DOI 10.31489/2022M1/127-135

UDC 510.67

A.R. Yeshkeyev, I.O. Tungushbayeva*, M.T. Kassymetova

Karagandy University of the name of academician E.A. Buketov, Karaganda, Kazakhstan
(E-mail: aibat.kz@gmail.com, intng@mail.ru, mairushaasd@mail.ru)

Connection between the amalgam and joint embedding properties

The paper aims to study the model-theoretic properties of differentially closed fields of zero and positive characteristics in framework of study of Jonsson theories. The main attention is paid to the amalgam and joint embedding properties of DCF theory as specific features of Jonsson theories, namely, the implication of JEP from AP. The necessity is identified and justified by importance of information about the mentioned properties for certain theories to obtain their detailed model-theoretic description. At the same time, the current apparatus for studying incomplete theories (Jonsson theories are generally incomplete) is not sufficiently developed. The following results have been obtained: The subclasses of Jonsson theories are determined from the point of view of joint embedding and amalgam properties. Within the exploration of one of these classes, namely the AP-theories, that the theories of differential and differentially closed fields of characteristic 0, differentially perfect and differentially closed fields of fixed positive characteristic are shown to be Jonsson and perfect. Along with this, the theory of differential fields of positive characteristic is considered as an example of an AP-theory that is not Jonsson, but has the model companion, which is perfect Jonsson theory, and the sufficient condition for the theory of differential fields is formulated in the context of being Jonsson.

Keywords: Jonsson theory, perfect Jonsson theory, differential field, differential closed field, differentially perfect field, amalgam property, joint embedding property, AP-theory, JEP-theory, strongly convex theory.

In Model Theory, when studying various examples of theories, information about the amalgam and joint embedding properties for considered theories is useful. The amalgam property and the joint embedding property are independent of each other. There are many examples of this fact. In particular, one can find some of them in [1; 270].

In this article, we examine the case when these two cases are dependent on each other. We call a theory AP-theory if the joint embedding property for this theory is a consequence of the amalgam property of this theory, i.e. when JEP follows from AP. At the same time, the amalgam property and the joint embedding property are necessary attributes of a class of Jonsson theories.

We consider a classic example of differentially closed fields of zero and positive characteristic within the study of AP-Jonsson theories.

As for differential algebra, the first works where differential algebra was separated into an independent branch of mathematics are the books of Ritt [2–3]. There are formulated many significant problems, many of them have not yet been solved. At the Moscow Congress of Mathematicians in 1966, Kolchin presented a report where the author formulated open problems that have determined the direction of differential algebra in recent years. The monograph [4] details the state of most of these sections. As Kaplansky wrote in his monograph [5], “differential algebra consists mostly of the works of Kolchin and Ritt”.

We begin by presenting the basic facts about differential rings, whose special case is namely differential fields, that will help us to reveal the algebraic essence of the theories and classes of their models considered in this article.

The differentiation of the ring R is a map

$$D : R \to R, \tag{1}$$

that satisfies the following conditions:

1) the mapping D is additive;

2) for any two elements x, y of the ring R, $D(xy) = xDy + yDx$ is executed.

*Corresponding author.
E-mail: intng@mail.ru

The element $D(x)$ will be called the derivative of the element a, whereas x itself is called the integral element $D(x)$. For derivatives $D^2(x), D^3(x), ..., D^n(x)$, the Leibniz rule can be written as

$$D^n(xy) = D^n(x)y + ... + C_n^i D^{n-i}(x)D^i(y) + ... + xD^n(y).$$

If the commutation property is observed for the element x and the derivative $D(x)$, we have $D^n(x) = nx^{n-1}D(x)$. In the case when the ring R has the unit and an inverse element x^{-1} for x,

$$D(x^{-1}) = -x^{-1}D(x)x^{-1}$$

holds. Moreover, $D(1) = 0$.

The following theorem is known:

Theorem 1 [5; 7]. For any differentiation in an arbitrary domain of integrity, there is a single extension to the corresponding field of relations.

Let a commutative ring with the unit be given and the differentiation D be introduced on it. Such a ring is called a differential ring. Here are some examples that reflect the essence of the differential ring.

