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Semi-integration of certain algebraic expressions

The theory of fractional calculus developed rapidly as the applications of this branch are extensive nowadays. There is no discipline of modern engineering and science that remains untouched by the techniques of fractional calculus. In fact, one could argue that real world processes are fractional order systems in general. In this article, we obtain the semi-integrals of certain algebraic functions in terms of difference of two complete elliptic integrals of different kinds by using series manipulation technique.

Keywords: hypergeometric functions, complete elliptic integrals, Pochhammer symbol, semi-integration.

1 Introduction, definitions and preliminaries

Fractional Calculus is the integration and differentiation of non-integer (fractional) order. The concept of fractional operators has been introduced almost simultaneously with the development of the classical ones. The idea of differentiation (and integration) to a non-integer order has appeared surprisingly early in the history of the Calculus. It is mentioned in a letter dated September 30, 1695, from G.W. Leibniz to G.A. L'Hôpital, and in another letter dated May 28, 1697, from Leibniz to J. Wallis. This consequently attracted the interest of many well-known mathematicians, including Euler, Liouville, Laplace, Riemann, Grünwald, Letnikov, and many others [1; 284].

In 1731, L. Euler extended the derivative formula in general form ([2; 80, Eq.(2.37)], [1; 285, Eq.(5)]):

$$D_x^{\mu}\left\{x^{\lambda}\right\} = \frac{d^{\mu}}{dx^{\mu}}x^{\lambda} = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda+1-\mu)}x^{\lambda-\mu},$$

where μ is not restricted to integer values and μ may be an arbitrary complex number and $\Gamma(1+\lambda),\ \Gamma(1+\lambda-\mu)$ are well-defined. When μ is positive real number, then above formula stands for fractional differentiation and when μ is negative real number, then above formula represents fractional integration.

In this paper, we shall use the following standard notations:

$$\mathbb{N} :=\ \{1,2,3,\cdots\}\,;\quad \mathbb{N}_0 := \mathbb{N}\bigcup\{0\}\,;\quad \mathbb{Z}_0^- := \mathbb{Z}^-\bigcup\{0\} = \{0,-1,-2,-3,\cdots\}\,.$$

The symbols $\mathbb{C}$, $\mathbb{R}$, $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}^+$ and $\mathbb{R}^-$ denote the sets of complex numbers, real numbers, natural numbers, integers, positive and negative real numbers, respectively.

The classical Pochhammer symbol $(\alpha)_p$ $(\alpha, p \in \mathbb{C})$ is defined by ([3; 22, Eq.(1), p.32, Q.N.(8) and Q.N.(9)], see also [1; 23, Eq.(22) and Eq.(23)]).

A natural generalization of the Gaussian hypergeometric series ${}_2F_1[\alpha,\beta;\gamma;z]$ is accomplished by introducing any arbitrary number of numerator and denominator parameters [1; 42, Eq.(1)].

Each of the following results will be needed in our present study.
Some complete elliptic integrals [4; 321, Eq.(25)]:

$$\mathbf{B}(x) = \int_0^{\frac{\pi}{2}} \frac{\cos^2\theta}{\sqrt{(1-x^2\ \sin^2\theta)}}\,d\theta = \frac{\pi}{4}\,{}_2F_1\left[\begin{array}{c} \frac{1}{2},\ \frac{1}{2}; \\ \\ 2; \end{array} x^2\right];\quad |x|<1,$$

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$$\mathbf{C}(x)=\int_0^{\frac{\pi}{2}}\frac{\sin^2\theta\cos^2\theta}{\left(\sqrt{(1-x^2\ \sin^2\theta)}\right)^3}\,d\theta=\frac{\pi}{16}\,{}_2F_1\left[\begin{array}{c}\frac{3}{2},\ \frac{3}{2};\\ \\ 3;\end{array}x^2\right];\quad |x|<1,$$

$$\mathbf{D}(x)=\int_0^{\frac{\pi}{2}}\frac{\sin^2\theta}{\sqrt{(1-x^2\ \sin^2\theta)}}\,d\theta=\frac{\pi}{4}\,{}_2F_1\left[\begin{array}{c}\frac{1}{2},\ \frac{3}{2};\\ \\ 2;\end{array}x^2\right];\quad |x|<1.$$

