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On classification of degenerate singular points of Ricci flows

We consider the normalized Ricci flow on generalized Wallach spaces that could be reduced to a system of nonlinear ODEs. As a main result we get the classification of degenerate singular points of the system under consideration in the important partial case $a_i = a_j, i, j \in \{1, 2, 3\}, i \neq j$. In general the problem can also be considered as two-parametric bifurcations of solutions of abstract dynamical systems. Thus the problem under investigation is interesting not only in geometrical sense, but concerns the theory of planar dynamical systems.

Key words: Riemannian invariant metric, Einstein metric, generalized Wallach space, Ricci flow, dynamical system, system of nonlinear ordinary differential equations, singular point, degenerate singular point, parametric bifurcations.

Introduction

In the present work we continue investigations started in [1–7]. Consider the autonomous system of nonlinear ODEs obtained in [6]:

$$\frac{dx_1}{dt} = f(x_1, x_2, x_3), \frac{dx_2}{dt} = g(x_1, x_2, x_3), \frac{dx_3}{dt} = h(x_1, x_2, x_3), x_i = x_i(t) > 0, \quad (1)$$

where

$$f(x_1, x_2, x_3) = -1 - a_1 x_1 \left(\frac{x_1}{x_2 x_3} - \frac{x_2}{x_1 x_3} - \frac{x_3}{x_1 x_2} \right) + x_1 B;$$

$$g(x_1, x_2, x_3) = -1 - a_2 x_2 \left(\frac{x_2}{x_1 x_3} - \frac{x_3}{x_1 x_2} - \frac{x_1}{x_2 x_3} \right) + x_2 B;$$

$$h(x_1, x_2, x_3) = -1 - a_3 x_3 \left(\frac{x_3}{x_1 x_2} - \frac{x_1}{x_2 x_3} - \frac{x_2}{x_1 x_3} \right) + x_3 B;$$

$$B := \left(\frac{1}{a_1 x_1} + \frac{1}{a_2 x_2} + \frac{1}{a_3 x_3} - \frac{x_1}{x_2 x_3} - \frac{x_2}{x_1 x_3} - \frac{x_3}{x_1 x_2} \right) \left(\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} \right)^{-1}, a_i \in (0, 1/2], i = 1, 2, 3.$$

Recall that system (1) arises at investigations of Ricci flows ([8], [9]) on generalized Wallach spaces (see details in [3–5]). As it was proved in [6], system (1) could be equivalently reduced to a system of two differential equations of the type

$$\frac{dx_1}{dt} = \tilde{f}(x_1, x_2), \frac{dx_2}{dt} = \tilde{g}(x_1, x_2), \quad (2)$$

where

$$\tilde{f}(x_1, x_2) = f(x_1, x_2, \varphi(x_1, x_2)), \tilde{g}(x_1, x_2) = g(x_1, x_2, \varphi(x_1, x_2)), \varphi(x_1, x_2) = x_1^{\frac{a_3}{a_1}} x_2^{\frac{a_3}{a_2}}.$$

In Theorems 1–3 of [2] we investigated the case $a_1 = a_2 = b$, $a_3 = c$, important from a geometrical point of view, where $b, c \in (0, 1/2]$, and determined all possible values of the parameters b and c ensuring the system (2) degenerate singular points with $x_1 = x_2$ (see [1] for detail).

In the present work these investigations are continued. More precisely, we offer a qualitative classification of such singular points. The main results of the present work are contained in Theorems 2–4.

The paper is organized as follows. In section 1 we reformulate some well-known facts. In section 2 we prove Lemma 1. In section 3 we prove Theorems 2–4.

