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Boundary value problem for fractional diffusion equation in a curvilinear angle domain

We consider a boundary value problem for the fractional diffusion equation in an angle domain with a curvilinear boundary. Existence and uniqueness theorems for solutions are proved. It is shown that Holder continuity of the curvilinear boundary ensures the existence of solutions. The uniqueness is proved in the class of functions that vanish at infinity with a power weight. The solution to the problem is constructed explicitly in terms of the solution of the Volterra integral equation.

Keywords: noncylindrical domain, curvilinear angle domain, boundary value problem, fractional diffusion equation.

Introduction and problem statement

Consider the equation

$$\left(\frac{\partial^\alpha}{\partial y^\alpha} - \frac{\partial^2}{\partial x^2}\right) u(x,y) = f(x,y), \qquad (0 < \alpha \le 1) \tag{1}$$

where $\frac{\partial^\alpha}{\partial y^\alpha}$ denotes a fractional derivative with respect to y of order α with the origin at the point $x = 0$.

The fractional diffusion equations and their generalizations have attracted great attention in recent years. Research of (1) began in works [1–4].

To give an idea of the variety of problems considered for this equation and the multiplicity of approaches to studying them, we mention [5–33]. An overview is provided in [31]. A more detailed survey can be found in the article [34]. We also point out the monographs [35–37], which reflect many of these approaches and contain vast bibliographies concerning the issue.

Interest in the study of fractional differential equations is also fueled by applications in physics and simulation (see e.g. [38–41]).

Fractional differentiation is given in the Riemann-Liouville sense [38], i.e.

$$\frac{\partial^\alpha}{\partial y^\alpha} u(x,y) = D^\alpha_{0y} u(x,y) = \frac{\partial}{\partial y} D^{\alpha-1}_{0y} u(x,y)$$

and

$$D^{\alpha-1}_{0y} u(x,y) = \frac{1}{\Gamma(1-\alpha)} \int_0^y u(t)(y-t)^{-\alpha}\, dt.$$

We will consider the equation (1) in a curivilinear angle domain Ω that is defined by

$$\Omega = \{(x,y) : \, y > 0, \; x > z(y)\},$$

where $z(y)$ is a non-decreasing continuous function such that $z(0) = 0$.

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A function $u(x, y)$ is called a regular solution of equation (1) in the domain Ω if $y^{1-\alpha}u(x, y) \in C\left(\overline{\Omega}\right)$, and, moreover, $u(x, y)$ has continuous derivatives in Ω with respect to x up to the second order, and the function $D_{0y}^{\alpha-1}u(x, y)$ is continuously differentiable as function of y for a fixed x at interior points of Ω, and $u(x, y)$ satisfies equation (1) at all points of Ω.

Our purpose is to solve the following problem: *find a regular solution of the equation* (1) *in the domain* Ω *satisfying initial and boundary value conditions*

$$\lim_{y\to 0} D_{0y}^{\alpha-1}u(x, y) = \tau(x) \qquad (x > 0) \tag{2}$$

and

$$u(z(y), y) = \varphi(y) \qquad (y > 0) \tag{3}$$

where $\tau(x)$ and $\varphi(y)$ are given continuous functions.

The problems in domains with curvilinear boundary were considered in [42], [26]. For the problems for parabolic equations in angles domains, we refer to [43–46].

1 Notations and preliminaries

In what follows, we use the denotations

$$w_\mu(x, y) = x^{\mu-1}\phi\left(-\beta, \mu; -|x|\, y^{-\beta}\right), \tag{4}$$

$$w(x, y) = w_0(x, y), \qquad \text{and} \qquad \beta = \frac{\alpha}{2}.$$

In (4), ϕ denotes the Wright function [47], [48],

$$\phi\,(a, b;\, z) = \sum_{k=0}^{\infty} \frac{z^k}{k!\,\Gamma(ak + b)} \qquad (a > -1).$$

The asymptotics of the Wright function can be given in the form [48]

$$\phi(-\beta, \mu; -t) = \exp\left(-\rho\, t^\sigma\right)\left[A\, t^\delta + O\left(t^{\delta-\sigma}\right)\right] \qquad (t \to \infty), \tag{5}$$

where $\beta \in (0, 1)$, $\sigma = \frac{1}{1-\beta}$, $\rho = (1-\beta)\beta^{\frac{\beta}{1-\beta}}$, and $\delta = \frac{1-2\mu}{2(1-\beta)}$.

In particular, formula (5) implies

$$|w_\mu(x, y)| \leq C\, |x|^{-\theta} y^{\beta\theta+\mu-1}, \tag{6}$$

where

$$\theta \geq \begin{cases} 0, & (-\mu) \notin \mathbb{N} \cup \{0\}, \\ -1, & (-\mu) \in \mathbb{N} \cup \{0\}, \end{cases} \qquad \text{and} \qquad C = C(\beta, \mu, \theta).$$

Here and subsequently, by C we denote positive constants, which may be different in different cases, indicating in parentheses the parameters on which they depend, if necessary, $C = C(\alpha, \beta, ...)$.