1) Any commutative ring with the unit can be represented as differential by considering zero differentiation on it ($\forall x \in R \;\; D(x) = 0$). We can conclude by this that the rings theory is a special case of the differential rings theory. It is worth mentioning that on the ring of integers and the field of rational numbers, it is impossible to introduce any differentiation other than zero.

2) The usual differentiation on the ring of infinitely differentiable functions on the real axis is also an example of the map (1). Moreover, infinitely differentiable functions form a ring closed with respect to differentiation.

3) On the ring of integer functions, it is also possible to introduce differentiation in the usual sense. There are no zero divisors available in the ring of infinitely differentiable functions, which makes it possible to form a field of relations.

4) If R is a differential ring, then there exists a ring of $R[x]$ polynomials formed with coefficients of R in variable x. If R is a field, then $R(x)$ denotes the field of rational functions of x. Using Theorem 1, we can continue differentiating the ring (field) A into the ring of polynomials $R[x]$ and the field $R(x)$. At the same time, we assume $D(x^n) = nx^{n-1}D(x)$ and then continue this mapping linearly.

5) If R is a differential ring, then in under $R[x_i]$ we mean the ring of polynomials in infinite number of variables $x_0, x_1, ...$, and each subsequent element x_{i+1} is a derivative of the previous x_i. Thus, some differentiation in the ring $R\{x_i\}$ is uniquely determined. Let us replace the designations with more suitable ones

$$x_0 = x, \;\; x_n = D^n(x).$$

The described process is called the adjunction of a differential indeterminate and gives us, as a result, a differential ring, the elements of which we call differential polynomials. These are ordinary polynomials from x and its derivatives.

In the case when R is a field, then the ring $R\{x\}$ is a differential domain of integrity, and Theorem 1 gives us the opportunity in the only way to continue differentiating into the corresponding field of relations $R\langle x\rangle$, whose elements are called differential rational functions of x.

In any differential ring R, the elements whose derivative is zero form a subring C called the ring of constants. Moreover, if R is a field, then C, respectively, is also a field. In addition, the constant field C contains within itself a subfield generated by the unit element R.

The characteristic of differential rings is of considerable importance. As the structure of the ring becomes more complex and gradually turns into a field, the characteristic plays an increasingly significant role. Differential fields with zero characteristic are well-studied, while the case with positive characteristic remains more sophisticated. One of them is described below.

Next, we consider the fields of the characteristic $p = 0$ and $p > 0$. We present important information about differential fields of characteristic 0 and consider some of their model-theoretic properties.

D. Marker [6] described differential and differentially closed fields as follows.

Definition 1 [6]. A differential field is a field K with the given differentiation operator $D : K \to K$, such that

$$\forall x \forall y D(x + y) = D(x) + D(y), \tag{2}$$

$$\forall x \forall y D(x, y) = xDy + yDx,$$

where $x, y \in K$. The language used to study differential fields is the language $L = \{+, -, \cdot, D, 0, 1\}$. Here the differentiation operator D plays the role of a single functional symbol.

Thus, the theory DF_0 of differential fields of characteristic 0 is given by the axioms of field theory and axioms (2).

As mentioned before, each differential field has a so-called subfield of constants C consisting of all elements x of the field for which $D(x) = 0$.

Let K be a differential field. Then, over K, the ring $K\{X_1, ..., X_n\}$ of differential polynomials can be defined as the ring of polynomials in infinite number of variables as follows:

$$K[X_1, ..., X_n, D(X_1), ..., D(X_n), ..., D^m(X_1), ..., D^m(X_n), ...],$$

where $D(D^n(X_i)) = D^{n+1}(X_i)$.

If $f \int \{X_1, ..., X_n\} \backslash K$, the order of f is the largest m such that $D^m(X^i)$ occurs in f for some i. If f is a constant, we say that f has order -1.

Definition 2 [6]. A differential field K is called differentially closed if whenever $f, g \in K\{X\}$, g has a nonzero value and the order of f is greater than the order of g, there exists $a \in K$ such that $f(a) = 0$ and $g(a) \neq 0$.

