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Numerical implementation of solving a control problem for loaded differential equations with multi-point condition

A linear boundary value problem with a parameter for loaded differential equations with multi-point condition is considered. The method of parameterization is used for solving the considered problem. We offer an algorithm for solving a control problem for the system of loaded differential equations with multi-point condition. The linear boundary value problem with a parameter for loaded differential equations with multi-point condition by introducing additional parameters at the partition points is reduced to equivalent boundary value problem with parameters. The equivalent boundary value problem with parameters consists of the Cauchy problem for the system of ordinary differential equations with parameters, multi-point condition, and continuity conditions. The solution of the Cauchy problem for the system of ordinary differential equations with parameters is constructed using the fundamental matrix of differential equation. The system of linear algebraic equations concerning the parameters is composed by substituting the values of the corresponding points in the built solutions to the multi-point condition and continuity conditions. The numerical method for finding the solution of the problem is suggested, which based on the solving the constructed system and solving Cauchy problem on the subintervals by Adams method and Bulirsch-Stoer method. The proposed numerical implementation is illustrated by example.

Keywords: problem with parameter, loaded differential equation, multi-point condition, numerical method, algorithm.

Introduction

The problem of constructing effective models finds its solution in many areas of science and technology. Therefore, a modern approach in the theory of control and identification of parameters should be directed to the development of new constructive methods and modifications of known methods for solving boundary value problems with parameters for ordinary and loaded differential equations with multi-point condition [1-7].

In recent years, an intensive study of loaded differential equations associated with various applications of problems has been observed. The problems of the applications described by these equations include the problems of long-term forecasting and regulation of the level of groundwater and soil resources, simulation of processes of transported particles, and some optimal control problems [8]. The theory of boundary value problems for the loaded differential equations with parameters is rapidly developing and is used in various fields of applied mathematics, biophysics, biomedicine, chemistry, etc. [8-13]. Despite this, the questions of finding the effective criterions of unique solvability and constructing the numerical algorithms for finding the solutions of boundary value problems for the system of loaded differential equations with parameters remain open. One of the constructive methods for investigating and solving the boundary value problems with parameters for the system of ordinary differential equations is the parameterization method [14].

The parameterization method was developed for the investigating and solving the boundary value problems for the system of ordinary differential equations. Later, this method was developed for the two-point boundary value problems for the Fredholm integro-differential equations [15-19]. The algorithms

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for finding the numerical solutions of the problems are considered. This approach are applied to two-point boundary value problems for system of ordinary and loaded differential equations with parameter [20, 21].

In the present paper, we offer numerical algorithm of parametrization method for solving the control problem for the loaded differential equation with multi-point condition.

So, we consider a linear boundary value problem with a parameter for loaded differential equation with multi-point condition

$$\frac{dx}{dt} = A(t)x + \sum_{j=1}^{N} K_j(t)x(\theta_j) + A_0(t)\mu + f(t), \quad x \in R^n, \ \mu \in R^m, \ t \in (0,T), \tag{1}$$

$$\sum_{i=0}^{N+1} C_i x(\theta_i) + B_0\mu = d, \quad d \in R^{n+m}. \tag{2}$$

Here the $(n \times n)$ matrices $A(t)$, $K_j(t)$ are continuous on $[0,T]$, $j = \overline{1,N}$; the $(n \times m)$ matrix $A_0(t)$ is continuous on $[0,T]$; the n vector $f(t)$ is continuous on $[0,T]$; the $((n+m) \times m)$ matrix B_0 and the $((n+m)\times n)$ matrices C_i, $i = \overline{0, N+1}$ are constants; $0 = \theta_0 < \theta_1 < \theta_2 < ... < \theta_{N-1} < \theta_N < \theta_{N+1} = T$; $\|x\| = \max\limits_{i=\overline{1,n}} |x_i|$.

$C([0,T], R^n)$ is the space of continuous functions $x : [0,T] \to R^n$ with the norm $||x||_1 = \max\limits_{t\in[0,T]} ||x(t)||$.

A pair $(x^*(t), \mu^*)$, with $x^*(t) \in C([0,T], R^n)$, $\mu^* \in R^m$, where n vector function $x^*(t)$ is continuously differentiable on $(0,T)$, is called a solution to problem (1), (2), if it satisfies the loaded differential equation (1) and condition (2) for $\mu = \mu^*$.

