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On a New Class of Singular Integro-differential Equations

In this paper for a new class of model and non-model partial integro-differential equations with singularity in the kernel, we obtained integral representation of family of solutions by aid of arbitrary functions. Such type of integro-differential equations are different from Cauchy-type singular integro-differential equations. Cauchy-type singular integro-differential equations are studied by the methods of the theory of analytic functions. In the process of our research the new types of singular integro-differential operators are introduced and main property of entered operators are learned. It is shown that the solution of studied equation is equivalent to the solution of system of two equations with respect to x and y, one of which is integral equation and the other is integro-differential equation. Further, non-model integro-differential equations are studied by regularization method. This regularization method for non-model equation is based on selecting and analysis of a model part of the equation and reduced to the solution of two second kind Volterra type integral equations with weak singularity in the kernel. It is shown that the presence of a non-model part in the equation does not affect to the general structure of the solutions. From here investigation of the model equations for given class of the integro-differential equations becomes important. In the cases, when the solution of given integro-differential equation depends on any arbitrary functions, a Cauchy type problems are investigated.

Keywords: singular integro-differential equation, model equation, non model equation, characteristic equation, Cauchy type problem.

Introduction

In addition to the theory of differential and integral equations, the theory of integro-differential (I-D) equations with regular and singular coefficients plays an important role in theoretical and applied research. There are many scientific publications where theoretical or applied aspects of the theory of I-D equations are studied. Of particular interest is research on I-D equations with singular kernels. During the last years the theory of the I-D equations basically developed in two directions. The first direction is connected with the study of approximate solutions of I-D equations [1–10]. The second direction is connected with construction of the general theory for a new classes of the I-D equations [11–18]. Study of the various aspects of I-D equations in Banach spaces also concerns to the second direction [19–21]. Also, in last years the methods of solving the direct and inverse problems and problems with small parameters for the I-D equations [22–24] were actively developed.

One of section in the theory of the I-D equations, which is not studied completely, is the section of I-D equations with singular and super singular coefficients. Some results in this direction are received by integral transform methods in [25–27]. But, we note that the singularity in dependent of the studied problems has different nature. Therefore, the approach of separate authors to study the singular problems happens in different ways.

In the classical singular integro-differential equations the integrals basically we understood in sense of a principal value of Cauchy. Consequently, in solving some equations the methods of analytic functions are used. Unlike this, we will investigate such singular I-D equations, in which integrals are understood in ordinary sense of Riemann. Therefore, our approaches in studying the given problem are also different from the works [13–15]. This work is a further continuation of our research in [28, 29].

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1. Formulation of problem and basic designation

We denote: $D = \{(x, y) : a < x < c,\ b < y < d\}$ and $\Gamma_1 = \{a < x < c,\ y = b\}$, $\Gamma_2 = \{x = a,\ b < y < d\}$. In the domain D we consider a partial integro-differential equation with singular coefficients:

$$\mathbf{K}\varphi \equiv \varphi'_x(x, y) + \frac{A(x)}{x-a}\varphi + \int_a^x \frac{B(t)}{(t-a)^2}\varphi(t, y)dt + \int_b^y \frac{E(s)}{s-b}\varphi'_x(x, s)ds+$$

$$+\int_b^y \frac{G(x, s)}{(x-a)(s-b)}\varphi(x, s)ds + \int_a^x \frac{dt}{(t-a)^2}\int_b^y \frac{H(t, s)}{s-b}\varphi(t, s)ds = f(x, y), \tag{1}$$

where $A(x)$, $B(x)$, $E(y)$, $G(x, y)$, $H(x, y)$ are given functions connecting to each other by equalities $G(x, y) = A(x)E(y), H(x, y) = B(x)E(y)$; $f(x, y)$ is the given function on the domain D; $\varphi(x, y)$ is a unknown function.

An importance of studying equation (1) consists in the following idea:

1 The degree of singularity in the kernel of the studied equation with respect to x is equal to 2 and with respect to y is equal to 1. Therefore the kernels of this equation are not Fredholm type kernels;

2 Singularity in this equation is not understood in the sense of a principal value of Cauchy;

3 For the solution of this equation we do not use methods of the theory of analytic functions;

4 We obtain the solution of the considering equation in the sense of generalized solution from the theory of regular I-D equations.