The differentiation formulas for the Wright function [48]

$$\frac{d}{dx}\,\phi(-\beta,\, \mu;\, x) = \phi(-\beta,\, \mu - \beta;\, x)$$

and [49]

$$D_{0y}^{\nu}\left[y^{\mu-1}\phi\left(-\beta,\, \mu;\, -\frac{c}{y^\beta}\right)\right] = y^{\mu-\nu-1}\phi\left(-\beta,\, \mu-\nu;\, -\frac{c}{y^\beta}\right) \quad (c > 0)$$

give

$$\frac{\partial}{\partial x}\, w_\mu(x,y) = -sign(x) w_{\mu-\beta}(x,y), \qquad D_{0y}^{\nu}\, w_\mu(x,y) = w_{\mu-\nu}(x,y), \tag{7}$$

and

$$\left(D_{0y}^{\alpha} - \frac{\partial^2}{\partial x^2}\right) w_\mu(x,y) = 0 \qquad (\mu \in \mathbb{R}, \quad |x| > 0, \quad y > 0).$$

We need later the following two statements.

Lemma 1. Let $\psi(t) \in L(0,y)$, let the function $\psi(t)$ be left continuous at the point $t = y$, and let

$$0 \le z(y) - z(t) \le C(y-y)^{\delta} \qquad (\delta > \beta,\ 0 < t < y).$$

Then

$$\lim_{\substack{x \to z(y) \\ x > z(y)}} \int_0^y \psi(t) w(x - z(t), y - t)\, dt = \psi(y) + \int_0^y \psi(t) w(z(y) - z(t), y - t)\, dt.$$

Lemma 2. Let $t^{1-\alpha} q(t) \in C[0,y]$, let $q(t)$ be Hölder continuous in a left neighborhood of y with an exponent $\delta > \alpha$,

$$|q(t) - q(y)| < C(y-t)^{\delta} \qquad (y - \varepsilon < t < y, \quad \varepsilon > 0),$$

and let

$$t^{1-\alpha} q(t) \le y^{1-\alpha} q(y) \qquad \text{for all} \quad t \in (0,y).$$

Then

$$\left(D_{0t}^{\alpha} q\right)_{t=y} \ge 0,$$

and $\left(D_{0t}^{\alpha} q\right)_{t=y} = 0$ if and only if $q(t) = C t^{\alpha-1}$.

The proof of Lemma 1 is given in [26]. Lemma 2 is proved in [42] (see also [50]).

2 Existence theorem

Theorem 1. Let

$$0 \le z(y_2) - z(y_1) \le C\,(y_2 - y_1)^{\delta} \qquad (0 \le y_1 < y_2,\ \delta > \beta), \tag{8}$$

$$\tau(x) \in C[0,\infty), \qquad y^{1-\alpha}\varphi(y) \in C[0,\infty), \qquad \tau(0) = \Gamma(\alpha)\left[y^{1-\alpha}\varphi(y)\right]_{y=0}, \tag{9}$$

$$y^{1-\alpha} f(x,y) \in C(\overline{\Omega}),$$

$$\lim_{x\to\infty} \tau(x) \exp\left(-\omega x^{\frac{2}{2-\alpha}}\right) = 0, \tag{10}$$

$$\lim_{x\to\infty} y^{1-\alpha} f(x,y) \exp\left(-\omega x^{\frac{2}{2-\alpha}}\right) = 0, \tag{11}$$

for every $\omega > 0$, and let the function $f(x,y)$ be representable in the form

$$f(x,y) = D_{0y}^{-\delta} g(x,y) \tag{12}$$

for some $\delta > \beta$ and $y^{1-\alpha+\delta} g(x,y) \in C(\overline{\Omega})$.

Then there exists a solution to problem (1), (2), and (3); it can be represented in the form:

$$u(x,y) = \int_0^y \psi(t) w(x - z(t), y - t)\, dt + T(x,y) + F(x,y), \tag{13}$$

where

$$T(x,y) = \frac{1}{2}\int_0^\infty \tau(s) w_\beta(x-s,y)\, ds, \qquad F(x,y) = \frac{1}{2}\int_0^y \int_{z(t)}^\infty f(s,t) w_\beta(x-s,y-t)\, ds\, dt,$$

and the function $\psi(x)$ is a solution of the integral equation

$$\psi(y) + \int_0^y \psi(t) w(z(y)-z(t), y-t)\, dt = \varphi(y) - T(z(y),y) - F(z(y),y). \tag{14}$$

Proof. Consider separately each of three summands on the right side of (13), namely $F(x,y)$, $T(x,y)$, and $\Psi(x,y)$, where

$$\Psi(x,y) = \int_0^y \psi(t) w(x-z(t), y-t)\, dt.$$

Let us start with $F(x,y)$. By (5), (6), and (7), with (11) and (12), we get

$$|F(x,y)| \leq C y^{3\beta-1},$$

and

$$\frac{\partial^2}{\partial x^2} F(x,y) = \frac{1}{2}\frac{\partial}{\partial x}\int_0^y \int_{z(t)}^\infty \operatorname{sign}(x-s)\, g(s,t) w_\delta(x-s,y-t)\, ds\, dt =$$

$$= -\int_0^y \frac{(y-t)^{\delta-1}}{\Gamma(\delta)} g(x,t) ds\, dt + \frac{1}{2}\int_0^y \int_{z(t)}^\infty g(s,t) w_{\delta-\beta}(x-s,y-t)\, ds\, dt =$$

$$= -f(x,y) + \frac{1}{2} D_{0y}^\alpha \int_0^y \int_{z(t)}^\infty f(s,t) w_\beta(x-s,y-t)\, ds\, dt.$$

