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On the solutions of some fractional q-differential equations with the Riemann-Liouville fractional q-derivative

This paper is devoted to explicit and numerical solutions to linear fractional q-difference equations and the Cauchy type problem associated with the Riemann-Liouville fractional q-derivative in q-calculus. The approaches based on the reduction to Volterra q-integral equations, on compositional relations, and on operational calculus are presented to give explicit solutions to linear q-difference equations. For simplicity, we give results involving fractional q-difference equations of real order $a > 0$ and given real numbers in q-calculus. Numerical treatment of fractional q-difference equations is also investigated. Finally, some examples are provided to illustrate our main results in each subsection.

Keywords: Cauchy type q-fractional problem, existence, uniqueness, q-derivative, q-calculus, fractional calculus, Riemann–Liouville fractional derivative, q-fractional derivative.

Introduction

During the last three decades, fractional differential equations have attracted great attention and have been wide range used in real world phenomena related to physics, chemistry, biology, signal-and image processing. Moreover, they are equipped with social sciences such as food supplement, climate and economics, see e.g. [1–9]. Hence, there has been a significant development in ordinary and partial differential equations involving fractional derivatives and a huge amount of papers, and also some books devoted to this subject in various spaces have appeared, see e.g. the monographs of T. Sandev and Z. Tomovski [7], A.A. Kilbas et al. [8], R. Hilfer [9], K.S. Miller and the B. Ross [10], the papers [11–19] and the references therein.

The origin of the q-difference calculus can be traced back to the works in [20, 21] by F. Jackson and R.D. Carmichael [22] from the beginning of the twentieth century. For more interesting theory results and scientific applications of the q-difference calculus, we cite the monographs [23, 24, 25] and the references therein. Recently, the fractional q-difference calculus has been proposed by W. Al-salam [26] and R.P. Agarwal [27] and P.M. Rajkovic', S.D. Marinkovic, and M.S. Stankovic [28]. Recently, many researchers got much interested in looking at fractional q-differential equations (FDEs) as new model equations for many physical problems. For example, some researchers obtained q-analogues of the integral and differential fractional operators properties, such as the q-Laplace transform and q-Taylor's formula [29], q-Mittage Leffler function [27] and so on.

We also pronounce that up to now, much attention has been focused on the fractional q-difference equations. There have been some papers dealing with the existence and uniqueness, or multiplicity of solutions to linear fractional q-difference equations by the use of some well-known fixed point theorems. For some recent developments on the subject, see e.g. [30–33] and the references therein. In Section 2 of this paper, we construct explicit solutions to linear fractional q-differential equations with the Riemann-Liouville fractional q-derivative $D^{\alpha}_{q,a+}f$ of order $\alpha > 0$ given by Definition 2, in the space $C^{\alpha}_{q,n-\alpha}[0,a]$, denned in (8). The main result, in this Section, is Theorem 1, but in order to prove this result we need to prove two results (Theorem 1 and 3) of independent interest.

The paper is organized as follows: the main results are presented and proved in subsection 2.1 and subsection 2.3, and the announced examples are given in subsection 2.2, 2.3, and 2.5. In order to not disturb these presentations, we include in Section 1 some necessary Preliminaries.

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1 Preliminaries

First, we start by recalling some elements of q-calculus, for more information see e.g. the books [23], [25], and [33]. Throughout this paper, we assume that $0 < q < 1$ and $0 < a < b < \infty$.

Let $\alpha \in \mathbb{R}$. Then a q-real number $[\alpha]_q$ is defined by

$$[\alpha]_q = \frac{1-q^\alpha}{1-q},$$

where $\lim\limits_{q\to 1} \frac{1-q^\alpha}{1-q} = \alpha$.

We introduce for $n \in \mathbb{N}$:

$$(a;q)_0 = 1,\ (a;q)_n = \prod_{k=0}^{n-1} \left(1-q^k a\right),\ (a;q)_\infty = \lim_{n\to\infty} (a,q)_n,\ (a;q)_\alpha = \frac{(a;q)_\infty}{(q^\alpha a;q)_\infty}.$$

The q-analogue of the power function $(a-b)_q^\alpha$ is defined by

$$(a-b)_q^\alpha{=}a^\alpha \frac{(b/a;q)_\infty}{(q^\alpha b/a;q)_\infty}.$$

Notice that $(a-b)_q^\alpha = a^\alpha (b/a;q)_\alpha$.

For any two real numbers α and β, we have

$$(a-b)_q^\alpha \left(a-q^\alpha b\right)_q^\beta = (a-b)_q^{\alpha+\beta}. \tag{1}$$

The q-analogue of the binomial coefficients $[n]_q!$ are defined by

$$[n]_q! = \begin{cases} 1, & \text{if n} = 0, \\ [1]_q \times [2]_q \times \dots \times [n]_q, & \text{if n} \in \text{N}, \end{cases}$$

The gamma function $\Gamma_q(x)$ is defined by

$$\Gamma_q(x) = \frac{(q;q)_\infty}{(q^x;q)_\infty}(1-q)^{1-x},$$

for any $x > 0$. Moreover, it yields that $\Gamma_q(x)[x]_q = \Gamma_q(x+1)$.