To solve the equation (1) we introduce some designations:

1 Through $\mathbb{C}_x^{\delta_1,\delta_2}(\overline{D})$ we denote a class of such functions, which have the first order continuous derivative with respect to the variable x with an asymptotic behaviour

$$f(x, y) = o[(x-a)^{\gamma_1}, (y-b)^{\gamma_2}], \gamma_1 > \delta_1, \gamma_2 > \delta_2. \tag{2}$$

2 We denote $\mathbb{C}^{\delta_1,0}(\overline{D}) = \mathbb{C}^{\delta_1}(\overline{\Gamma_1})$ and $\mathbb{C}^{0,\delta_2}(\overline{D}) = \mathbb{C}^{\delta_2}(\overline{\Gamma_2})$.

3 Through $\mathbb{C}^{\delta_1}_{f(a)}(\overline{\Gamma}_1)$ we designate a class of such functions, for difference of which there is true the following $f(x) - f(a) \to 0$ as $x \to a$. This type of functions has an asymptotic behaviour

$$f(x) - f(a) = o[(x-a)^{\gamma_1}], \gamma_1 > \delta_1.$$

If $f(a) = 0$, then we have denotation $\mathbb{C}^{\delta_1}_0(\overline{\Gamma}_1) = \mathbb{C}^{\delta_1}(\overline{\Gamma_1})$.

The solution of equation (1) we seek in the class $\mathbb{C}_x^{1,0}(\overline{D})$. For this purpose by $\Pi^{x,1,2}_{a,A(x),B(x)}$ and $\Pi^{y,1}_{b,E(y)}$ we designate operators acting on function $\varphi(x, y)$ by rules

$$\Pi^{x,1,2}_{a,A(x),B(x)}\varphi(x, y) \equiv \varphi'_x(x, y) + \frac{A(x)}{x-a}\varphi(x, y) + \int_a^x \frac{B(t)}{(t-a)^2}\varphi(t, y)dt, \tag{3}$$

$$\Pi^{y,1}_{b,E(y)}\varphi(x, y) \equiv \varphi(x, y) + \int_b^y \frac{E(s)}{s-b}\varphi(x, s)ds. \tag{4}$$

If $A(x) = A = \text{const}, B(x) = B = \text{const}$ and $E(x) = E = \text{const}$, then operators $\Pi^{x,1,2}_{a,A,B}\varphi(x, y)$ in (3) and $\Pi^{y,1}_{b,E}\varphi(x, y)$ in (4) we call model operators and the equations corresponding to these operators we call model equations.

So, equation (1) by means of just entered operators we represent as

$$\Pi^{x,1,2}_{a,A(x),B(x)}\Pi^{y,1}_{b,E(y)}\varphi(x, y) = f(x, y). \tag{5}$$

Then the solution of operator equation (5) is equivalent to the solution of the following system of I-D equations

$$\begin{cases} \Pi^{y,1}_{b,E(y)}\varphi(x, y) = \psi(x, y), \\ \Pi^{x,1,2}_{a,A(x),B(x)}\psi(x, y) = f(x, y), \end{cases} \tag{6}$$

where $\psi(x, y)$ is a new unknown function and the function $f(x, y)$ has an asymptotic behaviour (2).

By separating model part of system (6) we obtain

$$\begin{cases} \Pi^{y,1}_{b,E(b)}\varphi(x,y) = \Psi(x,y), \\ \Pi^{x,1,2}_{a,A(a),B(a)}\psi(x,y) = F(x,y), \end{cases} \tag{7}$$

where

$$\Psi(x,y) = \psi(x,y) - \int\limits_b^y \frac{E(s)-E(b)}{s-b}\varphi(x,s)ds, \tag{8}$$

$$F(x,y) = f(x,y) - \frac{A(x)-A(a)}{x-a}\psi(x,y) - \int\limits_a^x \frac{B(t)-B(a)}{(t-a)^2}\psi(t,y)dt. \tag{9}$$

It is obvious that the homogeneous equations of system (7) correspond to characteristic equations

$$1 + \frac{E(b)}{\lambda} = 0, \tag{10}$$

$$\mu + A(a) + \frac{B(a)}{\mu - 1} = 0. \tag{11}$$

In dependent of the roots of the characteristic equations (10), (11) we obtain the solution of the system of equations (7) in following form.