This means that

$$y^{1-\alpha} F(x,y) \in C\left(\overline{\Omega}\right), \qquad \lim_{y\to 0} D_{0y}^{\alpha-1} F(x,y) = 0, \tag{15}$$

and

$$\left(D_{0y}^\alpha - \frac{\partial^2}{\partial x^2}\right) F(x,y) = f(x,y).$$

Consider now $T(x,y)$. By (5), (7), and (10), one can check that

$$\left(D_{0y}^\alpha - \frac{\partial^2}{\partial x^2}\right) T(x,y) = 0.$$

Rewrite $T(x,y)$ in the form

$$T(x,y) = \frac{1}{2}\int_0^\infty \left[\tau(s) - \tau(x)\right] w_\beta(x-s,y)\, ds + \frac{\tau(x)}{2}\int_0^\infty w_\beta(x-s,y)\, ds.$$

Using (5), (6), and (7), leads to

$$\int_0^\infty w_\beta(x-s,y)\, ds = \left(\int_0^x + \int_x^\infty\right) w_\beta(x-s,y)\, ds = \frac{2y^{\alpha-1}}{\Gamma(\alpha)} - w_\alpha(x,y)$$

and

$$\left|\int_0^\infty \left[\tau(s)-\tau(x)\right] w_\beta(x-s,y)\, ds\right| \leq \left(\int_0^{x-\varepsilon} + \int_{x+\varepsilon}^\infty\right) |\tau(s)-\tau(x)|\, w_\beta(x-s,y)\, ds +$$

$$+C\sup_{t\in(x-\varepsilon,x+\varepsilon)}|\tau(t)-\tau(y)|\int_{x-\varepsilon}^{x+\varepsilon}w_\beta(x-s,y)\,ds\le Cy^{\alpha-1}\left[y^\theta+\omega(\varepsilon)\right],$$

where $\theta>0$ and $\omega(\varepsilon)\to 0$ as $\varepsilon\to 0$. This gives that

$$y^{1-\alpha}T(x,y)=C\left(\overline{\Omega}\setminus\{0,0\}\right)\qquad\text{and}\qquad D_{0y}^{\alpha-1}T(x,y)=\tau(x).$$

Let us examine the behavior of $T(x,y)$ in a neighborhood of $(0,0)$. Assume that $\sigma=\sigma(y)$ is a continuous function defined in a right neighborhood of the point $y=0$ such that $(\sigma(y),y)\in\Omega$ (i.e. $\sigma(y)>z(y)$) and $\sigma(y)\to 0$ as $y\to 0$. It is easy to see that

$$T(\sigma(y),y)=\frac{y^{\alpha-1}}{2}\int_0^{\sigma/y^\beta}\tau\left(sy^\beta\right)\phi\left(-\beta,\beta;\,s-\frac{\sigma}{y^\beta}\right)ds+$$

$$+\frac{y^{\alpha-1}}{2}\int_0^\infty\tau\left(sy^\beta+\sigma\right)\phi\left(-\beta,\beta;\,-s\right)ds$$

and

$$\lim_{y\to 0}y^{1-\alpha}T(\sigma(y),y)=\frac{\tau(0)}{\Gamma(\alpha)}-\frac{\tau(0)}{2}\lim_{y\to 0}\phi\left(-\beta,\alpha;\,-\frac{\sigma(y)}{y^\beta}\right). \tag{16}$$

Finally, consider $\Psi(x,y)$. By (6) and (8), we have

$$|w(z(y)-z(t),y-t)|\le C|z(y)-z(t)|(y-t)^{-\beta-1}\le C(y-t)^{\delta-\beta-1}.$$

This means that integral equation (14) has a unique solution $\psi(y)$. A simple computation gives

$$\left(D_{0y}^\alpha-\frac{\partial^2}{\partial x^2}\right)\Psi(x,y)=0,$$

and

$$\lim_{y\to 0}D_{0y}^{\alpha-1}\Psi(x,y)=0\qquad\text{and}\qquad y^{1-\alpha}\Psi(x,y)=C\left(\overline{\Omega}\setminus\{0,0\}\right).$$

Now we investigate the behavior of $\Psi(x,y)$ in a neighborhood of $(0,0)$. It follows from formulas (9), (14), (15), (16) that

$$y^{1-\alpha}\psi(y)\in C[0,\infty)\qquad\text{and}\qquad\lim_{y\to 0}y^{1-\alpha}\psi(y)=\frac{\tau(0)}{2\Gamma(\alpha)}. \tag{17}$$

Set $\psi_0(y)=y^{1-\alpha}\psi(y)$, $(\sigma(y),y)\in\Omega$ $(\sigma(y)>z(y))$, and $\sigma(y)\to 0$ as $y\to 0$. It is easy to see that

$$\Psi(\sigma(y),y)=\int_0^y\psi(t)\left[w(\sigma(y)-z(t),y-t)-w(\sigma(y)-z(y),y-t)\right]dt+$$

$$+\int_0^y t^{1-\alpha}\left[\psi_0(t)-\psi_0(y)\right]w(\sigma(y)-z(y),y-t)\,dt+$$

$$+\psi_0(y)\int_0^y t^{1-\alpha}w(\sigma(y)-z(y),y-t)\,dt.$$

This yields that

$$\lim_{y\to 0}y^{1-\alpha}\Psi(\sigma(y),y)=\psi_0(0)\lim_{y\to 0}\phi\left(-\beta,\alpha;\,-\frac{\sigma(y)-z(y)}{y^\beta}\right).$$