1. Preliminaries

It is obvious that the functions $\tilde{f}(x_1, x_2)$, $\tilde{g}(x_1, x_2)$ are analytic in a small neighborhood of an arbitrary point $(x_1^0, x_2^0) \neq (0, 0)$, and consequently the following representations are valid:

$$\begin{aligned} \tilde{f}(x_1, x_2) &= j_{11}(x_1 - x_1^0) + j_{12}(x_2 - x_2^0) + F(x_1, x_2); \\ \tilde{g}(x_1, x_2) &= j_{21}(x_1 - x_1^0) + j_{22}(x_2 - x_2^0) + G(x_1, x_2), \end{aligned}$$

where j_{ij} are entries of the Jacobian matrix $J = J(x_1^0, x_2^0) = \begin{pmatrix} \tilde{f}_{x_1} & \tilde{f}_{x_2} \\ \tilde{g}_{x_1} & \tilde{g}_{x_2} \end{pmatrix} \Big|_{(x_1, x_2) = (x_1^0, x_2^0)}$, F, G are some analytic functions in a neighborhood of the point (x_1^0, x_2^0) and $F = G = F_{x_1} = F_{x_2} = G_{x_1} = G_{x_2} = 0$ at the point (x_1^0, x_2^0) .

Let λ_1, λ_2 be eigenvalues of the matrix $J = J(x_1^0, x_2^0)$ and let $|\lambda_1| \leq |\lambda_2|$ without loss of generality.

Recall some well-known definitions of the qualitative theory of ODEs.

(x_1^0, x_2^0) is called a singular point of (2) if $\tilde{f} = \tilde{g} = 0$ at (x_1^0, x_2^0) .

(x_1^0, x_2^0) is called degenerate singular point if $\delta := \det(J) = 0$.

In the qualitative theory of ODEs the degenerate case consists of the following subcases [10]:

Semi-hyperbolic case ($\lambda_1 = 0, \lambda_2 \neq 0, J \neq 0$). There exist 3 types of phase portraits: saddles, nodes and saddle-nodes;

Nilpotent case ($\lambda_1 = 0, \lambda_2 = 0, J \neq 0$). In this case 13 topologically different types of phase portraits are possible (saddle, node, saddle-node, focus, center, cusp, et.c.);

Linearly zero case ($\lambda_1 = 0, \lambda_2 = 0, J = 0$). This case is more difficult for investigations and contains 65 different types of phase portraits due to classifications of [11].

One can easily obtain from general results of [7] that nilpotent case does not occur for (2), and linearly zero case may appear only at $a_1 = a_2 = a_3 = 1/4$.

In semi-hyperbolic cases we will use the following theorem.

Theorem 1 (Theorem 2.19 in [10]). Let $(0, 0)$ be an isolated singular point of the system

$$\frac{dx}{dt} = X(x, y), \quad \frac{dy}{dt} = \lambda y + Y(x, y), \quad \lambda > 0, \tag{3}$$

where X and Y are analytic in a neighborhood of the origin $(0, 0)$ with

$$X(0, 0) = Y(0, 0) = X_x(0, 0) = Y_x(0, 0) = X_y(0, 0) = Y_y(0, 0) = 0.$$

Let $y = \varphi(x)$ be the solution of the equation $\lambda y + Y(x, y) = 0$ in a neighborhood of $(0, 0)$, and suppose that the function $\psi(x) = X(x, \varphi(x))$ has the expression $\psi(x) = e_m x^m + o(x^m)$,

where $m \geq 2$ and $e_m \neq 0$.

Then

i) if m is odd and $e_m < 0$ (respectively $e_m > 0$), then $(0, 0)$ is a saddle (respectively an unstable node);

ii) if m is even, then $(0, 0)$ is a saddle-node.

Put $p_{n-i,i} := \frac{1}{i!(n-i)!} \frac{\partial^n X(0,0)}{\partial x^{n-i} \partial y^i}$, $q_{n-i,i} := \frac{1}{i!(n-i)!} \frac{\partial^n Y(0,0)}{\partial x^{n-i} \partial y^i}$.

Remark 1. By the implicit function theorem the equation $\lambda y + Y(x, y) = 0$ has an unique analytic solution $y = \varphi(x)$, $\varphi(0) = \varphi'(0) = 0$, in a sufficiently small neighborhood of $(0, 0)$.

