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On atomic and algebraically prime models obtained by closure of definable sets

This article discusses the properties of atomic and prime models obtained with the some closure operator given on definable subsets of the semantic model some fixed Jonsson theory. The main result is to obtain the equivalence of the thus defined atomic and prime models, and this coincidence follows the assumption that there is some model with nice-defined properties.

Keywords: Jonsson theory, semantic model, prime model, atomic model, algebraically prime model, pregeometry, definable subset.

The paper considered the syntactic and semantic characteristics of prime and atomic models [1]. A. Robinson defined a natural generalization of a prime model, and he called such a model an algebraically prime model. In work [2] the corresponding notions of atomicity and their connection with an algebraically prime model were systematically studied. We propose several new types of atomic models and refine these concepts for algebraically prime models within the framework of these types of atomic. We have previously obtained some results in connection with these new concepts in works [3–6].

With these concepts of types of atomic and primary models we can work in fixed classes of Jonsson theories, depending on the conditions of the problem under consideration. In work [7] generalizations of the concept of isomorphic embedding were considered and within the framework of this definition results were obtained connecting the concepts of atomic and algebraically prime within the framework of this generalization. Thus, this work is a synthesis of new results obtained using ideas and concepts of works [3–6] and [7]. In [8–13] some new directions related to the study of Jonsson theories and their companions were considered and studied. The results of this work can be useful for studying the properties of countable models related to the above topics from the list of papers [3–6], [8–13].

Remind some concepts from [7].

Let $\alpha \leq \omega$, $\mathfrak{A}, \mathfrak{B}$ are models first order of L. Then the mapping $f : \mathfrak{A} \to \mathfrak{B}$ is called α an embedding if for any formula $\varphi(\bar{x}) \in \Pi_\alpha$ and any tuple $\bar{a} \in A$ from the fact that $\mathfrak{A} \models \varphi(\bar{a})$, it follows $\mathfrak{B} \models \varphi(f(\bar{a}))$. A model $\mathfrak{A}$ of the theory T is called α-algebraically prime if $\mathfrak{A}$ α-embeddable into any model of the theory T.

From the above definitions it is easy to see that the concepts of an algebraically prime model and a prime model are obtained from the concept of an α-algebraically prime models for $\alpha = 0$ and $\alpha = \omega$ respectively. If Γ is a set of formulas, then we put $\Gamma^* \rightleftharpoons \{\neg\varphi/\varphi \in \Gamma\}$. If $\bar{a} =< a_0...a_n >$, $\mathfrak{A}$ is a model, then $\bar{a} \in \mathfrak{A}$ means that $a_i \in A, i < n$. A type p is called a Γ-type if $p \subseteq \Gamma$. Further, $t_\Gamma^{\mathfrak{A}}(\bar{a}) \rightleftharpoons \{\varphi(\bar{x})/\varphi(\bar{x}) \in L, \mathfrak{A} \models \varphi(\bar{a})\}$ is called a Γ- type $\bar{a}$ in $\mathfrak{A}$. Γ_2-type p is called a Γ_1-the main type if there is a Γ_1 is formula $\varphi(\bar{x})$ such that $T \models \forall(\bar{x})(\varphi(\bar{x}) \to \psi(\bar{x}))$ forall $\psi(\bar{x}) \in p$. In this case $\varphi(\bar{x})$ is said to generate p.

It is easy to see the following fact. Let $\mathfrak{A}$ is a model of the theory T, then $\mathfrak{A}$ is (Γ_1, Γ_2)-atomic model T if and only if for any $\bar{a} \in A$ there is such a formula $\varphi(\bar{x}) \in \Gamma_1$, which is true:

a) $\mathfrak{A} \models \varphi(\bar{x})$;

b) $\varphi(\bar{x})$ generates $t_{\Gamma \cup \Gamma^*}^{\mathfrak{A}}(\bar{a})$.