Taking into account (16), (17), and

$$\lim_{y\to 0}\phi\left(-\beta,\alpha;\,-\frac{\sigma(y)-z(y)}{y^\beta}\right)=\lim_{y\to 0}\phi\left(-\beta,\alpha;\,-\frac{\sigma(y)}{y^\beta}\right),$$

we get
$$y^{1-\alpha}\left[\Psi(x,y)+T(x,y)\right]\in C(\overline{\Omega})$$
and, consequently,
$$y^{1-\alpha}u(x,y)\in C(\overline{\Omega}).$$

To complete the proof, it remains to show that the function (13) satisfies (3). Indeed, by Lemma 1 we have
$$u(z(y),y)=\lim_{x\to z(y)}\Psi(x,y)+T\left(z(y),y\right)+F\left(z(y),y\right)=$$
$$=\psi(y)+\Psi(z(y),y)+T\left(z(y),y\right)+F\left(z(y),y\right).$$
Due to (14) we get
$$\psi(y)+\Psi(z(y),y)=\varphi(y)-T\left(z(y),y\right)-T\left(z(y),y\right).$$
Combining the last two equations leads to (3).

3 Solution uniqueness

Theorem 2. Let $z(y)\in C[0,\infty)$. There exists at most one regular solution to the problem (1), (2), and (3), satisfying
$$\lim_{x\to\infty}\sup_{0<y<T}\left|y^{1-\alpha}u(x,y)\right|=0 \tag{18}$$
for every $T>0$.

Proof. Let $u(x,y)$ be a solution of the homogeneous problem
$$\left(\frac{\partial^\alpha}{\partial y^\alpha}-\frac{\partial^2}{\partial x^2}\right)u(x,y)=0,\qquad \left(D_{0y}^{\alpha-1}u\right)_{y=0}=0,\qquad u(z(y),y)=0. \tag{19}$$
Set
$$\Omega_T=\Omega\cap\{y\le T\}\quad\text{for some}\quad T>0,\quad\text{and}\quad v(x,y)=y^{1-\alpha}u(x,y).$$
By (18), (19), and the equality $\left(D_{0y}^{\alpha-1}u\right)_{y=0}=\Gamma(\alpha)v(x,0)$, we can conclude that there is a point $(\xi,\eta)\in\Omega_T$ such that
$$v(\xi,\eta)\ge v(x,y)\qquad\text{for all}\quad(x,y)\in\Omega_T. \tag{20}$$

Lemma 2, with (19) and (20), yields that
$$\left[D_{0y}^{\alpha}u(\xi,y)\right]_{y=\eta}\ge 0\quad\text{and}\quad u_{xx}(\xi,\eta)\le 0.$$
This means that $\left[D_{0y}^{\alpha}u(\xi,y)\right]_{y=\eta}=0$, and consequently $u(\xi,y)=Cy^{\alpha-1}$. Taking into account that $v(x,0)=0$, this implies that $C=0$. Thus, we get
$$u(x,y)\le 0\qquad\text{for all}\quad(x,y)\in\Omega_T.$$
Similarly, considering the function $v(x,y)=-y^{1-\alpha}u(x,y)$ gives
$$u(x,y)\ge 0\qquad\text{for all}\quad(x,y)\in\Omega_T.$$
Thus, we can conclude that $u(x,y)=0$ for all $(x,y)\in\Omega_T$. The arbitrary choice of the number T implies that $u(x,y)=0$ in Ω.

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References

1 Wyss W. The fractional diffusion equation / W. Wyss // J. Math. Phys. — 1986. — № 27. — P. 2782–2785.

2 Schneider W.R. Fractional diffusion and wave equations / W.R. Schneider, W. Wyss // J. Math. Phys. — 1989. — № 30. — P. 134–144.

3 Kochubei A.N. Diffusion of fractional order / A.N. Kochubei // Differ. Equ. — 1990. — № 26. — P. 485–492.

4 Fujita Ya. Integrodifferential equation which interpolates the heat equation and the wave equation I, II / Ya. Fujita // Osaka J. Math. — 1990. — № 27. — P. 309–321, 797–804.

5 Mainardi F. The fundamental solutions for the fractional diffusion-wave equation / F. Mainardi // Appl. Math. Lett. — 1996. — № 9. — P. 23–28.

6 Engler H. Similiraty solutions for a class of hyperbolic integrodifferential equations / H. Engler // Differential Integral Equations. — 1997. — № 10. — P. 815–840.

7 Шогенов В.Х. Обобщенное уравнение переноса и дробные производные / В.Х. Шогенов, С.К. Кумыкова, М.Х. Шхануков-Лафишев // Докл. Нац. акад. наук Украины. — 1997. — №‘12. — С. 47–54.

8 Buckwar E. Invariance of a partial differential equation of fractional order under the Lie group of scaling transformations / E. Buckwar, Yu. Luchko // J. Math. Anal. Appl. — 1998. — № 227. — P. 81–97.