Since $Y(x, y)$ is represented by Taylor series then $y = \varphi(x)$ is represented by power series $y = \varphi(x) = \sum_{n=2}^{\infty} v_n x^n$. Moreover, $v_2 = -\frac{1}{\lambda} q_{2,0}$, $v_3 = -\frac{1}{\lambda} (v_2 q_{1,1} + q_{3,0})$, By the same reasons $\psi(x) = \sum_{n=2}^{\infty} e_n x^n$ with

$$\psi(0) = \psi'(0) = 0, e_2 = p_{2,0}, e_3 = v_2 p_{1,1} + p_{3,0}, \dots$$

It is clear that there exists a first nonzero term e_m in $\psi(x) = \sum_{n=2}^{\infty} e_n x^n$. Otherwise we have $\psi(x) \equiv 0$, i.e. we have the family of non-isolated singular points of (2) along the line $y = \varphi(x)$ in spite of our conditions.

Remark 2. The case $\lambda < 0$ can be reduced to (3) by the transformation $t \mapsto -t$. It is clear that $e_2 = -p_{2,0}$, $e_3 = -v_2 p_{1,1} - p_{3,0}$,

2. Auxiliary results

As the calculations show the direct substituting $x_3 = \varphi(x_1, x_2)$ into f and g leads to very complicated expressions for partial derivatives of \tilde{f} and \tilde{g} . We offer an effective way to avoid it.

Lemma 1. Let $(x_1^0, x_2^0, x_3^0) = (\gamma_1 q, \gamma_2 q, \gamma_3 q)$, where q, γ_i are positive real numbers. Then for partial derivatives of $\tilde{f}(x_1, x_2)$ and $\tilde{g}(x_1, x_2)$ the following formulas are valid at the point (x_1^0, x_2^0) :

$$\begin{aligned} \tilde{z}_{x_i} &= z_{x_i} + z_{x_3} \varphi_{x_i}; \\ \tilde{z}_{x_i x_j} &= z_{x_i x_j} + z_{x_i x_3} \varphi_{x_j} + (z_{x_3 x_j} + z_{x_3 x_3} \varphi_{x_j}) \varphi_{x_i} + z_{x_3} \varphi_{x_i x_j}; \\ \tilde{z}_{x_i x_j x_j} &= z_{x_i x_j x_j} + z_{x_i x_j x_3} \varphi_{x_j} + 2(z_{x_j x_j x_3} + z_{x_j x_3 x_3} \varphi_{x_j}) \varphi_{x_i} + 2z_{x_i x_3} \varphi_{x_j x_j} + (z_{x_j x_3 x_3} + z_{x_3 x_3 x_3} \varphi_{x_j}) (\varphi_{x_i})^2 + \\ &\quad + 2z_{x_3 x_3} \varphi_{x_i} \varphi_{x_j x_j} + (z_{x_3 x_j} + z_{x_3 x_3} \varphi_{x_j}) \varphi_{x_i x_j} + z_{x_3} \varphi_{x_i x_j x_j}, \end{aligned}$$

where

$$\begin{aligned} z \in \{f, g\}, \varphi_{x_i} &= -\frac{a_3 \gamma_3}{a_i \gamma_i}, \varphi_{x_i x_j} = \frac{a_3}{a_i} \left(\frac{a_3}{a_j} + \delta_{ij} \right) \frac{\gamma_3}{\gamma_i \gamma_j} \frac{1}{q}; \\ \varphi_{x_i x_j x_j} &= -\frac{a_3}{a_i} \left(\frac{a_3}{a_i} + 1 \right) \left(\frac{a_3}{a_j} + 2\delta_{ij} \right) \frac{\gamma_3}{\gamma_i^2 \gamma_j} \frac{1}{q^2}, \end{aligned}$$

δ_{ij} is the Kronecker's symbol, $i, j \in \{1, 2\}$.