Similarly, if $\mathfrak{A} \models T$, then $\mathfrak{A}$ is weakly (Γ_1, Γ_2) atomic model of T if and only if for any $\bar{a} \in A$ there is a formula $\varphi(\bar{x}) \in \Gamma_1$ that is true:

a) $\mathfrak{A} \models \varphi(\bar{x})$;

b) $\varphi(\bar{x})$ generates $t_{\Gamma_2}^{\mathfrak{A}}(\bar{a})$.

In papers [3–6] the properties of atomic models were considered with the help of the closure operator specifying some pregeometry on subsets of the semantic model of a fixed Jonsson theory.

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Let cl is some closure operator defining a pregeometry over C (for example cl $=$ acl or cl $=$ dcl). It is clear that such operator is a special case of the closure operator and its example is a closure operator defined on any linear space as a linear shell. Further, the concepts under consideration are produced within the framework of a perfect Jonsson theory and if the contrary is not specified then the considered Jonsson theories are assumed to be complete for existential sentences.

Let us give definitions related to the atomic and prime model considered in this theory.

Definition 1. A set A will be called the (Γ_1, Γ_2)-*cl* atomic in T if:

1) $\forall \bar{a} \in A$, $\exists \varphi(\bar{x}) \in \Gamma_1$ such that $\mathfrak{A} \models \varphi(\bar{a})$;

2) $\varphi(\bar{x})$ generates $t_{\Gamma \cup \Gamma^*}^{\mathfrak{A}}(\bar{a})$;

3) $cl(A) = M, M \in E_T$, where E_T class of existentially closed models of the theory T;

and obtained model M is said to be the (Γ_1, Γ_2)-*cl* atomic model of the theory T.

Definition 2. A set A is said to be weakly the (Γ_1, Γ_2)-*cl* atomic in T, if $\forall \bar{a} \in A$ $\exists \varphi(\bar{x}) \in \Gamma_1$ such that:

1) $\varphi(\bar{x}) \cup T$ is consistent;

2) $\varphi(\bar{x})$ generates $t_{\Gamma_2 \cup \Gamma_2^*}^{\mathfrak{A}}(\bar{a})$;

3) $cl(A) = M, M \in E_T$, where E_T class of existentially closed models of the theory T; And obtained model M is said to be weakly (Γ_1, Γ_2)-*cl* atomic model of the theory T.

Definition 3. A set A is said to be almost-weakly (Γ_1, Γ_2)-*cl* atomic in T if for any $\bar{a} \in A$ there exists a formula $\varphi(\bar{x}) \in \Gamma_1$ such that:

1) $\varphi(\bar{x}) \cup T$ is consistent;

2) $\varphi(\bar{x})$ generates $t_{\Gamma_2}^{\mathfrak{A}}(\bar{a})$;

3) $cl(A) = M, M \in E_T$, where E_T is the class of existentially closed models of theory T; And obtained model M is said to be almost-weakly (Γ_1, Γ_2)-*cl* atomic model of the theory T.

Definition 4. A set A is said to be the (Γ_1, Γ_2)-*cl* algebraically prime of the theory T, if $cl(A) = M$, M is (Γ_1, Γ_2)-*cl* atomic model of the theory T, $M \in E_T \cap AP_T$, where $AP_T \bigcap E_T \neq \varnothing$ and obtained model M is said to be (Γ_1, Γ_2)-*cl* algebraically prime of the theory T.

Definition 5. A set A is said to be almost (Γ_1, Γ_2)-*cl* algebraically prime of the theory T, if $cl(A) = M$, M is been almost (Γ_1, Γ_2)-*cl* atomic model of the theory T, $M \in E_T \cap AP_T$, where $AP_T \bigcap E_T \neq 0$ and obtained model M is said to be almost the (Γ_1, Γ_2)-*cl* algebraically prime of the theory T.

