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An inverse problem for Hilfer type differential equation of higher order

In three-dimensional domain, an identification problem of the source function for Hilfer type partial differential equation of the even order with a condition in an integral form and with a small positive parameter in the mixed derivative is considered. The solution of this fractional differential equation of a higher order is studied in the class of regular functions. The case, when the order of fractional operator is $0 < \alpha < 1$, is studied. The Fourier series method is used and a countable system of ordinary differential equations is obtained. The nonlocal boundary value problem is integrated as an ordinary differential equation. By the aid of given additional condition, we obtained the representation for redefinition (source) function. Using the Cauchy–Schwarz inequality and the Bessel inequality, we proved the absolute and uniform convergence of the obtained Fourier series.

Keywords: fractional order, Hilfer operator, inverse source problem, Fourier series, integral condition, unique solvability.

Introduction

The theory of the inverse boundary value problems is currently one of the most important fields of the modern theory of differential equations. Consequently, a large number of research works are devoted to study the different kind of inverse problems for differential and integro-differential equations (see, for example, [1–10]). In cases where the boundary of the flow domain of a physical process is unavailable for measurements, nonlocal conditions in an integral form can serve as additional information sufficient for unique solvability of the problem. Therefore, researches on the study of nonlocal boundary value problems for differential and integro-differential equations with integral conditions have been intensified (see, for example, [11–20]). In addition, we note that studies of many problems of gas dynamics, theory of elasticity, theory of plates and shells are described by higher-order partial differential equations.

Fractional calculus plays an important role for the mathematical modeling in many natural and engineering sciences [21]. In [22], it is considered problems of continuum and statistical mechanics. In [23] is studied the mathematical problems of Ebola epidemic model. In [24] and [25], it is studied the fractional model for the dynamics of tuberculosis infection and novel coronavirus (nCoV-2019), respectively. The construction of various models of theoretical physics by the aid of fractional calculus is described in [26, Vol. 4, 5], [27], [28]. A detailed review of the application of fractional calculus in solving problems of applied sciences is given in [29, Vol. 6-8], [30]. In [31], the unique solvability of boundary value problem for weak nonlinear partial differential equations of mixed type with fractional Hilfer operator is studied. In [32], the solvability of nonlocal problem for a mixed type fourth-order differential equation with Hilfer fractional operator is studied. In the direction of applications of fractional derivatives to solving partial differential equations the interesting results were also obtained in [33–42].

We recall some basic terms of fractional integro-differentiation, which have been used during the study. Let $(t_0; T) \subset \mathbb{R}^+ \equiv [0; \infty)$ be an interval on the set of positive real numbers, where $0 \le t_0 < T < \infty$. The Riemann–Liouville $0 < \alpha$-order fractional integral of a function $\eta(t)$ is defined as follows:

$$I^{\alpha}_{t_0+}\eta(t) = \frac{1}{\Gamma(\alpha)} \int\limits_{t_0}^{t} (t-s)^{\alpha-1}\eta(s)ds, \quad \alpha > 0, \quad t \in (t_0; T),$$

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where $\Gamma(\alpha)$ is the Gamma function.

Let $n-1<\alpha\le n,\ \ n\in\mathbb{N}$. The Riemann–Liouville α-order fractional derivative of a function $\eta(t)$ is defined as follows:

$$D_{t_0+}^{\alpha}\eta(t)=\frac{d^n}{dt^n}I_{t_0+}^{n-\alpha}\eta(t),\quad t\in\big(t_0;\,T\big).$$

The Gerasimov-Caputo α-order fractional derivative of a function $\eta(t)$ is defined by

$${}_*D_{t_0+}^{\alpha}\eta(t)=I_{t_0+}^{n-\alpha}\eta^{(n)}(t)=\frac{1}{\Gamma(n-\alpha)}\int\limits_{t_0}^{t}\frac{\eta^{(n)}(s)\,d\,s}{(t-s)^{\alpha-n+1}},\ \ t\in\big(t_0;\,T\big).$$

These derivatives are reduced to the n-th order derivatives for $\alpha=n\in\mathbb{N}$:

$$D_{t_0+}^{n}\eta(t)={}_*\,D_{t_0+}^{n}\eta(t)=\frac{d^n}{dt^n}\eta(t),\quad t\in\big(t_0;\,T\big).$$

The Hilfer fractional derivatives of α-order $(n-1<\alpha\le n,\ \ n\in\mathbb{N})$ and β-type $(0\le\beta\le1)$ are defined by the following composition of three operators:

$$D_{t_0+}^{\alpha,\,\beta}\eta(t)=I_{t_0+}^{\beta(n-\alpha)}\frac{d^n}{dt^n}I_{t_0+}^{(1-\beta)(n-\alpha)}\eta(t),\quad t\in\big(t_0;\,T\big).$$

For $\beta=0$, this operator is reduced to the Riemann–Liouville fractional derivative $D_{t_0+}^{\alpha,\,0}=D_{t_0+}^{\alpha}$and the case $\beta=1$ corresponds to the Gerasimov–Caputo fractional derivative $D_{t_0+}^{\alpha,\,1}={}_*\,D_{t_0+}^{\alpha}$. Let $\gamma=\alpha+\beta\,n-\alpha\,\beta$. It is easy to see, that $\alpha\le\gamma\le n$. Then it is convenient to use another designation for the operator $D^{\,\alpha,\,\gamma}\eta(t)=D_{t_0+}^{\alpha,\,\beta}\eta(t)$. The generalized Riemann–Liouville operator was introduced by R. Hilfer based on time evolutions that arise during the transition from the microscopic scale to the macroscopic time scale (see [26]).