1. Scheme of the method

Points $0 = \theta_0 < \theta_1 < \theta_2 < ... < \theta_{N-1} < \theta_N < \theta_{N+1} = T$ are given and the interval $[0,T)$ is divided into N subintervals: $[0,T) = \bigcup\limits_{r=1}^{N+1} [\theta_{r-1}, \theta_r)$.

$C([0,T], \theta_N, R^{n(N+1)})$ is the space of systems functions $x[t] = (x_1(t), x_2(t), ..., x_{N+1}(t))$, where x_r: $[\theta_{r-1}, \theta_r) \to R^n$ are continuous and have finite left-sided $\lim\limits_{t\to\theta_r-0} x_r(t)$ for all $r = \overline{1, N+1}$, with the norm $||x[\cdot]||_2 = \max\limits_{r=\overline{1,N+1}} \sup\limits_{t\in[\theta_{r-1},\theta_r)} ||x_r(t)||$.

Let $x_r(t)$ be the restriction of function $x(t)$ to the $r-$th interval $[\theta_{r-1}, \theta_r)$, i.e. $x_r(t) = x(t)$ for $t \in [\theta_{r-1}, \theta_r)$, $r = \overline{1, N+1}$. Then we reduce problem (1), (2) to the equivalent multipoint boundary value problem

$$\frac{dx_r}{dt} = A(t)x_r + \sum_{j=1}^{N} K_j(t)x_{j+1}(\theta_j) + A_0(t)\mu + f(t), \quad t \in [\theta_{r-1}, \theta_r), \quad r = \overline{1, N+1}, \tag{3}$$

$$\sum_{i=0}^{N} C_i x_{i+1}(\theta_i) + C_{N+1} \lim_{t\to T-0} x_{N+1}(t) + B_0\mu = d, \tag{4}$$

$$\lim_{t\to\theta_p-0} x_p(t) = x_{p+1}(\theta_p), \qquad p = \overline{1,N}, \tag{5}$$

where (5) are conditions for matching the solution at the interior points of partition.
The solution of problem (3)-(5) is the pair $(x^*[t], \mu^*)$ with elements $x^*[t] = (x_1^*(t), x_2^*(t), ..., x_{N+1}^*(t)) \in$ $\in C([0,T], \theta_N, R^{n(N+1)})$, $\mu \in R^m$, where functions $x_r^*(t)$, $r = \overline{1, N+1}$, are continuously differentiable

on $[\theta_{r-1},\theta_r)$, which satisfies system of loaded differential equations (3) and condition (4) with $\mu=\mu*$ and continuity conditions (5).

We introduce the additional parameters λ_r as a values of required functions at the points of partition: $\lambda_r=x_r(\theta_{r-1})$, $r=\overline{1,N+1}$, the $(N+2)-$th component is assigned the original parameter μ, i.e. $\lambda_{N+2}=\mu$. Making the substitution $x_r(t)=u_r(t)+\lambda_r$ on every r-th interval $[\theta_{r-1},\theta_r)$, $r=\overline{1,N+1}$, we obtain multipoint boundary value problem with parameters

$$\frac{du_r}{dt}=A(t)(u_r+\lambda_r)+\sum_{j=1}^{N}K_j(t)\lambda_{j+1}+A_0(t)\lambda_{N+2}+f(t),\quad t\in[\theta_{r-1},\theta_r),\quad r=\overline{1,N+1},\tag{6}$$

$$u_r(\theta_{r-1})=0,\qquad r=\overline{1,N+1},\tag{7}$$

$$\sum_{i=0}^{N}C_i\lambda_{i+1}+C_{N+1}\lambda_{N+1}+C_{N+1}\lim_{t\to T-0}u_{N+1}(t)+B_0\lambda_{N+2}=d,\tag{8}$$

$$\lambda_p+\lim_{t\to\theta_p-0}u_p(t)=\lambda_{p+1},\qquad p=\overline{1,N},\tag{9}$$

A solution to problem with parameters (6)–(9) is a pair of functions $(u^*[t],\lambda^*)$, where the function $u^*[t]=\big(u_1^*(t),u_2^*(t),\dots,u_{N+1}^*(t)\big)\in C([0,T],\theta_N,R^{n(N+1)})$ with continuously differentiable on $[\theta_{r-1},\theta_r)$ components $u_r^*(t)$, $r=\overline{1,N+1}$, and $\lambda^*=(\lambda_1^*,\lambda_2^*,\dots,\lambda_{N+1}^*,\lambda_{N+2}^*)\in R^{n(N+1)+m}$, satisfies system of ordinary differential equations (6), initial conditions (7), and relations (8), (9) for $\lambda_j=\lambda_j^*$, $j=\overline{1,N+2}$.