Definition 6. A set A is said to be almost-weakly (Γ_1, Γ_2)-*cl* algebraically prime of theory T, if $cl(A) = M$, M is been almost-weakly (Γ_1, Γ_2)-*cl* atomic model of the theory T, $M \in E_T \cap AP_T$, where $AP_T \bigcap E_T \neq 0$ and obtained model M is said to be almost-weakly (Γ_1, Γ_2)-*cl* algebraically prime of the theory T.

For the convenience of expression

"$\mathfrak{A}$ is (Γ_1, Γ_2)-*cl* atomic model of the theory T";

"$\mathfrak{A}$ is weakly (Γ_1, Γ_2)-*cl* atomic model of theory T";

"$\mathfrak{A}$ is an almost (Γ_1, Γ_2)-*cl* atomic model of theory T";

"$\mathfrak{A}$ is an almost-weakly (Γ_1, Γ_2)-*cl* atomic model of theory T";

and denote by (1), (2), (3), (4), respectively.

Lemma 1.

1. If $(\Gamma_2 = \Gamma_2^*)$, then (1) $\Leftrightarrow$ (2), (3) $\Leftrightarrow$ (4).

2. If $(\Gamma_1^* \subset \Gamma_2)$, then (1) $\Leftrightarrow$ (3), (2) $\Leftrightarrow$ (4).

3. If $(\Gamma_2 \cup \Gamma_2^*) \subset \Gamma_3$, then if

a) $\mathfrak{A}$ is weakly (Γ_1, Γ_2)-*cl* atomic model of the theory T, then it is true (1);

b) $\mathfrak{A}$ is an almost-weakly (Γ_1, Γ_2)-*cl* atomic model of the theory T, then it is true (3).

4. If $(\Gamma_1^* \subset \Gamma_2 \subset \Gamma_2^*)$, then (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4).

5. If $(\Gamma_1 = \Gamma_2 = \Gamma_2^*)$, then (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4).

6. If $(\Gamma_1 \subset \Gamma_1')$, $(\Gamma_2 \subset \Gamma_2')$, then $\tau - (\Gamma_1, \Gamma_2')$-*cl* atomic model of the theory $T \Rightarrow \tau - (\Gamma_1', \Gamma_2)$-*cl* atomic model of the theory T, where $\tau \in \{\emptyset, \text{weakly, almost, almost-weakly}\}$.

Proof. The proof follows easily from the definition.

Lemma 2. If T is complete for $\exists \Gamma_2$ (i.e., if $\psi(\bar{x}) \cup T$ consistent and $\psi(\bar{x}) \in \Gamma_2$, then it is true that $T \models \exists \bar{x} \psi(\bar{x})$) and $(\Gamma_1 \cup \Gamma_1^*) \subset \Gamma_2$ then it is true (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4).

Proof. Since $\Gamma_1^* \subset \Gamma_2$, then, by part 2) of Lemma 1, it suffices to show (2) $\Leftrightarrow$ (1). Let (2) $\bar{a} \in A$, $\mathfrak{A} \models \psi(\bar{a})$, $\psi(\bar{x}) \in \Gamma_1$, $\psi(\bar{x})$ generates $t_{\Gamma_2}^{\mathfrak{A}}(\bar{a})$.

Let $\neg \psi(\bar{x}) \in \Gamma_2^*$ and $\mathfrak{A} \models \neg \psi(\bar{a})$. Let us show that $T \models \psi(\bar{x}) \rightarrow \neg \varphi(\bar{x})$.

Suppose the opposite: $T \cup \{\psi(\bar{x}) \wedge \varphi(\bar{x})\}$ consistent. Since T is complete for $\exists \Gamma_2$, then $T \models \exists \bar{x}(\psi(\bar{x}) \wedge \varphi(\bar{x}))$. So there is $\bar{b} \in A$ such that $\mathfrak{A} \models \psi(\bar{b}) \wedge \varphi(\bar{b})$. Let $\theta(\bar{x}) \in \Gamma_1$, $\mathfrak{A} \models \theta(\bar{b})$ and $\theta(\bar{x})$ generates $t^{\mathfrak{A}}_{\Gamma_2}(\bar{b})$ by (2). Note that $T \models \theta(\bar{x}) \wedge \varphi(\bar{x})$ (1) as well as $T \vdash \psi(\bar{x}) \wedge \neg\theta(\bar{x})$ (2).