In this paper, for $0<\alpha<\gamma\le1$ we study the regular solvability of an inverse boundary value problem for a Hilfer type partial differential equation of even order with positive small parameter. The source function is in the integral condition containing the Riemann–Liouville $0<\alpha<1$-order fractional integral. The stability of the solution from the given functions is proved.

In the three-dimensional domain $\Omega=\{(t,\,x,\,y)\,|\,0<t<T,\ 0<x,\,y<l\}$ a partial differential equation of the following form is considered

$$D_{\varepsilon}^{\alpha,\,\gamma}\,[U]=a\,(t)\,b\,(x,\,y) \tag{1}$$

with a nonlocal condition on the integral form containing the Riemann–Liouville $0<\alpha<1$-order fractional integral

$$U\,(T,\,x,\,y)+\big(I_{0+}^{\rho}U\,(t,\,x,\,y)\big)\,|_{t=T}\,=\varphi(x,\,y),\quad 0\le x\,,\,y\le l, \tag{2}$$

where ρ, T and l are given positive real numbers,

$$D_{\varepsilon}^{\alpha,\,\gamma}\,[U]=\left[D^{\alpha,\,\gamma}+\varepsilon D^{\alpha,\,\gamma}\left(\frac{\partial^{\,4k}}{\partial\,x^{\,4k}}+\frac{\partial^{\,4k}}{\partial\,y^{\,4k}}\right)+\omega\left(\frac{\partial^{\,4k}}{\partial\,x^{\,4k}}+\frac{\partial^{\,4k}}{\partial\,y^{\,4k}}\right)\right]\,U\,(t,\,x,\,y),$$

ω is a positive parameter, ε is a positive small parameter, $0<\alpha<\gamma\le1$, k is a given positive integer, $a\,(t\,)\in C\,(\Omega_T)$, $\Omega_{\,T}\equiv[0;\,T]$, $\Omega_{\,l}\equiv[0;\,l]$, $b\,(x,\,y)\in C\,\left(\Omega_{\,l}^{\,2}\right)$ is a known function, $\varphi(x,\,y)$ is a source (redefinition) function, $\Omega_{\,l}^{\,2}\equiv\Omega_l\times\Omega_l$. We assume that for given functions are true the following boundary conditions

$$\varphi(0,\,y)=\varphi(l,\,y)=\varphi(x,\,0)=\varphi\,(x,\,l)=0,$$

$$b\,(0,\,y)=b\,(l,\,y)=b\,(x,\,0)=b\,(x,\,l)=0.$$

Problem Statement. We find the pair of unknown functions $\{U\,(t,\,x,\,y);\,\varphi\,(x,\,y)\}$, first of them satisfies differential equation (1), nonlocal integral condition (2), zero boundary value conditions

$$U\,(t,\,0,\,y)=U\,(t,\,l,\,y)=U\,(t,\,x,\,0)=U\,(t,\,x,\,l)=$$

$$=\frac{\partial^{\,2}}{\partial\,x^2}U\,(t,\,0,\,y)=\frac{\partial^{\,2}}{\partial\,x^2}U\,(t,\,l,\,y)=\frac{\partial^{\,2}}{\partial\,x^2}U\,(t,\,x,\,0)=\frac{\partial^{\,2}}{\partial\,x^2}U\,(t,\,x,\,l)=$$

$$= \frac{\partial^2}{\partial y^2} U(t,\, 0,\, y) = \frac{\partial^2}{\partial y^2} U(t,\, l,\, y) = \frac{\partial^2}{\partial y^2} U(t,\, x,\, 0) = \frac{\partial^2}{\partial y^2} U(t,\, x,\, l) = \ldots =$$

$$= \frac{\partial^{4k-2}}{\partial x^{4k-2}} U(t,\, 0,\, y) = \frac{\partial^{4k-2}}{\partial x^{4k-2}} U(t,\, l,\, y) = \frac{\partial^{4k-2}}{\partial x^{4k-2}} U(t,\, x,\, 0) = \frac{\partial^{4k-2}}{\partial x^{4k-2}} U(t,\, x,\, l) =$$

$$= \frac{\partial^{4k-2}}{\partial y^{4k-2}} U(t,0,y) = \frac{\partial^{4k-2}}{\partial y^{4k-2}} U(t,\, l,\, y) = \frac{\partial^{4k-2}}{\partial y^{4k-2}} U(t,\, x,\, 0) = \frac{\partial^{4k-2}}{\partial y^{4k-2}} U(t,\, x,\, l) = 0, \tag{3}$$

properties of the class of functions

$$\left[\begin{array}{l} t^{1-\gamma} U(t,\, x,\, y) \in C(\overline{\Omega}), \\ D^{\alpha,\gamma} U(t,\, x,\, y) \in C^{4k,\,4k}_{x,\,y}(\Omega) \cap C^{4k+0}_{x,\,y}(\Omega) \cap C^{0+4k}_{x,\,y}(\Omega) \end{array} \right. \tag{4}$$

and the additional condition

$$U\big(t_1,\, x,\, y\big) = \psi(x,\, y), \quad 0 < t_1 < T, \;\; 0 \le x,\, y \le l, \tag{5}$$