9 Gorenflo R. Wright functions as scale-invariant solutions of the diffusion-wave equation / R. Gorenflo, Yu. Luchko, F. Mainardi // J. Comput. Appl. Math. — 2000. — № 118. — P. 175–191.

10 Gorenflo R. Mapping between solutions of fractional diffusion-wave equations / R. Gorenflo, A. Iskenderov, Yu. Luchko // Fract. Calcul. and Appl. Anal. — 2000. — № 3. — P. 75–86.

11 Agrawal O.P. Solution for a fractional diffusion-wave equation defined in a bounded domain / O.P. Agrawal // Nonlinear Dynam. — 2002. — № 29. — P. 145–155.

12 Pskhu A.V. Solution of Boundary Value Problems for the Fractional Diffusion Equation by the Green Function Method / A.V. Pskhu // Differ. Equ. — 2003. — № 39. — P. 1509–1513.

13 Pskhu A.V. Solution of the First Boundary Value Problem for a Fractional-Order Diffusion Equation / A.V. Pskhu // Differ. Equ. — 2003. — № 39. — P. 1359–1363.

14 Eidelman S.D. Cauchy problem for fractional diffusion equations / S.D. Eidelman, A.N. Kochubei // Journal of Differential Equations. — 2004. — № 199. — P. 211–255.

15 Voroshilov A.A. The Cauchy problem for the diffusion-wave equation with the Caputo partial derivative / A.A. Voroshilov, A.A. Kilbas // Differ. Equ. — 2006. — № 42. — P. 638–649.

16 Pskhu A.V. The fundamental solution of a diffusion-wave equation of fractional order / A.V. Pskhu // Izv. Math. – 2009. – № 73. – P. 351–392

17 Liu J.J. A backward problem for the time-fractional diffusion equation / J.J. Liu, M. Yamamoto // Applicable Analysis. — 2010. — № 89(11). — P. 1769–1788.

18 Sakamoto K. Initial value boundary value problems for fractional diffusion-wave equations and applications to some inverse problems / K. Sakamoto, M. Yamamoto // J. Math. Anal. Appl. — 2011. — № 382. — P. 426–447.

19 Kemppainen J. Properties of the single layer potential for the time fractional diffusion equation / J. Kemppainen // Journal of Integral Equations and Applications. — 2011. — № 23. — P. 541–563.

20 Kochubei A.N. Asymptotic properties of solutions of the fractional diffusion-wave equation / A.N. Kochubei // Fract. Calcul. and Appl. Anal. — 2014. — № 17. — P. 881–896.

21 Al-Refai M. Asymptotic properties of solutions of the fractional diffusion-wave equation / M. Al-Refai, Yu. Luchko // Fract. Calcul. and Appl. Anal. — 2014. — № 17. — P. 483–498.

22 Мамчуев М.О. Необходимые нелокальные условия для диффузионно-волнового уравнения / М.О. Мамчуев // Вестн. Самар. ун-та. Естественнонауч. сер. — 2014. — № 7. — С. 45–59.

23 Tuan N.H. A regularization method for time-fractional linear inverse diffusion problems / N.H. Tuan, M. Kirane, V.C.H. Luu, Bin-B. Mohsin // Electronic Journal of Differential Equations. — 2016. — № 290. — P. 1–18.

24 Псху А.В. Уравнение дробной диффузии с оператором дискретно распределенного дифференцирования / А.В. Псху // Сибир. мат. электрон. изв. — 2016. — № 13. — С. 1078–1098.

25 Mamchuev M.O. Solutions of the main boundary value problems for a loaded second-order parabolic equation with constant coefficients / M.O. Mamchuev // Differ. Equ. — 2016. — № 52. — P. 789–797.

26 Псху А.В. Первая краевая задача для дробного диффузионно-волнового уравнения в нецилиндрической области / А.В. Псху // Изв. РАН. Сер. мат. — 2017. — № 81. — С. 1212–1233.

27 Mamchuev M.O. Solutions of the main boundary value problems for the time-fractional telegraph equation by the green function method / M.O. Mamchuev // Fract. Calc. Appl. Anal. — 2017. — № 20. — P. 190–211.

28 Масаева О.Х. Задача Дирихле для нелокального волнового уравнения с производной Римана-Лиувилля / О.Х. Масаева // Вестн. КРАУНЦ. Физ.-мат. науки. — 2019. — № 27. — С. 6–11.

29 Псху А.В. Функция Грина первой краевой задачи для дробного диффузионно-волнового уравнения в многомерной прямоугольной области / А.В. Псху // Итоги науки и техники. Сер. Современная математика и ее приложения. Тематические обзоры. — 2019. — № 167. — С. 52–61.

30 Alimov S. Inverse problem of determining an order of the Caputo time-fractional derivative for a subdiffusion equation / S. Alimov, R. Ashurov // Journal of Inverse and Ill-posed Problems. — 2020. — № 28. — P. 651–658.

31 Pskhu A.V. Green Functions of the First Boundary-Value Problem for a Fractional Diffusion–Wave Equation in Multidimensional Domains / A.V. Pskhu // Mathematics. — 2020. — № 8(4). — P. 464.

32 Pskhu A.V. Stabilization of solutions to the Cauchy problem for fractional diffusion-wave equation / A.V. Pskhu // Journal of Mathematical Sciences. — 2020. — № 250. — P. 800–810.