Since $\neg\theta(\bar{x}) \in \Gamma_1^* \subset \Gamma_2$ it follows from (2) that $\neg\theta(\bar{x}) \notin t^{\mathfrak{A}}_{\Gamma_2}(\bar{a})$, i.e., $\mathfrak{A} \models \theta(\bar{a})$. According to (1), in this case, $\mathfrak{A} \models \varphi(\bar{a})$ must be true. Contradiction. Recall that $\triangle_\kappa \rightleftharpoons \Sigma_\kappa \cap \Pi_\kappa$.

Corollary 3.

1) If $\Gamma_1 = \Gamma_2 = \Sigma_\omega$, then $(1) \Leftrightarrow (2) \Leftrightarrow (3) \Leftrightarrow (4)$.

2) For any $\alpha \leq \beta \leq \omega$, if $\Gamma_1 = \triangle$, $\Gamma_2 = \Sigma$, T is complete for Σ, then it is true $(1) \Leftrightarrow (2) \Leftrightarrow (3) \Leftrightarrow (4)$.

Proof. 1) follows from part 5) of Lemma 1;

2) from Lemma 2.

If $\Gamma \in \{\Sigma; \Pi\}$, then $\Gamma(\mathfrak{A}, \bar{a})_{\bar{a} \in A}$ denotes the set of sentences of the form Γ in the language L that are true on $(\mathfrak{A}, \bar{a})_{\bar{a} \in A}$.

Lemma 4. If $\mathfrak{A}$ is a model of T, then following conditions are equivalent:

1) $\mathfrak{A}$ is (Γ_1, Γ_2)-*cl* algebraically prime of the theory T.

2) Every model T can be enriched to the model $T \cup \Pi(\mathfrak{A}, \bar{a})_{\bar{a} \in A}$.

3) Every model T can be enriched to the model $T \cup \Sigma(\mathfrak{A}, \bar{a})_{\bar{a} \in A}$.

Proof $3) \Leftrightarrow 2) \Leftrightarrow 1)$ obviously.

Let's show $1) \Leftrightarrow 3)$.

Let $f : \mathfrak{A} \to \mathfrak{B}$ be isomorphic embedding $\bar{a} \in A$, $\varphi(\bar{x}) \in \Sigma_1$: $\mathfrak{A} \models \varphi(\bar{a})$, $\varphi(\bar{x}) = \exists \bar{y} \psi(\bar{y}, \bar{x})$, $\psi(\bar{y}, \bar{x}) \in \Pi$, $\bar{a_1} \in A$, $\mathfrak{A} \models \psi(\bar{a_1}, \bar{a})$. Then $\mathfrak{B} \models \psi(f(\bar{a_1}), f(\bar{a}))$ due to the fact f is an isomorphic embedding.

Further, we have $\mathfrak{B} \models \exists y \psi(\bar{y}, f(\bar{a}))$ i.e., $\mathfrak{B} \models \varphi(f(\bar{a}))$. Hence $(\mathfrak{B}, f(\bar{a}))_{\bar{a} \in A}$ are the model of $T \cup \Sigma_1(\mathfrak{A}, \bar{a})_{\bar{a} \in A}$

Definition 7. Let $\Phi(x_1...x_\kappa)$ be some set of formulas of the language L from variables $x_1...x_\kappa$. We say that Γ_1 locally omitted Φ if for any formula consistent with T formulas $\varphi(x_1...x_\kappa) \in \Gamma_1$ there is such a formula $\theta(x_1...x_\kappa) \in \Phi$ such that $\varphi \wedge \neg\theta$ consistent with T.