$\varphi(x) \in C\,[0;\, l]$, where $\psi(x,\, y)$ are given smooth function and

$$\psi(0,\, y) = \psi(l,\, y) = \psi(x,\, 0) = \psi(x,\, l) = 0,$$

$C^{4k+0}_{x,\,y}(\Omega)$ is the class of continuous functions $\frac{\partial^{4k} U(t,\,x,\,y)}{\partial x^{4k}}$ on Ω, while $C^{0+4k}_{x,\,y}(\Omega)$ is the class of continuous functions $\frac{\partial^{4k} U(t,\,x,\,y)}{\partial y^{4k}}$ on Ω, $\frac{\partial^{4k-2}}{\partial y^{4k-2}} U(t,\, x,\, l)$ we understand as $\frac{\partial^{4k-2}}{\partial y^{4k-2}} U(t,\, x,\, y)\Big|_{y=l}$, $\overline{\Omega} = \{(t,\, x,\, y) | 0 \le t \le T,$ $0 \le x,\, y \le l\}$.

1 Expansion of the solution in a Fourier series

We seek nontrivial solutions of the problem in the form of Fourier series

$$U(t,\, x,\, y) = \sum_{n,\, m=1}^{\infty} u_{n,\,m}(t)\, \vartheta_{n,\,m}(x,\, y), \tag{6}$$

where

$$u_{n,\,m}(t) = \int_0^l \int_0^l U(t,\, x,\, y)\, \vartheta_{n,\,m}(x,\, y)\, dx\, dy, \tag{7}$$

$$\vartheta_{n,\,m}(x,\, y) = \frac{2}{l} \sin \frac{\pi n}{l} x \sin \frac{\pi m}{l} y, n,\, m = 1,\, 2,\, \ldots$$

We also suppose that the following function is expand in a Fourier series

$$b(x,\, y) = \sum_{n,\, m=1}^{\infty} b_{n,\,m} \vartheta_{n,\,m}(x,\, y), \tag{8}$$

where

$$b_{n,\,m} = \int_0^l \int_0^l b(x,\, y)\, \vartheta_{n,\,m}(x,\, y)\, dx\, dy. \tag{9}$$

Substituting Fourier series (6) and (8) into given partial differential equation (1), we obtain the countable system of ordinary differential equations of a fractional $0 < \alpha,\, \gamma < 1$-order

$$D^{\alpha,\gamma} u_{n,\,m}(t) + \lambda^{2k}_{n,\,m}(\varepsilon)\, \omega\, u_{n,\,m}(t) = \frac{a(t)\, b_{n,\,m}}{1 + \varepsilon\, \mu^{4k}_{n,\,m}}, \tag{10}$$

where

$$\lambda^{2k}_{n,\,m}(\varepsilon) = \frac{\mu^{4k}_{n,\,m}}{1 + \varepsilon\, \mu^{4k}_{n,\,m}}, \;\; \mu^{k}_{n,\,m} = \left(\frac{\pi}{l}\right)^k \sqrt{n^{2k} + m^{2k}}.$$

The general solution of the countable system of differential equations (10) has the form [30]

$$u_{n,m}(t) = C_{n,m} t^{\gamma-1} E_{\alpha,\gamma}\left(-\lambda_{n,m}^{2k}(\varepsilon)\,\omega\, t^{\alpha}\right) + b_{n,m} h_{n,m}(t), \tag{11}$$

where

$$E_{\alpha,\gamma}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \gamma)}, \quad z,\, \alpha,\, \gamma \in \mathbb{C}, \quad Re\,(\alpha) > 0$$

is the Mittag–Leffler function [26, 269–295] and

$$h_{n,m}(t) = \frac{1}{1 + \varepsilon\, \mu_{n,m}^{4k}} \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}\left(-\lambda_{n,m}^{2k}(\varepsilon)\,\omega\,(t-s)^{\alpha}\right) a(s)\, ds,$$

$C_{n,m}$ is an arbitrary constant.

By Fourier coefficients (7), we rewrite integral condition (2) for the countable system (10)

$$u_{n,m}(T) + \left(I_{0+}^{\rho} u_{n,m}(t)\right)|_{t=T} = \int_0^l \int_0^l \left(U(T,\, x,\, y) + \left(I_{0+}^{\rho} U(t,\, x,\, y)\right)|_{t=T}\right) \vartheta_{n,m}(x,\, y)\, dx\, dy =$$

$$= \int_0^l \int_0^l \varphi(x,\, y)\, \vartheta_{n,m}(x,\, y)\, dx\, dy = \varphi_{n,m}. \tag{12}$$

To find the unknown coefficients $C_{n,m}$ in (11), we use condition (12) and from (11) we have

$$C_{n,m} = \frac{1}{\sigma_{0\,n,m}} \left[\varphi_{n,m} - b_{n,m} \sigma_{1\,n,m}\right], \tag{13}$$

where

$$\sigma_{0\,n,m} = T^{\gamma-1}\left[E_{\alpha,\gamma}\left(-\lambda_{n,m}^{2k}(\varepsilon)\,\omega\, T^{\alpha}\right) + T^{\rho} E_{\alpha,\rho+\gamma}\left(-\lambda_{n,m}^{2k}(\varepsilon)\,\omega\, T^{\alpha}\right)\right],$$
$$\sigma_{1\,n,m} = h_{n,m}(T) + \left(I_{0+}^{\rho} h_{n,m}(t)\right)_{|\,t=T}.$$

Hereinafter, we use the following properties of the Mittag–Leffler function:

1) The function $E_{\alpha,\beta}(-t)$ with $\alpha \in (0;\, 1]$, $\beta \geq \alpha$ is completely monotonic for $t > 0$, i.e.

$$(-1)^n \left[E_{\alpha,\beta}(-t)\right]^{(n)} \geq 0, \quad n = 0, 1, 2, ...$$

2) For all $\alpha \in (0;\, 2)$, $\beta \in \mathbb{R}$ and $\arg z = \pi$ there takes place the following estimate

$$|\,E_{\alpha,\gamma}(z)\,| \leq \frac{M_1}{1 + |z|},$$

where $0 < M_1 = \text{const}$ does not depend on z.

