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D.N. Nurgabyl[1,2,3], S.S. Nazhim[3,*]

[1] *Institute of Mathematics and Mathematical Modeling, Almaty, Kazakhstan;*
[2] *Zhetysu State University named after I.Zhansugurov, Taldykorgan, Kazakhstan;*
[3] *Kazakh National Women's Teacher Training University, Almaty, Kazakhstan*
(E-mail: kebek.kz@mail.ru, n.saniya_96@mail.ru)

Recovery problem for a singularly perturbed differential equation with an initial jump

The article investigates the asymptotic behavior of the solution to reconstructing the boundary conditions and the right-hand side for second-order differential equations with a small parameter at the highest derivative, which have an initial jump. Asymptotic estimates of the solution of the reconstruction problem are obtained for singularly perturbed second-order equations with an initial jump. The rules for the restoration of boundary conditions and the right parts of the original and degenerate problems are established. The asymptotic estimates of the solution of the perturbed problem are determined as well as the difference between the solution of the degenerate problem and the solution of the perturbed problem. A theorem on the existence, uniqueness, and representation of a solution to the reconstruction problem from the position of singularly perturbed equations is proved. The results obtained open up possibilities for the further development of the theory of singularly perturbed boundary value problems for ordinary differential equations.

Keywords: Perturbed problems, degenerate problems, small parameter, boundary value problem, initial jump, asymptotic behavior.

Introduction

At the end of the last century and over the past decade (ten years), approximate methods for solving differential equations, asymptotics construction, solution for singularly perturbed differential equations attracted the attention of many researches. This interest is caused by the needs of numerical methods for solving differential equations. Since in many cases the asymptotic approximation of solution of boundary value problem is useful to use as a first approximation in numerical calculations [1,2].

The difficulty of constructing an asymptotic approximation to the solution of initial and boundary value problems for differential and integro-differential equations is related to the perturbation feature. In this regard, the researchers proposed various asymptotic methods for constructing the asymptotic behavior of the solution of initial and boundary value problems. However, without a preliminary study of the asymptotic behavior of the solution of singularly perturbed initial problems with singular initial conditions and boundary value problems with boundary jumps phenomena, the greatest difficulty is the selection of a suitable asymptotic method for solving these problems [3-8].

The most general cases of the existence of the phenomenon of initial jumps were investigated in the works of Dauylbaev [9], Kasymov and Nurgabyl [10,11,12]. In the works of Nurgabyl [13,14], Mirzakulova [15] for the third-order equation with a small parameter at higher derivatives, the phenomena of boundary jumps were studied.

The constructions of an approximate solution of a boundary value problem defined with various additional conditions were studied by Hikosaka-Nobory [16], Kibenko and Perov [17], Dzhumabaev [18]. In these works, using the well-known structure of the differential equation and additional information, the problem of restoring the right side of the differential equation is solved.

[*]Corresponding author.
E-mail: n.saniya_96@mail.ru

So in the work of A.V. Kibenko and A.M. Perov [17] problem of simultaneously finding the function $y(t)$ and the parameter λ from the relations $y' = \lambda \cdot f(t,y)$, $y(0,\varepsilon) = \alpha$, $y(t_1,\varepsilon) = \beta$, with given $\alpha,\beta,t_1 \in \mathbb{R}$ were considered.

On the other hand, in some singularly perturbed boundary value problems, it might turn out that the number of additional conditions exceeds the order of the equation, and the equation contains unknown parameters. Interest in such problems is caused by the problems of the optimal management.

So, for example, [19], the restoration problem for singularly perturbed differential equations was investigated in the work, in case when the right-hand side of the differential equation and boundary conditions linearly depend on an unknown parameter. A priori estimates were established to solve a parametrized singularly perturbed boundary value problem for a second-order equation, by Mustafa Kudu et al. [20].

The proposed work is devoted to the study of the solution of a singularly perturbed boundary value problem with initial a jump in case when the boundary conditions depend on an unknown parameter in a nonlinear way.

1. Set the problem's condition

Let $\mathbb{R} = (-\infty,+\infty)$, $J = [0,1]$ and $\Lambda-$ be some bounded set from $\mathbb{R}$.