If the pair $(x^*(t),\mu^*)$ is a solution to problem (1), (2), then the pair $(u^*[t],\lambda^*)$ with elements $u^*[t]=\big(u_1^*(t),u_2^*(t),\dots,u_{N+1}^*(t)\big)\in C([0,T],\theta_N,R^{n(N+1)})$, $\lambda^*=(\lambda_1^*,\lambda_2^*,\dots,\lambda_{N+1}^*,\lambda_{N+2}^*)\in R^{n(N+1)+m}$, where $u_r^*(t)=x^*(t)-x^*(\theta_{r-1})$, $t\in[\theta_{r-1},\theta_r)$, $\lambda_r^*=x^*(\theta_{r-1})$, $r=\overline{1,N+1}$, $\lambda_{N+2}^*=\mu^*$ is a solution to problem (6)–(9). Conversely, if the pair $(\widetilde{u}[t],\widetilde{\lambda})$ with elements $\widetilde{u}[t]=\big(\widetilde{u}_1(t),\widetilde{u}_2(t),\dots,\widetilde{u}_{N+1}(t)\big)\in$ $\in C([0,T],\theta_N,R^{n(N+1)})$, $\widetilde{\lambda}=(\widetilde{\lambda}_1,\widetilde{\lambda}_2,\dots,\widetilde{\lambda}_{N+1},\widetilde{\lambda}_{N+2})\in R^{n(N+1)+m}$, is a solution to problem (6)–(9), then the pair $(\widetilde{x}(t),\widetilde{\mu})$ defined by the equalities $\widetilde{x}(t)=\widetilde{u}_r(t)+\widetilde{\lambda}_r$, $t\in[\theta_{r-1},\theta_r)$, $r=\overline{1,N+1}$, and $\widetilde{x}(T)=\lim_{t\to T-0}\widetilde{u}_{N+1}(t)+\widetilde{\lambda}_{N+1}$, and $\widetilde{\mu}=\widetilde{\lambda}_{N+2}$, is a solution to the origin problem with parameter (1), (2).

Let $X_r(t)$ be a fundamental matrix to the differential equation $\dfrac{dx}{dt}=A(t)x$ on $[\theta_{r-1},\theta_r]$, $r=\overline{1,N+1}$. Then the unique solution to the Cauchy problem for the system of ordinary differential equations (6), (7) at the fixed values $\lambda=(\lambda_1,\lambda_2,...,\lambda_{N+1},\lambda_{N+2})$, has the following form

$$u_r(t)=X_r(t)\int_{\theta_{r-1}}^{t}X_r^{-1}(\tau)A(\tau)d\tau\lambda_r+X_r(t)\int_{\theta_{r-1}}^{t}X_r^{-1}(\tau)A_0(\tau)d\tau\lambda_{N+2}+$$

$$+X_r(t)\int_{\theta_{r-1}}^{t}X_r^{-1}(\tau)\sum_{j=1}^{N}K_j(\tau)d\tau\lambda_{j+1}+X_r(t)\int_{\theta_{r-1}}^{t}X_r^{-1}(\tau)f(\tau)d\tau,\quad t\in[\theta_{r-1},\theta_r),\quad r=\overline{1,N+1}.\tag{10}$$

Substituting the corresponding right-hand sides of (10) into the conditions (8), (9), we obtain a system of linear algebraic equations with respect to the parameters λ_r, $\quad r=\overline{1,N+2}$:

$$\sum_{i=0}^{N}C_i\lambda_{i+1}+C_{N+1}\lambda_{N+1}+C_{N+1}\Big\{X_{N+1}(T)\int_{\theta_N}^{T}X_{N+1}^{-1}(\tau)A(\tau)d\tau\lambda_{N+1}+$$

$$+X_{N+1}(T)\int\limits_{\theta_N}^{T} X_{N+1}^{-1}(\tau)A_0(\tau)d\tau\lambda_{N+2}+X_{N+1}(T)\int\limits_{\theta_N}^{T} X_{N+1}^{-1}(\tau)\sum_{j=1}^{N}K_j(\tau)d\tau\lambda_{j+1}\Big\}+B_0\lambda_{N+2}=$$