Then, from here follows that there exists numbers $M_2,\, M_3 > 0$ such that $0 < M_2 \leq \sigma_{0\,n,m} \leq M_3$.

Further, substituting the defined coefficients (13) into representation (11), we derived that

$$u_{n,m}(t) = \varphi_{n,m} A_{n,m}(t) + b_{n,m} B_{n,m}(t), \tag{14}$$

where

$$A_{n,m}(t) = \frac{1}{\sigma_{0\,n,m}} t^{\gamma-1} E_{\alpha,\gamma}\left(-\lambda_{n,m}^{2k}(\varepsilon)\,\omega\, t^{\alpha}\right), \; B_{n,m}(t) = h_{n,m}(t) - \frac{\sigma_{1\,n,m}}{\sigma_{0\,n,m}} t^{\gamma-1} E_{\alpha,\gamma}\left(-\lambda_{n,m}^{2k}(\varepsilon)\,\omega\, t^{\alpha}\right).$$

Substituting the representation of Fourier coefficients (14) of main unknown function into Fourier series (6), we obtain

$$U(t,\, x,\, y) = \sum_{n,\, m=1}^{\infty} \vartheta_{n,m}(x,\, y)\left[\varphi_{n,m} A_{n,m}(t) + b_{n,m} B_{n,m}(t)\right]. \tag{15}$$

Fourier series (15) is a formal solution of the direct problem (1)–(4).

2 Determination of source function

Using additional condition (5) and taking into account (12), we obtain from Fourier series (15) following countable system for Fourier coefficients of the source function

$$\varphi_{n,m} A_{n,m}(t_1) + b_{n,m} B_{n,m}(t_1) = \psi_{n,m}, \tag{16}$$

where

$$\psi_{n,m} = \int\limits_0^l \int\limits_0^l \psi(x, y)\, \vartheta_{n,m}(x, y)\, dx\, dy. \tag{17}$$

From relation (16) we find the source function as

$$\varphi_{n,m} = \psi_{n,m} \chi_{1\,n,m} + b_{n,m} \chi_{2\,n,m}, \tag{18}$$

where

$$\chi_{1\,n,m} = \frac{1}{A_{n,m}(t_1)}, \quad \chi_{2\,n,m} = -\frac{B_{n,m}(t_1)}{A_{n,m}(t_1)}.$$

$$A_{n,m}(t_1) = \frac{1}{\sigma_{0\,n,m}} t_1^{\gamma-1} E_{\alpha,\gamma}\left(-\lambda_{n,m}^{2k}(\varepsilon)\,\omega\, t_1^{\alpha}\right) \neq 0, \quad 0 < t_1 < T.$$

Since $\varphi_{n,m}$ are Fourier coefficients (see (12)), we substitute representation (18) into the Fourier series

$$\varphi(x, y) = \sum_{n,m=1}^{\infty} \vartheta_{n,m}(x, y) \left[\psi_{n,m}\, \chi_{1\,n,m} + b_{n,m}\, \chi_{2\,n,m}\right]. \tag{19}$$

We prove absolutely and uniformly convergence of Fourier series (19) for the source function. We need to use the concepts of the following Banach spaces:
the Hilbert coordinate space ℓ_2 of number sequences $\{\varphi_{n,m}\}_{n,m=1}^{\infty}$ with norm

$$\|\varphi\|_{\ell_2} = \sqrt{\sum_{n,m=1}^{\infty} |\varphi_{n,m}|^2} < \infty;$$

the space $L_2(\Omega_l^2)$ of square-summable functions on the domain $\Omega_l^2 = \Omega_l \times \Omega_l$ with norm

$$\|\vartheta(x, y)\|_{L_2(\Omega_l^2)} = \sqrt{\int\limits_0^l \int\limits_0^l |\vartheta(x, y)|^2\, dx\, dy} < \infty.$$

Conditions of smoothness. Let for functions

$$\psi(x, y),\ b(x, y) \in C^{4k}(\Omega_l^2)$$

there exist piecewise continuous $4k+1$ order derivatives. Then by integrating in parts the functions (9) and (17) $4k+1$ times over every variable x, y, we obtain the following relations

$$|\psi_{n,m}| = \left(\frac{l}{\pi}\right)^{8k+2} \frac{\left|\psi_{n,m}^{(8k+2)}\right|}{n^{4k+1} m^{4k+1}}, \quad |b_{n,m}| = \left(\frac{l}{\pi}\right)^{8k+2} \frac{\left|b_{n,m}^{(8k+2)}\right|}{n^{4k+1} m^{4k+1}}, \tag{20}$$

$$\left\|\psi_{n,m}^{(8k+2)}\right\|_{\ell_2} \leq \frac{2}{l} \left\|\frac{\partial^{8k+2}\psi(x, y)}{\partial x^{4k+1}\, \partial y^{4k+1}}\right\|_{L_2(\Omega_l^2)}, \tag{21}$$