The q-analogue differential operator $D_q f(x)$ is

$$D_q f(x) = \frac{f(x) - f(qx)}{x(1-q)},$$

and the q-derivatives $D_q^n(f(x))$ of higher order are:

$$D_q^0(f(x)) = f(x), \quad D_q^n(f(x)) = D_q\left(D_q^{n-1} f(x)\right), \quad (n = 1, 2, 3, \dots)$$

The q-integral (or Jackson integral) $\int\limits_0^a f(x) d_q x$ is defined by

$$\int\limits_0^a f(x) d_q x = (1-q)a \sum_{m=0}^\infty q^m f(aq^m)$$

and

$$\int\limits_a^b f(x) d_q x = \int\limits_0^b f(x) d_q x - \int\limits_0^a f(x) d_q x,$$

for $0 < a < b$. Notice that

$$\int_a^b D_q f(x) d_q x = f(b) - f(a).$$

For any $t, s > 0$ the definition of q-Beta function is that:

$$B_q(t,s) = \frac{\Gamma_q(t)\Gamma_q(s)}{\Gamma_q(t+s)} = \int_0^1 x^{t-1}(qx;q)_{s-1} d_q x. \tag{2}$$

The (Mittag-Leffler) q-function $E_{\alpha,\beta}(z;q)$ is defined by

$$E_{\alpha,\beta,a}\left[zx^{\alpha}(a/x;q)_{\alpha};q\right] = \sum_{k=0}^{\infty} \frac{z^k x^{k\alpha}(a/x;q)_{k\alpha}}{\Gamma_q(\alpha k+\beta)} \tag{3}$$

and

$$E_{\alpha,m,l}\left[z;q\right] = \sum_{k=0}^{\infty} c_k z^k \tag{4}$$

where c_0 and $c_k = \prod_{j=0}^{k-1} \frac{\Gamma_q[\alpha(jm+l)+1]}{\Gamma_q[\alpha(jm+l+1)+1]} (k \in \mathbb{N})$.

A q-analogue of the classical exponential function e^x is

$$e_q^x = \sum_{j=0}^{\infty} \frac{x^j}{[j]!}. \tag{5}$$

Moreover, the multiple q-integral $\left(I_{q,a+}^n f\right)(x)$ is

$$\begin{aligned}\left(I_{q,a+}^n f\right)(x) &= \int_a^x \int_a^t \int_a^{t_{n-1}} \ldots \int_a^{t_2} d_q t_1 d_q t_2 \ldots d_q t_{n-1} d_q t \\ &= \frac{1}{\Gamma_q(n)} \int_a^x (x-qt)_q^{n-1} f(t) d_q t.\end{aligned}$$

Definition 1. The Riemann-Liouville q-fractional integrals $I_{q,a+}^{\alpha} f$ of order $\alpha > 0$ are defined by

$$\left(I_{q,a+}^{\alpha} f\right)(x) := \frac{x^{\alpha-1}}{\Gamma_q(\alpha)} \int_a^x (qt/x;q)_{\alpha-1} f(t) d_q t.$$

*Definition 2.*The Riemann-Liouville fractional q-derivative $D_{q,a+}^{\alpha} f$ of order $\alpha > 0$ is defined by

$$\left(D_{q,a+}^{\alpha} f\right)(x) := \left(D_{q,a+}^{[\alpha]} I_{q,a+}^{[\alpha]-\alpha} f\right)(x).$$

Notice that

$$\left(I_{q,a+}^{\alpha} x^{\lambda}(a/x;q)_{\lambda}\right)(x) = \frac{\Gamma_q(\lambda+1)}{\Gamma_q(\alpha+\lambda+1)} x^{\alpha+\lambda}(a/x)_q^{\alpha+\lambda}, \tag{6}$$

for $\lambda \in (-1, \infty)$.

For $1 \leq p < \infty$ we define the space $L_q^p = L_q^p[a,b]$ by

$$L_q^p[a,b] := \left\{ f : \left(\int_a^b |f(x)|^p d_q x \right)^{\frac{1}{p}} < \infty \right\}.$$

Let $\alpha > 0$, $\beta > 0$ and $1 \le p < \infty$. Then the q–fractional integration has the following semigroup property

$$\left(I^{\alpha}_{q,a+} I^{\beta}_{q,a+} f\right)(x) = \left(I^{\alpha+\beta}_{q,a+} f\right)(x), \tag{7}$$

for all $x \in [a, b]$ and $f(x) \in L^p_q[a, b]$.

Let $0 < a < b < \infty$ and $0 \le \lambda \le 1$. Then we introduce the space $C_{q,\lambda}[a, b]$ of functions f given on $[a, b]$, such that the functions with the norm

$$\|f\|_{C_{q,\lambda}[a,b]} := \max_{x\in[a,b]} \left|x^{\lambda} (qa/x; q)_{\lambda} f(x)\right| < \infty.$$

The space $C^{\alpha}_{q,n-\alpha}[0, a]$ defined for $n - q < \alpha \le n$, $n \in \mathbb{N}$ by

$$C^{\alpha}_{q,n-\alpha}[0, a] := \{f(x) : f(x) \in C_{q,n-\alpha}[a, b], \left(D^{\alpha}_{q,a+} f\right)(x) \in C_{q,n-\alpha}[a, b]\}. \tag{8}$$