Let $\lambda = -E(b) > 0$ and the roots of the characteristic equations (11) be real and different, $1 < \mu_1 < \mu_2$. Then the solution of nonhomogeneous equations (7) gives by formula:

$$\begin{cases} \varphi(x,y) = (y-b)^\lambda c_1(x) + \Psi(x,y) + \left(\Pi^{y,\lambda+1}_{b,\lambda}\right)^{-1}\Psi(x,y) \equiv E_1, \\ \psi(x,y) = (x-a)^{\mu_1}c_2(y) + (x-a)^{\mu_2}c_3(y) + \left(\Pi^{x,\mu_1,\mu_2}_{a,1-\mu_1,1-\mu_2}\right)^{-1}F(x,y) \equiv E_2, \end{cases} \tag{12}$$

where $E_1 = E_1[c_1(x), \Psi(x,y)]$, $E_2 = E_2[c_2(y), c_3(y), F(x,y)]$, operators $\left(\Pi^{y,\lambda+1}_{b,\lambda}\right)^{-1}$ and $\left(\Pi^{x,\mu_1,\mu_2}_{a,1-\mu_1,1-\mu_2}\right)^{-1}$ are inverse to operators $\Pi^{y,1}_{b,E(b)}$ and $\Pi^{x,1,2}_{a,A(a),B(a)}$, respectively, and the explicit form of these operators are

$$\left(\Pi^{y,\lambda+1}_{b,\lambda}\right)^{-1}\Psi(x,y) = \lambda\int\limits_b^y \left(\frac{y-b}{s-b}\right)^\lambda \frac{\Psi(x,y)}{s-b}ds,$$

$$\left(\Pi^{x,\mu_1,\mu_2}_{a,1-\mu_1,1-\mu_2}\right)^{-1}F(x,y) =$$

$$= \frac{1}{\sqrt{D_1}}\int\limits_a^x \left[(1-\mu_1)\left(\frac{x-a}{t-a}\right)^{\mu_1} - (1-\mu_2)\left(\frac{x-a}{t-a}\right)^{\mu_2}\right]F(t,y)dt.$$

Substituting the value of $\Psi(x,y)$ and $F(x,y)$ from (8), (9) into (12), after some transformations we come to solving the following Volterra type integral equations

$$\begin{cases} \varphi(x,y) + \int\limits_b^y K_1(y,s)\varphi(x,s)ds = E_1[c_1(x), \psi(x,y)], \\ \psi(x,y) + \frac{1}{\sqrt{D_1}}\int\limits_a^x K_2(x,t)\psi(x,y)dt = E_2[c_2(y), c_3(y), f(x,y)], \end{cases} \tag{13}$$

where

$$\begin{cases} K_1(y,s) = \left(\frac{y-b}{s-b}\right)^\lambda \frac{E(s)-E(b)}{s-b}, \\ K_2(x,t) = \left[(1-\mu_1)\left(\frac{x-a}{t-a}\right)^{\mu_1} - (1-\mu_2)\left(\frac{x-a}{t-a}\right)^{\mu_2}\right]\frac{A(t)-A(a)}{t-a} + \\ + \left[\left(\frac{x-a}{t-a}\right)^{\mu_1} - \left(\frac{x-a}{t-a}\right)^{\mu_2}\right]\frac{B(t)-B(a)}{t-a}. \end{cases}$$