$$\left\|b_{n,m}^{(8k+2)}\right\|_{\ell_2} \leq \frac{2}{l} \left\|\frac{\partial^{8k+2} b(x, y)}{\partial x^{4k+1} \partial y^{4k+1}}\right\|_{L_2(\Omega_l^2)}, \tag{22}$$

where

$$\psi_{n,m}^{(8k+2)}=\int\limits_0^l\int\limits_0^l \frac{\partial^{8k+2}\psi(x,\,y)}{\partial\, x^{4k+1}\,\partial\, y^{4k+1}}\,\vartheta_{n,m}(x,\,y)\,d\,x\,d\,y,$$

$$b_{n,m}^{(8k+2)}=\int\limits_0^l\int\limits_0^l \frac{\partial^{8k+2}b\,(x,\,y)}{\partial\, x^{4k+1}\,\partial\, y^{4k+1}}\,\vartheta_{n,m}(x,\,y)\,d\,x\,d\,y.$$

In obtaining estimates for the solution, we have used these formulas (20)–(22) and the above indicated properties of the Mittag–Leffler function. Then it is easy to see that

$$\sigma_2=\max_{n,\,m}\,\{|\,\chi_{1\,n,\,m}\,|\,;\;|\,\chi_{2\,n,\,m}\,|\}<\infty, \tag{23}$$

where

$$\chi_{1\,n,\,m}=\frac{1}{A_{\,n,\,m}(t_1)},\quad \chi_{2\,n,\,m}=-\frac{B_{\,n,\,m}(t_1)}{A_{\,n,\,m}(t_1)},\quad 0<t_1<T,$$

$$A_{\,n,\,m}(t)=\frac{1}{\sigma_{0\,n,\,m}}t^{\gamma-1}E_{\alpha,\,\gamma}\left(-\lambda_{n,\,m}^{2k}(\varepsilon)\,\omega\,t^{\alpha}\right),\; B_{n,\,m}(t)=h_{n,\,m}(t)-\frac{\sigma_{1\,n,\,m}}{\sigma_{0\,n,\,m}}\,t^{\gamma-1}E_{\alpha,\,\gamma}\left(-\lambda_{n,\,m}^{2k}(\varepsilon)\,\omega\,t^{\alpha}\right).$$

Theorem 1. Suppose that the conditions of smoothness and (23) are fulfilled. Then Fourier series (19) convergence absolutely and uniformly in the domain Ω_l^2.

Proof. We use formulas (20)–(22) and estimate (23). Using the Cauchy–Schwartz inequality for series (19), we obtain the estimate

$$\begin{aligned}
|\,\varphi\,(x,\,y)\,|&\le \sum_{n,\,m=1}^{\infty}|\,\vartheta_{\,n,\,m}(x,\,y)\,|\cdot|\,\psi_{n,\,m}\chi_{\,1\,n,\,m}+b_{\,n,\,m}\chi_{\,2\,n,\,m}\,|\le\\
&\le\frac{2}{l}\sigma_{\,2}\left[\sum_{n,\,m=1}^{\infty}|\,\psi_{n,\,m}\,|+\sum_{n,\,m=1}^{\infty}|\,b_{n,\,m}\,|\right]\le\\
&\le\frac{2}{l}\left(\frac{l}{\pi}\right)^{8k+2}\sigma_{\,2}\left[\sum_{n,\,m=1}^{\infty}\frac{\left|\psi_{n,\,m}^{(8k+2)}\right|}{n^{\,4k+1}m^{\,4k+1}}+\sum_{n,\,m=1}^{\infty}\frac{\left|b_{n,\,m}^{(8k+2)}\right|}{n^{\,4k+1}m^{\,4k+1}}\right]\le\\
&\le\frac{2}{l}\left(\frac{l}{\pi}\right)^{8k+2}\sigma_{\,2}C_{\,01}\left[\left\|\psi_{n,\,m}^{(8k+2)}\right\|_{\ell_{\,2}}+\left\|b_{n,\,m}^{(8k+2)}\right\|_{\ell_{\,2}}\right]\le\\
&\le\gamma_1\left[\left\|\frac{\partial^{8k+2}\psi\,(x,\,y)}{\partial\,x^{4k+1}\partial\,y^{4k+1}}\right\|_{L_{\,2}(\Omega_l^2)}+\left\|\frac{\partial^{8k+2}b\,(x,\,y)}{\partial\,x^{4k+1}\partial\,y^{4k+1}}\right\|_{L_{\,2}(\Omega_l^2)}\right]<\infty,
\end{aligned} \tag{24}$$

where

$$\gamma_1=\sigma_{\,2}C_{\,0\,1}\left(\frac{2}{l}\right)^2\left(\frac{l}{\pi}\right)^{8k+2},\quad C_{\,0\,1}=\sqrt{\sum_{n,\,m=1}^{\infty}\frac{1}{n^{\,8k+2}\,m^{\,8k+2}}}<\infty.$$

From estimate (24) the absolutely and uniformly convergence of Fourier series (19) implies. The Theorem 1 is proved.