2 On the solutions of some fractional q-differential equations with the Riemann-Liouville fractional q-derivative

2.1 The Cauchy type problem for the fractional q-differential equation

First, we consider the Cauchy type problem for the fractional q-differential equation in the following form:

$$\left(D^{\alpha}_{q,0+} y\right)(x) - \lambda y(x) \;=\; f(x), \quad 0 < x \le a, \alpha > 0; \lambda \in \mathbb{R}, \tag{9}$$

with the initial conditions:

$$\left(D^{\alpha-k}_{q,0+} y\right)(0+) = b_k, \;\; b_k \in \mathbb{R}, \;\; k = 0, 1, 2, \ldots, n = -[-\alpha]. \tag{10}$$

Next we construct the explicit solutions to linear fractional q-differential equations. In the classical case, several authors have considered such problems even in linear cases, see e.g. [8, Section 4] and the references therein.

Theorem 1. (See [34, Theorem 8.1]) Let $n-1 < \alpha \le n; n \in \mathbb{N}$, G be an open set in $\mathbb{R}$ and $f(.,.) : (0, a]\times G \to \mathbb{R}$ be a function such that $F(x, y(x)) = f(x) + \lambda y(x) \in L^1_q[0, a]$ for any $y \in G$. If $y(x) \in L^1_q[0, a]$, then $y(t)$ satisfies a.e. the relations (9)-(10) if and only if $y(x)$ satisfies a.e. the integral equation

$$y(x) := \sum_{k=0}^{n-1} \frac{b_k}{[k]_q} x^{\alpha-k} + \left(I^{\alpha}_{q,0} f(t, y(t))\right)(x), \forall x \in (0, c]. \tag{11}$$

Theorem 2. Let $n - 1 < \alpha < n (n \in \mathbb{N})$ and let $0 \le \gamma < 1$ be such that $\gamma \geqq \alpha$. Also, let $\lambda \in \mathbb{R}$ and $g(x) \in C^{\lambda}_q[0, b]$.If $f_0(x, y(x)) = \lambda y(x) + f(x)$, then the Cauchy problem (9)-(10) has unique solution $y(x) \in C^{\alpha,n-1}_{\gamma}[a, b]$ and this solution is given by

$$\begin{aligned} y(x) \;&:=\; \sum_{k=1}^{n} b_k x^{\alpha-k} E_{\alpha,\alpha-k+1,0}\left[\lambda x^{\alpha}; q\right] \\ &+ \int_0^x x^{\alpha-1} (qt/x; q)_{\alpha-1} E_{\alpha,\alpha,t}\left[\lambda x^{\alpha}(q^{\alpha}t/x; q)_{\alpha}; q\right] f(t) d_q t. \end{aligned} \tag{12}$$

Proof. First, we solve the Volterra q-integral equation (11), and apply the method of successive approximations by setting

$$y_0(x) = \sum_{k=1}^{n} \frac{b_k}{\Gamma_q(\alpha - k + 1)} x^{\alpha-k}$$

and

$$\begin{aligned} y_i(x) \;&=\; y_0(x) + \frac{\lambda x^{\alpha-1}}{\Gamma_q(\alpha)} \int_0^x (qt/x; q)_{\alpha-1} y_{i-1}(t) d_q t \\ &+ \frac{x^{\alpha-1}}{\Gamma_q(\alpha)} \int_0^x (qt/x; q)^{\alpha-1}_q f(t) d_q t. \end{aligned} \tag{13}$$

Using Definition 1 and (6), (13), we find $y_1(x)$:

$$y_1(x) = y_0(x) + \lambda \left(I^{\alpha}_{q,0+} y_0\right)(x) + \left(I^{\alpha}_{q,0+} f\right)(x)$$

that is,

$$\begin{aligned} y_1(x) &= \sum_{k=1}^{n} \frac{b_k}{\Gamma_q(\alpha-k+1)} x^{\alpha-k} + \lambda \sum_{k=1}^{n} \frac{b_k}{\Gamma_q(\alpha-k+1)} \left(I^{\alpha}_{q,0+} t^{\alpha-k}\right)(x) + \left(I^{\alpha}_{q,0+} f\right)(x) \\ &= \sum_{k=1}^{n} \frac{b_k}{\Gamma_q(\alpha-k+1)} x^{\alpha-k} + \lambda \sum_{k=1}^{n} \frac{b_k x^{2\alpha-k}}{\Gamma_q(2\alpha-k+1)} + \left(I^{\alpha}_{q,0+} f\right)(x) \\ &= \sum_{k=1}^{n} b_k \sum_{m=1}^{2} \frac{\lambda^{m-1} x^{m\alpha-k}}{\Gamma_q(\alpha m-k+1)} + \left(I^{\alpha}_{q,0+} f\right)(x). \end{aligned} \tag{14}$$

