and study of the displacement problems for various types of partial differential equations is provided in [26–32]. In this paper, sufficient conditions are found for the given functions $\alpha_i(x)$, $\beta_i(x)$, $i=\overline{1,3}$; $\varphi_j(x)$, $\psi_j(x)$, $j=\overline{1,2}$; $f(x,y)$ *that insure a unique regular solution of investigated problems 1 and 2*. In particular cases, when the relation $a(x)=\frac{\alpha_2(x)+\gamma_1\alpha_1(x)}{\alpha_3(x)-\gamma_1\alpha_1(x)}=a=const$ or $b(x)=\frac{\gamma_1\beta_1(x)+\beta_3(x)}{\gamma_2\beta_1(x)-\beta_2(x)}=b=const$ the regular solutions of *problems 1 and 2* are written explicitly.

Research task 1

The following Theorem holds.

Theorem 1. Let the given functions $\alpha_1(x)$, $\alpha_2(x)$, $\alpha_3(x)$, $\psi_1(x)$, $\psi_2(x)$ and $f(x,y)$ be such that

$$\alpha_1(x), \alpha_2(x), \alpha_3(x), \psi_2(x) \in C\,[0,\, r] \cap C^2\,(0,\, r)\,, \tag{8}$$

$$\psi_1(x) \in C^1\,[0,\, r] \cap C^3\,(0,\, r)\,, \tag{9}$$

$$f(x,y) \in C^1\left(\bar{\Omega}_2\right), \tag{10}$$

$$\alpha_3(x) - \gamma_1\alpha_1(x) \neq 0 \quad \forall\, x \in [0, r]\,. \tag{11}$$

Then there is a unique regular solution of Problem 1 in the domain Ω.

Proof. Assume there is a solution of problem (1), (4), (5) and assume that

$$u(x,0) = \tau(x), \quad 0 \le x \le r, \tag{12}$$

$$\lim_{y\to 0} u_y(x,y) = u_y(x,0) = \nu(x), \quad 0 < x < r. \tag{13}$$

Find the relations between the functions $\tau(x)$ and $\nu(x)$ brought from Ω_1 and Ω_2 of the domain Ω onto the line I. The solution to problem (12), (13), when $|\lambda| \le \frac{m}{2}$ for equation (2), is written out according to one of the formulas [33]:

$$u(x,y) = \frac{1}{B\left(\varepsilon_1, \varepsilon_2\right)} \int\limits_0^1 \tau\left[x + (1-\varepsilon)\,(-y)^{\frac{1}{1-\varepsilon}}\,(2t-1)\right]\, t^{\varepsilon_2 - 1}\,(1-t)^{\varepsilon_1 - 1}\,dt +$$

$$+\frac{y}{B\left(1-\varepsilon_1,\, 1-\varepsilon_2\right)} \int\limits_0^1 \nu\left[x + (1-\varepsilon)\,(-y)^{\frac{1}{1-\varepsilon}}\,(2t-1)\right]\, t^{-\varepsilon_1}\,(1-t)^{-\varepsilon_2}\,dt, \;\; |\lambda| < \frac{m}{2}, \tag{14}$$

$$u(x,y) = \tau\left[x + (1-\varepsilon)\,(-y)^{\frac{1}{1-\varepsilon}}\right] +$$

$$+\,(1-\varepsilon)\; y \int\limits_0^1 \nu\left[x + (1-\varepsilon)\,(-y)^{\frac{1}{1-\varepsilon}}\,(2t-1)\right]\;(1-t)^{-\varepsilon}\,dt, \;\; \lambda = \frac{m}{2}, \tag{15}$$

$$u(x,y) = \tau\left[x - (1-\varepsilon)\,(-y)^{\frac{1}{1-\varepsilon}}\right] +$$

$$+\,(1-\varepsilon)\; y \int\limits_0^1 \nu^{-}\left[x + (1-\varepsilon)\,(-y)^{\frac{1}{1-\varepsilon}}\,(1-2t)\right]\;(1-t)^{-\varepsilon}\,dt, \;\; \lambda = -\frac{m}{2}. \tag{16}$$

First, consider the case for $|\lambda| < \frac{m}{2}$. In this instance, employing (14) we get

$$u\left[\theta_{r0}(x)\right] = u\left(\frac{r+x}{2},\, -\,(2-2\varepsilon)^{\varepsilon-1}\,(r-x)^{1-\varepsilon}\right) = \frac{1}{B\left(\varepsilon_1,\, \varepsilon_2\right)} \int\limits_0^1 \tau\left[x + (r-x)\,t\right]\, t^{\varepsilon_2-1}\,(1-t)^{\varepsilon_1-1}\,dt -$$

$$-\frac{1}{B\left(1-\varepsilon_1,\, 1-\varepsilon_2\right)}\,(2-2\varepsilon)^{\varepsilon-1}\,(r-x)^{1-\varepsilon} \int\limits_0^1 \nu\left[x + (r-x)\,t\right]\, t^{-\varepsilon_1}\,(1-t)^{-\varepsilon_2}\,dt.$$