3 Determination of main unknown function

We determined the source function as a Fourier series (19). So, the source function is known. Using representation (16), Fourier series (15), we can present the main unknown function as

$$U\,(t,\,x,\,y)=\sum_{n,\,m=1}^{\infty}\vartheta_{\,n,\,m}(x,\,y)\,[\psi_{\,n,\,m}P_{\,n,\,m}(t)+b_{\,n,\,m}Q_{\,n,\,m}(t)]\,, \tag{25}$$

where

$$P_{\,n,\,m}(t)=\chi_{\,1\,n,\,m}A_{\,n,\,m}(t),\quad Q_{\,n,\,m}(t)=\chi_{\,2\,n,\,m}A_{\,n,\,m}(t)+B_{\,n,\,m}(t).$$

To establish the uniqueness of the function $U(t, x, y)$ we suppose that there are two functions U_1 and U_2 satisfying given conditions (1)–(5). Then their difference $U = U_1 - U_2$ is a solution of differential equation (1), satisfying conditions (2)–(5) with the function $\psi(x, y) \equiv 0$. By virtue of relations (9) and (16), we have $\psi_{n,m} = 0$. Hence, from formulas (7) and (25) in the domain Ω we obtain the zero identity

$$\int_0^l \int_0^l t^{1-\gamma} U(t, x, y)\, \vartheta_{n,m}(x, y)\, dx\, dy \equiv 0.$$

By virtue of the completeness of the systems of eigenfunctions $\left\{\sqrt{\frac{2}{l}} \sin \frac{\pi n}{l} x\right\}$, $\left\{\sqrt{\frac{2}{l}} \sin \frac{\pi m}{l} y\right\}$ in $L_2\left(\Omega_l^2\right)$ we deduce that $U(t, x, y) \equiv 0$ for all $x \in \Omega_l^2 \equiv [0; l]^2$ and $t \in \Omega_T \equiv [0; T]$.

Since $t^{1-\gamma} U(t, x, y) \in C\left(\overline{\Omega}\right)$ then $t^{1-\gamma} U(t, x, y) \equiv 0$ in the domain $\overline{\Omega}$. Therefore, the solution to problem (1)–(5) is unique in the domain $\overline{\Omega}$.

Theorem 2. Let the conditions of the Theorem 1 be fulfilled. Then the series (25) converges in the domain Ω. At the same time, in this domain their term-by-term differentiation is possible.

Proof. By virtue of conditions of the theorem 1 and properties of Mittag–Leffler function, as in the case of (23), the functions $t^{1-\gamma} P_{n,m}(t)$, $t^{1-\gamma} Q_{n,m}(t)$ are uniformly bounded on the segment $[0; T]$. So, for any positive integers n, m there exists a finite constant σ_3, that there takes place the following estimate

$$\max_{n,m} \left\{ \max_{0 \le t \le T} \left| t^{1-\gamma} P_{n,m}(t) \right|; \max_{0 \le t \le T} \left| t^{1-\gamma} Q_{n,m}(t) \right| \right\} \le \sigma_3. \tag{26}$$

Using estimates (20)–(22) and (26), analogously to estimate (24), for series (25) we obtain

$$\left| t^{1-\gamma} U(t, x, y) \right| \le \sum_{n,m=1}^{\infty} \left| \vartheta_{n,m}(x, y) \right| \cdot \left| \psi_{n,m} t^{1-\gamma} P_{n,m}(t) + b_{n,m} t^{1-\gamma} Q_{n,m}(t) \right| \le$$

$$\le \gamma_2 \left[\left\| \frac{\partial^{8k+2} \psi(x, y)}{\partial x^{4k+1} \partial y^{4k+1}} \right\|_{L_2\left(\Omega_l^2\right)} + \left\| \frac{\partial^{8k+2} b(x, y)}{\partial x^{4k+1} \partial y^{4k+1}} \right\|_{L_2\left(\Omega_l^2\right)} \right] < \infty, \tag{27}$$

where $\gamma_2 = C_{01} \sigma_3 \left(\frac{2}{l}\right)^2 \left(\frac{l}{\pi}\right)^{8k+2}$.

From estimate (27) the absolutely and uniformly convergence of Fourier series (25) implies. We differentiate the required number of times function (25)

$$\frac{\partial^{4k}}{\partial x^{4k}} t^{1-\gamma} U(t, x, y) = \sum_{n,m=1}^{\infty} \left(\frac{\pi n}{l}\right)^{4k} \vartheta_{n,m}(x, y) \left[\psi_{n,m} t^{1-\gamma} P_{n,m}(t) + b_{n,m} t^{1-\gamma} Q_{n,m}(t)\right], \tag{28}$$

$$\frac{\partial^{4k}}{\partial y^{4k}} t^{1-\gamma} U(t, x, y) = \sum_{n,m=1}^{\infty} \left(\frac{\pi m}{l}\right)^{4k} \vartheta_{n,m}(x, y) \left[\psi_{n,m} t^{1-\gamma} P_{n,m}(t) + b_{n,m} t^{1-\gamma} Q_{n,m}(t)\right]. \tag{29}$$

The expansions of the following functions in a Fourier series are defined in a similar way

$$t^{1-\gamma} D^{\alpha,\gamma} U(t, x, y), \quad \frac{\partial^{4k}}{\partial x^{4k}} t^{1-\gamma} D^{\alpha,\gamma} U(t, x, y), \quad \frac{\partial^{4k}}{\partial y^{4k}} t^{1-\gamma} D^{\alpha,\gamma} U(t, x, y).$$