Complete elliptic integral of second kind [4; 317, Eq.(2)]:

$$\mathbf{E}(x)=\int_0^{\frac{\pi}{2}}\sqrt{(1-x^2\ \sin^2\theta)}\,d\theta=\frac{\pi}{2}\,{}_2F_1\left[\begin{array}{c}\frac{1}{2},\ -\frac{1}{2};\\ \\ 1;\end{array}x^2\right];\quad |x|<1.$$

Complete elliptic integral of first kind [4; 317, Eq.(1)]:

$$\mathbf{K}(x)=\int_0^{\frac{\pi}{2}}\frac{d\theta}{\sqrt{(1-x^2\ \sin^2\theta)}}=\frac{\pi}{2}\,{}_2F_1\left[\begin{array}{c}\frac{1}{2},\ \frac{1}{2};\\ \\ 1;\end{array}x^2\right];\quad |x|<1.$$

See [3; 70, Q.N.(10)]:

$${}_2F_1\left[\begin{array}{c}\alpha,\ \alpha-\frac{1}{2};\\ \\ 2\alpha;\end{array}z\right]=\left(\frac{2}{1+\sqrt{(1-z)}}\right)^{2\alpha-1},\tag{1}$$

where $|z|<1$ and $2\alpha\in\mathbb{C}\backslash\mathbb{Z}_0^-$.

$${}_2F_1\left[\begin{array}{c}\alpha,\ \alpha+\frac{1}{2};\\ \\ 2\alpha;\end{array}z\right]=\frac{1}{\sqrt{(1-z)}}\left(\frac{2}{1+\sqrt{(1-z)}}\right)^{2\alpha-1},\tag{2}$$

where $|z|<1$ and $2\alpha\in\mathbb{C}\backslash\mathbb{Z}_0^-$.

Special value of the hypergeometric function [5; 474, Entry(100)]:

$${}_2F_1\left[\begin{array}{c}2,\ \frac{1}{2};\\ \\ 3;\end{array}x\right]=\frac{4}{3x^2}\left\{2-(2+x)\sqrt{1-x}\right\}.$$

Considering the work of Abramowitz *et al.* [6], Andrews [7, 1], Gradshteyn *et al.* [8], Magnus *et al.* [9], Srivastava *et al.* [10] and Qureshi *et al.* [11], we aim at obtaining the semi-integrals of certain algebraic functions. In Section 2, semi-integrals of some algebraic functions are mentioned in terms of difference of two complete elliptic integrals of different kinds. In Section 3, their proofs are given by using series manipulation technique.

2 Some results involving semi-integration

In this section, we obtain the semi-integration of some algebraic expressions in terms of difference of two complete elliptic integrals of different kinds.

$$\frac{d^{-\frac{1}{2}}}{dx^{-\frac{1}{2}}}\left\{\frac{-\sqrt{x}}{\sqrt{(1-x)}\ \left(1+\sqrt{(1-x)}\right)^2}\right\}=\frac{2}{\sqrt{\pi}}\ \left\{\mathbf{B}(\sqrt{x})-\mathbf{D}(\sqrt{x})\right\}.\tag{3}$$

$$\frac{d^{-\frac{1}{2}}}{dx^{-\frac{1}{2}}}\left\{\frac{-1}{x^{\frac{5}{2}}}\left(\frac{20-17x+x^2}{\sqrt{(1-x)}}-20+7x\right)\right\}=\frac{2}{\sqrt{\pi}}\ \left\{\mathbf{B}(\sqrt{x})-4\mathbf{C}(\sqrt{x})\right\}.\tag{4}$$

$$\frac{d^{-\frac{1}{2}}}{dx^{-\frac{1}{2}}}\left\{\frac{2-(2+x)\sqrt{(1-x)}}{x^{\frac{3}{2}}}\right\}=\frac{4}{\sqrt{\pi}}\left\{\mathbf{B}(\sqrt{x})-\frac{1}{2}\mathbf{E}(\sqrt{x})\right\}. \tag{5}$$

$$\frac{d^{-\frac{1}{2}}}{dx^{-\frac{1}{2}}}\left\{\frac{-\sqrt{x}}{\sqrt{(1-x)}\,\left(1+\sqrt{(1-x)}\right)^2}\right\}=\frac{4}{\sqrt{\pi}}\left\{\mathbf{B}(\sqrt{x})-\frac{1}{2}\mathbf{K}(\sqrt{x})\right\}. \tag{6}$$