Consider the boundary value problem:

$$L_\varepsilon y \equiv \varepsilon y" + A(t)y' + B(t)y = \lambda h(t), \tag{1}$$

$$y(0,\varepsilon) = \alpha_0,\ y(1,\varepsilon) = \beta_0(\lambda), \quad y'(1,\varepsilon) = \beta_1(\lambda), \tag{2}$$

where $\alpha_0,\beta_0,\beta_1 \in R, \lambda$ is an unknown parameter, $\varepsilon > 0$ is a small parameter.

The problem is to determine the pair $(y(t,\lambda(\varepsilon),\varepsilon),\lambda(\varepsilon))$, where the function $y(t,\lambda,\varepsilon)$ at $0 \le t \le 1$ satisfies equation (1) and the boundary conditions (2), to establish an asymptotic estimate for the solution to problem (1), (2), to formulate a degenerate problem, to define condition for the occurrence of a jump.

A pair (λ, y) is called a solution to problem (1), (2) if, for each fixed value of $\lambda = \overline{\lambda} \in \Lambda$, the function $y(t,\varepsilon)$ is a solution to problem (1) (2).

In this article, using the results of the research[11], an analytical representation of the problem solution (1), (2) will be constructed and on its basis the existence and uniqueness of a solution are proved, a degenerate problem is formulated, an proximity of a solutions of the original and degenerate problem are proved at $\varepsilon \to 0$, the nature of the derivative growth of the solution of the problem (1), (2), the condition for the appearance of a jump, and asymptotic estimates of the solution of problem will be established (1), (2).

Let be:

1^0. $A(t), B(t), h(t) \in C^1(J)$;

2^0. $A(t) \ge \nu > 0, t \in J$;

3^0. The equation $R_0(\lambda_0) = -\beta_0 \dfrac{B(1)}{A(1)} + \lambda_0 \dfrac{F(1)}{A(1)} - \beta_1(\lambda_0) = 0$ has a unique solution $\lambda_0 = \bar{\lambda}_0$, at that $R_0'(\bar{\lambda}_0) = \dfrac{F(1)}{A(1)} - \beta_1'(\bar{\lambda}_0) \neq 0$.

2. Construction of the initial function

We consider the homogeneous equation

$$L_\varepsilon y\,(t,\varepsilon) \equiv \varepsilon y''\,(t,\varepsilon) + A\,(t)\,y'\,(t,\varepsilon) + B\,(t)\,y\,(t,\varepsilon) = 0, \tag{3}$$

which corresponds to the inhomogeneous equation (1).

The following lemma holds [4].

Lemma 1. Let conditions 1^0 - 2^0 be satisfied. Then for the fundamental system of solutions $y_1(t,\varepsilon)$, $y_2(t,\varepsilon)$ homogeneous equation (3) the following asymptotic representations are valid at $\varepsilon \to 0$:

$$y_1^{(j)}(t,\varepsilon) = y_0^{(j)}(t) + O(\varepsilon), y_2^{(j)}(t,\varepsilon) = \frac{1}{\varepsilon^j}\exp\left(\frac{1}{\varepsilon}\int_0^t \mu(x)\,dx\right) u_0(t)\,\mu^j(t)\,[1+O(\varepsilon)],\ j=0,1 \quad (4)$$

where $y_0(t) = \exp\left(-\int_0^t \frac{B(s)}{A(s)}ds\right)$, $\mu(t) = -A(t)$, $u_0(t) = \frac{A(0)}{A(t)} \cdot \exp\left(-\int_0^t \frac{B(x)}{A(x)}dx\right)$.
We introduce the initial function

$$K(t,s,\varepsilon) = \frac{W(t,s,\varepsilon)}{W(s,\varepsilon)}, \quad (5)$$

where $W(t,\varepsilon)$ is the Wronskian of the fundamental system of solutions $y_1(t,\varepsilon)$, $y_2(t,\varepsilon)$ of equation (3), $W(t,s,\varepsilon)$ is a second-order determinant obtained from $W(s,\varepsilon)$ by replacing the second row with a row with elements$y_1(t,\varepsilon)$, $y_2(t,\varepsilon)$.
Obviously, the function $K(t,s,\varepsilon)$ satisfies by at the variable t the homogeneous equation (3) and the initial conditions $K(t,t,\varepsilon)=0$, $\quad K_t'(t,t,\varepsilon)=1$, and does not depend on the choice of the fundamental system of equation solutions (4). Therefore, the initial function $K(t,s,\varepsilon)$ for equation (4) is uniquely determined.
From (5) taking into account (4) we obtain:

$$W(s,\varepsilon) = \frac{1}{\varepsilon}\exp\left(\frac{1}{\varepsilon}\int_0^s \mu(x)\,dx\right) u_0(s)\,y_0(s)\,\mu(s)\,[1+O(\varepsilon)] \neq 0; \quad (6)$$

$$W^{(q)}(t,s,\varepsilon) = u_0(s)e^{\frac{1}{\varepsilon}\int_0^s \mu(x)dx}\left[y_0^{(q)}(t) + \frac{1}{\varepsilon^q}e^{\frac{1}{\varepsilon}\int_s^t \mu(x)dx}\frac{u_0(t)\mu^q(t)}{u_0(s)} + O\left(\varepsilon + \frac{\varepsilon}{\varepsilon^q}e^{\frac{1}{\varepsilon}\int_s^t \mu(x)dx}\right)\right]. \quad (7)$$

Now using these estimates, it is easy to verify the validity of the following lemma.

Lemma 2. If conditions 1), 2) are satisfied, then the initial function $K(t,s,\varepsilon)$ for $0 \le s \le t \le 1$ and sufficiently small $\varepsilon > 0$ is representable in the form

$$K(t,s,\varepsilon) = \frac{\varepsilon}{\mu(s)}\left(\frac{y_0(t)}{y_0(s)} + \frac{u_0(t)}{u_0(s)}\exp\left(\frac{1}{\varepsilon}\int_s^t \mu(x)dx\right) + O(\varepsilon)\right), \quad (8)$$

$$K_t'(t,s,\varepsilon) = \frac{\varepsilon}{\mu(s)}\left(\frac{y_0'(t)}{y_0(s)} + \frac{1}{\varepsilon}\cdot\frac{u_0(t)\mu(t)}{u_0(s)}\exp\left(\frac{1}{\varepsilon}\int_s^t \mu(x)dx\right) + O\left(\varepsilon + e^{\frac{1}{\varepsilon}\int_s^t \mu(x)dx}\right)\right).$$

Proof. Estimating function (5) with regard for (6) and (7), we obtain estimate (8).

3. Solution representation of the auxiliary boundary value problem

Since the initial function $K(t,s,\varepsilon)$ satisfies the homogeneous equation (3) and the initial conditions $K(t,t,\varepsilon)=0$, $\quad K_t'(t,t,\varepsilon)=1$,so at fixed value of the parameterλ, the general solution of the inhomogeneous equation (1) can be represented in the form

$$y(t,\varepsilon) = \tilde{c}_1 y_1(t,\varepsilon) + \tilde{c}_2 y_2(t,\varepsilon) + \frac{\lambda}{\varepsilon}\int_0^t K(t,s,\varepsilon)F(s)ds, \quad (9)$$

where $\tilde{c}_1,\ \ \tilde{c}_2$ are arbitrary constants. Then the boundary value problem (1), (2) is solvable, and can be represented in the form (9), if and only if, when the coefficients $\tilde{c}_1,\ \ \tilde{c}_2$ can be chosen so that (9) satisfies the boundary conditions (2). Thus, substituting (9) in (2), we find

$$\tilde{c}_1 y_1(0,\varepsilon)+\tilde{c}_2 y_2(0,\varepsilon)=\alpha_0,$$

$$\tilde{c}_1 y_1(1,\varepsilon)+\tilde{c}_2 y_2(1,\varepsilon)+\frac{\lambda}{\varepsilon}\int\limits_0^1 K(1,s,\varepsilon)F(s)ds=\beta_0(\lambda), \tag{10}$$

$$\tilde{c}_1 y_1'(1,\varepsilon)+\tilde{c}_2 y_2'(1,\varepsilon)+\frac{\lambda}{\varepsilon}\int\limits_0^1 K_t'(1,s,\varepsilon)F(s)ds=\beta_1(\lambda).$$

From the first two equations of system (10) we have:

$$J(\varepsilon)=\begin{vmatrix} y_1(0,\varepsilon) & y_2(0,\varepsilon)\\ y_1(1,\varepsilon) & y_2(1,\varepsilon)\end{vmatrix}=-y_0(1)+O(\varepsilon)\neq 0, \tag{11}$$

$$\tilde{c}_1=\frac{1}{J(\varepsilon)}\begin{vmatrix} \alpha_0 & y_2(0,\varepsilon)\\ \beta_0-\sigma_0 & y_2(1,\varepsilon)\end{vmatrix},\quad \tilde{c}_2=\frac{1}{J(\varepsilon)}\begin{vmatrix} y_1(0,\varepsilon) & \alpha_0\\ y_1(1,\varepsilon) & \beta_0-\sigma_0\end{vmatrix}. \tag{12}$$