We show the convergence of series (28) and (29). As in the case of estimate (27), applying the Cauchy–Schwarz inequality, we obtain:

$$\left| \frac{\partial^{4k}}{\partial x^{4k}} t^{1-\gamma} U(t, x, y) \right| \le \sum_{n,m=1}^{\infty} \left(\frac{\pi n}{l}\right)^{4k} \left| t^{1-\gamma} u_{n,m}(t) \right| \cdot \left| \vartheta_{n,m}(x, y) \right| \le$$

$$\le \frac{2}{l} \left(\frac{\pi}{l}\right)^{4k} \sigma_3 \left[\sum_{n,m=1}^{\infty} n^{4k} \left| \psi_{n,m} \right| + \sum_{n,m=1}^{\infty} n^{4k} \left| b_{n,m} \right| \right] \le$$

$$\leq \frac{2}{l}\left(\frac{l}{\pi}\right)^{4k+2}\sigma_3\left[\sum_{n,\,m=1}^{\infty}\frac{\left|\psi_{n,\,m}^{(8k+2)}\right|}{n\,m^{4k+1}}+\sum_{n,\,m=1}^{\infty}\frac{\left|b_{n,\,m}^{(8k+2)}\right|}{n\,m^{4k+1}}\right]\leq$$

$$\leq \frac{2}{l}\left(\frac{l}{\pi}\right)^{4k+2}\sigma_3 C_{02}\left[\left\|\psi_{n,\,m}^{(8k+2)}\right\|_{\ell_2}+\left\|b_{n,\,m}^{(8k+2)}\right\|_{\ell_2}\right]\leq$$

$$\leq \gamma_3\left[\left\|\frac{\partial^{8k+2}\psi(x,\,y)}{\partial x^{4k+1}\partial y^{4k+1}}\right\|_{L_2(\Omega_l^2)}+\left\|\frac{\partial^{8k+2}b\,(x,\,y)}{\partial x^{4k+1}\partial y^{4k+1}}\right\|_{L_2(\Omega_l^2)}\right]<\infty, \tag{30}$$

where $\gamma_3=\left(\frac{2}{l}\right)^2\left(\frac{l}{\pi}\right)^{4k+2}\sigma_3 C_{02}$, $C_{0\,2}=\sqrt{\sum\limits_{n,\,m=1}^{\infty}\frac{1}{n\,m^{8k+2}}}<\infty$;

$$\left|\frac{\partial^{4k}}{\partial y^{4k}}t^{1-\gamma}U\,(t,\,x,\,y)\right|\leq\sum_{n,\,m=1}^{\infty}\left(\frac{\pi\,m}{l}\right)^{4k}\left|\,t^{1-\gamma}u_{n,\,m}(t)\right|\cdot\left|\,\vartheta_{n,\,m}(x,\,y)\,\right|\leq$$

$$\leq \frac{2}{l}\left(\frac{\pi}{l}\right)^{4k}\sigma_3\left[\sum_{n,\,m=1}^{\infty}m^{4k}\,|\,\psi_{n,\,m}|+\sum_{n,\,m=1}^{\infty}m^{4k}\,|\,b_{n,\,m}\,|\right]\leq$$

$$\leq \frac{2}{l}\left(\frac{l}{\pi}\right)^{4k+2}\sigma_3\left[\sum_{n,\,m=1}^{\infty}\frac{\left|\psi_{n,\,m}^{(8k+2)}\right|}{n^{4k+1}m}+\sum_{n,\,m=1}^{\infty}\frac{\left|b_{n,\,m}^{(8k+2)}\right|}{n^{4k+1}m}\right]\leq$$

$$\leq \frac{2}{l}\left(\frac{l}{\pi}\right)^{4k+2}\sigma_3 C_{03}\left[\left\|\psi_{n,\,m}^{(8k+2)}\right\|_{\ell_2}+\left\|b_{n,\,m}^{(8k+2)}\right\|_{\ell_2}\right]\leq$$

$$\leq \gamma_4\left[\left\|\frac{\partial^{8k+2}\psi(x,\,y)}{\partial x^{4k+1}\partial y^{4k+1}}\right\|_{L_2(\Omega_l^2)}+\left\|\frac{\partial^{8k+2}b\,(x,\,y)}{\partial x^{4k+1}\partial y^{4k+1}}\right\|_{L_2(\Omega_l^2)}\right]<\infty, \tag{31}$$

where

$$\gamma_4=\left(\frac{2}{l}\right)^2\left(\frac{l}{\pi}\right)^{4k+2}\sigma_3 C_{03},\ \ C_{0\,3}=\sqrt{\sum_{n,\,m=1}^{\infty}\frac{1}{n^{8k+2}m}}<\infty.$$

It is easy to prove the convergence of Fourier series for functions

$$t^{1-\gamma}D^{\alpha,\,\gamma}U\,(t,\,x,\,y),\ \ \frac{\partial^{4k}}{\partial x^{4k}}t^{1-\gamma}D^{\alpha,\,\gamma}U\,(t,\,x,\,y),\ \ \frac{\partial^{4k}}{\partial y^{4k}}t^{1-\gamma}D^{\alpha,\,\gamma}U\,(t,\,x,\,y),$$

since the necessary estimates can be obtained by a similar way as for the cases of estimates (29), (30) and (31). Therefore, the function $U\,(t,\,x,\,y)$ belongs to the class of functions (4). Theorem 2 is proved.

4 Stability of the solution $U(t,x,y)$ with respect to the given functions and the source function

Theorem 3. Suppose that all the conditions of Theorem 2 are fulfilled. Then, the function $U\,(t,\,x,\,y)$ as a solution to problem (1)–(5) is stable with respect to a given function $\psi(x,\,y)$.