Theorem 5. Let T be Π_2-axiomatizable consistent theory of a countable language L and for any $n < \omega$ let $\Phi(x_1...x_{m_n})$ be the set of the Π_1 are formulas of m_n variables. If T Σ_1 locally omitted every $\Phi^n, n < \omega$, then T has a countable model which omitted every set $\Phi^n, n < \omega$.

The proof can be taken from [15].

Theorem 6. Let T be a perfect Jonsson theory complete for Π_2 sentences. Then every (Σ, Σ)-*cl* algebraically prime model of theory T is an almost-weakly (Σ, Σ)-*cl* atomic model of the theory T.

Proof. Let $\mathfrak{A}$ be the (Σ, Σ)-*cl* algebraically prime model of theory T. Suppose there is a $\bar{a} \in A$, such that $t^{\mathfrak{A}}_{\Sigma}(\bar{a})$ is not be the Σ_2 is principle type. Since $\Sigma_1 \subset \Pi_2$ then by Theorem 5, there exists a model $\mathfrak{B}$ of the theory T, which omits $t^{\mathfrak{A}}_{\Sigma_1}(\bar{a})$. Let $f : \mathfrak{A} \to \mathfrak{B}$ be an embedding. Then by Lemma 4 we have $t^{\mathfrak{A}}_{\Sigma_1}(\bar{a}) \subseteq t^{\mathfrak{B}}_{\Sigma_1}(f(\bar{a}))$. It follows that $f(\bar{a})$ implements $t^{\mathfrak{A}}_{\Sigma_1}(\bar{a})$ to $\mathfrak{B}$. This contradicts Theorem 5.

Definition 8. Let t_1 be the Γ_1-type, t_2 be the Γ_2-type, then they say that t_1 and t_2 T-equivalent if $T \cup t_1 \vdash t_2$ & $T \cup t_2 \vdash t_1$. In this case, write $t_1 \sim_T t_2$. The following is known next lemma.

Lemma 7. Let T be perfect Jonsson theory complete for Π_2 sentences and $\mathfrak{A} \models T$, then there is a model $\mathfrak{B}$, such that:

1) $\mathfrak{B} \models T$;

2) $\mathfrak{A}$ is isomorphically embeddable in $\mathfrak{B}$;

3) for any $\bar{b} \in B$ $t^{\mathfrak{B}}_{\Sigma_1}(\bar{b}) \sim_T t^{\mathfrak{B}}_{\Sigma_2}(\bar{b})$.

Proof. The proof follows from [14] and the above definitions.

Theorem 8. Let T be the perfect Jonsson theory complete for Π_2 sentences. Then every (Σ, Σ)-*cl* algebraically prime model of the theory T is an almost-weakly (Σ, Σ)-*cl* atomic model of the theory T.

Proof. Firstly, we prove the following fact (F). If $\varphi(\bar{x}) \in \Sigma_1$ and $\varphi(\bar{x}) \cup T$ is consistent, then there is a formula $\psi(\bar{x}) \in \Sigma_1$ such that $T \cup \psi(\bar{x})$ is consistent and $T \models \psi(\bar{x}) \to \varphi(\bar{x})$. Indeed, let $\varphi(\bar{x}) \in \Sigma_1$ and $\varphi(\bar{x}) \cup T$ are consistent. Since that T is complete for Π_2 sentences we have $T \vdash \exists \bar{x} \varphi(\bar{x})$. Since T Π_2 is axiomatizable, then by Lemma 7 there exists a model $\mathfrak{B} \models T$, such that for any $\bar{b} \in B$ is holds

$$t^{\mathfrak{B}}_{\Sigma_1}(\bar{b}) \sim_T t^{\mathfrak{B}}_{\Sigma_2}(\bar{b}) \qquad (*)$$

Let $\bar{b} \in B$ such that $\mathfrak{B} \models \varphi(\bar{b})$. Due to (*) and the closedness concerning the conjunction of the type $t^{\mathfrak{B}}_{\Sigma_1}(\bar{b})$ there is a formula $\psi(\bar{x}) \in t^{\mathfrak{B}}_{\Sigma_1}(\bar{b})$, such that $T \vdash \psi(\bar{x}) \to \varphi(\bar{x})$. Fact (F) is proved.