$$\frac{d^{-\frac{1}{2}}}{dx^{-\frac{1}{2}}}\left\{\frac{3\sqrt{x}}{\sqrt{(1-x)}\,\left(1+\sqrt{(1-x)}\right)^3}\right\}=\frac{8}{\sqrt{\pi}}\left\{\mathbf{C}(\sqrt{x})-\frac{1}{4}\mathbf{D}(\sqrt{x})\right\}. \tag{7}$$

$$\frac{d^{-\frac{1}{2}}}{dx^{-\frac{1}{2}}}\left\{\sqrt{x}\;{}_4F_3\left[\begin{array}{c}\frac{1}{2},\ 2,\ \frac{73+\sqrt{145}}{32},\ \frac{73-\sqrt{145}}{32};\\ 4,\ \frac{41+\sqrt{145}}{32},\ \frac{41-\sqrt{145}}{32};\end{array} x\right]\right\}=\frac{8}{\sqrt{\pi}}\left\{\mathbf{C}(\sqrt{x})-\frac{1}{8}\mathbf{E}(\sqrt{x})\right\}. \tag{8}$$

$$\frac{d^{-\frac{1}{2}}}{dx^{-\frac{1}{2}}}\left\{\sqrt{\left(\frac{x}{1-x}\right)}\;\frac{(8\sqrt{(1-x)}-x+8)}{\left(1+\sqrt{(1-x)}\right)^4}\right\}=\frac{16}{\sqrt{\pi}}\left\{\mathbf{C}(\sqrt{x})-\frac{1}{8}\mathbf{K}(\sqrt{x})\right\}. \tag{9}$$

$$\frac{d^{-\frac{1}{2}}}{dx^{-\frac{1}{2}}}\left\{\frac{x^2-x+2\left(1-\sqrt{(1-x)}\right)}{x^{\frac{3}{2}}\sqrt{(1-x)}}\right\}=\frac{4}{\sqrt{\pi}}\left\{\mathbf{D}(\sqrt{x})-\frac{1}{2}\mathbf{E}(\sqrt{x})\right\}. \tag{10}$$

$$\frac{d^{-\frac{1}{2}}}{dx^{-\frac{1}{2}}}\left\{\frac{\sqrt{x}}{\sqrt{(1-x)}\,\left(1+\sqrt{(1-x)}\right)^2}\right\}=\frac{4}{\sqrt{\pi}}\left\{\mathbf{D}(\sqrt{x})-\frac{1}{2}\mathbf{K}(\sqrt{x})\right\}. \tag{11}$$

$$\frac{d^{-\frac{1}{2}}}{dx^{-\frac{1}{2}}}\left\{\sqrt{\left(\frac{x}{1-x}\right)}\right\}=\frac{2}{\sqrt{\pi}}\left\{\mathbf{K}(\sqrt{x})-\mathbf{E}(\sqrt{x})\right\}. \tag{12}$$

3 Demonstration of the semi-integrals

Proof of the result (3):

$$\frac{d^{-\frac{1}{2}}}{dx^{-\frac{1}{2}}}\left\{\frac{-\sqrt{x}}{\sqrt{1-x}\,(1+\sqrt{1-x})^2}\right\}=\frac{-1}{4}\frac{d^{-\frac{1}{2}}}{dx^{-\frac{1}{2}}}\left\{\sqrt{x}\,\frac{1}{\sqrt{1-x}}\left(\frac{2}{1+\sqrt{1-x}}\right)^2\right\}. \tag{13}$$

Using equation (2) in equation (13), we get

$$\frac{d^{-\frac{1}{2}}}{dx^{-\frac{1}{2}}}\left\{\frac{-\sqrt{x}}{\sqrt{1-x}\,(1+\sqrt{1-x})^2}\right\}=\frac{-1}{4}\frac{d^{-\frac{1}{2}}}{dx^{-\frac{1}{2}}}\left\{\sqrt{x}\;{}_2F_1\left[\begin{array}{c}\frac{3}{2},\ 2;\\ 3;\end{array} x\right]\right\}=$$

$$=\frac{-1}{4}\sum_{n=0}^{\infty}\frac{\left(\frac{3}{2}\right)_n(2)_n}{(3)_n\;n!}\;\frac{d^{-\frac{1}{2}}}{dx^{-\frac{1}{2}}}\;x^{n+\frac{1}{2}}=$$