33 Pskhu A.V. Fractional Diffusion-Wave Equation with Application in Electrodynamics / A.V. Pskhu, S. Rekhviashvili // Journal of Mathematical Sciences. — 2020. — № 8. — P. 2086.

34 Kilbas A.A. Partial fractional differential equations and some of their applications / A.A. Kilbas // Analysis. — 2010. — № 30. — P. 35–66.

35 Pskhu A.V. Partial differential equations of fractional order / A.V. Pskhu. — Nauka: Moscow, 2005.

36 Kilbas A.A. Theory and applications of fractional differential equations / A.A. Kilbas, H.M. Srivastava, J.J. Trujillo. — North-Holland Math. Stud. Elsevier: Amsterdam, Vol. 204, 2006.

37 Kochubei A. Handbook of Fractional Calculus with Applications. Volume 2: Fractional Differential Equations / A. Kochubei, Y. Luchko (Eds.). — De Gruyter: Berlin, Germany, 2019.

38 Nakhushev A.M. Fractional calculus and its applications / A.M. Nakhushev. — Fizmatlit: Moscow, 2003.

39 Uchaikin V.V. Method of Fractional Derivatives / V.V. Uchaikin. — Artishok: Ulyanovsk, 2008.

40 Atanacković T.M. Fractional Calculus with Applications in Mechanics / T.M. Atanacković, S. Pilipović, B. Stanković, D. Zorica. — ISTE, Wiley: London, Hoboken, 2014.

41 Tarasov V. (Ed.) Handbook of Fractional Calculus with Applications. Volumes 4, 5: Applications in Physics, Parts A, B / V. Tarasov (Ed.). — De Gruyter: Berlin, Germany, 2019.

42 Pskhu A.V. On a boundary value problem for a fractional partial differential equation in a domain with curvilinear boundary / A.V. Pskhu // Differential Equations. — 2015. — № 51(8). — P. 1072–1078.

43 Рамазанов М.И. О сингулярном интегральном уравнении Вольтерра краевой задачи теплопроводности в вырождающейся области / М.И. Рамазанов, Н.К. Гульманов // Вестн. Удмурт. ун-та. Математика. Механика. Компьютерные науки. — 2021. — № 31(2). — С. 241–252.

44 Ramazanov M.I. Solution of a two-dimensional boundary value problem of heat conduction in a degenerating domain / M.I. Ramazanov, N.K. Gulmanov // Journal of Mathematics, Mechanics and Computer Science. — 2021. — № 111(3). — P. 65–78.

45 Amangalieva M.M. On one homogeneous problem for the heat equation in an infinite angular domain / M.M. Amangalieva, M.T. Dzhenaliev, M.T. Kosmakova, M.I. Ramazanov // Siberian Mathematical Journal. — 2015. — № 56(6). — P. 982–995.

46 Jenaliyev M.T. On a Homogeneous Parabolic Problem in an Infinite Corner Domain / M.T. Jenaliyev, M.I. Ramazanov // Filomat. — 2018. — № 32(3). — P. 965–974.

47 Wright E.M. On the coefficients of power series having exponential singularities / E.M. Wright // J. London Math. Soc. — 1933. — № 8. — P. 71–79.

48 Wright E.M. The generalized Bessel function of order greater than one / E.M. Wright // Quart. J. Math., Oxford Ser. — 1940. — № 11. — P. 36–48.

49 Pskhu A.V. Solution of a Boundary Value Problem for a Fractional Partial Differential Equation / A.V. Pskhu // Differential Equations. — 2003. — № 39. — P. 1092–1099.

50 Pskhu A.V. Nakhushev extremum principle for a class of integro-differential operators / A.V. Pskhu // Fract. Calc. Appl. Anal. — 2020. — № 23. — P. 1712–1722.

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Қисық сызықты бұрыштық облыстағы бөлшек ретті диффузиялық теңдеу үшін шеттік есеп

Мақалада қисық сызықты бұрыштық облыстағы бөлшек ретті диффузиялық теңдеудің шеттік есебі зерттелген. Қарастырылып отырған облыстағы есептің шешімінің бар болуы мен жалғыздығы туралы теоремалар дәлелденді. Гёльдер бойынша қисық сызықты шекараның үздіксіздігі шешімдердің бар болуын қамтамасыз ететіндігін көрсеткен. Шешімнің жалғыздығы шексіздікте дәрежелік салмақпен нөлге айналатын функциялар класында дәлелденді. Есептің шешуі Вольтерраның интегралдық теңдеуінің шешуі арқылы айқын түрде құрылады.

Кілт сөздер: цилиндрлік емес облыс, қисық сызықты бұрыштық облыс, шеттік есеп, бөлшек ретті диффузиялық теңдеу.

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Краевая задача для дробного диффузионного уравнения в криволинейной угловой области

В статье рассмотрена и доказана теорема существования и единственности рассматриваемой краевой задачи. Показано, что непрерывность криволинейной границы по Гёльдеру обеспечивает существование решений. Единственность решения задачи доказана в классе функций, обращающихся в нуль на бесконечности со степенным весом. Вычисление ответа задачи построено в явном виде через решение интегрального уравнения Вольтерра.

Ключевые слова: нецилиндрическая область, криволинейная угловая область, краевая задача, дробное диффузионное уравнение.