$$=d-C_{N+1}X_{N+1}(T)\int\limits_{\theta_N}^{T} X_{N+1}^{-1}(\tau)f(\tau)d\tau, \tag{11}$$

$$\lambda_p+X_p(\theta_p)\int\limits_{\theta_{p-1}}^{\theta_p} X_p^{-1}(\tau)A(\tau)d\tau\lambda_p+X_p(\theta_p)\int\limits_{\theta_{p-1}}^{\theta_p} X_p^{-1}(\tau)A_0(\tau)d\tau\lambda_{N+2}+$$

$$+X_p(\theta_p)\int\limits_{\theta_{p-1}}^{\theta_p} X_p^{-1}(\tau)\sum_{j=1}^{N}K_j(\tau)d\tau\lambda_{j+1}-\lambda_{p+1}=-X_p(\theta_p)\int\limits_{\theta_{p-1}}^{\theta_p} X_p^{-1}(\tau)f(\tau)d\tau, \qquad p=\overline{1,N}. \tag{12}$$

Denoting by $Q_*(\theta_N)$ the matrix corresponding to the left-hand side of system (11), (12) which is consist of the coefficients at the parameters λ_r, $\quad r=\overline{1,N+2}$, and then introducing the vector

$$F_*(\theta_N)=\begin{pmatrix} d-C_{N+1}X_{N+1}(T)\int\limits_{\theta_N}^{T} X_{N+1}^{-1}(\tau)f(\tau)d\tau \\ -X_1(\theta_1)\int\limits_{0}^{\theta_1} X_1^{-1}(\tau)f(\tau)d\tau \\ \cdots \\ -X_N(\theta_N)\int\limits_{\theta_{N-1}}^{\theta_N} X_N^{-1}(\tau)f(\tau)d\tau, \end{pmatrix}$$

we write the system (11), (12) as:

$$Q_*(\theta_N)\lambda=F_*(\theta_N), \quad \lambda\in R^{n(N+1)+m}. \tag{13}$$

It is not difficult to establish that the solvability of the boundary value problem (1), (2) is equivalent to the solvability of the system (13). The solution of the system (13) is a vector $\lambda^*=(\lambda_1^*,\lambda_2^*,...,\lambda_{N+1}^*,$ $\lambda_{N+2}^*)\in R^{n(N+1)+m}$, consists of the values of the solutions of the original problem (1), (2) in the initial points of subintervals, i.e. $\lambda_r^*=x^*(\theta_{r-1})$, $\quad r=\overline{1,N+1}$, $\lambda_{N+2}^*=\mu^*$.

Further we consider the Cauchy problems for ordinary differential equations on subintervals

$$\frac{dz}{dt}=A(t)z+P(t), \quad z(\theta_{r-1})=0, \quad t\in[\theta_{r-1},\theta_r], \quad r=\overline{1,N+1}, \tag{14}$$

where $P(t)$ is either $(n\times n)$ matrix, or n vector, both continuous on $[\theta_{r-1},\theta_r], r=\overline{1,N+1}$. Consequently, solution to problem (14) is a square matrix or a vector of dimension n. Denote by a(P,t) the solution to the Cauchy problem (14). Obviously,

$$a(P,t)=X_r(t)\int\limits_{\theta_{r-1}}^{t} X_r^{-1}(\tau)P(\tau)d\tau, \quad t\in[\theta_{r-1},\theta_r], \quad r=\overline{1,N+1},$$

where $X_r(t)$ is a fundamental matrix of differential equation (14) on the r-th interval.

3. Algorithm for finding of solution to problem (1), (2)

We offer the following numerical implementation of algorithm. This algorithm is based on the the Adams method and the Bulirsch-Stoer method to solve the Cauchy problems for ordinary differential equations.

1. Suppose we have a partition: $0 = \theta_0 < \theta_1 < \theta_2 < ... < \theta_{N-1} < \theta_N < \theta_{N+1} = T$. Divide each r-th interval $[\theta_{r-1}, \theta_r]$, $r = \overline{1, N+1}$, into N_r parts with step $h_r = (\theta_r - \theta_{r-1})/N_r$. Assume on each interval $[\theta_{r-1}, \theta_r]$, $r = \overline{1, N+1}$, the variable $\widehat{\theta}$ takes its discrete values: $\widehat{\theta} = \theta_{r-1}$, $\widehat{\theta} = \theta_{r-1} + h_r$, ..., $\widehat{\theta} = \theta_{r-1} + (N_r - 1)h_r$, $\widehat{\theta} = \theta_r$, and denote by $\{\theta_{r-1}, \theta_r\}$, $r = \overline{1, N+1}$, the set of such points.