If we introduce new unknown functions as $\varphi_1(x,y) = \frac{\varphi(x,y)}{(y-b)^{\lambda}}$, $\psi_1(x,y) = \frac{\psi(x,y)}{(x-a)^{\mu_1}}$, then instead the integral equations (13) we solve the following integral equations

$$\begin{cases} \varphi_1(x,y) + \int\limits_b^y K_{1,1}(y,s)\varphi_1(x,s)ds = \dfrac{E_1[c_1(x),\psi(x,y)]}{(y-b)^{\lambda}}, \\ \psi_1(x,y) + \dfrac{1}{\sqrt{D_1}}\int\limits_a^x K_{2,1}(x,t)\psi_1(x,y)dt = \dfrac{E_2[c_2(y),c_3(y),f(x,y)]}{(x-a)^{\mu_1}}, \end{cases} \tag{14}$$

where

$$\begin{cases} K_{1,1}(y,s) = \dfrac{E(s)-E(b)}{s-b}, \\ K_{2,1}(x,t) = \left[1-\mu_1-(1-\mu_2)\left(\dfrac{x-a}{t-a}\right)^{\mu_2-\mu_1}\right]\dfrac{A(t)-A(a)}{t-a}+ \\ +\left[1-\left(\dfrac{x-a}{t-a}\right)^{\mu_2-\mu_1}\right]\dfrac{B(t)-B(a)}{t-a}. \end{cases}$$

If the following conditions are fulfilled

$$\begin{aligned} &A(x) \in \mathbb{C}^{\mu_2-\mu_1}_{A(a)}(\overline{\Gamma_1}), B(x) \in \mathbb{C}^{\mu_2-\mu_1}_{B(a)}(\overline{\Gamma_1}), E(y) \in \mathbb{C}^{\varepsilon}_{E(b)}(\overline{\Gamma_2}), \\ &f(x,y) \in \mathbb{C}^{\mu_2-1,\lambda-1}(\overline{D}), \{c_2(y), c_3(y)\} \in \mathbb{C}^{\lambda-1}(\overline{\Gamma_2}), \end{aligned} \tag{15}$$

then the integral equations (14) become a Volterra type integral equations with weak singularity in the kernel and with the continuous right-hand side function. The solution of equations (14) by means of resolvent we write as

$$\begin{cases} \varphi_1(x,y) = \dfrac{E_1[c_1(x),\psi(x,y)]}{(y-b)^{\lambda}} - \int\limits_b^y \Gamma_1(y,s)\dfrac{E_1[c_1(x),\psi(x,s)]}{(s-b)^{\lambda}}ds, \\ \psi_1(x,y) = \dfrac{E_2[c_2(y),c_3(y),f(x,y)]}{(x-a)^{\mu_1}} - \dfrac{1}{\sqrt{D_1}}\int\limits_a^x \Gamma_2(x,t)\dfrac{E_2[c_2(y),c_3(y),f(t,y)]}{(t-a)^{\mu_1}}dt, \end{cases}$$

where $\Gamma_1(y,s)$, $\Gamma_2(x,t)$ are corresponding resolvent of integral equations (14).

Now coming back to our unknown functions $\varphi(x,y), \psi(x,y)$ we find the solution of equation (5) (or equivalent equation (1)) as follow:

$$\begin{cases} \varphi(x,y) = E_1[c_1(x),\psi(x,y)] - \int\limits_b^y \Gamma_1(y,s)\left(\dfrac{y-b}{s-b}\right)^{\lambda} E_1[c_1(x),\psi(x,s)]ds, \\ \psi(x,y) = E_2[c_2(y),c_3(y),f(x,y)]- \\ -\dfrac{1}{\sqrt{D_1}}\int\limits_a^x \Gamma_2(x,t)\left(\dfrac{x-a}{t-a}\right)^{\mu_1} E_2[c_2(y),c_3(y),f(t,y)]dt. \end{cases} \tag{16}$$

So we proved:

Theorem 1. We assume that conditions (15) are fulfilled; $G(x,y) = A(x)E(y), H(x,y) = B(x)E(y)$ in I-D equation (1) and the roots of the characteristic equations (10) and (11) are such that $\lambda = -E(b) > 0$, $1 < \mu_1 < \mu_2$. Then the solution of equation (1) in the class $C^{1,0}_x(\overline{D})$ is presented by formula (16).

From the validity of this Theorem 1, the following corollary holds.