References

1 Wyss, W. (1986). The fractional diffusion equation. *J. Math. Phys., 27*, 2782–2785.

2 Schneider, W.R., & Wyss, W. (1989). *Fractional diffusion and wave equations. J. Math. Phys., 30*, 134–144.

3 Kochubei, A.N. (1990). Diffusion of fractional order. *Differ. Equ., 26*, 485–492.

4 Fujita, Ya. (1990). Integrodifferential equation which interpolates the heat equation and the wave equation I, II. *Osaka J. Math., 27*, 309–321, 797–804.

5 Mainardi, F. (1996). The fundamental solutions for the fractional diffusion-wave equation. *Appl. Math. Lett., 9*, 23–28.

6 Engler, H. (1997). Similiraty solutions for a class of hyperbolic integrodifferential equations. *Differential Integral Equations, 10*, 815–840.

7 Shogenov, V.Kh., Kumykova, S.K., & Shkhanukov-Lafishev, M.Kh. (1997). Obobshchennoe uravnenie perenosa i drobnye proizvodnye [The generalized transport equation and fractional derivatives]. *Doklady Natsionalnoi akademii nauk Ukrainy — Reports of the National Academy of Sciences of Ukraine, 12*, 47–54 [in Russian].

8 Buckwar, E., & Luchko, Yu. (1998). Invariance of a partial differential equation of fractional order under the Lie group of scaling transformations. *J. Math. Anal. Appl., 227*, 81–97.

9 Gorenflo, R., Luchko, Yu., & Mainardi, F. (2000). Wright functions as scale-invariant solutions of the diffusion-wave equation. *J. Comput. Appl. Math., 118*, 175–191.

10 Gorenflo, R., Iskenderov, A. & Luchko, Yu. (2000). Mapping between solutions of fractional diffusion-wave equations. *Fract. Calcul. and Appl. Anal., 3*, 75–86.

11 Agrawal, O.P. (2002). Solution for a fractional diffusion-wave equation defined in a bounded domain. *Nonlinear Dynam., 29*, 145–155.

12 Pskhu, A.V. (2003). Solution of Boundary Value Problems for the Fractional Diffusion Equation by the Green Function Method. *Differ. Equ., 39*, 1509–1513.

13 Pskhu, A.V. (2003). Solution of the First Boundary Value Problem for a Fractional-Order Diffusion Equation. *Differ. Equ., 39*, 1359–1363.

14 Eidelman, S.D., & Kochubei, A.N. (2004). Cauchy problem for fractional diffusion equations. *Journal of Differential Equations, 199*, 211–255.

15 Voroshilov, A.A., & Kilbas, A.A. (2006). The Cauchy problem for the diffusion-wave equation with the Caputo partial derivative. *Differ. Equ., 42*, 638–649.

16 Pskhu, A.V. (2009). The fundamental solution of a diffusion-wave equation of fractional order. *Izv. Math., 73*, 351–392.

17 Liu, J.J., & Yamamoto, M. (2010). A backward problem for the time-fractional diffusion equation. *Applicable Analysis., 89*(11), 1769–1788.

18 Sakamoto, K., & Yamamoto, M. (2011). Initial value boundary value problems for fractional diffusion-wave equations and applications to some inverse problems. *J. Math. Anal. Appl., 382*, 426–447.

19 Kemppainen, J. (2011). Properties of the single layer potential for the time fractional diffusion equation. *Journal of Integral Equations and Applications, 23*, 541–563.

20 Kochubei, A.N. (2014). Asymptotic properties of solutions of the fractional diffusion-wave equation. *Fract. Calcul. and Appl. Anal., 17*, 881–896.

21 Al-Refai, M., & Luchko, Yu. (2014). Asymptotic properties of solutions of the fractional diffusion-wave equation. *Fract. Calcul. and Appl. Anal., 17*, 483–498.

22 Mamchuev, M.O. (2014). Neobkhodimye nelokalnye usloviia dlia diffuzionno-volnovogo uravneniia [Necessary non-local conditions for a diffusion-wave equation]. *Vestnik Samarskogo universiteta. Estestvennonauchnaia seriia — Journal of Samara University. Natural science Series, 7*, 45–59 [in Russian].

23 Tuan, N.H., Kirane, M., Luu, V.C.H., & Mohsin, Bin-B. (2016). A regularization method for time-fractional linear inverse diffusion problems. *Electronic Journal of Differential Equations, 290*, 1–18.

24 Pskhu, A.V. (2016). Uravnenie drobnoi diffuzii s operatorom diskretno raspredelennogo differentsirovaniia [Fractional diffusion equation with discretely distributed differentiation operator]. *Sibirskie matematicheskie elektronnye izvestiia — Siberian Electronic Mathematical Reports, 13*, 1078–1098 [in Russian].

25 Mamchuev, M.O. (2016). Solutions of the main boundary value problems for a loaded second-order parabolic equation with constant coefficients. *Differ. Equ., 52*, 789–797.

26 Pskhu, A.V.(2017). Pervaia kraevaia zadacha dlia drobnogo diffuzionno-volnovogo uravneniia v netsilindricheskoi oblasti [The first boundary-value problem for a fractional diffusion-wave equation in a non-cylindrical domain]. *Izvestiia RAN. Seriia matematicheskaia — Izvestiya: Mathematics, 81*, 1212–1233 [in Russian].