Similarly, using Definition 1 and (6), (7), (14) we have for $y_2(x)$ that

$$\begin{aligned} y_2(x) &= y_0(x) + \lambda \left(I^{\alpha}_{q,0+} y_1\right)(x) + \left(I^{\alpha}_{q,0+} f\right)(x) \\ &= \sum_{k=1}^{n} \frac{b_k}{\Gamma_q(\alpha-k+1)} x^{\alpha-k} + \frac{\lambda}{\Gamma_q(\alpha)} \sum_{k=1}^{n} b_k \sum_{m=1}^{2} \frac{\lambda^{m-1}}{\Gamma_q(\alpha m-k+1)} \left(I^{\alpha}_{q,0+} t^{m\alpha-k}\right)(x) \\ &+ \lambda \left(I^{\alpha}_{q,0+} I^{\alpha}_{q,0+} f(t)\right)(x) + \left(I^{\alpha}_{q,0+} f\right)(x) \\ &= \sum_{k=1}^{n} \frac{b_k}{\Gamma_q(\alpha-k+1)} x^{\alpha-k} + \lambda \sum_{k=1}^{n} b_k \sum_{m=1}^{2} \frac{\lambda^{m-1}}{\Gamma_q(\alpha(m+1)-k+1)} x^{\alpha(m+1)-k} \\ &+ \lambda \left(I^{2\alpha}_{q,0+} f(t)\right)(x) + \left(I^{\alpha}_{q,0+} f\right)(x) \\ &= \sum_{k=1}^{n} \frac{b_k}{\Gamma_q(\alpha-k+1)} x^{\alpha-k} + \lambda \sum_{k=1}^{n} b_k \sum_{m=1}^{2} \frac{\lambda^{m-1}}{\Gamma_q(\alpha(m+1)-k+1)} x^{\alpha(m+1)-k} \\ &+ \frac{\lambda x^{2\alpha-1}}{\Gamma(2\alpha)} \int_0^x f(t)\,(qt/x;q)_{2\alpha-1}\, d_q t + \left(I^{\alpha}_{q,0+} f\right)(x). \end{aligned}$$

Thus,

$$\begin{aligned} y_2(x) &= \sum_{k=1}^{n} b_k \sum_{m=1}^{3} \frac{\lambda^{m-1} x^{\alpha m-k}}{\Gamma_q(\alpha m-k+1)} \\ &+ \int_0^x \left[\sum_{m=1}^{2} \frac{\lambda^{m-1} x^{\alpha m-1} (qt/x;q)_{\alpha m-1}}{\Gamma_q(\alpha m)}\right] f(t) d_q t. \end{aligned}$$

Continuing this process, we derive the following relation for $y_i(x)$:

$$\begin{aligned} y_i(x) &= \sum_{k=1}^{n} b_k \sum_{m=1}^{i+1} \frac{\lambda^{m-1} x^{\alpha m-k}}{\Gamma_q(\alpha m-k+1)} \\ &+ \int_0^x \left[\sum_{m=1}^{i} \frac{\lambda^{m-1} x^{\alpha m-1} (qt/x;q)_{\alpha m-1}}{\Gamma_q(\alpha m)}\right] f(t) d_q t \\ &= \sum_{k=1}^{n} b_k \sum_{m=0}^{i} \frac{\lambda^{m} x^{\alpha(m+1)-k}}{\Gamma_q(\alpha(m+1)-k+1)} \\ &+ \int_0^x \left[\sum_{m=0}^{i-1} \frac{\lambda^{m} x^{\alpha(m+1)-1} (qt/x;q)_{\alpha(m+1)-1}}{\Gamma_q(\alpha(m+1))}\right] f(t) d_q t. \end{aligned}$$

Taking the limit as $i \to \infty$ and using (1), we obtain the following explicit solution $y(x)$ to the q-integral equation (11):

$$\begin{aligned} y(x) &= \sum_{k=1}^{n} b_k x^{\alpha-k} \sum_{m=0}^{\infty} \frac{\lambda^m x^{\alpha m}}{\Gamma_q(\alpha m + \alpha - k + 1)} \\ &+ \int_0^x x^{\alpha-1}(qt/x;q)_{\alpha-1} \left[\sum_{m=0}^{\infty} \frac{\lambda^m x^{\alpha m}(qt/x;q)_{\alpha m}}{\Gamma_q(\alpha m + \alpha)} \right] f(t) d_q t. \end{aligned}$$

On the basis of Theorem 1 and (3) an explicit solution to the Volterra q-integral equation (11) and hence, to the Cauchy type problem (9)-(10).

2.2 Miscellaneous Examples

In this subsection, we present some examples and discuss these examples in connection with the results obtained in Theorem 2. Our examples are q-analogues of examples given in [8, Examples 3.1-3.2].

Example 1. Let $0 < \alpha < 1$ and $\lambda, b \in \mathbb{R}$. Then, the solution to the Cauchy type problem in the following form:

$$\left(D^{\alpha}_{q,0+}y\right)(x) - \lambda y(x) = f(x), \quad \left(D^{\alpha-1}_{q,0+}y\right)(0+) = b$$

has the explicit solution

$$y(x) = bt^{\alpha-1} E_{\alpha,\alpha,0}\left[\lambda t^{\alpha}; q\right] + x^{\alpha-1} \int_0^x (qt/x;q)_{\alpha-1} E_{\alpha,\alpha,t}\left[\lambda x^{\alpha}(q^{\alpha}t/x;q)_{\alpha}; q\right] f(t) d_q t.$$