2. Solving the Cauchy problems for ordinary differential equations

$$\frac{dz}{dt} = A(t)z + A(t), \quad z(\theta_{r-1}) = 0, \quad t \in [\theta_{r-1}, \theta_r], \quad r = \overline{1, N+1},$$

$$\frac{dz}{dt} = A(t)z + K_j(t), \quad z(\theta_{r-1}) = 0, \quad t \in [\theta_{r-1}, \theta_r], \quad j = \overline{1, N}, \quad r = \overline{1, N+1},$$

$$\frac{dz}{dt} = A(t)z + A_0(t), \quad z(\theta_{r-1}) = 0, \quad t \in [\theta_{r-1}, \theta_r], \quad r = \overline{1, N+1},$$

$$\frac{dz}{dt} = A(t)z + f(t), \quad z(\theta_{r-1}) = 0, \quad t \in [\theta_{r-1}, \theta_r], \quad r = \overline{1, N+1},$$

by using the Adams method or the Bulirsch-Stoer method, we find the values of $(n \times n)$ matrices $a_r(A, \widehat{\theta})$, $a_r(K_j, \widehat{\theta})$, $j = \overline{1, N}$, $(n \times m)$ matrices $a_r(A_0, \widehat{\theta})$ and n vector $a_r(f, \widehat{\theta})$ on $\{\theta_{r-1}, \theta_r\}$, $r = \overline{1, N+1}$.

3. Construct the system of linear algebraic equations with respect to parameters

$$Q_*^{\widetilde{h}}(\theta_N)\lambda = F_*^{\widetilde{h}}(\theta_N), \quad \lambda \in R^{n(N+1)+m}. \tag{15}$$

Solving the system (15), we find $\lambda^{\widetilde{h}}$. As noted above, the elements of $\lambda^{\widetilde{h}} = (\lambda_1^{\widetilde{h}}, \lambda_2^{\widetilde{h}}, ..., \lambda_{N+1}^{\widetilde{h}}, \lambda_{N+2}^{\widetilde{h}})$ are the values of approximate solution to problem (1), (2) in the starting points of subintervals: $x^{\widetilde{h}_r}(\theta_{r-1}) = \lambda_r^{\widetilde{h}}$, $r = \overline{1, N+1}$, $\mu^* = \lambda_{N+2}^*$.

4. To define the values of approximate solution at the remaining points of set $\{\theta_{r-1}, \theta_r\}$, $r = \overline{1, N+1}$, we solve the Cauchy problems

$$\frac{dx}{dt} = A(t)x + \sum_{j=1}^{N} K_j(t)\lambda_{j+1}^{\widetilde{h}} + A_0(t)\lambda_{N+2}^{\widetilde{h}} + f(t),$$

$$x(\theta_{r-1}) = \lambda_r^{\widetilde{h}}, \quad t \in [\theta_{r-1}, \theta_r], \quad r = \overline{1, N+1}.$$

And the solutions to Cauchy problems are found by the Adams method or the Bulirsch-Stoer method. Thus, the algorithm allows us to find the numerical solution to the problem (1), (2). To illustrate the proposed approach for the numerical solving linear boundary value problem with a parameter for an loaded differential equation with multipoint condition (1), (2) on the basis of parameterization method, let us consider the following example.

4. Example

Consider a linear boundary value problem with a parameter for loaded differential equation with multipoint condition

$$\frac{dx}{dt} = A(t)x + \sum_{j=1}^{N} K_j(t)x(\theta_j) + A_0(t)\mu + f(t), \quad x \in R^2, \ \mu \in R^3, \ t \in (0, 1), \tag{16}$$

$$\sum_{i=0}^{N+1} C_i x(\theta_i) + B_0\mu = d, \quad d \in R^5. \tag{17}$$

where $A(t) = \begin{pmatrix} t & cos(t) \\ 1 & t^2-2 \end{pmatrix}$, $A_0(t) = \begin{pmatrix} t & e^t & t-3 \\ 0 & t^2 & t+7 \end{pmatrix}$, $B_0 = \begin{pmatrix} 1 & 4 & 6 \\ -4 & 6 & 2 \\ 1 & 7 & -2 \\ -4 & 3 & 11 \\ 1 & 0 & 4 \end{pmatrix}$.