Proof. This follows by using the chain rule several time:

$$\frac{\partial}{\partial x_i} z(x_1, x_2, \varphi(x_1, x_2)) = \frac{\partial}{\partial x_i} z(x_1, x_2, x_3) + \frac{\partial}{\partial x_3} z(x_1, x_2, x_3) \frac{\partial \varphi(x_1, x_2)}{\partial x_i}, \quad i = 1, 2.$$

3. Main results

Now we shall formulate and proof main results of the present work. Denote $D := 1 - 4(1 - 2c)(b + c)$, $\mu := 1 \pm \sqrt{D}$ as in [4], and recall the special values of the parameters b, c used in [2]:

$$\begin{aligned} b_1 &= (\sqrt{3} - 1) / 4, \quad b_2 = (\sqrt{2} - 1) / 2, \quad b_3 = (\sqrt{5} - 1) / 4, \quad b_4 = \sqrt{2} / 4; \\ c_1 &= (1 - 2b - \sqrt{4b^2 + 4b - 1}) / 4, \quad c_2 = (1 - 2b + \sqrt{4b^2 + 4b - 1}) / 4; \\ c_3 &= (16b^3 - 4b + 1) / (2 - 16b^2), \quad c_4 = (1 - 8b^2) / (8b). \end{aligned}$$

We will separately consider the cases when $\delta = 0$ by $D = 0$ and $\delta = 0$ despite the fact $D \neq 0$ in the formula

$$\delta = \frac{D \pm \sqrt{D}}{4(b+c)^2 \mu^2 q^2} (8b(b+c) - \mu) \text{ obtained in Theorem 1 of [1].}$$

Case 1. Degenerate singular points with $D = 0$

As it has proved in Theorem 1 of [2] only for fixed $b \in [b_2, 1/4)$, $c = c_1$ or $b \in [b_2, 1/2]$, $c = c_2$ the system (2) has an isolated singular point of the kind $(x_1^0, x_2^0) = (2(b+c)q, 2(b+c)q)$ with $D = 0$, where $q = q(c) = (2(b+c))^{-2d/b} > 0$, $d = (2b^{-1} + c^{-1})^{-1}$.

Table 1

Types of degenerate singular points of the system (2) in the case $D = 0$

Values of b	Values of c , and corresponding types of degenerate singular points of (2)		
	Semi-hyperbolic type, $\lambda_1 = 0, \lambda_2 \neq 0, J \neq 0$	Nilpotent type, $\lambda_1 = \lambda_2 = 0, J \neq 0$	Linearly zero type, $\lambda_1 = \lambda_2 = 0, J = 0$
$b \in (0, b_2)$	–	–	–
$b \in [b_2, 1/4)$	$c = c_1, c = c_2,$ saddle-node	–	–
$b = 1/4$	–	–	$c = c_2 = 1/4$, saddle
$b \in (1/4, 1/2]$	$c = c_2$, saddle-node	–	–

Theorem 2. Let $D = 0$. Then for the singular point $(x_1^0, x_2^0) = (2(b+c)q, 2(b+c)q)$ of the system (2) only the following types of singularities are possible shown in Table 1:

- (a) (x_1^0, x_2^0) is a semi-hyperbolic saddle-node only for $b \in [b_2, 1/4)$, $c = c_1$ or $b \in [b_2, 1/4) \cup (1/4, 1/2]$, $c = c_2$;
- (b) (x_1^0, x_2^0) is a linear zero saddle only at $b = 1/4$, $c = 1/4$;
- (c) There are no values of b, c such that (x_1^0, x_2^0) could be a nilpotent singular point.

Proof. (a) Case $b \in [b_2, 1/4)$, $c = c_1$.

Then we have

$$(x_1^0, x_2^0) = (2(b+c_1)q_1, 2(b+c_1)q_1), \tag{4}$$

where $q_1 := q(c_1) > 0$.

By Theorem 1 of [1] at the point (4) the matrix of linear part of (2) takes the form: $J = k_1 \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$,

where $k_1 = \frac{1-2b + \sqrt{4b^2 + 4b - 1}}{2q_1} > 0$ whenever $b \in [b_2, 1/4)$.