In 1959, A.Robinson [7] showed that the theory of differential fields has a model completion. Robinson also introduced the concept of a differentially closed field. However, as noted by Mikhalev A.V. and Pankratiev E.V. in their review [8], in Robinson's works the theory DCF_0 received a specific description in the sense of axiomatization, which was corrected by L. Blum. In her PhD thesis [9], she formulates two missing axioms (in addition to the DF_0 axioms) for the theory of differentially closed fields of characteristic zero as follows:
1) Each nonconstant polynomial in one variable has a solution.
2) If $f(x)$ and $g(x)$ are differential equations, such that the order of $f(x)$ is higher in the order of $g(x)$, then $f(x)$ has a solution, not a solution of $g(x)$. B. Poizat, in his work [10], proved that DCF_0 is complete and is the model completion of the DF_0.

To study the model-theoretic properties of the theories DF_0 and DCF_0, we need the following definitions:

Definition 3 [11; 99]. The theory of T has the joint embedding property(JEP), if for any models U, B of the theory T there exists a model M of the theory T and isomorphic embeddings $f : U \to M$, $g : B \to M$.

Definition 4 [11; 99]. The theory of T has the amalgam property (AP), if for any models U, B_1, B_2 of the theory T and isomorphic embeddings $f_1 : U \to B_1$, $f_2 : U \to B_2$ there are $M \models T$ and isomorphic embeddings $g_z : B_1 \to M$, $B_2 \to M$, such that $g_1 \circ f_1 = g_2 \circ f_2$.

We will consider the theories of DF_0 and DCF_0 from the point of view of Jonssonness.

To begin with, let us recall the definitions of the Jonsson theory and some related concepts.

Definition 5 [11; 144]. The theory of T is called a Jonsson theory if:
1. The theory T has infinite models;
2. T is an inductive theory;
3. The theory T has the amalgam property (AP).
4. The theory T has the joint embedding property (JEP).

Examples of Jonsson theories are:
1) group theory;
2) abelian groups theory;
3) boolean algebras theory;
4) linear order theory;
5) the theory of fields of characteristic p, where p is zero or a prime number;
6) ordered fields theory;
7) modules theory.

One can found the proofs in [12-13].

Definition 6. [14] It is said that C_T is a semantic model of the Jonsson theory of T if C_T is a ω^+-homogeneous ω^+-universal model of the theory T.

Theorem 2 [11; 152]. The theory T is Jonsson if and only if it has the semantic model C_T. Many facts concerning semantic models and related concepts of cosemanticity and similarity of the Jonsson theories are described in [15].

Definition 7 [16]. A Jonsson theory T is called perfect if its semantic model C_T is saturated.

Definition 8 [16]. An elementary theory of the semantic model of Jonsson theory T is called to be the center of this theory. Denoted through T^*, i.e. $Th(C) = T^*$.

Theorem 3 [11; 155]. Let T be an arbitrary Jonsson theory. Then the following conditions are equivalent:

1) The theory T is perfect;

2) $T^* = Th(C)$ is a model companion of the theory T. More information about the concept of Jonsson perfection can be found in [17].

We define the following subclasses of Jonsson theories. Focusing on AP and JEP properties for certain theories, we distinguish the following four types of theories:

Definition 9. A theory T is called to be

1) AP-theory if in theory T amalgam property entails joint embedding property;

2) JEP-theories if in theory T joint embedding property entails amalgam property;

3) AJ-theories if in theory T both properties are equivalent.

Otherwise, we say that for the theory of T, the properties of AP and JEP are independent of each other.

The described types form corresponding subclasses in the class of Jonsson theories, on which our interest is focused. However, there are theories relating to some of the types 1–3, which are not Jonsson. An example of such a theory will be discussed later in this paper.

We need the following definition.

Definition 10 [18]. A theory T is called convex if for any model of A and any family $\{A_i | i \in I\}$ submodels A which are models of the theory T the intersection $\bigcap\limits_{i \in I}$ is also a submodel of T, if it is nonempty. If the intersection is never empty, T is said to be strongly convex.

It is important to note that, according to this definition, the theory of differential fields of any characteristic is strongly convex. Based on this fact the theory can be attributed to AP-theories.

As for Definitions 3 and 4, B. Jonsson [19] was engaged in the study of "amalgam properties", who cited DF_0 as examples of theories with these properties. The proof in [5] was presented by I. Kaplansky. Robinson [7] noted that this is the result of the existence of a model companion for the considered theory:

Theorem 4 [20; 157]. The theory of T admits the amalgam property if and only if it has a model completion.