Corollary. We assume that conditions (15) are fulfilled; $G(x,y) = A(x)E(y), H(x,y) = B(x)E(y)$ in the I-D equation (1) and roots of the characteristic equations (10) and (11) are such that $\lambda = -E(b) < 0,\ \mu_1 < \mu_2 < 1$. Then the solution of equation (1) in the class $C^{1,0}_x(\overline{D})$ is presented by following formula

$$\varphi(x,y) = E_1[0,\psi(x,y)] - \int\limits_b^y \Gamma_1(y,s)\left(\dfrac{y-b}{s-b}\right)^{\lambda} E_1[0,\psi(x,s)]ds,$$

where

$$\psi(x,y) = E_2[0,0,f(x,y)] - \frac{1}{\sqrt{D_1}}\int\limits_a^x \Gamma_2(x,t)\left(\frac{x-a}{t-a}\right)^{\mu_1} E_2[0,0,f(t,y)]dt.$$

Let $\lambda = -E(b) > 0$ and the roots of the characteristic equation (11) be real and equal, $\mu_1 = \mu_2 = \mu > 1$. Then the solution to the second equation of system (7) we represent as:

$$\psi(x,y) = (x-a)^\mu c_4(y) + (x-a)^\mu \ln(x-a) c_5(y) +$$
$$+\left(\Pi^{x,\mu}_{a,\mu-1,1}\right)^{-1} F(x,y) \equiv E_3[c_4(y), c_5(y), F(x,y)],$$

where

$$\left(\Pi^{x,\mu}_{a,\mu-1,\,1}\right)^{-1} F(x,y) = \int\limits_a^x \left(\frac{x-a}{t-a}\right)^\mu \left[(\mu-1)\ln\left(\frac{x-a}{t-a}\right)+1\right] F(t,y)dt.$$

Here the operator $\left(\Pi^{x,\mu}_{a,\mu-1,1}\right)^{-1}$ is inverse to I-D operator $\Pi^{x,1,2}_{a,A(a),B(a)}$, when the roots of the characteristic equation (11) are real and equal.

In this case repeating the above-stated scheme without stopping in details we obtain the general solution of equation (11) in a following form:

$$\varphi(x,y) = E_1[c_1(x), \psi(x,y)] - \int\limits_b^y \Gamma_1(y,s)\left(\frac{y-b}{s-b}\right)^\lambda E_1[c_1(x),\psi(x,s)]ds, \tag{17}$$

where

$$\psi(x,y) = E_3[c_4(y), c_5(y), f(x,y)] - \int\limits_a^x \Gamma_3(x,t)\left(\frac{x-a}{t-a}\right)^\mu E_3[c_4(y), c_5(y), f(t,y)]dt$$

and $\Gamma_3(x,t)$ is the resolvent of the integral equation which is written in an explicit form.

Thus the following theorem takes place:

Theorem 2. We assume that the following conditions are fulfilled:

1) $G(x,y) = A(x)E(y), H(x,y) = B(x)E(y)$ in an I-D equation (1);

2) The roots of the characteristic equations (10) and (11) are such that $\lambda = -E(b) > 0$, $\mu > 1$;

3) The following inclusions take place

$$A(x) \in \mathbb{C}^\varepsilon_{A(a)}(\overline{\Gamma_1}), B(x) \in \mathbb{C}^\varepsilon_{B(a)}(\overline{\Gamma_1}), E(y) \in \mathbb{C}^\varepsilon_{E(b)}(\overline{\Gamma_2}),$$
$$f(x,y) \in \mathbb{C}^{\mu-1,\lambda-1}(\overline{D}), \{c_4(y), c_5(y)\} \in \mathbb{C}^{\lambda-1}(\overline{\Gamma_2}).$$

Then the solution of equation (1) in the class $C^{1,0}_x(\overline{D})$ can be represented by formula (17).

