UDC 510.67

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Properties of hybrids of Jonsson theories

This work is an introduction to the study of the properties of a new concept, as a hybrid of Jonsson theories. We define the basic concepts and framework for studying the model-theoretic properties of these concepts. The main goal of this paper is to study the model-theoretic properties of companions of hybrids of Jonsson theories. The main objects of study are the Jonsson hybrids and their classes of models. In this paper, the main task is to consider the various links between such theories. In order to understand more deeply these connections and ultimately the connection with the primary theory itself, special algebraic constructions of semantic models of the considered fragments were identified and on this basis hybrids of these fragments were determined. In this paper, such algebraic constructions are called semantic hybrids.

Keywords: Jonsson theory, semantic model, hybrid, existentially prime, pregeometry, model companion.

This work refers to the model theory a branch of mathematics that is the language of mathematical logic studies the laws of a general nature of various types of mathematical structures. The most advanced ideas and concepts of the model theory were interpreted using examples from classical algebra and set theory. Subsequently, with their help, profound scientific results were obtained in various algebraic structures. It is no coincidence, because model theory was originally defined as symbiosis of universal algebra and mathematical logic.

In the modern model theory there is a conditional division into «Western» and «Eastern» themes. This convention appeared by the remark of a well-known expert in the area of model theory J. Keisler. Thus he divided the studies related to the works of A. Tarski and A. Robinson. The first author lived on the west coast of the USA, and A. Robinson respectively on the east coast. As a rule, studies related to the eastern theme are connected with first-order formulas, the prenex of which does not exceed two, and the role of morphisms that preserve these formulas and compare various models using mappings between themselves is played by either isomorphic embeddings or various homomorphisms. Moreover, the semantic aspect of such problems is determined by studying the behavior of class of existentially closed models of some fixed inductive theory.

In the class of inductive theories, a special place is occupied by Jonsson theories, i.e. theories that admit the properties of amalgam and joint embedding.

It is well known that many algebraic examples are related to Jonsson theories, for example, fields of a fixed characteristic, Boolean algebras, groups, Abelian groups, various types of rings, polygons, various types of lattices, various types of orders.

Progress in the study of Jonsson theories was achieved in the case of a perfect Jonsson theory. It turned out that such theories have a model companion as their center. For example, the center of a field of a fixed characteristic is an algebraic closed field of the same characteristic. In addition, it is necessary to note the following semantic fact that in a perfect Jonsson theory the class of existentially closed models coincides with the class of all models of its center.

Recently, experts on the model theory of the western direction attach great importance to the study of model-theoretic properties of structural problems of small models in enrichment signature. At the same time, these enrichments should retain some properties of the objects under study. As a rule, in the case of the study of the specific model-theoretic properties of the complete theory, they are very rarely transferred to the study of Jonsson theories. This is primarily due to the fact that the Jonsson theories are not complete theories. Therefore, the ability to find model-theoretic conditions, when possible, is an important task. In this regard, the introduction of new concepts and the corresponding technical apparatus is an important moment for studying the properties of Jonsson theory.

In this work are considered the model-theoretic properties of a new class of Jonsson theories, namely the theories obtained using various algebraic constructions of semantic models of two different Jonsson theories of the same language.

Earlier in the study of Jonsson theories, it was noted that due to the incompleteness of the considered theory, the requirement of $\forall\exists$ completeness, or $\exists$ completeness is a necessary condition for obtaining analogs of theorems obtained for complete theories. And also, due to the fact that in a perfect Jonsson theory the class of the center models coincides with the class of its existentially closed models, the so-called Jonsson sets were defined, i.e. special subsets of the semantic model of the considered Jonsson theory, the definable closures of which were some existentially closed submodels of this semantic model. It is well known that the set of all $\forall\exists$ consequences of the true ones in some existentially closed model form the Jonsson theory. The set of all $\forall\exists$ consequences that are true on a definable closure of a Jonsson subset forms the Jonsson theory and is called a fragment of this special subset.