Hence, we can rewrite as follows:

$$\left(D^{\alpha}_{q,0+}y\right)(x) - \lambda y(x) = 0, \quad \left(D^{\alpha-1}_{q,0+}y\right)(0+) = b,$$

and the solution of this problem

$$y(x) = bt^{\alpha-1} E_{\alpha,\alpha,0}\left[\lambda t^{\alpha}; q\right].$$

In particular, for $\alpha = 1/2$ the Cauchy type problem

$$\left(D^{1/2}_{q,0+}y\right)(x) - \lambda y(x) = f(x), \left(I^{1/2}_{q,0+}y\right)(0+) = b$$

has the solution given by

$$\begin{aligned} y(x) &= \frac{b}{t^{\frac{1}{2}}} E_{1/2,1/2,0}\left[\lambda t^{1/2}; q\right] \\ &+ x^{1/2} \int_0^x (qt/x;q)_{1/2} E_{1/2,1/2,t}\left[\lambda x^{1/2}(q^{1/2}t/x;q)_{1/2}; q\right] f(t) d_q t \end{aligned}$$

and the solution to the problem

$$\left(D^{1/2}_{q,0+}y\right)(x) - \lambda y(x) = 0, \quad \left(I^{1/2}_{q,0+}y\right)(0+) = b$$

is given by

$$y(x) = \frac{b}{t^{1/2}} E_{1/2,1/2,0}\left[\lambda t^{1/2}\right].$$

Example 2. We assume that $1 < \alpha < 2$ and $\lambda, b, d \in \mathbb{R}$. Then the Cauchy type problem in the following form:

$$\left(D^{\alpha}_{q,0+}y\right)(x) - \lambda y(x) = f(x), \quad \left(D^{\alpha-1}_{q,0+}y\right)(0+) = b, \quad \left(D^{\alpha-2}_{q,0+}y\right)(0+) = d,$$

and its solution has the form:

$$\begin{aligned} y(x) &= bt^{\alpha-1} E_{\alpha,\alpha,0}\left[\lambda t^{\alpha}; q\right] + dt^{\alpha-2} E_{\alpha,\alpha-1,0}\left[\lambda t^{\alpha}; q\right] \\ &+ x^{\alpha-1} \int_0^x (qt/x;q)_{\alpha-1} E_{\alpha,\alpha,t}\left[\lambda x^{\alpha}(q^{\alpha}t/x;q)_{\alpha}; q\right] f(t) d_q t. \end{aligned}$$

Particularly, the solution to the problem

$$\left(D^{\alpha}_{q,0+}y\right)(x)-\lambda y(x)=0,\left(D^{\alpha-1}_{q,0+}y\right)(0+)=b,$$
$$\left(D^{\alpha-2}_{q,0+}y\right)(0+)=d,$$

is given by

$$y(x) \quad = \quad bt^{\alpha-1}E_{\alpha,\alpha,0}\left[\lambda t^{\alpha};q\right]+dt^{\alpha-2}E_{\alpha,\alpha-1,0}\left[\lambda t^{\alpha};q\right].$$

2.3 General homogeneous fractional q-differential equation

In subsection, we consider in the following more general homogeneous fractional q-differential equation than (9):

$$\left(D^{\alpha}_{q,0+}y\right)(x)-\lambda x^{\beta}y(x)=0,\quad 0<x\le a<\infty,\alpha>0,\lambda\in\mathbb{R}, \tag{15}$$

with initial date

$$\left(D^{\alpha-k}_{q,0+}y\right)(0+)=b_k,b_k\in\mathbb{R},k=0,1,2,\dots,n=-[-\alpha]. \tag{16}$$

Theorem 3. Let $\alpha>0$, $n=-[-\alpha]$, $\lambda\in\mathbb{R}$ and $\beta\ge 0$. Then the Cauchy type problem (15)-(16) has a unique solution $y(x)$ in the space $C^{\alpha}_{q,n-\alpha}[0,a]$ and this solution is given by

$$y(x) \quad = \quad \sum_{j=1}^{n}\frac{b_j}{\Gamma_q(\alpha-j+1)}t^{\alpha-j}E_{\alpha,1+\frac{\beta}{\alpha},1+\frac{(\beta-j)}{\alpha}}\left[\lambda t^{\alpha+\beta};q\right]. \tag{17}$$

Proof. Let $\beta>-\alpha$. Then basic on Theorem 1 the problem (15)-(16) is equivalent in the space $C^{\alpha}_{q,n-\alpha}[0,a]$ to the Volterra q-integral equation of the second kind in the following form:

$$y(x)=\sum_{j=1}^{n}\frac{b_j}{\Gamma_q\left(\alpha-j+1\right)}t^{\alpha-j}+\frac{\lambda x^{\alpha-1}}{\Gamma_q\left(\alpha\right)}\int_0^x t^{\beta}\left(qt/x;q\right)_{\alpha-1}y(t)d_qt. \tag{18}$$

Similarity, we again apply the method of successive approximations to solve this q-integral equation (18). We assume that $y_0(x)=\sum\limits_{j=1}^{n}\frac{b_j}{\Gamma_q(\alpha-j+1)}x^{\alpha-j}$ and

$$y_m(x)=y_0(x)+\frac{\lambda x^{\alpha-1}}{\Gamma_q\left(\alpha\right)}\int_0^x t^{\beta}\left(qt/x;q\right)_{\alpha-1}y_{m-1}(t)d_qt. \tag{19}$$