Introducing a new variable $z = x + (r - x)\, t$ we rewrite the last equality as

$$u\left[\theta_{r0}(x)\right] = \frac{(r-x)^{1-\varepsilon}}{B\left(\varepsilon_1,\, \varepsilon_2\right)} \int\limits_x^r \frac{\tau\left(z\right)\,\left(r-z\right)^{\varepsilon_1 - 1}}{\left(z-x\right)^{1-\varepsilon_2}}\, dz - \frac{\left(2-2\varepsilon\right)^{\varepsilon - 1}}{B\left(1-\varepsilon_1,\, 1-\varepsilon_2\right)} \int\limits_x^r \frac{\nu\left(z\right)\,\left(r-z\right)^{-\varepsilon_2}}{\left(z-x\right)^{\varepsilon_1}}\, dz.$$

In terms of fractional (in the sense of Riemann-Liouville) integro-differentiation operator, the previous equality can be rewritten as

$$u\left[\theta_{r0}(x)\right] = \frac{\Gamma\left(\varepsilon\right)}{\Gamma\left(\varepsilon_1\right)} \left(r-x\right)^{1-\varepsilon}\; D_{rx}^{-\varepsilon_2}\left[\tau(t)\left(r-t\right)^{\varepsilon_1 - 1}\right] -$$

$$-\frac{\Gamma\left(2-\varepsilon\right)}{\Gamma\left(1-\varepsilon_2\right)} \left(2-2\varepsilon\right)^{\varepsilon-1}\; D_{rx}^{\varepsilon_1 - 1}\left[\nu(t)\left(r-t\right)^{-\varepsilon_2}\right]. \tag{17}$$

Further we use the following properties of the weighted composition of operators for fractional differentiation and integration with the same origins [5], [6], [9]:

$$D_{cx}^{-\beta}\, D_{ct}^{\beta}\varphi\left(s\right) = \varphi(x), \tag{18}$$

$$D_{cx}^{\alpha}\left|t-c\right|^{\alpha+\beta}\; D_{ct}^{\beta}\varphi\left(s\right) = \left|x-c\right|^{\beta}\; D_{cx}^{\alpha+\beta}\left|t-c\right|^{\alpha}\varphi(t), \tag{19}$$

where $0 < \alpha \le 1$, $\beta < 0$, $\alpha + \beta > -1$; $\varphi(x) \in L\left[a,\, b\right]$, and when $\alpha + \beta > 0$ the function $\varphi(x)$ contains the fractional derivative $D_{cx}^{\alpha+\beta}\varphi(t)$.

Applying to both sides of equality (17) the operator $D_{rx}^{1-\varepsilon_1}$ and using the above composition properties (18) and (19) we find

$$D_{rx}^{1-\varepsilon_1} u\left[\theta_{r0}(t)\right] = \gamma_1\;\left(r-x\right)^{-\varepsilon_2} D_{rx}^{1-\varepsilon}\tau(t) - \gamma_2\;\left(r-x\right)^{-\varepsilon_2}\nu(x). \tag{20}$$

Substituting the value $D_{rx}^{1-\varepsilon_1}\, u\left[\theta_{r0}(t)\right]$ from (20) into (5) we come to the ratio

$$\left[\alpha_2(x) + \gamma_1\alpha_1(x)\right]\; D_{rx}^{1-\varepsilon}\tau(t) + \left[\alpha_3(x) - \gamma_1\alpha_1(x)\right]\,\nu(x) = \psi_2(x). \tag{21}$$

The obtained relation (21) is the first fundamental relation between the functions $\tau(x)$ and $\nu(x)$ taken from the domain Ω_1 onto the line I.

Next, we find the fundamental relationship between the functions $\tau(x)$ and $\nu(x)$ taken from the domain Ω_2 onto the line I. The solution of problem (12), (13) for equation (3) is written by the d'Alembert formula [34]:

$$u(x,y) = \frac{\tau\left(x+y\right) + \tau\left(x-y\right)}{2} + \frac{1}{2}\int\limits_{x-y}^{x+y}\nu(t)dt + \frac{1}{2}\int\limits_0^y\int\limits_{x-y+t}^{x+y-t} f\left(t,\, s\right)\, dsdt. \tag{22}$$

Satisfying (22) to condition (4), we obtain

$$u\left[\theta_{r1}(x)\right] = u\left(\frac{r+x}{2},\, \frac{r-x}{2}\right) = \frac{\tau\left(r\right) + \tau(x)}{2} + \frac{1}{2}\int\limits_x^r \nu(t)dt + \frac{1}{2}\int\limits_0^{(r-x)/2}\int\limits_{x+t}^{r-t} f\left(s,\, t\right)\, dsdt = \psi_1(x). \tag{23}$$

Differentiating (23) we arrive at the relation

$$\nu(x) = \tau'(x) - 2\psi_1'(x) - \int\limits_0^{(r-x)/2} f\left(x+t,\, t\right)\, dt. \tag{24}$$

Relation (24) is the second fundamental relation between the functions $\tau(x)$ and $\nu(x)$ taken from the domain Ω_2 onto the line I.