Property 1. [9; 130] DCF_0 allows quantifier elimination.

Property 2. [9; 131] DF_0 has the joint embedding and the amalgam properties.

Note that originally [9] in the formulation and proof of these properties, L. Blum refers to Theorem (0.3.7), which states that if a universal theory T has a model companion, the theory T has "amalgam properties", which mean both the amalgam property and the joint embedding property in our sense.

Theorem 5 [9; 128]. The DCF_0 theory is a model completion of the DF_0 theory.

Finally, we proceed to consider the model-theoretic properties of the described fields from the point of view of Jonssonness. Let DF_0 be the theory of differential fields of characteristic 0.

Theorem 6. DF_0 is a Jonsson theory.

Proof. (1) It is easy to see that DF_0 has infinite models.

(2) Since DF_0 is a $\forall$-axiomatizable theory, it is also $\forall\exists$-theory. Hence, it is inductive.

(3),(4) As already noted in [9] Blum, DF_0 has the amalgam property (AP) and the joint embedding property (JEP) due to the presence of a model replenishment of DCF_0. Moreover, in the case under consideration, the property JEP follows from AP: Two differential fields F_1 and F_2 always have a nonempty intersection, which will also be a differential field, isomorphically embedded in both of these fields. Then, by virtue of AP, there are isomorphic embeddings of F_1 and F_2 in some differential field F. The role of F can be played, for example, by a composite of fields F_1 and F_2 – the intersection of all differential fields of characteristic 0 containing F_1 and F_2, on which differentiation is continued accordingly. Thus, the result of having JEP follows from posession of AP in the theory of differential fields of characteristic 0. This is a consequence of the fact that DF_0 is a strongly convex theory.

Theorem 7. DF_0 is a perfect Jonsson theory.

Proof. The proof follows from the fact that DF_0 has a model completion, which is DCF_0. Let us conduct it in detail. According to Theorem 5, the theory of differential fields of characteristic 0 has a model completion – the theory of differentially closed fields of characteristic 0, which is also its model companion. In addition, as Theorem 2 states, DF_0 due to its Jonssonness, there must be the semantic model C_T and, accordingly, the center $DF_0^* = Th(C_T)$. If $DF_0^* = DCF_0$, then, by virtue of Theorem 3, DF_0 will be perfect. Let us show it.

The proof will be carried out from the opposite: let us say $DF_0^* \neq DCF_0$. In this case, since DF_0^* is complete and DCF_0 is model complete, for any sentence ψ of the signature in question, either

$$\psi \in DF_0 \text{ and } \neg\psi \in DCF_0, \tag{3}$$

or

$$\psi \in DF_0 \text{ and } \psi \notin DCF_0,\ \neg\psi \notin DCF_0. \tag{4}$$

However, DF_0, DCF_0, and DF_0^* are obviously model consistent, and, at the same time, are be embedded into the semantic model of the theory of DF_0. Using this fact, we can easily get a contradiction for both cases (3) and (4), which means that $DF_0^* = DCF_0$. Therefore, DF_0 is a perfect Jonsson theory.

Theorem 8. DCF_0 is a perfect Jonsson theory.

Proof. To begin with, let us show the Jonssonness of DCF_0 theory.

(1)DCF_0 has infinite models;

(2) DCF_0 is a $\forall\exists$-theory, and therefore it is inductive;

(3) DCF_0 is a model-complete theory, which means that by the Theorem 4 has AP;

(4) From (3) it follows JEP, since the nonempty intersection of two differentially closed fields of characteristic 0 always exists and is their submodel embedded into both fields. This fact is confirmed by the well-known Robinson criterion (Theorem 8).

The perfectness of DCF_0 follows again from the fact that the theory in question is model complete, which means it represents a model companion for itself.

Here again, it is important to note that due to the strong convexity of DCF_0, we have the result obtained, namely, that DCF_0 (along with the DF_0 described above) is an AP-Jonsson theory.

Now consider the differential fields of characteristic $p > 0$. To define such a field, we, similarly, add axiom (2) to the axiomatics of the field theory of characteristic p again, thus obtaining the theory of DF_p.