Substituting the found values $\tilde{c}_1$, $\tilde{c}_2$ from (12) into (9), we find that for each fixed value of the parameter λ, the solution of the auxiliary boundary value problem is representable in the form

$$y=\alpha_0\Phi_1(t,\varepsilon)+\beta_0\Phi_2(t,\varepsilon)-\Phi_2(t,\varepsilon)\frac{\lambda}{\varepsilon}\int\limits_0^1 K(1,s,\varepsilon)F(s)ds+\frac{\lambda}{\varepsilon}\int\limits_0^t K(t,s,\varepsilon)F(s)ds, \tag{13}$$

where the functions

$$\Phi_1(t,\varepsilon)=\frac{J_1(t,\varepsilon)}{J(\varepsilon)},\ \Phi_2(t,\varepsilon)=\frac{J_2(t,\varepsilon)}{J(\varepsilon)} \tag{14}$$

are called boundary functions of the boundary value problem (1), (2), which satisfy the homogeneous equation (3) and the boundary conditions:

$$\Phi_1(0,\varepsilon)=1,\quad \Phi_1(1,\varepsilon)=0,\quad \Phi_2(0,\varepsilon)=0,\quad \Phi_2(1,\varepsilon)=1, \tag{15}$$

where the determinant $J_1(t,\varepsilon)$ is obtained from the determinant $J(\varepsilon)$ by replacing the first row with the row $y_1(t,\varepsilon)$, $y_2(t,\varepsilon)$, and $J_2(t,\varepsilon)$ is obtained from $J(\varepsilon)$ by replacing the second row with the row $y_1(t,\varepsilon)$, $y_2(t,\varepsilon)$.

Based on (14), it can be proved that the boundary functions $\Phi_k(t,\varepsilon),\ \ k=1,2$ satisfy the boundary conditions (15) do not depend on the fundamental system of solutions $y_1(t,\varepsilon),\ y_2(t,\varepsilon)$ of equation (3). Then, for sufficiently small $\varepsilon>0$, the boundary functions $\Phi_k(t,\varepsilon),\ \ k=1,2$ on the interval [0,1] exist are unique, and are expressed by formula (14).

Lemma 3. If conditions 1^0-3^0, are satisfied, then at sufficiently small $\varepsilon>0$, *for the boundary functions* $\Phi_i^{(q)}(t,\varepsilon)$ *on the interval* $0\le t\le 1$ *the following estimates are true:*

$$\Phi_1(t,\varepsilon)=u_0(t)\,e^{\frac{1}{\varepsilon}\int\limits_0^t\mu(x)dx}+O(\varepsilon),\Phi_2(t,\varepsilon)=\frac{y_0(t)}{y_0(1)}+u_0(t)\,e^{\frac{1}{\varepsilon}\int\limits_0^t\mu(x)dx}+O(\varepsilon),$$

$$\Phi_1'(t,\varepsilon)=\frac{1}{\varepsilon}u_0(t)\mu(t)\,e^{\frac{1}{\varepsilon}\int\limits_0^t\mu(x)dx}+O\left(\varepsilon+e^{\frac{1}{\varepsilon}\int\limits_0^t\mu(x)dx}\right), \tag{16}$$

$$\Phi_2'(t,\varepsilon) = \frac{y_0'(t)}{y_0(1)} + u_0(t)\mu(t)\, e^{\frac{1}{\varepsilon}\int\limits_0^t \mu(x)dx} + O\left(\varepsilon + e^{\frac{1}{\varepsilon}\int\limits_0^t \mu(x)dx}\right).$$

The proof of the lemma directly follows from (14), taking into account estimates (11) and (4).