Excluding the sought function $\nu(x)$ from (21) and (24) in view of the matching condition $\tau(r) = \psi_1(r)$, with respect to $\tau(x)$, we arrive at the following problem for the first-order ordinary differential equation with a fractional derivative in the lower terms

$$[\alpha_3(x) - \gamma_1\alpha_1(x)]\,\tau'(x) + [\alpha_2(x) + \gamma_1\alpha_1(x)]\,D_{rx}^{1-\varepsilon}\tau(t) =$$

$$= 2\,[\alpha_3(x) - \gamma_1\alpha_1(x)]\,\psi_1'(x) - \psi_2(x) + [\alpha_3(x) - \gamma_1\alpha_1(x)]\int\limits_0^{(r-x)/2} f\,(x+t,t)\,dt, \tag{25}$$

$$\tau\,(r) = \psi_1\,(r)\,. \tag{26}$$

If condition (11) of Theorem 1 is satisfied, then by dividing each term in equation (25) by $\alpha_3(x) - \gamma_1\,\alpha_1(x)$ with the subsequent integration of the resulting equation over x ranging from x to r we come to the integral equation

$$\tau(x) - \frac{1}{\Gamma\,(\varepsilon)}\int\limits_x^r K(x,t)\,\tau(t)dt = F_1(x), \tag{27}$$

which corresponds to problem (25), (26).

It follows from properties (8), (9), (10) that equation (27) is a Volterra integral equation of the second kind with the kernel $K(x,t) \in L_2\,(\,[0,\,r]\times[0,\,r]\,)$ and with the right-hand side $F_1(x) = C^1\,[0,\,r]\cap C^3\,(0,\,r)$. According to the general theory on Volterra integral equations the solution of equation (27) exists, is unique and can be written out by the formula:

$$\tau(x) = F_1(x) + \frac{1}{\Gamma\,(\varepsilon)}\int\limits_x^r R(x,t)F_1(t)dt, \tag{28}$$

where $R(x,t) = \sum\limits_{n=0}^{\infty}\Gamma^{-n}\,(\varepsilon)\;K_n(x,t)$ is a kernel resolvent $K(x,t)$; $K_0(x,t) = K(x,t)$, $K_{n+1}(x,t) = \int\limits_x^t K\,(x,s)\;K_n\,(s,t)\,ds$ are iterated kernels of the basic kernel $K(x,t)$; moreover, the resolvent $R(x,t)$, as well as the basic kernel $K(x,t)$ of equation (27), will belong to the class $R(x,t) \in L_2\,([0,r]\times[0,r])$, and the solution $\tau(x)$ of equation (27), as well as its right side $F_1(x)$, will belong to the class $\tau(x) \in C^1\,[0,\,r]\cap C^3\,(0,\,r)$.

The solution of equation (27) for $a(x) = a = const$ is written explicitly by the formula:

$$\tau(x) = F_1(x) + a\int\limits_x^r (t-x)^{\varepsilon-1}\,E_{1/\varepsilon}\left[a\,(t-x)^{\varepsilon}\,;\varepsilon\right]F_1(t)dt. \tag{29}$$

The sought function $\tau(x)$ for $\lambda = \pm\frac{m}{2}$ is found again employing formulas (28) or (29), but $\varepsilon_2 = 0$, $\varepsilon = \varepsilon_1 = \frac{m}{m+2}$, $\gamma_1 = 1$, $\gamma_2 = 2^{\varepsilon-1}\,(1-\varepsilon)^{\varepsilon}\,\Gamma\,(1-\varepsilon)$ at $\lambda = -\frac{m}{2}$ and $\varepsilon_1 = 0$, $\varepsilon = \varepsilon_2 = \frac{m}{m+2}$, $\gamma_1 = 0$, $\gamma_2 = 2^{\varepsilon-1}\,(1-\varepsilon)^{\varepsilon}$ at $\lambda = \frac{m}{2}$.

Once the function $\tau(x)$ has been found, the second sought function $\nu(x)$ is found employing formulas (21) or (24). Then the solution of the studied problem 1 in the domain Ω_1 is written out according to one of the (14), (15) or (16) formulas and in the domain Ω_2 the problem (12), (13) for equation (3) is solved by formula (22).