J has the eigenvalues $\lambda_1 = 0$ and $\lambda_2 = 2k_1 \neq 0$, there fore, we deal with a singular point of semi-hyperbolic type. Using the transformations

$$x_1 = x_1(x, y) = x_1^0 + (x+y)/2, \quad x_2 = x_2(x, y) = x_2^0 + (x-y)/2, \tag{5}$$

one can move (4) to the origin $(0, 0)$. Thus the system (2) can be transformed to the form as in (3):

$$\frac{dx}{dt} = X(x, y), \quad \frac{dy}{dt} = 2k_1 y + Y(x, y), \tag{6}$$

where

$$\begin{aligned} X(x, y) &= F(x_1(x, y), x_2(x, y)) + G(x_1(x, y), x_2(x, y)); \\ Y(x, y) &= F(x_1(x, y), x_2(x, y)) - G(x_1(x, y), x_2(x, y)). \end{aligned}$$

Since F and G are analytic in a neighborhood of (4), then X and Y are analytic in a neighborhood of $(0, 0)$.

Obviously, $F_{x_i x_j} = \tilde{f}_{x_i x_j}$, $G_{x_i x_j} = \tilde{g}_{x_i x_j}$.

Taking into account (5) and using Lemma 1 we get

$$p_{2,0} := \frac{1}{2} X_{xx}(0,0) = \frac{1}{8} \left(\tilde{f}_{x_1 x_1} + 2\tilde{f}_{x_1 x_2} + \tilde{f}_{x_2 x_2} \right) \Big|_{(x_1, x_2) = (x_1^0, x_2^0)} + \frac{1}{8} \left(\tilde{g}_{x_1 x_1} + 2\tilde{g}_{x_1 x_2} + \tilde{g}_{x_2 x_2} \right) \Big|_{(x_1, x_2) = (x_1^0, x_2^0)} = \frac{1 + 4c_1 - 8c_1^2}{16b(b + c_1)q_1^2}.$$

By Lemma 1 in [2] the function $c_1 = c_1(b)$ satisfies the condition $0 < c_1 < 1/2$ at $b \in [b_2, 1/4)$.

Then $1 + 4c_1 - 8c_1^2 > 0$, and consequently $p_{2,0} > 0$ for all $b \in [b_2, 1/4)$.

In other words, the system has saddle-node at the point $(0,0)$ according to Theorem 1 and Remark 1.

Going back to the system (2) we conclude that (4) is a saddle-node for (2).

Case $b \in [b_2, 1/4) \cup (1/4, 1/2]$, $c = c_2$.

In this case we have $(x_1^0, x_2^0) = (2(b + c_2)q_2, 2(b + c_2)q_2)$, where $q_2 := q(c_2) > 0$.

Using the formulas of Theorem 1 in [1] we find that $J = k_2 \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$, where

$$k_2 = \frac{1 - 2b - \sqrt{4b^2 + 4b - 1}}{2q_2}.$$

It is not difficult to see that $k_2 > 0$ at $b \in [b_2, 1/4)$ and $k_2 < 0$ at $b \in (1/4, 1/2]$.

In the same manner as in the previous case we can easily find that

$$p_{2,0} := \frac{1}{2} X_{xx}(0,0) = \frac{1 + 4c_2 - 8c_2^2}{16b(b + c_2)q_2^2} > 0,$$

because of $0 < c_2 < 1/2$ for all $b \in [b_2, 1/2]$ by Lemma 1 in [2].

By Remarks 1 and 2 (depending on the fact whether $k_2 > 0$ or $k_2 < 0$) and by Theorem 1 such point $(x_1^0, x_2^0) = (2(b + c_2)q_2, 2(b + c_2)q_2)$ is a saddle-node for (2).

(b) Let now $b = 1/4$, $c = c_2$.

Then $c = c_2 = 1/4$, $k_2 = 0$ and $(x_1^0, x_2^0) = (1,1)$.

Therefore, $J = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, and according to [10] we have a linear zero case.

Using the results of [11] it has proved in [6] that $(x_1^0, x_2^0) = (1,1)$ is a saddle point with six hyperbolic sectors around it (see fig.).