Using the same arguments as above, by using (2), (7), and (19) we find $y_1(x)$:

$$\begin{aligned}
y_1(x) &= y_0(x)+\frac{\lambda x^{\alpha-1}}{\Gamma_q(\alpha)}\int_0^x t^{\beta}\left(qt/x;q\right)_{\alpha-1}y_0(x)d_qt\\
&= y_0(x)+\frac{\lambda}{\Gamma_q(\alpha)}\sum_{j=1}^{n}\frac{b_jx^{\alpha-1}}{\Gamma_q(\alpha-j+1)}\int_0^x t^{\alpha+\beta-j}\left(qt/x;q\right)_{\alpha-1}d_qt\\
&= y_0(x)+\frac{\lambda}{\Gamma_q(\alpha)}\sum_{j=1}^{n}\frac{b_jx^{\alpha-1}}{\Gamma_q(\alpha-j+1)}\int_0^1 (xy)^{\alpha+\beta-j}\left(qxy/x;q\right)_{\alpha-1}xd_qy\\
&= y_0(x)+\frac{\lambda}{\Gamma_q(\alpha)}\sum_{j=1}^{n}\frac{b_jx^{2\alpha+\beta-j}}{\Gamma_q(\alpha-j+1)}\int_0^1 y^{\alpha+\beta-j}\left(qy;q\right)_{\alpha-1}d_qy\\
&= \sum_{j=1}^{n}\frac{b_j}{\Gamma_q(\alpha-j+1)}x^{\alpha-j}+\frac{\lambda}{\Gamma_q(\alpha)}\sum_{j=1}^{n}\frac{b_jx^{2\alpha+\beta-j}}{\Gamma_q(\alpha-j+1)}B_q\left(\alpha+\beta-j+1,\alpha\right)\\
&= \sum_{j=1}^{n}\frac{b_j}{\Gamma_q(\alpha-j+1)}x^{\alpha-j}\\
&+ \lambda\sum_{j=1}^{n}\frac{b_jx^{2\alpha+\beta-j}}{\Gamma_q(\alpha-j+1)}\frac{\Gamma_q(\alpha+\beta-j+1)}{\Gamma_q(2\alpha+\beta-j+1)}.
\end{aligned} \tag{20}$$

Similarly, for $m = 2$ using (2), (19) and taking (20) into account, we derive

$$\begin{aligned} y_2(x) &= y_0(x) + \lambda \left(I^{\alpha}_{q,0+} y_1\right)(x) \\ &= y_0(x) + \lambda \sum_{j=1}^{n} \frac{b_j}{\Gamma_q(\alpha - j + 1)} \left(I^{\alpha}_{q,0+} t^{\alpha-j}\right)(x) \\ &+ \lambda^2 \sum_{j=1}^{n} \frac{b_j \Gamma_q(\alpha+\beta-j+1)}{\Gamma_q(\alpha-j+1)\,\Gamma_q(2\alpha+\beta-j+1)} \left(I^{\alpha}_{q,0+} t^{2\alpha+\beta-j}\right)(x) \\ &= \sum_{j=1}^{n} \frac{b_j t^{\alpha-j}}{\Gamma_q(\alpha-j+1)} \left[1 + c_1 \left(\lambda t^{\alpha+\beta}\right) + c_2 \left(\lambda t^{\alpha+\beta}\right)^2\right], \end{aligned}$$

where

$$\begin{aligned} c_1 &= \frac{\Gamma_q(\alpha+\beta-j+1)}{\Gamma_q(2\alpha+\beta-j+1)}, \\ c_2 &= \frac{\Gamma_q(\alpha+\beta-j+1)\Gamma_q(2\alpha+2\beta-j+1)}{\Gamma_q(2\alpha+\beta-j+1)\Gamma_q(3\alpha+2\beta-j+1)}. \end{aligned}$$

Continuing this process for $m \in \mathbb{N}$, we have $y_m(x)$:

$$y_m(x) = \sum_{j=1}^{n} \frac{b_j}{\Gamma_q(\alpha-j+1)} t^{\alpha-j} \left[1 + \sum_{k=1}^{m} c_k \left(\lambda t^{\alpha+\beta}\right)^k\right], \tag{21}$$

where

$$c_k = \prod_{r=1}^{k} \frac{\Gamma_q[r(\alpha+\beta)-j+1]}{\Gamma_q[r(\alpha+\beta)+\alpha-j+1]}, \qquad k \in \mathbb{N}.$$

Taking the limit as $m \to \infty$ tow site of (21), we obtain the following explicit solution $y(x)$ to the Cauchy type problem (15)-(16):

$$y(x) = \sum_{j=1}^{n} \frac{b_j}{\Gamma_q(\alpha-j+1)} t^{\alpha-j} \left[1 + \sum_{k=1}^{\infty} c_k \left(\lambda t^{\alpha+\beta}\right)^k\right].$$

According to the relations (4), we rewrite this solution (18) in terms of the generalized Mittag-Leffler q-function $E_{\alpha,m,l}\,[z;q]$.

2.4 Further Examples

In subsection, we present some examples and discuss them in connection with the results obtained in Section 2.3.