4. On the unique solvability of the solution of the recovery problem

Now, substituting (13) in the third equation of system (10), we obtain

$$R(\lambda,\varepsilon) \equiv \alpha_0\Phi_1'(1,\varepsilon) + \beta_0\Phi_2'(1,\varepsilon) - \Phi_2'(1,\varepsilon)\frac{\lambda}{\varepsilon}\int\limits_0^1 K(1,s,\varepsilon)F(s)ds + \tag{17}$$

$$+\frac{\lambda}{\varepsilon}\int\limits_0^1 K_t'(1,s,\varepsilon)F(s)ds - \beta_1(\lambda) = 0.$$

Thus, the solution of the boundary value problem (1), (2) is uniquely solvable if and only if the equation

$$R(\lambda,\varepsilon) = 0 \tag{18}$$

regarding a parameter λ has a unique solution.
Let us prove that equation (17) is solvable with respect to λ. To this end, from (17) we find $R'(\lambda,\varepsilon)$ and we study the asymptotic behavior of the functions $R(\lambda,\varepsilon)$ and

$$R'(\lambda,\varepsilon) \equiv -\frac{\Phi_2'(1,\varepsilon)}{\varepsilon}\int\limits_0^1 K(1,s,\varepsilon)F(s)ds + \frac{1}{\varepsilon}\int\limits_0^1 K_t'(1,s,\varepsilon)F(s)ds - \beta_1'(\lambda). \tag{19}$$

at $\varepsilon \to 0$.

Estimating the expressions from (17) and (18) at sufficiently small $\varepsilon > 0$, we obtain:

$$\Phi_1'(1,\varepsilon) = o(\varepsilon), \Phi_2'(1,\varepsilon) = \frac{y_0'(1)}{y_0(1)} + O(\varepsilon)$$

$$K(1,s,\varepsilon) = \frac{\varepsilon}{\mu(s)}\left(\frac{y_0(1)}{y_0(s)} + \frac{u_0(1)}{u_0(s)}\exp\left(\frac{1}{\varepsilon}\int\limits_s^1 \mu(x)dx\right) + O\left(\varepsilon\right)\right),$$

$$K_t'(1,s,\varepsilon) = \frac{\varepsilon}{\mu(s)}\left(\frac{y_0'(1)}{y_0(s)} + \frac{1}{\varepsilon}\frac{u_0(1)\mu(1)}{u_0(s)}\exp\left(\frac{1}{\varepsilon}\int\limits_s^1 \mu(x)dx\right) + O\left(\varepsilon + e^{\frac{1}{\varepsilon}\int\limits_s^{\lambda 1} \mu(x)dx}\right)\right),$$

$$\frac{\lambda}{\varepsilon}\int\limits_0^1 K(1,s,\varepsilon)F(s)ds = \lambda\int\limits_0^1 \frac{y_0(1)}{y_0(s)\mu(s)}F(s)ds + O(\varepsilon), \tag{20}$$

$$\frac{\lambda}{\varepsilon}\int\limits_0^1 K_t'(1,s,\varepsilon)F(s)ds = \lambda\int\limits_0^1 \frac{y_0'(1)F(s)}{y_0(s)\mu(s)}ds - \lambda\frac{F(1)}{\mu(1)} + O(\varepsilon).$$

$$\frac{\lambda}{\varepsilon}\int\limits_0^t K(t,s,\varepsilon)F(s)ds = \lambda\int\limits_0^t \frac{y_0(t)}{y_0(s)\mu(s)}F(s)ds + O(\varepsilon).$$

$$\frac{\lambda}{\varepsilon}\int\limits_0^t K'(t,s,\varepsilon)F(s)ds = \lambda\int\limits_0^t \frac{y_0'(t)F(s)}{y_0(s)\mu(s)}ds - \lambda\frac{F(t)}{\mu(t)} + \lambda\frac{u_0(t)\mu(t)}{u_0(0)\mu^2(0)}e^{\frac{1}{\varepsilon}\int\limits_0^t \mu(x)dx} + O(\varepsilon)$$

Taking into account estimates (20) for the functions $R(\lambda,\varepsilon)$ and $R'(\lambda,\varepsilon)$ at sufficiently small $\varepsilon > 0$, the following representations are valid:

$$R(\lambda,\varepsilon) = -\beta_0\frac{B(1)}{A(1)} + \lambda\frac{F(1)}{A(1)} - \beta_1(\lambda) + O(\varepsilon) = R_0(\lambda) + O(\varepsilon).$$

$$R'(\lambda,\varepsilon) = \frac{F(1)}{A(1)} - \beta'(\lambda) + O(\varepsilon) = R_0'(\lambda) + O(\varepsilon)$$