$$=\frac{-1}{4}\sum_{n=0}^{\infty}\frac{\left(\frac{3}{2}\right)_n(2)_n}{(3)_n}\;\frac{\Gamma\left(n+\frac{3}{2}\right)}{\Gamma\left(n+2\right)}\;\frac{x^{n+1}}{n!}=$$

$$=\frac{-x\sqrt{\pi}}{8}\sum_{n=0}^{\infty}\frac{\left(\frac{3}{2}\right)_n\left(\frac{3}{2}\right)_n}{(3)_n}\;\frac{x^n}{n!}=$$

$$= \frac{-x\sqrt{\pi}}{8} \; {}_2F_1\left[\begin{matrix} \frac{3}{2}, \; \frac{3}{2}; \\ \\ 3; \end{matrix} \; x \right]. \tag{14}$$

From right-hand side of equation (3), we have

$$\frac{2}{\sqrt{\pi}} \; \{\mathbf{B}(\sqrt{x}) - \mathbf{D}(\sqrt{x})\} = \frac{\sqrt{\pi}}{2} \left\{ {}_2F_1\left[\begin{matrix} \frac{1}{2}, \; \frac{1}{2}; \\ \\ 2; \end{matrix} \; x \right] - {}_2F_1\left[\begin{matrix} \frac{1}{2}, \; \frac{3}{2}; \\ \\ 2; \end{matrix} \; x \right] \right\} =$$

$$= \frac{\sqrt{\pi}}{2} \left\{ \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{2}\right)_n \; x^n}{(2)_n \, n!} - \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n \left(\frac{3}{2}\right)_n \; x^n}{(2)_n \, n!} \right\} =$$

$$= \frac{\sqrt{\pi}}{2} \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n \; x^n}{(2)_n \, n!} \; \left\{ \left(\frac{1}{2}\right)_n - \left(\frac{3}{2}\right)_n \right\} =$$

$$= -\sqrt{\pi} \; \sum_{n=1}^{\infty} \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{2}\right)_n \; x^n}{(2)_n \, (n-1)!}.$$

Replacing n by $n+1$, we get

$$\frac{2}{\sqrt{\pi}} \; \{\mathbf{B}(\sqrt{x}) - \mathbf{D}(\sqrt{x})\} = \frac{-x\sqrt{\pi}}{8} \; \sum_{n=0}^{\infty} \frac{\left(\frac{3}{2}\right)_n \left(\frac{3}{2}\right)_n}{(3)_n} \; \frac{x^n}{n!} =$$

$$= \frac{-x\sqrt{\pi}}{8} \; {}_2F_1\left[\begin{matrix} \frac{3}{2}, \; \frac{3}{2}; \\ \\ 3; \end{matrix} \; x \right]. \tag{15}$$

From equations (14) and (15), we arrive at the result (3).

Proof of the results (4) to (8):

The proof of the results (4) to (8) follow the same steps as in the proof of the result (3). So we omit the details.

Proof of the result (9):

$$\frac{d^{-\frac{1}{2}}}{dx^{-\frac{1}{2}}} \; \left\{ \sqrt{\frac{x}{1-x}} \; \frac{(8\,\sqrt{1-x} - x + 8)}{\left(1+\sqrt{1-x}\right)^4} \right\} = \frac{d^{-\frac{1}{2}}}{dx^{-\frac{1}{2}}} \; \left\{ \sqrt{x} \; \left(\frac{8\sqrt{1-x} - (x-8)}{\left(1+\sqrt{1-x}\right)^4 \sqrt{1-x}} \right) \right\} =$$

$$= \frac{d^{-\frac{1}{2}}}{dx^{-\frac{1}{2}}} \; \left\{ \sqrt{x} \; \left(\frac{8\sqrt{1-x}\left(1+\sqrt{1-x}\right) + 7x}{\left(1+\sqrt{1-x}\right)^4 \sqrt{1-x}} \right) \right\} =$$

$$= \frac{d^{-\frac{1}{2}}}{dx^{-\frac{1}{2}}} \; \left\{ \sqrt{x} \; \left(\frac{8}{\left(1+\sqrt{1-x}\right)^3} + \frac{7x}{\left(1+\sqrt{1-x}\right)^4 \sqrt{1-x}} \right) \right\} =$$

$$= \frac{d^{-\frac{1}{2}}}{dx^{-\frac{1}{2}}} \; \left\{ \sqrt{x} \; \left(\frac{2}{1+\sqrt{1-x}} \right)^3 + \frac{7x}{16\sqrt{1-x}} \left(\frac{2}{1+\sqrt{1-x}} \right)^4 \right\}. \tag{16}$$