Example 3. Let $0 < \alpha < l$, $\beta > -\alpha$ and $\lambda \in \mathbb{R}$ and $b \in \mathbb{R}$. Then the solution to the Cauchy type problem

$$\left(D^{\alpha}_{q,0+} y\right)(x) - \lambda t^{\beta} y(x) = 0, \left(D^{\alpha-1}_{q,0+} y\right)(0+) = b,$$

is given by

$$y(x) = \frac{b t^{\alpha-1}}{\Gamma_q(\alpha)} E_{\alpha,1+\frac{\beta}{\alpha},1+\frac{(\beta-1)}{\alpha}} \left[\lambda t^{\alpha+\beta}; q\right].$$

In particular, for $\alpha = 1/2$ the Cauchy type problem in the following form

$$\left(D^{1/2}_{q,0+} y\right)(x) - \lambda t^{\beta} y(x) = 0, \left(D^{-1/2}_{q,0+} y\right)(0+) = b,$$

has a unique solution given by

$$y(x) = \frac{b}{\Gamma_q(1/2)} t^{-\frac{1}{2}} E_{\frac{1}{2},1+2\beta,2\beta-1}\left[\lambda t^{\beta+\frac{1}{2}}; q\right].$$

Example 4. The solution to the Cauchy type problem

$$\left(D^{\alpha}_{q,0+}y\right)(x) - \lambda t^{\beta} y(x) = 0, \left(D^{\alpha-1}_{q,0+}y\right)(0+) = b,$$

$$\left(D^{\alpha-2}_{q,0+}y\right)(0+) = d,$$

with $b, d \in \mathbb{R}$, $1 < \alpha < 2$, $\beta \in \mathbb{R}$ $(\beta > -\alpha)$ and $\lambda \in \mathbb{R}$ has the form

$$\begin{aligned} y(x) &= \frac{b}{\Gamma_q(\alpha)} t^{\alpha-1} E_{\alpha,1+\frac{\beta}{\alpha},1+\frac{(\beta-1)}{\alpha}}\left[\lambda t^{\alpha+\beta}; q\right] \\ &+ \frac{d}{\Gamma_q(\alpha)} t^{\alpha-2} E_{\alpha,1+\frac{\beta}{\alpha},1+\frac{(\beta-2)}{\alpha}}\left[\lambda t^{\alpha+\beta}; q\right]. \end{aligned}$$

2.5 The Cauchy Problems for Ordinary q-Differential Equations

In this subsection, we use the results of subsection 3.1 when $\alpha \in \mathbb{N}$, and we derive the explicit solutions to the Cauchy problems for ordinary q-differential equations of order n on $[0, a]$.

Let $\lambda, b_k \in \mathbb{R}, n, k \in \mathbb{N}$ such that $k \le n$. Then we consider the ordinary q-differential equation:

$$D^{(n)}_q(x) - \lambda y(x) = f(x), \tag{22}$$

with inial data

$$D^{(n-k)}_q(0+) = b_k, \tag{23}$$

which is a particular case of the Cauchy problem (9)-(10) with $\alpha \in \mathbb{N}$. Therefore, from (12) we derive the solution to (22)-(23) in the following form:

$$\begin{aligned} y(x) : &= \sum_{j=1}^{n} b_j t^{n-j} E_{n,n-j+1,0}\left[(\lambda t)^n; q\right] + \\ &+ x^{n-1} \int_0^x (qt/x; q)_{n-1} E_{n,n,t}\left[\lambda x^n (q^n t/x; q)_n; q\right] f(t) d_q t. \end{aligned} \tag{24}$$

which is the unique explicit solution of the Cauchy problem (24) in the space $C^n_{q,0}[0, a]$.

Example 5. Let $\alpha = 1$ and $b \in \mathbb{R}$. Then the solution to the Cauchy type problem in the following form

$$\left(D_q y\right)(x) - \lambda y(x) = f(x); \;\; y(0) = b,$$

has a unique solution given by

$$\begin{aligned} y(x) &= bE_{1,1,0}\left[\lambda t; q\right] \\ &+ \int_0^x x E_{1,1,t}\left[\lambda(qt/x); q\right] f(t) d_q t. \end{aligned}$$

Form (3) and (5) it follows that

$$\begin{aligned} y(x) &= b\sum_{k=0}^{\infty} \frac{\lambda^k t^k}{\Gamma_q(k+1)} + \\ &+ \int_0^x \sum_{k=0}^{\infty} \frac{(\lambda(x-qt))^k}{\Gamma_q(k+1)} f(t) d_q t, \\ &= be_q^{\lambda t} + \int_0^x e_q^{\lambda(x-qt)} f(t) d_q t. \end{aligned}$$

Example 6. Let $b, d \in \mathbb{R}$. Then the solution to the Cauchy type problem

$$\left(D_q^2 y\right)(x) - \lambda y(x) = f(x), \;\; y(0) = b, \;\; \left(D_q y\right)(x) = d,$$

is given by

$$\begin{aligned} y(x) &= btE_{2,2,0}\left[\lambda t^2; q\right] + dE_{2,1,0}\left[\lambda t^2; q\right] \\ &+ x\int_0^x (qt/x;q)E_{2,2,t}\left[x^2\lambda(q^2t/x;q)_2\right] f(t)d_qt, \end{aligned}$$