Remark 1. In this case, if $\lambda = -E(b) < 0$, $\mu < 1$, we obtain the unique solution of equation (1) in the form

$$\varphi(x,y) = E_1[0, \psi(x,y)] - \int\limits_b^y \Gamma_1(y,s)\left(\frac{y-b}{s-b}\right)^\lambda E_1[0,\psi(x,s)]ds,$$

where

$$\psi(x,y) = E_3[0,0,f(x,y)] - \int\limits_a^x \Gamma_3(x,t)\left(\frac{x-a}{t-a}\right)^\mu E_3[0,0,f(t,y)]dt.$$

Let be $\lambda = -E(b) > 0$ and the roots of the characteristic equation (11) are complex, conjugate and

$$\mu_{1,2} = \frac{1-A(a)}{2} \pm \frac{\sqrt{4B(a)-(1+A(a))^2}}{2} i \equiv \alpha \pm \beta i.$$

Then we can write the solution to the second equation of system (7) as:

$$\psi(x,y) = (x-a)^\alpha \left\{\cos\left[\beta\ln(x-a)\right] c_6(y) + \sin\left[\beta\ln(x-a)\right] c_7(y)\right\} +$$
$$+\left(\Pi^{x,\alpha}_{a,\alpha-1,\beta}\right)^{-1} F(x,y) \equiv E_4[c_6(y), c_7(y),\, F(x,y)],$$

where

$$\left(\Pi^{x,\alpha}_{a,\alpha-1,\beta}\right)^{-1} F(x,y) = \frac{1}{\beta}\int\limits_a^x \left(\frac{x-a}{t-a}\right)^{\alpha} \left\{(\alpha-1)\sin\left[\beta \ln\left(\frac{x-a}{t-a}\right)\right] + \right.$$

$$\left. +\beta\cos\left[\beta\ln\left(\frac{x-a}{t-a}\right)\right]\right\} f(x,s)dt.$$

Here the operator $\left(\Pi^{x,\alpha}_{a,\alpha-1,\beta}\right)^{-1}$ is inverse to I-D operator $\Pi^{x,1,2}_{a,A(a),B(a)}$, when the roots of the characteristic equation (11) are complex and conjugate.

In this case too repeating the above-stated scheme by means of resolvent of corresponding Volterra type integral equation we obtain the general solution of equation (1) in a following form:

$$\varphi(x,y) = E_1[c_1(x),\psi(x,y)] - \int\limits_b^y \Gamma_1(y,s)\left(\frac{y-b}{s-b}\right)^{\lambda} E_1[c_1(x),\psi(x,s)]ds, \tag{18}$$

where

$$\psi(x,y) = E_4[c_6(y),c_7(y),f(x,y)] - \frac{1}{\beta}\int\limits_a^x \Gamma_4(x,t)\left(\frac{x-a}{t-a}\right)^{\alpha} E_4[c_6(y),c_7(y),f(t,y)]dt.$$

Thus, the following theorem takes place.

Theorem 3. We assume that the following conditions are fulfilled:

1) $G(x,y) = A(x)\,E(y),\; H(x,y) = B(x)\,E(y)$ in an I-D equation (1);

2) The roots of the characteristic equations (10) and (11) are such that $\lambda = -E(b) > 0,\; Re\,\mu_{1,2} = \alpha > 1$;

3) The following inclusions take place

$$A(x) \in \mathbb{C}^{\varepsilon}_{A(a)}(\overline{\Gamma_1}), B(x) \in \mathbb{C}^{\varepsilon}_{B(a)}(\overline{\Gamma_1}), E(y) \in \mathbb{C}^{\varepsilon}_{E(b)}(\overline{\Gamma_2}),$$
$$f(x,y) \in \mathbb{C}^{\alpha-1,\lambda-1}(\overline{D}), \{c_6(y), c_7(y)\} \in \mathbb{C}^{\lambda-1}(\overline{\Gamma_2}).$$

Then the solution of equation (1) in the class $C^{1,0}_x(\overline{D})$ can be represented by formula (18).