Case 1. Let $N = 1$. $\theta_0 = 0$, $\theta_1 = \frac{1}{2}$, $\theta_2 = 1$, $K_1(t) = \begin{pmatrix} t & t-1 \\ 3t+1 & t \end{pmatrix}$,

$$f(t) = \begin{pmatrix} -t^4 + 7t^2 - \frac{21t}{8} + \frac{47}{8} - 19e^t - cos(\pi t)cos(t) - tcos(t) \\ -2t^3 - 19t^2 + \frac{17t}{8} - \frac{481}{8} - t^2cos(\pi t) - \pi sin(\pi t) + 2cos(\pi t) \end{pmatrix},$$

$$C_0 = \begin{pmatrix} 2 & 5 \\ 0 & 2 \\ 6 & -4 \\ 1 & 0 \\ 3 & 1 \end{pmatrix}, \quad C_1 = \begin{pmatrix} 1 & 6 \\ 4 & 5 \\ 3 & 2 \\ 0 & -4 \\ 3 & 5 \end{pmatrix}, \quad C_2 = \begin{pmatrix} -2 & 5 \\ 0 & 4 \\ 6 & 4 \\ 8 & 0 \\ 3 & -1 \end{pmatrix}, \quad d = \begin{pmatrix} \frac{1097}{8} \\ 149 \\ \frac{667}{8} \\ 150 \\ \frac{159}{8} \end{pmatrix}.$$

We use the numerical implementation of algorithm. Accuracy of solution depends on the accuracy of solving the Cauchy problem on subintervals. We provide the results of the numerical implementation of algorithm based on the Adams method and the Bulirsch-Stoer method by partitioning the subintervals [0, 0.5], [0.5, 1] with step h = 0.05.

Solution to problem with parameter (16), (17) is pair $(x^*(t), \mu^*)$, where $x^*(t) = \begin{pmatrix} t^3 - 4t \\ t + cos(\pi t) \end{pmatrix}$, $\mu^* = \begin{pmatrix} -5 \\ 19 \\ 9 \end{pmatrix}$. Table 1 provides the numerical solution values $(\widetilde{x}(t), \widetilde{\mu})$.

The following estimates are true:
using the Adams method for solving the Cauchy problems for ordinary differential equations

$$\max \|\mu^* - \widetilde{\mu}\| < 0.00005, \qquad \max_{j=\overline{0,20}} \|x^*(t_j) - \widetilde{x}(t_j)\| < 0.00008;$$

using the Bulirsch-Stoer method for solving the Cauchy problems for ordinary differential equations

$$\max \|\mu^* - \widetilde{\mu}\| < 0.00000002, \qquad \max_{j=\overline{0,20}} \|x^*(t_j) - \widetilde{x}(t_j)\| < 0.00000002.$$

Case 2. Let $N = 3$. $\theta_0 = 0$, $\theta_1 = \frac{1}{4}$, $\theta_2 = \frac{1}{2}$, $\theta_3 = \frac{3}{4}$, $\theta_4 = 1$, $K_2(t) = \begin{pmatrix} t^2 & 1 \\ t & t^2-1 \end{pmatrix}$,

$$K_3(t) = \begin{pmatrix} 2 & 2t \\ \frac{t^2}{2} & 3 \end{pmatrix}, \quad C_3 = \begin{pmatrix} -1 & 6 \\ 4 & -5 \\ 7 & 2 \\ 0 & 4 \\ -3 & 5 \end{pmatrix} \quad C_4 = \begin{pmatrix} 3 & 5 \\ 1 & 2 \\ 0 & -4 \\ -1 & 0 \\ 3 & -1 \end{pmatrix} \quad d = \begin{pmatrix} \frac{4315}{32} \\ 5\sqrt{2} + \frac{545}{4} \\ \frac{311}{4} \\ 166 - 4\sqrt{2} \\ \frac{853}{32} \end{pmatrix},$$

$$f(t) = \begin{pmatrix} -t^4 + \frac{71t^2}{8} - \frac{305}{64} + \frac{\sqrt{2}t}{2} + \frac{\sqrt{2}}{2} + \frac{893}{32} - 19e^t - cos(\pi t)cos(t) - tcos(t) \\ -2t^3 - \frac{2331t^2}{128} + \frac{101t}{64} - \frac{\sqrt{2}t}{2} + \frac{3\sqrt{2}}{2} - \frac{4017}{64} - t^2cos(\pi t) - \pi sin(\pi t) + 2cos(\pi t) \end{pmatrix}.$$