Proof. We show that the solution $U\,(t,\,x,\,y)$ of differential equation (1) is stable with respect to a given function $\psi(x,\,y)$. Let $U_1(t,\,x,\,y)$ and $U_2(t,\,x,\,y)$ be two different solutions of the inverse boundary value problem (1)–(5), corresponding to two different values of the function $\psi_1(x,\,y)$ and $\psi_2(x,\,y)$, respectively.

We put that $|\,\psi_{1\,n,\,m}-\psi_{2\,n,\,m}\,|<\delta_{n,\,m}$, where $0<\delta_{n,\,m}$ is a sufficiently small positive quantity and the series $\sum\limits_{n,\,m=1}^{\infty}|\,\delta_{n,\,m}\,|$ is convergent. Then, considering this fact by virtue of the conditions of the theorem, from Fourier series (25), it is easy to obtain that

$$\left\|\,t^{1-\gamma}\left[U_1(t,\,x,\,y)-U_2(t,\,x,\,y)\right]\right\|_{C\,(\overline{\Omega})}\leq\frac{2}{l}\sigma_3\sum_{n,\,m=1}^{\infty}|\,\psi_{1\,n,\,m}-\psi_{2\,n,\,m}\,|<\frac{2}{l}\sigma_3\sum_{n,\,m=1}^{\infty}|\,\delta_{n,\,m}\,|<\infty.$$

We put $\varepsilon = \frac{2}{l}\sigma_3 \sum\limits_{n,\, m=1}^{\infty} |\, \delta_{n,\, m} \,| < \infty$. Then, from last estimate we finally obtain assertions about the stability of the solution of the differential equation (1) with respect to a given function $\psi(x,\, y)$ in (5). The Theorem 3 is proved.

By a similar way we have have proved that there hold the following two theorems.

Theorem 4. Suppose that all conditions of Theorem 2 are fulfilled. Then, the function $U\,(t,\, x,\, y)$ as a solution to problem (1)–(5) is stable with respect to the given function $b\,(x,\, y)$ in the right-hand side of equation (1).

Theorem 5. Suppose that all conditions of Theorem 2 are fulfilled. Then, the function $U\,(t,\, x,\, y)$ as a solution to problem (1)–(5) is stable with respect to the source function $\varphi\,(x,\, y)$.

Remark. It is easy to study the stability of function $U\,(t,\, x,\, y)$ with respect to asmall parameter ε (see [43]).

Conclusions

In three-dimensional domain, an inverse problem of identification of a source function for Hilfer type partial differential equation (1) of the higher even order with integral form condition (2) and a small positive parameter in mixed derivative is considered. Suppose that the conditions of smoothness are fulfilled. Then the solution to this fractional differential equation of the higher order for $0 < \alpha < \gamma \leq 1$ is studied in the class of regular functions. The Fourier series method have been used and a countable system of ordinary differential equations has been obtained (10). The nonlocal inverse boundary value problem is integrated as an ordinary differential equation. By the aid of given additional condition, we obtained the representation for the source function. Using the Cauchy–Schwarz inequality and the Bessel inequality, we proved the absolute and uniform convergence of the obtained Fourier series (19) for the source function $\varphi\,(x,\, y)$ and (25) for the unknown function $U\,(t,\, x,\, y)$ and its derivatives. It is proved that solution of problem (1)–(5) $U\,(t,\, x,\, y)$ is stable with respect to the given functions $\psi(x,\, y),\ b\,(x,\, y)$ and the source function $\varphi\,(x,\, y)$.

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Жоғары ретті Хильфер типінің жартылай туындылы дифференциалдық теңдеудің кері есебі

Үшөлшемді облыста интегралдық формадағы және аралас туындылы кіші оң параметрі бар жұп ретті Хильфер типінің жартылай туындылы теңдеу үшін функция көзін анықтау есебі қарастырылған. Бұл жоғары ретті бөлшекті дифференциалдық теңдеудің шешімі тұрақты функциялар класында зерттелген. Бөлшекті оператордың реті $0 < \alpha < 1$ болатын жағдай қарастырылды. Фурье қатарлары әдісі қолданылды және қарапайым дифференциалдық теңдеулердің есептеу жүйесі алынды. Локалды емес шеттік есебі қарапайым дифференциалдық теңдеу ретінде интегралданады. Қосымша шарт арқылы қайта анықтау функциясы туралы түсінік берілген. Коши-Шварц теңсіздігі мен Бессель теңсіздігін қолдана отырып, алынған Фурье қатарларының абсолютті және бірқалыпты жинақтылығы дәлелденді.

Кілт сөздер: бөлшекті рет, Хилфер операторы, функция көзі туралы кері есеп, Фурье қатарлары, интегралдық шарт, бірмәнді шешілуі.

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Обратная задача для дифференциального уравнения в частных производных типа Хильфера высшего порядка

В трехмерной области рассмотрена задача идентификации функции источника для уравнения в частных производных типа Хильфера четного порядка с условием в интегральной форме и малым положительным параметром в смешанной производной. Решение этого дробного дифференциального уравнения высшего порядка получено в классе регулярных функций. Авторами изучен случай для порядка дробного оператора $0 < \alpha < 1$. Применен метод рядов Фурье, и получена счетная система обыкновенных дифференциальных уравнений. Нелокальная краевая задача интегрирована как обыкновенное дифференциальное уравнение. С помощью дополнительного условия получено представление для функции переопределения. С помощью неравенств Коши–Шварца и Бесселя доказана абсолютная и равномерная сходимость полученных рядов Фурье.

Ключевые слова: дробный порядок, оператор Хильфера, обратная задача об источнике, ряды Фурье, интегральное условие, однозначная разрешимость.

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