27 Mamchuev, M.O. (2017). Solutions of the main boundary value problems for the time-fractional telegraph equation by the Green function method. *Fract. Calc. Appl. Anal., 20*, 190–211.

28 Masaeva, O.Kh. (2019). Zadacha Dirikhle dlia nelokalnogo volnovogo uravneniia s proizvodnoi Rimana–Liuvillia [Dirichlet problem for a nonlocal wave equation with Riemann–Liouville derivative]. *Vestnik KRAUNC. Fiziko-matematicheskie nauki — Bulletin KRASEC. Physical and Mathematical Sciences, 27*, 6–11 [in Russian].

29 Pskhu, A.V. (2019). Funktsiia Grina pervoi kraevoi zadachi dlia drobnogo diffuzionno-volnovogo uravneniia v mnogomernoi priamougolnoi oblasti [Green function of the first boundary-value

problem for the fractional diffusion wave equation in a multidimensional rectangular domain]. *Itogi nauki i tekhniki. Seriia Sovremennaia matematika i ee prilozheniia. Tematicheskie obzory — Results of science and technology. Series "Modern mathematics and its applications. Thematic reviews", 167*, 52–61 [in Russian].

30 Alimov, S., & Ashurov, R. (2020). Inverse problem of determining an order of the Caputo time-fractional derivative for a subdiffusion equation. *Journal of Inverse and Ill-posed Problems, 28*, 651–658.

31 Pskhu, A.V. (2020). Green Functions of the First Boundary-Value Problem for a Fractional Diffusion–Wave Equation in Multidimensional Domains. *Mathematics, 8*(4), 464.

32 Pskhu, A.V. (2020). Stabilization of solutions to the Cauchy problem for fractional diffusion-wave equation. *Journal of Mathematical Sciences, 250*, 800–810.

33 Pskhu, A.V., & Rekhviashvili, S. (2020). Fractional Diffusion-Wave Equation with Application in Electrodynamics. *Journal of Mathematical Sciences, 8*, 2086.

34 Kilbas, A.A. (2010). Partial fractional differential equations and some of their applications. *Analysis, 30*, 35–66.

35 Pskhu, A.V. (2005). *Partial differential equations of fractional order*. Nauka: Moscow.

36 Kilbas, A.A., Srivastava, H.M., & Trujillo, J.J. (2006). *Theory and applications of fractional differential equations, 204*. North-Holland Math. Stud. Elsevier: Amsterdam.

37 Kochubei, A., & Luchko, Y. (Eds.). (2019). *Handbook of Fractional Calculus with Applications. Volume 2: Fractional Differential Equations*. De Gruyter: Berlin, Germany.

38 Nakhushev, A.M. (2003). *Fractional calculus and its applications*. Fizmatlit: Moscow.

39 Uchaikin, V.V. (2008). *Method of Fractional Derivatives*. Artishok: Ulyanovsk.

40 Atanacković, T.M., Pilipović, S., Stanković, B., & Zorica, D. *Fractional Calculus with Applications in Mechanics*. ISTE, Wiley: London, Hoboken.

41 Tarasov, V. (Ed.). (2019). *Handbook of Fractional Calculus with Applications. Volumes 4, 5: Applications in Physics, Parts A, B*. De Gruyter: Berlin, Germany.

42 Pskhu, A.V. (2015). On a boundary value problem for a fractional partial differential equation in a domain with curvilinear boundary. *Differential Equations, 51*(8), 1072–1078.

43 Ramazanov, M.I., & Gulmanov, N.K. (2021). O singuliarnom integralnom uravnenii Volterra kraevoi zadachi teploprovodnosti v vyrozhdaiushcheisia oblasti [On the singular Volterra integral equation of the boundary value problem for heat conduction in a degenerating domain]. *Vestnik Udmurtskogo universiteta. Matematika. Mekhanika. Kompiuternye nauki — The Bulletin of Udmurt University. Mathematics. Mechanics. Computer Science, 31*(2), 241–252 [in Russian].

44 Ramazanov, M.I., & Gulmanov, N.K. (2021). Solution of a two-dimensional boundary value problem of heat conduction in a degenerating domain. *Journal of Mathematics, Mechanics and Computer Science, 111*(3), 65–78.

45 Amangalieva, M.M., Dzhenaliev, M.T., Kosmakova, M.T., & Ramazanov, M.I. On one homogeneous problem for the heat equation in an infinite angular domain. *Siberian Mathematical Journal, 56*(6), 982–995.

46 Jenaliyev, M.T., & Ramazanov, M.I. (2018). On a Homogeneous Parabolic Problem in an Infinite Corner Domain *Filomat, 32*(3), 965–974.

47 Wright, E.M. (1933). On the coefficients of power series having exponential singularities. *J. London Math. Soc., 8*, 71–79.

48 Wright, E.M. (1940). The generalized Bessel function of order greater than one. *Quart. J. Math., Oxford Ser., 11*, 36–48.

49 Pskhu, A.V. (2003). Solution of a Boundary Value Problem for a Fractional Partial Differential Equation. *Differential Equations, 39*, 1092–1099.

50 Pskhu, A.V. (2020). Nakhushev extremum principle for a class of integro-differential operators. *Fract. Calc. Appl. Anal., 23*, 1712–1722.