For differential fields of characteristic p, the relation $F^p \subseteq C$ is fulfilled, where F^p are all elements of the field raised to the power of p, C is a subfield of constants. The relation is true because $D(a^p) = pa^{p-1}D(a)$ for any $a \in F$.

In the works [21, 22], C. Wood obtained the following results regarding differential fields of characteristic p:

Theorem 9 [21]. The theory DF_p of differential fields of characteristic p does not admit the amalgam property.

The author notes that the main reason is the absence of the p-th roots for some constant elements of the field.

The consequence of the absence of AP C. Wood also highlights the following important theorem: *Theorem 10* [21]. The DF_p theory has no model completion.

In fact, to prove it, it is enough to refer to the Theorem 9.

To obtain a theory that allows the elimination of quantifiers, which has the amalgam property and model completion, C. Wood [21, 22] modifies the theory of DF_p, supplementing it with the axiom

$$\forall x \exists y\ (D(x) = 0 \to y^p = x),$$

and obtains the so-called theory DPF of differentially perfect fields:

Definition 11. A differentially perfect field F is a differential field such that $F^p = C$.

The DPF models are the DF_p models in which the fields of constants are closed with respect to the operation of extracting p-th root. Thus, the following theorem holds:

Theorem 11 [21]. The theory of differentially perfect fields of characteristic p admits the amalgam property.

Let us now define the theory of DCF_p differentially closed fields of characteristic p: To the axioms of DF_p we will add the following definition of a differentially closed field.

Definition 12. A differential field of characteristic p is called differentially closed if for each positive integer n in the language L we can determine the sentence φ_n stating that there is a solution for $f(x) = 0, g(x) \neq 0$ for each pair of differential polynomials in one differential variable such that f and g have order and total degree at most n, and the order of f is higher than the order of g.

The most important model-theoretic properties of DCF_p are completeness and model completeness. The key place is occupied by the following statement:

Theorem 12 [21]. DCF_p is a model companion for DF_p and a model completion for DPF.

The following results demonstrate the behavior of the theories DF_p, DPF and DCF_p from the point of view of studying the Jonsson theories.

Theorem 13. DF_p is not a Jonsson theory.

Proof. According to Theorem 9, since, DF_p does not have the amalgam property, it, following the Definition 5, is not a Jonsson. In addition, DF_p does not have JEP since it is AP-theory.

This fact is noteworthy because, as mentioned earlier, the fields theory F_p in case of characteristic p is a Jonsson, whereas the introduction of the functional symbol D into the signature F_p deprives DF_p of this property. At the same time, the Jonssonness appears when the differential field of characteristic 0 is transformed into a differentially perfect:

Theorem 14. DPF is a Jonsson theory.

Proof. Again, we will carry out the proof following the definition of Jonsson theory.

(1) DPF has infinite models;

(2) DPF is $\forall\exists$-axiomatizable, which means it is inductive;

(3) DPF has model completion, hence has AP;

(4) As mentioned before, any field of constants in a differential field contains a subfield generated by a unit element. Such a field is differentially perfect by definition and can serve as a DPF model that is embedded in any two differentially perfect fields F_1 and F_2. Further, by property (3), there is a model DPF in which F_1 and F_2 are embedded.

Here again we see the manifestation of the property of being AP-theory: Possession of the amalgam property allowed DPF to also have the joint embedding property .

Moreover,

Theorem 15. DPF is a perfect Jonsson theory.

Proof. The proof is similar to the proof of Theorem 7 and follows from the fact that DPF has a model complement (and, accordingly, a model companion), which is the theory of DCF_p, as stated by Theorem 11.

Theorem 16. DCF_p is a perfect Jonsson theory.

Proof. Let us show the Jonssonness of the DCF_p theory.

(1) DCF_p has infinite models;

(2) DCF_p is $\forall\exists$-axiomatizable, hence inductive;

(3) DCF_p, by Theorem 13, the model complete and, therefore, has AP.

(4) From [21] we can find out that theory DF_p has the prime model F_p which is unique. Since every differentially closed field is differential, this means that for any two models F_1 and F_2 of DCF_p, there exists a model F that can be embedded into F_1 and F_2, and, further by (3), there is a model F' such that F_1 and F_2 are embedded into F'.