Research task 2

Similarly as above, satisfying (14) to condition (7) we find the first fundamental relation between the sought functions $\tau(x)$ and $\nu(x)$ taken in the domain Ω_1 onto the line I:

$$[\gamma_1\beta_1(x)+\beta_3(x)]\,\tau(x)=[\gamma_2\beta_1(x)-\beta_2(x)]\,D_{rx}^{\varepsilon-1}\nu(t)+\varphi_2(x). \tag{30}$$

Employing (22) under condition (6), we find the second fundamental relation between $\tau(x)$ and $\nu(x)$ taken in the domain Ω_2 onto the line I:

$$\tau(x)=2\varphi_1(x)-\tau\,(0)-\int\limits_0^x\nu(t)dt-\int\limits_0^{x/2}\int\limits_t^{x-t}f\,(s,t)\,dsdt. \tag{31}$$

Excluding in (30) and (31) the sought function $\tau(x)$ in view of the matching condition $\tau\,(0)=\varphi_1\,(0)$ with respect to $\nu(x)$ we obtain the equation

$$[\gamma_2\beta_1(x)-\beta_2(x)]\,D_{rx}^{\varepsilon-1}\nu(t)+[\gamma_1\beta_1(x)+\beta_3(x)]\int\limits_0^x\nu(t)dt=$$

$$=2\,[\gamma_1\beta_1(x)+\beta_3(x)]\,[\varphi_1(x)-\varphi_1(0)]-\varphi_2(x)-[\gamma_1\beta_1(x)+\beta_3(x)]\int\limits_0^{x/2}\int\limits_t^{x-t}f(s,t)\,dsdt.$$

Denoting $\upsilon(x)=\int\limits_0^x\nu(t)dt$ provided that $\gamma_2\beta_1(x)-\beta_2(x)\neq 0\;\forall\,x\in[0,r]$ the recent equality is rewritten as follows

$$D_{rx}^{\varepsilon-1}\upsilon'(t)+b(x)\upsilon(x)=F_2(x),\quad 0<x<r, \tag{32}$$

while

$$\upsilon\,(0)=0. \tag{33}$$

Once the operator $D_{rx}^{1-\varepsilon}$ has been applied to both sides of the equation (32) it could be represented as follows

$$\upsilon'(x)+D_{rx}^{1-\varepsilon}b(t)\upsilon(t)=D_{rx}^{1-\varepsilon}F_2(t),\quad 0<x<r. \tag{34}$$

Integrating equation (34) over x ranging from x to r taking into account (33) we arrive at the equation of the form

$$\upsilon(x)+\frac{1}{\Gamma\,(\varepsilon)}\int\limits_0^r b(t)K(x,t)\upsilon(t)dt=-\frac{1}{\Gamma\,(\varepsilon)}\int\limits_0^r K(x,t)F_2(t)dt, \tag{35}$$

equivalent to problem (32)–(33), where $K(x,t)=\begin{cases} t^{\varepsilon-1}-(t-x)^{\varepsilon-1}, & 0\le t<x,\\ t^{\varepsilon-1}, & x<t\le r.\end{cases}$

If $b(x)\in C^1\,[0,r]\cap C^3\,(0,r)$ is a positive non-decreasing function, then there is a unique regular solution of equation (35) [6; 133]. Then $\nu(x)=\upsilon'(x)$ and $\tau(x)$ are found by one of the formulas (30) or (31).

In the case, when $b(x)=b=const$ the solution of equation (34) is written explicitly by the formula:

$$\upsilon(x)=-\frac{1+b\int\limits_x^r(t-x)^{\varepsilon-1}\;E_{1/\varepsilon}\left[b\;(t-x)^{\varepsilon}\,;\,\varepsilon\right]dt}{1+b\int\limits_0^r t^{\varepsilon-1}\,E_{1/\varepsilon}\left(b\,t^{\varepsilon};\,\varepsilon\right)dt}\left[D_{0r}^{-\varepsilon}F_2(t)+b\int\limits_0^r t^{\varepsilon-1}\,E_{1/\varepsilon}\left(b\,t^{\varepsilon};\,\varepsilon\right)\;D_{rt}^{-\varepsilon}F_2\,(s)\,dt\right]+$$

$$+D_{rx}^{-\varepsilon}F_2(t)+b\int\limits_x^r (t-x)^{\varepsilon-1}\ E_{1/\varepsilon}\left[b\ (t-x)^{\varepsilon}\,;\,\varepsilon\right]D_{rt}^{-\varepsilon}F_2\,(s)\,dt$$

provided that $1+b\int\limits_0^r t^{\varepsilon-1}\,E_{1/\varepsilon}\,(b\,t^{\varepsilon};\,\varepsilon)\,dt\neq 0$. If $b\geq 0$, then the fulfillment of this inequality is obvious.

Once the functions $\tau(x)$ and $\nu(x)$ have been found similarly as for the previous problem 1, the solution to problem 2 in the domain Ω_1 is written employing one of the formulas (14), (15) or (16), and in the domain Ω_2 the problem (12)–(13) for equation (3) is solved by the formula (22).

References

1 Смирнов М.М. Уравнения смешанного типа / М.М. Смирнов. – М.: Наука, 1970. – 296 с.