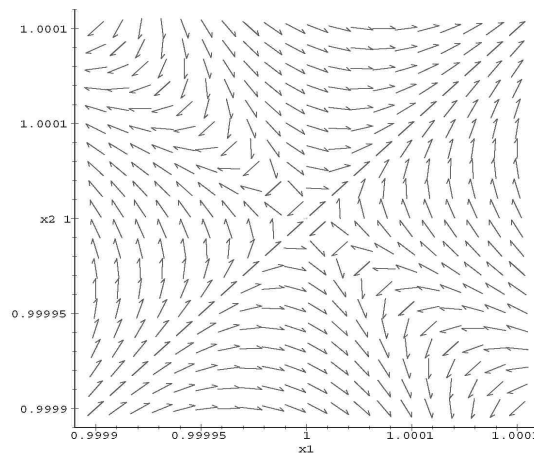


Figure. The phase portrait of (2) at $b = c = 1/4$

(c) We considered all possible values of b, c ensuring $D = 0$. Therefore, under the conditions of Theorem 2 the system (2) has no singular points of nilpotent type. The theorem is proved.

Case 2. Degenerate singular points with $D > 0, \mu = 1 \pm \sqrt{D}$

According to Theorems 2 and 3 in [2] the system (2) has an isolated degenerate singular point of the kind

$$(x_1^0, x_2^0) = (2(b + c_3)q_3, 2(b + c_3)q_3) \tag{7}$$

for all fixed $b \in (1/4, b_3], c = c_3$ or $b \in (0, 1/4), c = c_3$, where

$$q_3 = (2(b + c_3))^{-2d/b} \tilde{\mu}^{-d/c_3} > 0, \tilde{\mu} = \frac{4b(2b-1)}{8b^2-1} > 0, d = (2b^{-1} + c_3^{-1})^{-1}.$$

Remind that $1 + \sqrt{D} = \tilde{\mu}$ at $b \in (1/4, b_3], c = c_3$ and $1 - \sqrt{D} = \tilde{\mu}$ at $b \in (0, 1/4), c = c_3$.

Table 2

Types of degenerate singular points of the system (2) in the case $0 < D < 1, \mu = 1 - \sqrt{D}$

Values of b	Values of c , and corresponding types of degenerate singular points of (2)		
	Semi-hyperbolic type, $\lambda_1 = 0, \lambda_2 \neq 0, J \neq 0$	Nilpotent type, $\lambda_1 = \lambda_2 = 0, J \neq 0$	Linearly zero type, $\lambda_1 = \lambda_2 = 0, J = 0$
$b \in (0, 1/4)$	$c = c_3$, saddle	–	–
$b \in [1/4, 1/2]$	–	–	–

In this case the following two theorems are valid.

Theorem 3. Let $0 < D < 1, \mu = 1 - \sqrt{D}$. Then for the singular point (7) of the system (2) only the following types of singularities are possible shown in Table 2:

- (a) (x_1^0, x_2^0) is a semi-hyperbolic saddle only at $b \in (0, 1/4), c = c_3$;
- (b) There are no values of b, c such that (x_1^0, x_2^0) could be nilpotent or linearly zero singular point.

Proof. (a) Let $b \in (0, 1/4), c = c_3$. Then by Theorem 1 in [1] we have $J = k_3 \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, where

$$k_3 = \frac{4b-1}{(2b-1)q_3} > 0, \text{ i.e. there is a semi-hyperbolic case again } (\lambda_1 = 0, \lambda_2 = 2k_3 \neq 0).$$

Put

$$x_1 = x_1(x, y) = x_1^0 + (x + y) / 2, x_2 = x_2(x, y) = x_2^0 + (y - x) / 2. \tag{8}$$

Then the system (2) can be transformed to the form:

$$\frac{dx}{dt} = X(x, y), \frac{dy}{dt} = 2k_3y + Y(x, y), \tag{9}$$

where

$$\begin{aligned} X(x, y) &= F(x_1(x, y), x_2(x, y)) - G(x_1(x, y), x_2(x, y)), \\ Y(x, y) &= F(x_1(x, y), x_2(x, y)) + G(x_1(x, y), x_2(x, y)). \end{aligned}$$