Although DF_p is not a Jonsson theory, note that, it has a Jonsson model completion DCF_p (which is perfect in Jonsson sense). At the same time, another important remark that we can make based on the results obtained is the following fact: The perfectness (in the field sence) based on the differential field is a sufficient condition for the theory of differential fields of characteristic p to be perfect Jonsson theory.

Acknowledgments

This research was funded by the Science Committee of the Ministry of Education and Science of the Republic of Kazakhstan (Grant No. AP09260237).

References

1 Forrest W.K. Model theory for universal classes with the amalgamation property: A study in the foundations of Model Theory and algebra / W.K. Forrest // Annals of Mathematical Logic. — 1977. — 11(3). — P. 263–366.

2 Ritt J.F. Integration in finite terms / J.F. Ritt. — Ann Arbor (MI): UMI, 1995.

3 Ritt J.F. Differential algebra / J.F. Ritt. — New York: Publ. by American Mathematical Society. — 1950.

4 Kolchin E.R. Differential algebra and algebraic groups / E.R. Kolchin. — New York: Academic Press. — 1973.

5 Kaplansky I. An introduction to differential algebra / I. Kaplansky. — Paris: Hermann. — 1957. — 85 p.

6 Marker D. Model Theory of Differential Fields / D. Marker // Model Theory, Algebra, and Geometry. — MSRI Publications. — 2000. — V.39. — P. 53–63.

7 Robinson A. Introduction to Model Theory and to the metamathematics of algebra / A. Robinson. — Amsterdam: North-Holland. — 1986.

8 Михалёв А.В. Дифференциальная и разностная алгебра / А.В Михалёв, Е.В. Панкратьев // Итоги науки и техн. Сер. Алгебра. Тополгия Геометрия. — 1987. — Т. 25. — С. 67–139.

9 Blum L.C. Generalized algebraic theories: A model theoretic approach: PhD thesis / Lenore Carol Blum. — Massachusetts: MIT. — 1968. — 180 p.

10 Poizat B. Rangs des types dans les corps differentiels. Group etude theor. stables / B. Poizat // Univ Pierre et Marie Curie. — 1977-1978. — 1. — P. 6/01-6/13.

11 Ешкеев А.Р. Йонсоновские теории и их классы моделей: моногр. / А.Р. Ешкеев, М.Т. Касыметова. — Караганда: Изд-во КарГУ, 2016. — 370 с.

12 Ешкеев А.Р. JSp-косемантичность и JSB-свойство абелевых групп / А.Р.Ешкеев, О.И. Ульбрихт // Сиб. электрон. мат. изв. — 2016. — Т. 13. — С. 861–864.

13 Ешкеев А.Р. JSp-косемантичность R-модулей / А.Р.Ешкеев, О.И. Ульбрихт // Сиб. электрон. мат. изв. — 2019. — Т. 16. — С. 1233–1244.

14 Yeshkeyev A. R. Properties of hybrids of Jonsson theories / A. Yeshkeyev, N.Mussina // Bulletin of the Karaganda University-Mathematics. — 2018. — № 4(92). — P. 99–104.

15 Yeshkeyev A. R. The Properties of Similarity for Jonsson's Theories and Their Models / A. R. Yeshkeyev // Bullutin of the Karaganda University-Mathematics. — 2015. — № 4(80). — P. 52–59.

16 Yeshkeyev A. R. The atomic definable subsets of semantic model / A. Yeshkeyev, A. Issaeva, N. Mussina // Bulletin of the Karaganda University-Mathematics. — 2019. — № 2(94). — P. 84–91.

17 Yeshkeyev A. R. Companions of the fragments in the Jonsson enrichment / A. R. Yeshkeyev // Bulletin of the Karaganda University-Mathematics. — 2017. — № 1(85). — P. 41–45.

18 Yeshkeyev A. R. Model-theoretical questions of the Jonsson spectrum / A. R. Yeshkeyev // Bulletin of the Karaganda University-Mathematics. — 2020. — № 2(98). — P. 165–173.

19 Jonsson B. Extensions of relational structures / B. Jonsson. — The Theory of Models, North Holland, Amsterdam. — 1965. — P. 146–157.