2 Protter M.H. The Cauchy problem for a hyperbolic second-order equation with data on the parabolic line / M.H. Protter // Canad. J. of Math. – 1954. – 6. – P. 542-553.

3 Бицадзе А.В. Уравнения смешанного типа / А.В. Бицадзе. – М.: Изд-во АН СССР, 1959. – 164 с.

4 Лыков А.В. Применение методов термодинамики необратимых процессов к исследованию тепло- и массообмена / А.В. Лыков // Инж.-физ. журн. – 1955. – 9. – №3. – С. 287-304.

5 Нахушев А.М. Уравнения математической биологии / А.М. Нахушев. – М.: Высш. шк., 1995. – 301 с.

6 Нахушев А.М. Дробное исчисление и его применение / А.М. Нахушев. – М.: Физматлит, 2003. – 272 с.

7 Берс Л. Математические вопросы дозвуковой и околозвуковой газовой динамики / Л. Берс. – М.: Иностранная литература, 1961. – 208 с.

8 Франкль Ф.И. Избранные труды по газовой динамике / Ф.И. Франкль. – М.: Наука, 1973. – 771 с.

9 Самко С.Г. Интегралы и производные дробного порядка и некоторые их приложения / С.Г. Самко, А.А. Килбас, О.И. Маричев. – Минск: Наука и техника, 1987. – 688 с.

10 Псху А.В. Уравнения в частных производных дробного порядка / А.В. Псху. – М.: Наука, 2005. – 199 с.

11 Джрбашян М.М. Интегральные преобразования и представления функций в комплексной плоскости / М.М. Джрбашян. – М.: Наука, 1966. – 672 с.

12 Кальменов Т.Ш. Критерий единственности решения задачи Дарбу для одного вырождающегося гиперболического уравнения / Т.Ш. Кальменов // Дифференц. уравнения. – 1971. – 7. – №1. – С. 178-181.

13 Балкизов Ж.А. Краевая задача для вырождающегося внутри области гиперболического уравнения / Ж.А. Балкизов // Изв. высш. учеб. зав. Северо-Кавказский регион. Сер. Естественные науки. – 2016. – №1(189). – С. 5-10.

14 Балкизов Ж.А. Первая краевая задача для вырождающегося внутри области гиперболического уравнения / Ж.А. Балкизов // Владикавказ. мат. журн. – 2016. – 18. – № 2. – С. 19-30.

15 Кумыкова С.К. Об одной краевой задаче для гиперболического уравнения, вырождающегося внутри области / С.К. Кумыкова, Ф.Б. Нахушева // Дифференц. уравнения. – 1978. – 14. – № 1. – С. 50–65.

16 Балкизов Ж.А. Краевые задачи с данными на противоположных характеристиках для смешанно-гиперболического уравнения второго порядка // Докл. Адыгской (Черкесской) Междунар. акад. наук. – 2020. – 20. – № 3. – С. 6–13.

17 Балкизов Ж.А. Краевые задачи для смешанно-гиперболического уравнения / Ж.А. Балкизов // Вестн. Дагестан. гос. ун-та. Сер. 1: Естественные науки. – 2021. – 36. – № 1. – С. 7–14.

18 Жегалов В.И. Краевая задача для уравнения смешанного типа с граничными условиями на обеих характеристиках и с разрывами на линии перехода / В.И. Жегалов // Ученые записки Казанского университета. – 1962. – 122. – 3. – С. 3–16.

19 Нахушев А.М. Новая краевая задача для одного вырождающегося гиперболического уравнения / А.М. Нахушев // Докл. Акад. наук СССР. – 1969. – 187. – № 4. – С. 736–739.

20 Нахушев А.М. О некоторых краевых задачах для гиперболических уравнений и уравнений смешанного типа / А.М. Нахушев // Дифференц. уравнения. – 1969. – 5. – № 1. – С. 44–59.

21 Салахитдинов М.С. О некоторых краевых задачах для гиперболического уравнения, вырождающегося внутри области / М.С. Салахитдинов, М. Мирсабуров // Дифференц. уравнения. – 1981. – 17. – № 1. – С. 129–136.

22 Салахитдинов М.С. О двух нелокальных краевых задачах для вырождающегося гиперболического уравнения / М.С. Салахитдинов, М. Мирсабуров // Дифференц. уравнения. – 1982. – 18. – № 1. – С. 116–127.

23 Репин О.А. Задача с нелокальными условиями на характеристиках для уравнения влагопереноса / С.В. Ефимова, О.А. Репин // Дифференц. уравнения. – 2004. – 40. – № 1. – С. 116–127.

24 Репин О.А. О задаче с операторами М. Сайго на характеристиках для вырождающегося внутри области гиперболического уравнения / О.А. Репин // Вестн. Самар. гос. техн. ун-та. Сер. физ.-мат. науки. – 2006. – Вып. 43. – С. 10–14.