Further $\mathfrak{A}$ be (Σ, Σ)-*cl* algebraically prime model of theory T, $\bar{a} \in A$, $t = t^{\mathfrak{A}}_{\Sigma}(\bar{a})$. By Theorem 6 $\mathfrak{A}$-almost-weakly (Σ, Σ)-*cl* atomic model of theory T. Therefore, there is a formula $\varphi(\bar{x}) \in \Sigma$ consistent with T, which generates $t^{\mathfrak{A}}_{\Sigma}(\bar{a})$. According to (F), there exists a formula $\psi(\bar{x}) \in \Sigma$ consistent with T, for which the following

holds: $T \vdash \psi(\bar{x}) \to \varphi(\bar{x})$. Obviously $\psi(\bar{x})$ generates $t_\Sigma^{\mathfrak{A}}(\bar{a})$. Due to the arbitrariness $\bar{a} \in A$ a model $\mathfrak{A}$ is almost-weakly (Σ, Σ)-cl atomic model of the theory T.

Remark. Let $\alpha, \beta \le \omega$, $\bar{x}^\alpha \rightleftharpoons < x_i : 1 \le i \le 1 + \alpha >$, $\bar{a}^\alpha = < a_i : 1 \le i \le 1 + \alpha >$.

Definition 9. 1) α-type is called any set of formulas consistent with T, the free variables of which are found in $\bar{x}$;

2) ω-type ρ is called Γ-ω-type, if $\rho \subseteq \Gamma$;

3) Γ-ω-type ρ is called Γ_1-principle type if there exists such a sequence $\langle \psi_n(\bar{x}^n) : 1 \le n < \omega \rangle$ Γ_1-formulas, such that:

a) $T \cup \psi_n(\bar{x}^n)$ is consistent, $1 \le n < \omega$;

b) $\psi_n(\bar{x}^n)$ generates $\rho \upharpoonright \bar{x}^n$, where $\rho \upharpoonright \bar{x}^n$ is set of formulas from ρ, the free variables of which are among $(x_1, ..., x_n)$, $1 \le n < \omega$;

c) $T \vdash \psi_n(\bar{x}^n) \leftrightarrow \exists \bar{x}_{n+1} \psi_{n+1}(\bar{x}^{n+1})$, $1 \le n < \omega$.

Definition 10. A model $\mathfrak{A}$ of the theory T is said to be the fine almost-weakly (Γ_1, Γ_2)-cl atomic model of T if each tuple of ω elements $\mathfrak{A}$ implements Γ_1-principle type Γ_2 ω-type.

Lemma 9. Let $\mathfrak{A}$ be a countable model of the perfect Jonsson theory T, $A = \bar{a}^\omega = < a_1 ..., a_n, ... >$ implements (Σ, Σ)-ω-type. $\mathfrak{B} \models T$, $\mathfrak{B}$ is the isomorphically embeddable in $\mathfrak{A}$. Then $\mathfrak{B}$ is a fine almost-weakly (Σ, Σ)-cl atomic model of T.

Proof. Let $\bar{b}^\omega = < b_1, ..., b_n, ... >$ be an arbitrary tuple of ω-elements $\mathfrak{B}$. Such that $\mathfrak{B}$ is the isomorphically embeddable in $\mathfrak{A}$, then $b_k = a_{i_k}$ for some $1 \le k < \omega$.

Let $n_k \rightleftharpoons ij : 1 \le j \le k$,

$Z_k = 1, 2, 3, .., n_k \setminus ij : 1 \le j \le k$; $\bar{y}^k = < y_1, ..., y_k >$.