Remark 2. If $\lambda = -E(b) < 0,\; \alpha < 1$, we obtain the unique solution of equation (1) in the form

$$\varphi(x,y) = E_1[0,\psi(x,y)] - \int\limits_b^y \Gamma_1(y,s)\left(\frac{y-b}{s-b}\right)^{\lambda} E_1[0,\psi(x,s)]ds,$$

where

$$\psi(x,y) = E_4[0,0,f(x,y)] - \int\limits_a^x \Gamma_4(x,t)\left(\frac{x-a}{t-a}\right)^{\alpha} E_4[0,0,f(t,y)]dt.$$

2. Boundary value problem

One of the most important results in the theory of I-D equations is solving the Cauchy type problem and boundary value problems. The Cauchy type problem, when x_0 does not the same as singular point, is solved as in ordinary theory. But in the theory of singular I-D equations it is interesting when corresponding conditions are given at singular point. In this case such kind problems we call singular Cauchy type problems. The singular Cauchy type problems are different from ordinary Cauchy problems, that in our case the problem gives with some weights.

We introduce the following denotation:

$$\begin{cases} P^{y}_{b,\lambda}\left[\varphi(x,y)\right] = \dfrac{1}{(y-b)^{\lambda}}\varphi(x,y), \\ P^{x}_{a,\mu_1}\left[\varphi(x,y)\right] = \dfrac{1}{(x-a)^{\mu_1}}\left[\Pi^{y,1}_{b,E(y)}\varphi(x,y)\right], \\ P^{x}_{a,\mu_1,\mu_2}\left[\varphi'_x(x,y)\right] = \dfrac{1}{(x-a)^{\mu_1-\mu_2-1}}\dfrac{d}{dx}\left[\dfrac{1}{(x-a)^{\mu_1}}\Pi^{y,1}_{b,E(y)}\varphi(x,y)\right]. \end{cases}$$

Then the following remark holds:

Remark 3. The solution of type (16) has the following property:

$$\begin{cases} \left[P^y_{b,\lambda}\left[\varphi(x,y)\right]\right]_{y=b} = c_1(x), \\ \left[P^x_{a,\mu_1}\left[\varphi(x,y)\right]\right]_{x=a} = c_2(y), \\ \left[P^x_{a,\mu_1,\mu_2}\left[\varphi'_x(x,y)\right]\right]_{x=a} = (\mu_2-\mu_1)c_3(y). \end{cases} \tag{19}$$

Now we consider the following singular Cauchy type problem:

Singular Cauchy type problem. Let the roots of characteristic equations (10) and (11) be such that $\lambda = -E(b) > 0,\ 1 < \mu_1 < \mu_2$. It is required to find such solution of equation (1), which belongs to the class $C^{1,0}_x(\overline{D})$ and satisfies the following initial value conditions:

$$\begin{cases} \left[P^y_{b,\lambda}\left[\varphi(x,y)\right]\right]_{y=b} = \omega_1(x), \\ \left[P^x_{a,\mu_1}\left[\varphi(x,y)\right]\right]_{x=a} = \omega_2(y), \\ \left[P^x_{a,\mu_1,\mu_2}\left[\varphi'_x(x,y)\right]\right]_{x=a} = \omega_3(y), \end{cases}$$

where $\omega_1(x)$, $\omega_2(y)$, $\omega_3(y)$ are given functions on the domain D.

By virtue of integral representation (16) and its property (19), for equation (1) we find functions $c_1(x),\ c_2(y)$, $c_3(y)$ by means of the given functions $\omega_1(x),\ \omega_2(y),\ \omega_3(y)$: $c_1(x) = \omega_1(x),\ c_2(y) = \omega_2(y),\ c_3(y) = \frac{1}{\mu_2-\mu_1}\omega_3(y)$. Therefore substituting these values into (16), the unique solution of the singular Cauchy type problem be found as

$$\begin{cases} \varphi(x,y) = E_1[\omega_1(x),\psi(x,y)] - \int\limits_b^y \Gamma_1(y,s)\left(\frac{y-b}{s-b}\right)^{\lambda} E_1[\omega_1(x),\psi(x,s)]ds, \\ \psi(x,y) = E_2[\omega_2(y), \frac{1}{\mu_2-\mu_1}\omega_3(y), f(x,y)] - \\ -\frac{1}{\sqrt{D_1}}\int\limits_a^x \Gamma_2(x,t)\left(\frac{x-a}{t-a}\right)^{\mu_1} E_2\left[\omega_2(y), \frac{1}{\mu_2-\mu_1}\omega_3(y), f(t,y)\right]dt. \end{cases} \tag{20}$$

So we have proved the following theorem.