Using equations (1) and (2) in equation (16), we get

$$\frac{d^{-\frac{1}{2}}}{dx^{-\frac{1}{2}}} \; \left\{ \sqrt{\frac{x}{1-x}} \; \frac{(8\,\sqrt{1-x} - x + 8)}{\left(1+\sqrt{1-x}\right)^4} \right\} = \frac{d^{-\frac{1}{2}}}{dx^{-\frac{1}{2}}} \; \left\{ \sqrt{x} \; {}_2F_1\left[\begin{matrix} \frac{3}{2}, \; 2; \\ \\ 4; \end{matrix} \; x \right] + \frac{7x}{16} \; {}_2F_1\left[\begin{matrix} \frac{5}{2}, \; 3; \\ \\ 5; \end{matrix} \; x \right] \right\} =$$

$$= \sum_{n=0}^{\infty} \frac{\left(\frac{3}{2}\right)_n (2)_n}{(4)_n \;\; n!} \; \frac{d^{-\frac{1}{2}}}{dx^{-\frac{1}{2}}} \; x^{n+\frac{1}{2}} + \frac{7}{16} \; \sum_{n=0}^{\infty} \frac{\left(\frac{5}{2}\right)_n (3)_n}{(5)_n \;\; n!} \; \frac{d^{-\frac{1}{2}}}{dx^{-\frac{1}{2}}} \; x^{n+\frac{3}{2}} =$$

$$= \sum_{n=0}^{\infty} \frac{\left(\frac{3}{2}\right)_n (2)_n}{(4)_n} \frac{\Gamma\left(n+\frac{3}{2}\right)}{\Gamma(n+2)} \frac{x^{n+1}}{n!} + \frac{7}{16} \sum_{n=0}^{\infty} \frac{\left(\frac{5}{2}\right)_n (3)_n}{(5)_n} \frac{\Gamma\left(n+\frac{5}{2}\right)}{\Gamma(n+3)} \frac{x^{n+2}}{n!} =$$

$$= \frac{x\sqrt{\pi}}{2} \sum_{n=0}^{\infty} \frac{\left(\frac{3}{2}\right)_n \left(\frac{3}{2}\right)_n}{(4)_n} \frac{x^n}{n!} + \frac{21x^2\sqrt{\pi}}{128} \sum_{n=0}^{\infty} \frac{\left(\frac{5}{2}\right)_n \left(\frac{5}{2}\right)_n}{(5)_n} \frac{x^n}{n!} =$$

$$= \frac{x\sqrt{\pi}}{2}\, {}_2F_1\left[\begin{array}{c} \frac{3}{2},\ \frac{3}{2}; \\ \\ 4; \end{array} x\right] + \frac{21x^2\sqrt{\pi}}{128}\, {}_2F_1\left[\begin{array}{c} \frac{5}{2},\ \frac{5}{2}; \\ \\ 5; \end{array} x\right]. \tag{17}$$

From right-hand side of equation (9), we have

$$\frac{16}{\sqrt{\pi}} \left\{\mathbf{C}(\sqrt{x}) - \frac{1}{8}\mathbf{K}(\sqrt{x})\right\} = \sqrt{\pi}\left\{ {}_2F_1\left[\begin{array}{c} \frac{3}{2},\ \frac{3}{2}; \\ \\ 3; \end{array} x\right] - {}_2F_1\left[\begin{array}{c} \frac{1}{2},\ \frac{1}{2}; \\ \\ 1; \end{array} x\right]\right\} =$$

$$= \sqrt{\pi}\left\{ \sum_{n=0}^{\infty} \frac{\left(\frac{3}{2}\right)_n \left(\frac{3}{2}\right)_n x^n}{(3)_n\, n!} - \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{2}\right)_n x^n}{(1)_n\, n!} \right\} =$$

$$= \sqrt{\pi} \sum_{n=0}^{\infty} \frac{x^n}{n!} \left\{ \frac{\left(\frac{3}{2}\right)_n \left(\frac{3}{2}\right)_n}{(3)_n} - \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{2}\right)_n}{(1)_n} \right\} =$$

$$= \frac{5\sqrt{\pi}}{2} \sum_{n=1}^{\infty} \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{2}\right)_n \left(\frac{12}{7}\right)_n x^n}{\left(\frac{5}{7}\right)_n (3)_n (n-1)!}.$$