In particular, for $b, d \in \mathbb{R}$ and $f(x) = 0$ the solution to the problem

$$\left(D_q^2 y\right)(x) - \lambda y(x) = 0, \;\; y(0+) = b, \;\; \left(D_q y\right)(x) = d,$$

has the form

$$y(x) = btE_{2,2,0}\left[\lambda t^2; q\right] + dE_{2,1,0}\left[\lambda t^2; q\right].$$

Form (2) and (8) it follows that

$$\begin{aligned} y(x) &= bt\sum_{k=0}^{\infty}\frac{\lambda^k t^{2k}}{\Gamma_q(2k+2)} + d\sum_{k=0}^{\infty}\frac{\lambda^k t^{2k}}{\Gamma_q(2k+1)} \\ &= bsin_q(\sqrt{\lambda}t) + dcos_q(\sqrt{\lambda}t), \end{aligned}$$

where $sin_q(\sqrt{\lambda}t) = \sum\limits_{k=0}^{\infty}\frac{(\sqrt{\lambda}t)^{2k+1}}{\Gamma_q(2k+2)}$ and $cos_q(\sqrt{\lambda}t) = \sum\limits_{k=0}^{\infty}\frac{(\sqrt{\lambda}t)^{2k}}{\Gamma_q(2k+1)}$.

For $\lambda, b_k \in \mathbb{R}, n, k \in \mathbb{N}$ and $\beta \geq 0$ we consider the Cauchy problem in the following form:

$$y^{(n)}(x) - \lambda t^{\beta} y(x) = f(x); \;\; y^{(n-k)}(0+) = b_k,$$

which is a particular case of the problem (16)-(17) with $\alpha = n$. Using (4) we get the solution in the following form:

$$y(x) = \sum_{j=1}^{n}\frac{b_j}{\Gamma_q(n-j+1)}t^{n-j}E_{n,1+\frac{\beta}{n},1+\frac{(\beta-j)}{n}}\left[\lambda t^{n+\beta}; q\right].$$

Example 7. We assume that $\beta > -1$ and $b \in \mathbb{R}$. Then the solution to the Cauchy problem

$$\left(D_q y\right)(x) - \lambda t^{\beta} y(x) = f(x); \;\; y(0) = b$$

has the form

$$y(x) = bE_{1,1+\beta,\beta}\left[\lambda t^{1+\beta}; q\right].$$

Example 8. Let $b, d \in \mathbb{R}$ and $\beta > -2$. Then the solution to the Cauchy problem

$$\left(D_q^2 y\right)(x) - \lambda t^{\beta} y(x) = f(x), \;\; y(0+) = b, \;\; \left(D_q y\right)(0) = d,$$

is given by

$$y(x) = btE_{2,1+\frac{\beta}{2},\frac{(\beta+1)}{2}}\left[\lambda t^{2+\beta}; q\right] + dE_{2,1+\frac{\beta}{2},\frac{(\beta)}{2}}\left[\lambda t^{2+\beta}; q\right].$$

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Кейбір Риман-Лиувиль бөлшек q-туындылы q-бөлшек дифференциалдық теңдеулердің шешімдері туралы

Мақала бөлшек-сызықтық q-айырымдық теңдеулері мен бөлшек q-Риман–Лиувилл туындысымен байланысты Коши типтес есептерді нақты және сандық шешуге арналған. q-Вольтер интегралдық теңдеулеріне, композициялық қатынастарға және сызықтық q-айырымдық теңдеулерінің нақты шешімдерін алу үшін операциялық есептеуге редукцияға негізделген тәсілдер ұсынылған. Қарапайым бөлу үшін нақты $a > 0$ ретті q-бөлшек айырымдық теңдеулерін және q-есептеулеріндегі нақты сандарды қамтитын нәтижелер берілген. Сондай-ақ, бөлшек q-айырымдық теңдеулерінің сандық өңделуі зерттелді. Сонымен әр бөлімде негізгі нәтижелерді көрсететін бірнеше мысалдар келтірілген.

Кілт сөздер: Коши типтес q-бөлшек есеп, бар болуы, бірегейлігі, q-туынды, q-есептеу, бөлшек есептеу, Риман–Лиувилл бөлшек туындысы, q-бөлшек туынды.

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О решениях некоторых q-дробных дифференциальных уравнений с дробными q-производными Римана–Лиувилля

Статья посвящена явному и численному решению дробно-линейных q-разностных уравнений и задачи типа Коши, связанной с дробной q-производной Римана–Лиувилля в q-исчислении. Представлены подходы, основанные на редукции к q-интегральным уравнениям Вольтерра, композиционным соотношениям и операционному исчислению, для получения явных решений линейных q-разностных уравнений. Для простоты авторами приведены результаты, включающие дробные q-разностные уравнения действительного порядка $a > 0$ и заданные действительные числа в q-исчислении. Также исследована численная обработка дробных q-разностных уравнений. В итоге, в каждом подразделе представлены некоторые примеры, иллюстрирующие полученные основные результаты.

Ключевые слова: q-дробная задача типа Коши, существование, единственность, q-производная, q-вычисление, дробное исчисление, дробная производная Римана–Лиувилля, q-дробная производная.