25 Балкизов Ж.А. Задача со смещением для вырождающегося гиперболического уравнения первого рода / Ж.А. Балкизов // Вестн. Самар. гос. техн. ун-та. Сер. физ.-мат. науки. – 2021. – 25. – №1. – С. 21–34.

26 Салахитдинов М.С. Уравнения смешанно-составного типа / М.С. Салахитдинов. – Ташкент: ФАН, 1974. – 165 с.

27 Репин О.А. Краевые задачи со смещением для уравнений гиперболического и смешанного типов / О.А. Репин. – Самара: Изд-во Самар. филиала Саратов. ун-та, 1992. – 162 с.

28 Кальменов Т.Ш. Краевые задачи для линейных уравнений в частных производных гиперболического типа / Т.Ш. Кальменов. – Шымкент: Гылая, 1993. – 328 с.

29 Салахитдинов М.С. Краевые задачи для уравнений смешанного типа со спектральным параметром / М.С. Салахитдинов, А.К. Уринов. – Ташкент: ФАН, 1997. – 165 с.

30 Нахушев А.М. Задачи со смещением для уравнений в частных производных / А.М. Нахушев. – М.: Наука, 2006. – 287 с.

31 Нахушева З.А. Нелокальные краевые задачи для основного и смешанного типов дифференциальных уравнений / З.А. Нахушева. – Нальчик: КБНЦ РАН, 2011. – 196 с.

32 Сабитов К.Б. К теории уравнений смешанного типа / К.Б. Сабитов. – М.: Физматлит, 2014. – 304 с.

33 Смирнов М.М. Вырождающиеся гиперболические уравнения / М.М. Смирнов. – Минск: Выш. шк., 1977. – 160 с.

34 Тихонов А.Н. Уравнения математической физики / А.Н. Тихонов, А.А. Самарский. – М.: МГУ; Наука, 2004. – 798 с.

Ж.А. Балкизов[1], З.Х. Гучаева[2], А.Х. Кодзоков[2]

[1] *РҒА КБҒО Қолданбалы математика және автоматтандыру институты, Нальчик, Ресей;*
[2] *Х.М. Бербеков атындағы Кабардин-Балкар мемлекеттік университеті, Нальчик, Ресей*

Екінші ретті аралас-гиперболалық теңдеу үшін ығысуы бар ішкі-шеттік есептер

Мақалада облыстың бір бөлігінде толқындық оператордан, ал басқасында бірінші ретті өзгеше гиперболалық оператордан тұратын екінші ретті аралас-гиперболалық теңдеу үшін ығысуы бар ішкі-шеттік есептер зерттелген. Берілген функциялар бойынша зерттелетін есептердің шешімінің бар болуын, бірегейлігін қамтамасыз ететін жеткілікті шарттар анықталды. Кейбір дербес жағдайларда зерттелетін есептердің шешімдері айқын түрде жазылған.

Кілт сөздер: толқын теңдеуі, бірінші ретті өзгеше гиперболалық теңдеу, Вольтерраның интегралдық теңдеуі, екінші типті Фредгольм интегралдық теңдеуі, Трикоми әдісі, интегралдық теңдеулер әдісі, бөлшек есептеу теориясының әдісі.

Ж.А. Балкизов[1], З.Х. Гучаева[2], А.Х. Кодзоков[2]

[1] *Институт прикладной математики и автоматизации КБНЦ РАН, Нальчик, Россия;*
[2] *Кабардино-Балкарский государственный университет имени Х.М. Бербекова, Нальчик, Россия*

Внутренне-краевые задачи со смещением для смешанно-гиперболического уравнения второго порядка

В статье исследованы внутренне-краевые задачи со смещением для смешанно-гиперболического уравнения второго порядка, состоящего из волнового оператора в одной части области и вырождающегося гиперболического оператора первого рода — в другой. Найдены достаточные условия на заданные функции, обеспечивающие существование единственного регулярного решения исследуемых задач. В некоторых частных случаях решения исследуемых задач выписаны в явном виде.

Ключевые слова: волновое уравнение, вырождающееся гиперболическое уравнение первого рода, интегральное уравнение Вольтерра, интегральное уравнение Фредгольма второго рода, метод Трикоми, метод интегральных уравнений, методы теории дробного исчисления.

References

1 Smirnov, M.M. (1970). *Uravneniia smeshannogo tipa [Mixed type equations].* Moscow: Nauka [in Russian].

2 Protter, M.H. (1954). The Cauchy problem for a hyperbolic second-order equation with data on the parabolic line. *Canad. J. of Math., 6,* 542–553.

3 Bitsadze, A.V. (1959). *Uravneniia smeshannogo tipa [Mixed type equations].* Moscow: Izdatelstvo AN SSSR [in Russian].

4 Lykov, A.V. (1955). Primenenie metodov termodinamiki neobratimykh protsessov k issledovaniiu teplo- i massoobmena [Application of methods of thermodynamics of irreversible processes to the study of heat and mass transfer]. *Inzhenerno-fizicheskii zhurnal – Journal of Engineering Physics and Thermophysics, 9, 3,* 287–304 [in Russian].