Such that $\bar{a}^\omega$ implements the Σ-principal type Σ-ω-type, then there exists a sequence of Σ-formulas $\langle \psi_n(\bar{x}^n) : 1 \le n < \omega \rangle$, for which the following is true:

1) $\psi_n(\bar{x}^n) \cup T$ is consistent, $1 \le n < \omega$;

2) $\psi_n(\bar{x}^n)$ generates $t_\Sigma^{\mathfrak{A}}(\bar{a}^n)$ $1 \le n < \omega$;

3) $T \vdash \psi_n(\bar{x}^n) \leftrightarrow \exists \bar{x}^n \psi_{n+1}(\bar{x}^{n+1})$ $1 \le n < \omega$.

Let us denote by what

$$
S_k(\bar{y}^k) \rightleftharpoons \begin{cases} \psi_{n_k}(\bar{x}^{n_k}) \begin{pmatrix} x_{i1}, ..., x_{ik} \\ y_1, ..., y_k \end{pmatrix}, \text{ if } Z_k = \emptyset \\ \exists ... x_S ... \psi_{n_k}(\bar{x}^{n_k})_{S \in Z_k} \begin{pmatrix} x_{i1}, ..., x_{ik} \\ y_1, ..., y_k \end{pmatrix}, \text{ if } Z_k \ne \emptyset. \end{cases}
$$

Then it is clear that:

a) $S_k(\bar{y}^k) \in \Sigma_1$ $1 \le k < \omega$;

b) $S(\bar{y}^k)$ consistent with T, $1 \le k < \omega$;

c) $S(\bar{y}^k)$ generates $t_{\Sigma_1}^{\mathfrak{A}}(\bar{b}^k)$, $1 \le k < \omega$;

d) $T \vdash S_k(\bar{y}^k) \leftrightarrow \exists y_{k+1} S_{k+1}(\bar{y}^{k+1}), 1 \le k < \omega$.

Further such that $\mathfrak{B}$ is isomorphic embedding $\mathfrak{A}$, then $t_\Sigma^{\mathfrak{B}}(\bar{b}^k) \subseteq t_\Sigma^{\mathfrak{A}}(\bar{a}^k)$. Hence $S(\bar{y}^k)$ generates $t_\Sigma^{\mathfrak{B}}(\bar{b}^k)$ $1 \le k < \omega$. Thus, since $\bar{b}^\omega$ is arbitrary, the model $\mathfrak{B}$ is fine almost-weakly (Σ, Σ)-cl atomic model T.

Corollary 10. Let $\mathfrak{A} \models T$, $\bar{a}^\omega = A$. Then:

1) if $\bar{a}^\omega$ implements Σ is principle type Σ-ω-type, then any infinite a^ω implements some Σ-principle type Σ-ω-type;

2) if $\mathfrak{A}$ is the fine an almost-weekly (Σ, Σ)-cl atomic model T, then $\mathfrak{A}$ is almost-weakly (Σ, Σ)-cl atomic model T.

Proof. Follows from Lemma 9.

Lemma 11. Let T has a fine almost-weakly (Σ, Σ)-cl atomic model, then each (Σ, Σ)-cl algebraically prime model of theory T is a fine almost-weekly (Σ, Σ)-cl atomic model of the theory T.

Proof. Let $\mathfrak{B}$ be an arbitrary (Σ, Σ)-cl algebraically prime model of theory T, $\mathfrak{A}$ is fine an almost-weekly (Σ, Σ)-cl atomic model of the theory T, then there is an embedding $f : \mathfrak{A} \to \mathfrak{B}$. Let $\mathfrak{A}' = f[\mathfrak{B}]$. Obviously $\mathfrak{A}'$ embedded in $\mathfrak{A}$, and by Lemma 9 $\mathfrak{A}'$, therefore $\mathfrak{B}$ is also fine almost-weakly (Σ, Σ)-cl-atomic models of the theory T.