20 Barwise J. Handbook of mathematical logic / J. Barwise, H. J. Keisler. — Amsterdam: Elsevier. — 2006.

21 Wood C. The Model Theory of differential fields of characteristic $p \neq 0$ / C. Wood // Proceedings of the American. Mathematical Society. — 1973. — 40(2). — P. 577–584.

22 Wood C. Prime model extensions for differential fields of characteristic $p \neq 0$ / C. Wood // Journal of Symbolic Logic. — 1974. — 39(3). — P. 469–477.

А.Р. Ешкеев, И.О. Тунгушбаева, М.Т. Қасыметова

Академик Е.А. Бөкетов атындағы Қарағанды университеті, Қарағанды, Қазақстан

Амальгама мен үйлесімді енгізу қасиеттерінің байланысы

Зерттеудің мақсаты – йонсондық теорияларды зерттеу аясында нөлдік және оң сипаттамамен дифференциалдық тұйық өрістер теориясының модельді-теоретикалық қасиеттерін анықтау. Негізгі назар амальгама мен үйлесімді енгізу қасиеттерін зерттеуге және осы теорияны йонсондық теорияларының маңызды белгілері ретінде біріктіруге, атап айтқанда AP-тен JEP қасиетінің болуына байланысты. Қажеттілік белгілі бір теориялардың жоғарыда аталған қасиеттері туралы ақпаратты негүрлым толық модельді-теоретикалық сипаттаудың маңыздылығына байланысты. Сонымен қатар, бүгінгі таңда жалпы жағдайда йонсондық болып табылатын толық емес теорияларды зерттеу аппараты жеткіліксіз дамыған. Мына нәтижелер алынды: үйлесімді енгізу мен амальгама қасиеттерінің болуы тұрғысынан йонсондық теориялардың ішкі кластары анықталды. Осы кластардың бірін, атап айтқанда AP-теориялардың класын зерттеу аясында 0 сипаттамамен дифференциалды тұйық және дифференциалдық өрістерінің, бекітілген оң сипаттамамен дифференциалдық тұйық және дифференциалдық кемел өрістерінің теорияларының йонсондылығы мен кемелділігі көрсетілген. Сонымен қатар, йонсондық емес, бірақ кемел йонсондық модельді компаньоны бар AP теориясының мысалы ретінде оң сипаттамамен дифференциалдық өрістер теориясы қарастырылды, сондай-ақ, йонсондық болу қасиеті тұрғысынан дифференциалдық өрістер теориясы үшін жеткілікті шарт тұжырымдалды.

Кілт сөздер: йонсондық теория, йонсондық кемел теория, дифференциалдық өріс, дифференциалдық тұйық өріс, дифференциалдық кемел өріс, амальгама қасиеті, үйлесімді енгізу қасиеті, AP-теориясы, JEP-теориясы, қатты дөңес теория.

А.Р. Ешкеев, И.О. Тунгушбаева, М.Т. Касыметова

Карагандинский университет имени академика Е.А. Букетова, Караганда, Казахстан

Связь свойств амальгамы и совместного вложения

Цель исследования — изучение теоретико-модельных свойств теории дифференциально замкнутых полей нулевой и положительной характеристик в рамках исследования йонсоновских теорий. Основное внимание удалено свойствам амальгамы и совместному вложению данной теории как важнейших особенностей йонсоновских теорий, а именно следствия наличия свойства JEP из AP. Необходимость обусловлена важностью владения информацией об упомянутых выше свойствах у тех или иных теорий для их более полного теоретико-модельного описания. При этом на сегодняшний день аппарат изучения неполных теорий, которыми в общем случае являются йонсоновские, развит недостаточно. Получены следующие результаты: определены подклассы йонсоновских теорий с точки зрения наличия свойств совместного вложения и амальгамы. В рамках рассмотрения класса AP-теорий показаны йонсоновость и совершенность теорий дифференциальных и дифференциально замкнутых полей характеристики 0, дифференциально совершенных и дифференциально замкнутых полей фиксированной положительной характеристики. Наряду с этим, в качестве примера AP-теории, не являющейся йонсоновской, но имеющей совершенный йонсоновский модельный компаньон, изучена теория дифференциальных полей положительной характеристики, а также сформулировано достаточное условие для теории дифференциальных полей в контексте свойства быть йонсоновской.