Hence, by virtue of condition 3^0, we conclude that at the point λ_0 for sufficiently small $\varepsilon > 0$ the following asymptotic representations are valid:

$$R(\lambda_0,\varepsilon) = O(\varepsilon);$$

$$R'(\lambda_0,\varepsilon) = R_0'(\lambda_0) + O(\varepsilon) \neq 0.$$

Consequently, in a sufficiently small neighborhood of the point λ_0 there is a unique point $\tilde{\lambda}(\varepsilon)$ such that will be fulfilled equality

$$R(\tilde{\lambda}(\varepsilon),\varepsilon) = 0,$$

at that

$$\left|\tilde{\lambda}(\varepsilon) - \lambda_0\right| \leq K\varepsilon.$$

Thus, we proved that there a unique solution $(y(t,\varepsilon),\, \tilde{\lambda}(\varepsilon))$ of the boundary value problem (1), (2) exists. Thus, the following theorem holds.

Theorem 1. If conditions 1^0 – 3^0 are satisfied, then the boundary-value problem (1), (2) has a unique solution and this solution can be represented in the form

$$y(t,\tilde{\lambda}(\varepsilon)) = \alpha_0\Phi_1(t,\varepsilon) + \beta_0\Phi_2(t,\varepsilon) - \Phi_2(t,\varepsilon)\frac{\tilde{\lambda}(\varepsilon)}{\varepsilon}\int\limits_0^1 K(1,s,\varepsilon)F(s)ds +$$

$$+\frac{\tilde{\lambda}(\varepsilon)}{\varepsilon}\int\limits_0^t K(t,s,\varepsilon)F(s)ds. \quad (21)$$

5. Limit Transition Theorem. The phenomena of the initial jump

The following estimate holds for solution (21):

$$y(t,\varepsilon) = \beta_0\exp\left(-\int\limits_1^t \frac{B(x)}{A(x)}dx\right) - \lambda_0\int\limits_0^1 \exp\left(-\int\limits_s^t \frac{B(x)}{A(x)}dx\right)\frac{F(s)}{\mu(s)}ds +$$

$$+\lambda_0\int\limits_0^t \exp\left(-\int\limits_s^t \frac{B(x)}{A(x)}dx\right)\frac{F(s)}{\mu(s)}ds + O\left(\varepsilon + \exp\left(\frac{1}{\varepsilon}\int\limits_0^t \mu(x)dx\right)\right). \quad (22)$$

Now, we define a degenerate problem. Without any additional considerations, we cannot formulate the boundary conditions for the unperturbed (degenerate) equation

$$L_0\bar{y} \equiv A(t)\bar{y}' + B(t)\bar{y} = \lambda_0 F(t), \quad (23)$$

obtained from (1) at $\varepsilon = 0$.

Such an additional consideration we can obtain from estimate (22). It follows from (22) that the limit function $y(t,\varepsilon)$ does not contain α_0 and β_1 at $\varepsilon \to 0$. Therefore, the boundary conditions for the solution are defined in the form

$$\bar{y}(1) = \beta_0(\lambda_0). \tag{24}$$

Therefore, the solution to problem (23), (24) is representable in the form

$$\bar{y}(t) = \beta_0(\lambda_0) \exp\left(-\int\limits_1^t \frac{B(x)}{A(x)}dx\right) + \lambda_0 \int\limits_1^t \exp\left(-\int\limits_s^t \frac{B(x)}{A(x)}dx\right) \frac{F(s)}{\mu(s)}ds, \tag{25}$$

Theorem 2. Let conditions 1^0-3^0 be satisfied. Then, for sufficiently small $\varepsilon > 0$, the following estimate holds:

$$\left| y\left(t, \widetilde{\lambda}(\varepsilon), \varepsilon\right) - \overline{y}\left(t, \lambda_0\right)\right| = O\left(\varepsilon + \exp\left(-\frac{\gamma t}{\varepsilon}\right)\right). \tag{26}$$

The proof follows from representations (22), (25).