Lemma 12. Let T perfect Jonsson theory complete for Π_1-sentences. Then every fine almost-weakly (Σ, Σ)-cl-atomic models of the theory T is a (Σ, Σ)-cl algebraically prime model of T.

Proof. Let $\bar{a}^\omega = <a_1, ..., a_n, ...>$ are elements from A. Since $\bar{a}^\omega$ implements Σ_1-principal $\Sigma_1 - \omega$-type, there exists $\langle \psi_n(\bar{x}^n) : 1 \le n < \omega \rangle$- is a sequence of Σ_1-formulas for which the condition of item 3 of Definition 7 is true. Such that T is complete for Π_1-sentences, then $\mathfrak{B} \models \exists \bar{x}^n \psi_n(\bar{x}^n), 1 \le n < \omega$, where $\mathfrak{B} \models T$. Further, since $T \vdash \psi_n(\bar{x}^n) \leftrightarrow \exists \bar{x}^{n+1} \psi_{n+1}(\bar{x}^{n+1})$ for each $1 \le n < \omega$, then it is possible (step by step) to gradually find such $b_1, ..., b_n$ from B, such that $\mathfrak{B} \models \psi_n(\bar{x}^n), 1 \le n < \omega$, where $\bar{b}^n = \langle b_1, ..., b_n \rangle$. But $\psi_n(\bar{x}^n)$ generates $t_{\Sigma_1}^{\mathfrak{A}}(\bar{a}^n)$, so $t_{\Sigma_1}^{\mathfrak{A}}(\bar{a}^n) \subseteq t_{\Sigma_1}^{\mathfrak{B}}(\bar{b}^n), 1 \le n < \omega$.

Therefore, the mapping $f : \mathfrak{A} \to \mathfrak{B}$, where $f(a_n) = b_n, 1 \le n < \omega$, is an isomorphic embedding.

Theorem 13. Let T be the perfect Jonsson theory complete for Π_1-sentences and has fine almost-weakly (Σ, Σ)-*cl* atomic model. Then the following conditions are equivalent:

1) $\mathfrak{A}$ is the (Σ, Σ)-*cl* algebraically prime model of theory T.

2) $\mathfrak{A}$ is the fine almost-weakly (Σ, Σ)-*cl* atomic model of the theory T.

Proof. 1) $\Rightarrow$ 2) follows from Lemma 11. 2) $\Rightarrow$ 1) from Lemma 12.

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Анықталған жиынының тұйықтау операторы көмегімен алынған атомдық және алгебралық жай модельдер туралы

Мақалада қандай да бір бекітілген йонсондық теорияның семантикалық моделінің анықталған ішкі жиынында берілген қандайда бір тұйықтау операторының көмегімен алынған атомдық және жай модельдердің қасиеттері қарастырылған. Негізгі нәтиже ретінде атомдық және жай модельдерде анықталған эквиваленттілікті табу болып табылады, яғни бұл сәйкестік жақсы қасиеттерімен берілген қандай да бір модель бар деген шығады.

Кілт сөздер: йонсондық теориясы, семантикалық модель, жай модель, атомдық модель, алгебралық жай модель, предгеометрия, анықталған ішкі жиын.

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Об атомных и алгебраически простых моделях, полученных замыканием определимых множеств

В статье рассмотрены свойства атомных и простых моделей, полученных с помощью некоторого оператора замыкания, заданного на определимых подмножествах семантической модели некоторой фиксированной йонсоновской теории. Основным результатом явилось получение эквивалентности определенными таким образом атомной и простой моделей, причем это совпадение следует при предположении, что существует некоторая модель с хорошо заданными свойствами.

Ключевые слова: йонсоновская теория, семантическая модель, простая модель, атомная модель, алгебраически простая модель, предгеометрия, определимое подмножество.

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