Table 1

Results received by using MathCad15

t	$\widetilde{x}_1(t)$	$\widetilde{x}_2(t)$	$\widetilde{x}_1(t)$	$\widetilde{x}_2(t)$
	Adams method		Bulirsch-Stoer method	
0	-0.0000768202	1.0000483623	0.0000000206	0.9999999887
0.05	-0.1999451311	1.0377205669	-0.1998749798	1.0376883286
0.1	-0.3990642033	1.0510744329	-0.3989999803	1.0510565038
0.15	-0.5966839083	1.0410116624	-0.5966249809	1.0410065115
0.2	-0.7920541444	1.0090107586	-0.7919999817	1.0090169819
0.25	-0.984424811	0.9570903823	-0.9843749827	0.9571067693
0.3	-1.173045817	0.8877597447	-1.1729999838	0.8877852415
0.35	-1.3571670797	0.8039568233	-1.3571249852	0.8039904905
0.4	-1.5360385196	0.7089759575	-1.5359999867	0.709016987
0.45	-1.7089100632	0.6063867668	-1.7088749883	0.6064344599
0.5	-1.875031638	0.4999462346	-1.87499999	0.4999999973
0.55	-2.0336244073	0.393563497	-2.0336249926	0.3935655316
0.6	-2.1839925346	0.2909724033	-2.1839999946	0.2909830051
0.65	-2.3253604761	0.1959893476	-2.3253749967	0.1960095024
0.7	-2.4569780171	0.112185956	-2.4569999988	0.1122147524
0.75	-2.5780953578	0.0428566274	-2.578125001	0.0428932257
0.8	-2.6879621167	-0.0090590279	-2.6880000032	-0.0090169855
0.85	-2.7858287117	-0.0410520677	-2.7858750057	-0.0410065138
0.9	-2.8709446048	-0.0511055772	-2.8710000083	-0.0510565047
0.95	-2.9425595558	-0.0377405131	-2.9426250112	-0.037688328
1	-2.9999235398	-0.0000548018	-3.0000000144	0.0000000132
	$\widetilde{\mu}_1 = -4.9999451449$		$\widetilde{\mu}_1 = -5.0000000147$	
	$\widetilde{\mu}_2 = 19.0000319553$		$\widetilde{\mu}_2 = 18.9999999821$	
	$\widetilde{\mu}_3 = 8.9999764358$		$\widetilde{\mu}_3 = 9.0000000058$	

In this case we provide the results of the numerical implementation of algorithm by partitioning the interval $[0,1]$ with step $h = 0.25$ and partitioning the subintervals $[0, 0.25]$, $[0.25, 0.5]$, $[0.5, 0.75]$, $[0.75, 1]$ with step $h_1 = 0.025$. For the second case the following estimates are true:
The errors of using the Adams method

$$\max \|\mu^* - \widetilde{\mu}\| < 0.00002, \qquad \max_{j=\overline{0,40}} \|x^*(t_j) - \widetilde{x}(t_j)\| < 0.00002;$$

The errors of using the Bulirsch-Stoer method

$$\max \|\mu^* - \widetilde{\mu}\| < 0.000000001, \qquad \max_{j=\overline{0,40}} \|x^*(t_j) - \widetilde{x}(t_j)\| < 0.000000003.$$

As we can see, the numerical algorithm based on the Bulirsch-Stoer method proposed is effective and allows us to obtain the numerical solution to the the problem with a parameter for loaded differential equation with multipoint condition of higher order accuracy.

Below in the Figure 1, we plot graphs of the exact and numerical solutions to the problem (16), (17) on the interval [0, 1].