Thus, it directly follows from Theorem 2 that the solution $\left(y(t, \widetilde{\lambda}(\varepsilon), \varepsilon),\ \widetilde{\lambda}(\varepsilon)\right)$ of the singularly perturbed problem (1), (2) at tends small parameter ε to zero tends to the solution of $\bar{y}(t,\lambda_0)$:

$$\lim_{\varepsilon\to 0} y(t, \widetilde{\lambda}(\varepsilon)\varepsilon) = \bar{y}(t,\lambda_0), \quad 0 < t \le 1. \tag{27}$$

Hence, we conclude that

$$\lim_{\varepsilon\to 0} y(0, \tilde{\lambda}(\varepsilon), \varepsilon) - \bar{y}(0,\lambda_0) = \Delta, \tag{28}$$

where Δ is a some magnitude. We define magnitude of the jump Δ. Using formulas (23), (26), (27) and the condition $y(0,\varepsilon) = \alpha_0$, we determine the magnitude of the initial jump:

$$\Delta = \alpha_0 - \beta_0(\lambda_0) \exp\left(\int\limits_0^1 \frac{B(x)}{A(x)}\right) + \lambda_0 \int_0^1 \exp\left(-\int\limits_s^0 \frac{B(x)}{A(x)}dx\right) \frac{F(s)}{A(s)}ds.$$

Conclusions

The proposed algorithm serves as the basis for constructing asymptotic solutions of some linear and nonlinear singularly perturbed boundary value problems with parameters for higher order equations with more complex additional conditions such $U_i(y) = 0,\ i = \overline{1,.n}$ where $U_i(y)$ the linear form of $y^{(j)}(0,\varepsilon), y^{(j)}(1,\varepsilon), j = \overline{0.n-1}$.

In this work, the asymptotic behavior of the solution to the problem of reconstructing the boundary conditions and the right-hand side for second-order differential equations with a small parameter at the highest derivative were studied. At that the following new results were obtained:

- Asymptotic estimates are obtained for the solution of the reconstruction problem for singularly perturbed second-order equations with an initial jump;

- rules for the restoration of boundary conditions and the right side of the original and degenerate problems were established;

- Asymptotic estimates are obtained for the solution of the perturbed problem and the difference between the solution of the degenerate problem and the solution of the perturbed problem.

The results obtained open up possibilities for the further development of the theory of singularly perturbed boundary value problems for ordinary differential equations.

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Д.Н. Нургабыл, С.С. Нажим

Бастапқы секірумен сингулярлы ауытқыған дифференциалдық теңдеуді қалпына келтіру есебі

Мақалада бастапқы секіру құбылысына ие жоғары туынды кезінде кіші параметрі бар екінші ретті дифференциалдық теңдеулер үшін оң және шеттік жағдайларды қалпына келтіру есептерін шешудің асимптотикалық шешімі зерттелген. Бастапқы секірумен екінші ретті сингулярлы ауытқыған теңдеулер үшін қалпына келтіру есебін шешудің асимптотикалық бағалары алынды. Шеттік жағдайларды қалпына келтіру ережелері, бастапқы және қалыптасқан міндеттердің оң бөліктері белгіленген. Ауытқыған есептің шешімін асимптотикалық бағалау, сонымен қатар ауытқыған және азғындалған есептердің шешімдері арасындағы айырмашылық анықталды. Сингулярлы ауытқыған теңдеулер позициясындағы қалпына келтіру есебінің бар болуы, біртұтастығы және шешімін ұсыну туралы теорема дәлелденген. Алынған нәтижелер жай дифференциалдық теңдеулер үшін сингулярлы ауытқыған шеттік есептер теориясының одан әрі дамуына мүмкіндік береді.

Кілт сөздер: ауытқыған есептер, кіші параметр, шеттік есептер, бастапқы секіріс, асимптотикалық қасиет.

Д.Н. Нургабыл, С.С. Нажим

Задача восстановления сингулярно возмущенного дифференциального уравнения с начальным скачком

В статье исследовано асимптотическое поведение решения задачи восстановления краевых условий и правой части для дифференциальных уравнений второго порядка с малым параметром при старшей производной, обладающих явлением начального скачка. Получены асимптотические оценки решения задачи восстановления для сингулярно возмущенных уравнений второго порядка с начальным скачком. Установлены правила восстановления краевых условий и правые части исходной и вырожденной задач. Определены асимптотические оценки решения возмущенной задачи и разности между решением вырожденной задачи и решением возмущенной задачи. Доказана теорема о существовании, единственности и представлении решения задачи восстановления с позиции сингулярно возмущенных уравнений. Полученные результаты открывают возможности для дальнейшего развития теории сингулярно возмущенных краевых задач для обыкновенных дифференциальных уравнений.

Ключевые слова: возмущенные задачи, вырожденные задачи, малый параметр, краевая задача, начальный скачок, асимптотическое поведение.

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