5 Nakhushev, A.M. (1995). *Uravneniia matematicheskoi biologii [Equations of mathematical biology].* Moscow: Vysshaia shkola [in Russian].

6 Nakhushev, A.M. (2003). *Drobnoe ischislenie i ego primenenie [Fractional calculus and its application].* Moscow: Fizmatlit [in Russian].

7 Bers, L. (1961). *Matematicheskie voprosy dozvukovoi i okolozvukovoi gazovoi dinamiki [Mathematical problems of subsonic and transonic gas dynamics].* Moscow: Inostrannaia literatura [in Russian].

8 Frankl, F.I. (1973). *Izbrannye trudy po gazovoi dinamike [Selected papers on gas dynamics].* Moscow: Nauka [in Russian].

9 Samko, S.G., Kilbas, A.A., & Marichev, O.I. (1987). *Integraly i proizvodnye drobnogo poriadka i nekotorye ikh prilozheniia [Integrals and derivatives of fractional order and some of their applications].* Minsk: Nauka i tekhnika [in Russian].

10 Pskhu, A.V. (2005). *Uravneniia v chastnykh proizvodnykh drobnogo poriadka [Fractional partial differential equations].* Moscow: Nauka [in Russian].

11 Dzhrbashian, M.M. (1966). *Integralnye preobrazovaniia i predstavleniia funktsii v kompleksnoi ploskosti [Integral transformations and representations of functions in the complex plane].* Moscow: Nauka [in Russian].

12 Kalmenov, T.Sh. (1971). *Kriterii edinstvennosti resheniia zadachi Darbu dlia odnogo vyrozhdaiushchegosia giperbolicheskogo uravneniia [A criterion for unique solvability for Darboux problem for a degenerate hyperbolic equation]. Differentsialnye uravneniia – Differential Equations, 7, 1,* 178–181 [in Russian].

13 Balkizov, Zh.A. (2016). Kraevaia zadacha dlia vyrozhdaiushchegosia vnutri oblasti giperbolicheskogo uravneniia [A boundary value problem for a hyperbolic equation degenerating inside the domain]. *Izvestiia vysshikh uchebnykh zavedenii. Severo-Kavkazskii region. Seriia Yestestvennye nauki – Bulletin Of Higher Education Institutes North Caucasus Region. Natural Sciences, 1(189),* 5–10 [in Russian].

14 Balkizov, Zh.A. (2016). Pervaia kraevaia zadacha dlia vyrozhdaiushchegosia vnutri oblasti giperbolicheskogo uravneniia [The first boundary value problem for a hyperbolic equation degenerating inside the domain]. *Vladikavkazskii matematicheskii zhurnal – Vladikavkaz Mathematical Journal, 18, 2,* 19–30 [in Russian].

15 Kumykova, S.K., & Nakhusheva, F.B. (1978). Ob odnoi kraevoi zadache dlia giperbolicheskogo uravneniia, vyrozhdaiushchegosia vnutri oblasti [On a boundary value problem for a hyperbolic equation degenerating inside a domain]. *Differentsialnye uravneniia – Differential Equations, 14, 1,* 50–65 [in Russian].

16 Balkizov, Zh.A. (2020). Kraevye zadachi s dannymi na protivopolozhnykh kharakteristikakh dlia smeshanno-giperbolicheskogo uravneniia vtorogo poriadka [Boundary value problems with data on opposite characteristics for a second-order mixed-hyperbolic equation]. *Doklady Adygskoi (Cherkesskoi) Mezhdunarodnoi akademii nauk – Reports of Adyghe (Circassian) International Academy of Sciences, 20, 3,* 6–13 [in Russian].

17 Balkizov, Zh.A. (2021). Kraevye zadachi dlia smeshanno-giperbolicheskogo uravneniia [Boundary value problems for a mixed-hyperbolic equation]. *Vestnik Dagestanskogo gosudarstvennogo universiteta. Seriia 1: Yestestvennye nauki – Herald of Dagestan State University. Series 1. Natural Sciences, 36, 1,* 7–14 [in Russian].

18 Zhegalov, V.I. (1962). Kraevaia zadacha dlia uravneniia smeshannogo tipa s granichnymi usloviiami na obeikh kharakteristikakh i s razryvami na linii perekhoda [Boundary value problem for a mixed-type equation with boundary conditions on both characteristics and with discontinuities on the transition line]. *Uchenye zapiski Kazanskogo universiteta – Proceedings of Kazan University, 122, 3,* 3–16 [in Russian].