Replacing n by $n+1$, we get

$$\frac{16}{\sqrt{\pi}} \left\{\mathbf{C}(\sqrt{x}) - \frac{1}{8}\mathbf{K}(\sqrt{x})\right\} = \frac{x\sqrt{\pi}}{2} \sum_{n=0}^{\infty} \frac{\left(\frac{3}{2}\right)_n \left(\frac{3}{2}\right)_n}{(4)_n} \frac{x^n}{n!} \left(1 + \frac{7n}{12}\right) =$$

$$= \frac{x\sqrt{\pi}}{2} \left\{ \sum_{n=0}^{\infty} \frac{\left(\frac{3}{2}\right)_n \left(\frac{3}{2}\right)_n}{(4)_n} \frac{x^n}{n!} + \frac{7}{12} \sum_{n=1}^{\infty} \frac{\left(\frac{3}{2}\right)_n \left(\frac{3}{2}\right)_n}{(4)_n} \frac{x^n}{(n-1)!} \right\} =$$

$$= \frac{x\sqrt{\pi}}{2} \sum_{n=0}^{\infty} \frac{\left(\frac{3}{2}\right)_n \left(\frac{3}{2}\right)_n}{(4)_n} \frac{x^n}{n!} + \frac{21x^2\sqrt{\pi}}{128} \sum_{n=0}^{\infty} \frac{\left(\frac{5}{2}\right)_n \left(\frac{5}{2}\right)_n}{(5)_n} \frac{x^n}{n!} =$$

$$= \frac{x\sqrt{\pi}}{2}\, {}_2F_1\left[\begin{array}{c} \frac{3}{2},\ \frac{3}{2}; \\ \\ 4; \end{array} x\right] + \frac{21x^2\sqrt{\pi}}{128}\, {}_2F_1\left[\begin{array}{c} \frac{5}{2},\ \frac{5}{2}; \\ \\ 5; \end{array} x\right]. \tag{18}$$

From equations (17) and (18), we arrive at the result (9).

Proof of the results (10), (11) and (12):

The results (10), (11) and (12) are obtained in a similar manner by following the same steps as in the proof of the result (3) and making use of the equation (2). So we omit the details here.

4 Concluding remarks and observations

In this paper, we have obtained the semi-integrals of certain algebraic functions in terms of difference of two complete elliptic integrals of different kinds by using series manipulation technique. We conclude this paper with the remark that the semi-integrals of various other functions can be derived in an analogous manner. Moreover, the results deduced above are expected to lead some potential applications in several fields of Applied Mathematics, Statistics and Engineering Sciences.

Conflicts of interest: The authors declare that there are no conflicts of interest.

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Кейбір алгебралық өрнектерді жартылай интегралдау

Бөлшекті есептеу теориясы тез дамып келеді, себебі қазіргі уақытта осы облыстың қосымшалары өте кең. Қазіргі заманғы техника мен ғылымның бірде-бір пәні бөлшекті есептеу әдістерінен тыс қалған жоқ. Шын мәнінде, нақты әлем процестері бөлшекті ретті жүйелер деп айтуға болады. Мақалада алгебралық функциялардың жартылай интегралдарын қатарлармен манипуляциялау техникасын қолдана отырып, әр түрлі екі толық эллиптикалық интегралдардың айырымы тұрғысынан алынған.

Кілт сөздер: гипергеометриялық функциялар, толық эллиптикалық интегралдар, Похгаммер символы, жартылайинтегралдау.

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Полуинтегрирование некоторых алгебраических выражений

Теория дробного исчисления быстро развивается, так как приложения этой области в настоящее время очень широки. Ни одна дисциплина современной техники и науки, в целом, не остается нетронутой методами дробного исчисления. На самом деле можно утверждать, что процессы реального мира являются системами дробного порядка. В статье получены полуинтегралы некоторых алгебраических функций в терминах разности двух полных эллиптических интегралов разных видов с помощью техник и манипулирования рядами.

Ключевые слова: гипергеометрические функции, полные эллиптические интегралы, символ Похгаммера, полуинтегрирование.