From (8) we have

$$\begin{aligned} X_{xx}(0, 0) &= \frac{1}{4} \left(\tilde{f}_{x_1x_1} - 2\tilde{f}_{x_1x_2} + \tilde{f}_{x_2x_2} \right) \Big|_{(x_1, x_2) = (x_1^0, x_2^0)} - \frac{1}{4} \left(\tilde{g}_{x_1x_1} - 2\tilde{g}_{x_1x_2} + \tilde{g}_{x_2x_2} \right) \Big|_{(x_1, x_2) = (x_1^0, x_2^0)} ; \\ X_{xy}(0, 0) &= \frac{1}{4} \left(\tilde{f}_{x_1x_1} - \tilde{f}_{x_2x_2} \right) \Big|_{(x_1, x_2) = (x_1^0, x_2^0)} - \frac{1}{4} \left(\tilde{g}_{x_1x_1} - \tilde{g}_{x_2x_2} \right) \Big|_{(x_1, x_2) = (x_1^0, x_2^0)} , \\ Y_{xx}(0, 0) &= \frac{1}{4} \left(\tilde{f}_{x_1x_1} - 2\tilde{f}_{x_1x_2} + \tilde{f}_{x_2x_2} \right) \Big|_{(x_1, x_2) = (x_1^0, x_2^0)} + \frac{1}{4} \left(\tilde{g}_{x_1x_1} - 2\tilde{g}_{x_1x_2} + \tilde{g}_{x_2x_2} \right) \Big|_{(x_1, x_2) = (x_1^0, x_2^0)} ; \\ X_{xxx}(0, 0) &= \frac{1}{8} \left(\tilde{f}_{x_1x_1x_1} - 3\tilde{f}_{x_1x_1x_2} + 3\tilde{f}_{x_1x_2x_2} - \tilde{f}_{x_2x_2x_2} \right) \Big|_{(x_1, x_2) = (x_1^0, x_2^0)} - \\ &\quad - \frac{1}{8} \left(\tilde{g}_{x_1x_1x_1} - 3\tilde{g}_{x_1x_1x_2} + 3\tilde{g}_{x_1x_2x_2} - \tilde{g}_{x_2x_2x_2} \right) \Big|_{(x_1, x_2) = (x_1^0, x_2^0)} . \end{aligned}$$

Desired partial derivatives of the functions \tilde{f} and \tilde{g} we shall calculate using Lemma 1. Thus

$$\begin{aligned}
 p_{2,0} &= \frac{1}{2} X_{xx}(0,0) = 0, \\
 p_{1,1} &= X_{xy}(0,0) = \frac{1}{2} \frac{(8b^3 + 4b^2 - 4b + 1)(8b^2 - 1)}{b(2b-1)^2 q_3^2}; \\
 q_{2,0} &= \frac{1}{2} Y_{xx}(0,0) = -\frac{1}{2} \frac{(4b^2 + 2b - 1)(8b^2 - 1)}{(2b-1)^2 q_3^2}; \\
 p_{3,0} &= \frac{1}{6} X_{xxx}(0,0) = -\frac{1}{8} \frac{(24b^3 + 4b^2 - 6b + 1)(8b^2 - 1)^2}{b(2b-1)^3 q_3^3}.
 \end{aligned}$$

It easily follows that

$$p_{1,1}v_2 + p_{3,0} = -p_{1,1} \frac{q_{2,0}}{2k_3} + p_{3,0} = \frac{(8b^2 - 1)^3(2b+1)}{8b(2b-1)q_3^3} \frac{1}{4b-1}.$$

It is clear that $\frac{(8b^2 - 1)^3}{2b-1} > 0$ for $b \in (0, b_3]$. Therefore, $p_{1,1}v_2 + p_{3,0} < 0$ for $b \in (0, 1/4)$.

Since $k_3 > 0$ then $e_2 = p_{2,0} = 0$, $e_3 = p_{1,1}v_2 + p_{3,0} < 0$ for every $b \in (0, 1/4)$ by Remark 1.

Hence $(0,0)$ is a saddle of (9) by Theorem 1. Then (7) is a saddle point for the system (2) respectively.

(b) We checked all values of b, c leading to degenerate singular points in the case $0 < D < 1$, $\mu = 1 - \sqrt{D}$ by Theorem 3 in [2]. Hence nilpotent or linearly zero cases never can occur for (2) if $\mu = 1 - \sqrt{D}$, $0 < D < 1$. This proves the theorem.