Ключевые слова: йонсоновская теория, совершенная йонсоновская теория, дифференциальное поле, дифференциально замкнутое поле, дифференциально совершенное поле, свойство амальгамы, свойство совместного вложения, AP-теория, JEP-теория, сильно выпуклая теория.

References

1 Forrest, W.K. (1977). Model Theory for universal classes with the amalgamation property: A study in the foundations of Model Theory and Algebra. *Annals of Mathematical Logic, 11*(3), 263–366.

2 Ritt, J.F. (1995). *Integration in finite terms.* Ann Arbor (MI): UMI.

3 Ritt, J.F. (1950). *Differential algebra.* New York: Publ. by American Mathematical Society.

4 Kolchin, E.R. (1973). *Differential algebra and algebraic groups.* New York: Academic Press.

5 Kaplansky, I. (1957). *An introduction to differential algebra.* Paris: Hermann.

6 Marker, D. (2000) Model Theory of Differential Fields. *Model Theory, Algebra, and Geometry, 39*, 53–63.

7 Robinson, A. (1986). *Introduction to Model Theory and to the metamathematics of algebra.* Amsterdam: North-Holland.

8 Mikhalev, A.V., & Pankratiev, Ye.V. (1987). Differentsialnaia i raznostnaia algebra [Differential and difference algebra]. *Itogi nauki i tekhnoiki. Seriia Algebra. Topologiia. Geometriia – Results of Science and Technology. Algebra. Topology. Geometry, 25*, 67–139 [in Russian].

9 Blum, L.C. (1968). Generalized algebraic theories: A model theoretic approach (PhD thesis, B.S., Simone College, 1963). Massachusetts: MIT.

10 Poizat, B. (1977–1978). Rangs des types dans les corps differentiels. *Group etude theor. stables, 1*(6), 6-01–6-13 [in Frech].

11 Yeshkeyev, A.R. & Kassymetova, M.T. (2016). *Ionsonovskie teorii i ikh klassy modelei [Jonsson theories and their classes of models].* Karaganda: Izdatelstvo Karagandinskogo universiteta [in Russian].

12 Yeshkeyev, A.R., & Ulbrikht, O.I. (2016). JSp-kosemantichnost i JSB-svoistvo abelevykh grupp [*JSp*-cosemanticness and JSB property of Abelian groups]. *Sibirskie elektronnye matematicheskie izvestiia — Siberian Electronic Mathematical Collections, 13*, 861–864 [in Russian].

13 Yeshkeyev, A.R., & Ulbrikht, O.I. (2019). JSp-kosemantichnost R-modulei [JSp-cosemanticness of R-modules]. *Sibirskie elektronnye matematicheskie izvestiia — Siberian Electronic Mathematical Collections, 16,* 1233–1244 [in Russian].

14 Yeshkeyev, A.R., & Mussina, N. (2018). Properties of hybrids of Jonsson theories. *Bulletin of the Karaganda University-Mathematics, 92*(4), 99–104.

15 Yeshkeyev, A. R. (2015). The Properties of Similarity for Jonsson's Theories and Their Models . Bullutin of the Karaganda University-Mathematics, 4(80), 52–59.

16 Yeshkeyev, A., Issaeva, A., & Mussina, N. (2019). The atomic definable subsets of semantic model. *Bulletin of the Karaganda University-Mathematics, 94*(2), 84–91.

17 Yeshkeyev, A. (2017). Companions of the fragments in the Jonsson enrichment. *Bulletin of the Karaganda University-Mathematics, 85*(1), 41–45.

18 Yeshkeyev, A.R. (2020). Model-theoretical questions of the Jonsson spectrum. *Bulletin of the Karaganda University-Mathematics, 98*(2), 165–173.

19 Jonsson, B. (1965). *Extensions of relational structures.* The Theory of Models, North Holland, Amsterdam, 146–157.

20 Barwise, J., & Keisler, H. J. (2006). *Handbook of mathematical logic.* Amsterdam: Elsevier.

21 Wood, C. (1973). The Model Theory of differential fields of characteristic $p \neq 0$. *Proceedings of the American Mathematical Society, 40*(2), 577–584.

22 Wood, C. (1974). Prime model extensions for differential fields of characteristic $p \neq 0$. *Journal of Symbolic Logic, 39*(3), 469–477.