Theorem. 4 Let all conditions of Theorem 1 be fulfilled. Then the singular Cauchy type problem has a unique solution, which is given by (20).

Such kind of problems can be investigated, when the roots of characteristic equation (11) are real, equal, complex and conjugate.

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Т.К. Юлдашев, С.К. Зарифзода

Сингулярлық интегро-дифференциалдық теңдеулердің бір жаңа класы туралы

Мақалада ядрода ерекшелігі бар модельдік және модельдік емес дербес туындылы интегро-дифференциалдық теңдеулердің жаңа класы үшін еркін функциялар көмегімен интегралдық үйір түріндегі шешімі алынған. Интегро-дифференциалдық теңдеулердің бұл түрі Коши типтес сингулярлық интегро-дифференциалдық теңдеулерден өзгеше. Коши типіндегі сингулярлық интегро-дифференциалдық теңдеулер аналитикалық функциялар теориясының әдістерімен қарастырылған. Зерттеу барысында сингулярлық интегро-дифференциалдық операторлардың жаңа түрлері енгізіліп, осы енгізілген операторлардың негізгі қасиеттері зерттелді. Зерттелетін теңдеудің шешімі x және y-ке қатысты екі теңдеу жүйесінің шешіміне тең екендігі көрсетілген, олардың бірі – интегралдық теңдеу, ал екіншісі – интегро-дифференциалдық теңдеу болып табылады. Әрі қарай модельдік емес интегро-дифференциалдық теңдеулер регуляризациялау әдісімен зерттелген. Модельдік емес теңдеуді реттеудің бұл әдісі теңдеудің модельдік бөлігін таңдауға және талдауға негізделген және ядродағы әлсіз ерекшелігі бар екінші типтегі Вольтерр типтес екі интегралдық теңдеуді шешуге дейін азаяды. Теңдеуде модельдік емес бөліктің болуы шешімдердің жалпы құрылымына әсер етпейтіні көрсетілген. Сондықтан интегро-дифференциалдық теңдеулердің осы класы үшін модельдік теңдеулерді зерттеу маңызды. Осы интегро-дифференциалдық теңдеудің шешімі кез келген ерікті функциялардан тәуелді болған жағдайда Коши типтес есептер зерттеледі.

Кілт сөздер: сингулярлық интегро-дифференциалдық теңдеу, модельдік теңдеу, модельдік емес теңдеу, сипаттамалық теңдеу, Коши типтес есеп.

Т.К. Юлдашев, С.К. Зарифзода

Об одном новом классе сингулярных интегро-дифференциальных уравнений

В статье для нового класса модельных и немодельных интегро-дифференциальных уравнений в частных производных с особенностью в ядре получено решение в виде семейства интегрального представления с помощью произвольных функций. Такой тип интегро-дифференциальных уравнений отличается от сингулярных интегро-дифференциальных уравнений типа Коши. Сингулярные интегро-дифференциальные уравнения типа Коши изучаются методами теории аналитических функций. В процессе нашего исследования введены новые типы сингулярных интегро-дифференциальных операторов и изучены основные свойства этих введенных операторов. Показано, что решение исследуемого уравнения эквивалентно решению системы двух уравнений относительно x и y, одно из которых является интегральным уравнением, а другое — интегро-дифференциальным уравнением. Далее немодельные интегро-дифференциальные уравнения исследованы методом регуляризации. Этот метод регуляризации немодельного уравнения основан на выборе и анализе модельной части уравнения и сведен к решению двух интегральных уравнений типа Вольтерра второго рода со слабой особенностью в ядре. Показано, что наличие немодельной части в уравнении не влияет на общую структуру решений. Отсюда важное значение приобретает исследование модельных уравнений для данного класса интегро-дифференциальных уравнений. В случаях, когда решение данного интегро-дифференциального уравнения зависит от любых произвольных функций, исследованы задачи типа Коши.

Ключевые слова: сингулярное интегро-дифференциальное уравнение, модельное уравнение, немодельное уравнение, характеристическое уравнение, задача типа Коши.

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