Table 3

Types of degenerate singular points of the system (2) in the case $D > 0$, $\mu = 1 + \sqrt{D}$

Values of b	Values of c , and corresponding types of degenerate singular points of (2)		
	Semi-hyperbolic type, $\lambda_1 = 0, \lambda_2 \neq 0, J \neq 0$	Nilpotent type, $\lambda_1 = \lambda_2 = 0, J \neq 0$	Linearly zero type, $\lambda_1 = \lambda_2 = 0, J = 0$
$b \in (0, 1/4)$	–	–	–
$b \in (1/4, b_3]$	$c = c_3$, saddle	–	–
$b \in (b_3, 1/2]$	–	–	–

Theorem 4. Let $D > 0$, $\mu = 1 + \sqrt{D}$. Then for the singular point (7) of the system (2) only the following types of singularities are possible shown in Table 3:

(a) (x_1^0, x_2^0) is a semi-hyperbolic saddle only at $b \in (1/4, b_3]$, $c = c_3$;

(b) There are no values of b, c such that (x_1^0, x_2^0) could be nilpotent or linearly zero singular point.

Proof. This theorem can be proved in a similar way as above taking into account the fact that

$$p_{1,1}v_2 + p_{3,0} = \frac{(8b^2 - 1)^3(2b+1)}{8b(2b-1)q_3^3} \frac{1}{4b-1} > 0$$

for $b \in (1/4, b_3]$, $c = c_3$ determined from Theorem 2 in [2].

However, $k_3 = \frac{4b-1}{(2b-1)q_3} < 0$ for such b , therefore we have $e_3 = -p_{1,1}v_2 - p_{3,0} < 0$ again by Remark 2.

The theorem is proved.

Conclusion

In Theorems 2–4 we investigated the important (from a geometrical point of view, see [3–7] for detail) partial case $a_1 = a_2 = b$, $a_3 = c$ and gave a qualitative classification of degenerate ($\delta = 0$) singular points of the system (2) of the kind $x_1^0 = x_2^0$.

Our further publications will be devoted to non-degenerate cases ($\delta \neq 0$). Here we briefly announce some general results obtained in this direction: non-degenerate singular points of (2) could be only hyperbolic nodes or hyperbolic saddles at $D > 0$. Moreover, nodes are stable (respectively unstable), if $\mu = 1 + \sqrt{D}$ (respectively $\mu = 1 - \sqrt{D}$ and $D < 1$).

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Н.Ә.Әбиев

Риччи ағымдарының өзгешеленген ерекше нүктелерін классификациялау туралы

Мақалада сызықсыз қарапайым дифференциалдық теңдеулер жүйесіне келтірілетін жалпыланған Уоллах кеңістіктеріндегі нормалдастырылған Риччи ағымдары қарастырылды. Негізгі нәтиже ретінде зерттелініп жатқан жүйенің маңызды $a_i = a_j$, $i, j \in \{1, 2, 3\}$, $i \neq j$, дербес жағдайындағы өзгешеленген ерекше нүктелерінің классификациясы алынды. Жалпы жағдайда бұл есепті абстрактты динамикалық жүйе шешімдерінің екі параметрлі бифуркациясы ретінде де қарастыруға болады. Сонымен, зерттелінетін есеп тек геометриялық тұрғыдан ғана қызықты болмастан, жазық динамикалық жүйелер теориясына да қатысты болады.

Н.А.Абиев

О классификации вырожденных особых точек потоков Риччи

В статье рассмотрены нормализованные потоки Риччи на обобщенных пространствах Уоллаха, приводимые к системе нелинейных обыкновенных дифференциальных уравнений. В качестве основного результата получена классификация вырожденных особых точек исследуемой системы в важном частном случае $a_i = a_j$, $i, j \in \{1, 2, 3\}$, $i \neq j$. В общем случае эту задачу можно изучать как двухпараметрическую бифуркацию решений абстрактной динамической системы. Таким образом, задача интересна не только с геометрической точки зрения, но также имеет отношение и к теории плоских динамических систем.

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