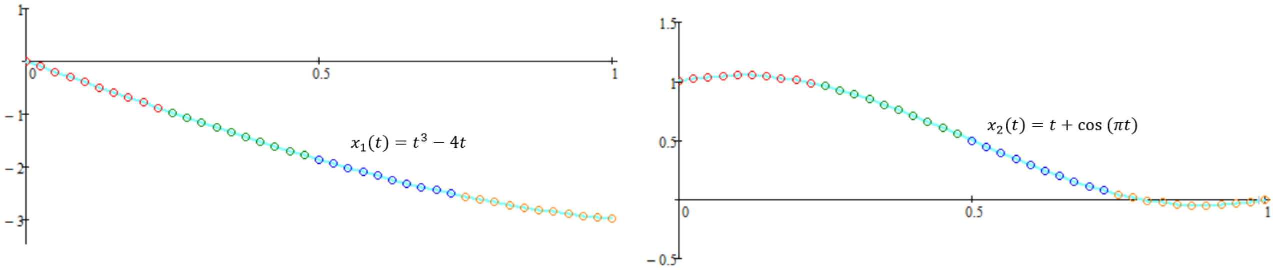

Figure 1. The exact solution values are indicated by the light blue solid line and the numerical solution values obtained by the Bulirsch-Stoer method are indicated by the symbol o

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Ж.М. Кадирбаева, А.Д. Джумабаев

Көпнүктелі шарты бар жүктелген дифференциалдық теңдеулер үшін басқару есебін шешудің сандық жүзеге асырылуы

Көпнүктелі шарты бар жүктелген дифференциалдық теңдеулер үшін параметрі бар сызықтық шеттік есеп зерттелді. Қарастырылып отырған есепті шешу үшін параметрлеу әдісі қолданылды. Көпнүктелі шарты бар жүктелген дифференциалдық теңдеулер үшін басқару есебін шешу алгоритмі ұсынылды. Көпнүктелі шарты бар жүктелген дифференциалдық теңдеулер үшін параметрі бар сызықтық шеттік есеп бөлу нүктелерінде қосымша параметрлер енгізу арқылы пара-пар параметрлері бар шеттік есепке келтірілді. Пара-пар параметрлері бар шеттік есеп жай дифференциалдық теңдеулер жүйесі үшін параметрлері бар Коши есебінен, көпнүктелі шартынан және үзіліссіздік шарттарынан тұрады. Параметрлері бар жай дифференциалдық теңдеулер жүйесі үшін Коши есебінің шешімі дифференциалдық теңдеудің фундаменталдық матрицасының көмегімен тұрғызылды. Тұрғызылған шешімнің сәйкес нүктелеріндегі мәндерін көпнүктелі шартқа және үзіліссіздік шарттарына қоя отырып, параметрлерге қатысты сызықты алгебралық теңдеулер жүйесі құрылды. Қарастырылып отырған есепті шешудің құрылған жүйені және ішкі аралықтардағы Коши есептерін Адамс және Булирш-Штёр әдістерін қолданып, шешуге негізделген сандық әдісі ұсынылды және ол жүзеге асырылу мысалмен көрнектелді.

Кілт сөздер: параметрі бар есеп, жүктелген дифференциалдық теңдеу, көпнүктелі шарт, сандық әдіс, алгоритм.

Ж.М.Кадирбаева, А.Д.Джумабаев

Численная реализация решения задачи управления для нагруженных дифференциальных уравнений с многоточечным условием

Исследована линейная краевая задача с параметром для нагруженных дифференциальных уравнений с многоточечным условием. Для решения рассматриваемой задачи применен метод параметризации. Предложен алгоритм решения задачи управления для системы нагруженных дифференциальных уравнений с многоточечным условием. Линейная краевая задача с параметром для нагруженных дифференциальных уравнений с многоточечным условием путем введения дополнительных параметров в точках разбиения сводится к эквивалентной краевой задаче с параметрами. Эквивалентная краевая задача с параметрами состоит из задачи Коши для системы обыкновенных дифференциальных уравнений с параметрами, многоточечного условия и условия склеивания. Решение задачи Коши для системы обыкновенных дифференциальных уравнений с параметрами строится с помощью фундаментальной матрицы дифференциального уравнения. Подставляя значения в соответствующих точках построенного решения в многоточечное условие и условия склеивания, составляется система линейных алгебраических уравнений относительно параметров. Предложен численный метод нахождения решения задачи, основанный на решении построенной системы и задачи Коши на подынтервалах по методам Адамса и Булирша-Штёра. Предлагаемая численная реализация проиллюстрирована примером.

Ключевые слова: задача с параметром, нагруженное дифференциальное уравнение, многоточечное условие, численный метод, алгоритм.

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