19 Nakhushev, A.M. (1969). *Novaia kraevaia zadacha dlia odnogo vyrozhdaiushchegosia giperbolicheskogo uravneniia [A new boundary value problem for a degenerate hyperbolic equation]. Doklady Akademii nauk SSSR – Proceedings of the USSR Academy of Sciences, 187, 4,* 736–739 [in Russian].

20 Nakhushev, A.M. (1969). O nekotorykh kraevykh zadachakh dlia giperbolicheskikh uravnenii i uravnenii smeshannogo tipa [On some boundary value problems for hyperbolic equations and equations of mixed type]. *Differentsialnye uravneniia – Differential Equations, 5, 1,* 44–59 [in Russian].

21 Salakhitdinov, M.S., & Mirsaburov, M. (1981). O nekotorykh kraevykh zadachakh dlia giperbolicheskogo uravneniia, vyrozhdaiushchegosia vnutri oblasti [On some boundary value problems for a hyperbolic equation degenerating inside a domain]. *Differentsialnye uravneniia – Differential Equations, 17, 1,* 129–136 [in Russian].

22 Salakhitdinov, M.S., & Mirsaburov, M. (1982). O dvukh nelokalnykh kraevykh zadachakh dlia vyrozhdaiushchegosia giperbolicheskogo uravneniia [On two nonlocal boundary value problems for a degenerate hyperbolic equation]. *Differentsialnye uravneniia – Differential Equations, 18, 1,* 116–127 [in Russian].

23 Efimova, S.V., & Repin, O.A. (2004). Zadacha s nelokalnymi usloviiami na kharakteristikakh dlia uravneniia vlagoperenosa [Problem with nonlocal conditions on characteristics for the moisture transfer equation]. *Differentsialnye uravneniia – Differential Equations, 40, 1,* 116–127 [in Russian].

24 Repin, O.A. (2006). O zadache s operatorami M. Saigo na kharakteristikakh dlia vyrozhdaiushchegosia vnutri oblasti giperbolicheskogo uravneniia [On the problem with M. Saigo's operators on characteristics for a hyperbolic equation degenerating inside the domain]. *Vestnik Samarskogo gosudarstvennogo tekhnicheskogo universiteta. Seriia Fiziko-matematicheskie nauki – Journal of Samara State Technical University, Ser. Physical and Mathematical Sciences, 43,* 10–14 [in Russian].

25 Balkizov, Zh.A. (2021). Zadacha so smeshcheniem dlia vyrozhdaiushchegosia giperbolicheskogo uravneniia pervogo roda [Problem with displacement for first kind degenerate hyperbolic equation]. *Vestnik Samarskogo gosudarstvennogo tekhnicheskogo universiteta. Seriia Fiziko-matematicheskie nauki – Journal of Samara State Technical University, Ser. Physical and Mathematical Sciences, 25, 1,* 21–34 [in Russian].

26 Salakhitdinov, M.S. (1974). *Uravneniia smeshanno-sostavnogo tipa [Mixed-compound equations].* Tashkent: FAN [in Russian].

27 Repin, O.A. (1992). *Kraevye zadachi so smeshcheniem dlia uravnenii giperbolicheskogo i smeshannogo tipov [Boundary value problems with displacement for hyperbolic and mixed type equations].* Samara: Izdatelstvo Samarskogo filiala Saratovskogo universiteta [in Russian].

28 Kalmenov, T.Sh. (1993). *Kraevye zadachi dlia lineinykh uravnenii v chastnykh proizvodnykh giperbolicheskogo tipa [Boundary value problems for hyperbolic type linear partial differential equations].* Shymkent: Gylaia [in Russian].

29 Salakhitdinov, M.S., & Urinov, A.K. (1997). *Kraevye zadachi dlia uravnenii smeshannogo tipa so spektralnym parametrom [Boundary value problems for equations of mixed type with a spectral parameter].* Tashkent: FAN [in Russian].

30 Nakhushev, A.M. (2006). *Zadachi so smeshcheniem dlia uravnenii v chastnykh proizvodnykh [Boundary value problems with shift for partial differential equations].* Moscow: Nauka [in Russian].

31 Nakhusheva, Z.A. (2011). *Nelokalnye kraevye zadachi dlia osnovnogo i smeshannogo tipov differentsialnykh uravnenii [Nonlocal boundary value problems for differential equations of basic and mixed types]*. Nalchik: KBNTS RAN [in Russian].

32 Sabitov, K.B. (2014). *K teorii uravnenii smeshannogo tipa [On the theory of mixed type equations]*. Moscow: Fizmatlit [in Russian].

33 Smirnov, M.M. (1977). *Vyrozhdaiushchiesia giperbolicheskie uravneniia [Degenerate hyperbolic equations]*. Minsk: Vysheishaia shkola [in Russian].

34 Tikhonov, A.N., & Samarskii, A.A. (2004). *Uravneniia matematicheskoi fiziki [Equations of mathematical physics]*. Moscow: Moskovskii gosudarstvennyi universitet; Nauka [in Russian].