In this article, the main task is to consideration the various connections between such theories. In order to understand more profound these connections and ultimately the connection with the original theory itself, and were determined special algebraic constructions of semantic models of the considered fragments and on this basis were determined hybrids of these fragments. Such algebraic constructions will be called semantic hybrids.

As an example of a semantic hybrid, we can consider the union and intersection, the Cartesian product, the direct sum, the product of filters and ultrafilters of semantic models of fragments of $\nabla - cl$-subsets of semantic model of in the considered Jonsson theory.

It is interesting to study the model-theoretic properties of various companions of a fixed hybrid. These properties of theories include almost all the classical attributes of the modern model theory, such as stability, categorical, strong minimality, model completeness, axiomatizability, interpretability, spectral issues, etc. As for the semantic aspect, there will be interesting various properties related to the concept of definable formula subsets of the semantic model of a hybrid with respect to the following concepts: atomicity, algebraic primeness, existential closure, convexity, existential primeness.

Thus, given the above, we can note that the results of this work in their content are related to the «Eastern» model theory.

We give the necessary definitions of the basic concepts of this article.

Let given a countable language of the first order.

The following definition describes one of the basic concepts of this article.

Definition 1. A theory T is Jonsson if:

1) theory T has infinite models;
2) theory T is inductive;
3) theory T has the joint embedding property (JEP);
4) theory T has the property of amalgam (AP).

Examples of Jonsson theories are:

1) group Theory;
2) theory of Abelian groups;
3) theory of fields of fixed characteristics;
4) theory of Boolean algebras;
5) theory of polygons over a fixed monoid;
6) theory of modules over a fixed ring;
7) theory of linear order.

The following definition of the universality and homogeneity of model allocates semantic invariant of any Jonsson theory, namely its semantic model. Moreover, it turned out that the saturation or non-saturation of this model significantly changes the structural properties of both the Jonsson theory itself and its class of models.

Definition 2. Let $\kappa \geq \omega$. Model M of theory T is called κ-universal for T, if each model T with the power strictly less κ isomorphically imbedded in M; κ-homogeneous for T, if for any two models A and A_1 of theory T, which are submodels of M with the power strictly less then κ and for isomorphism $f : A \to A_1$ for each extension B of model A, wich is a submodel of M and is model of T with the power strictly less then κ there is exist the extension B_1 of model A_1, which is a submodel of M and an isomorphism $g : B \to B_1$ which extends f.

Definition 3. Model C of Jonsson theory T is called semantic model, if it is ω^+-homogeneous-universal.

As can be seen from the definition of the Jonsson theory, this theory is not complete. But nevertheless, with the help of its semantic invariant (semantic model) we can always determine the center of Jonsson theory, which is a complete theory.

Definition 4. The center of Jonsson theory T is called an elementary theory of the its semantic model. Denoted through T^*, i.e. $T^* = Th(C)$.

The following two facts speak about the «good» exclusivity of the semantic model.

Fact 1 [1; 160]. Each Jonsson theory T has k^+-homogeneous-universal model of power 2^k. Conversely, if a theory T is inductive and has infinite model and ω^+-homogeneous-universal model then the theory T is a Jonsson theory.

Fact 2 [1; 160]. Let T is a Jonsson theory. Two k-homogeneous-universal models M and M_1 of T are elementary equivalent.

It is well known from the course of model theory that a saturated model is always a homogeneous-universal model, the reverse is also true. But this definition of homogeneous-universal model [2; 299] is considered as a rule in the framework in the study of complete theory. In the framework of the study of Jonsson theory, we are dealing with a particular case of the definition of a homogeneous-universal model belonging to B. Jonsson [3]. The concept of a saturated model is the same in both cases. By virtue of a more general situation of homogeneous-universality in the case of Jonsson theory, we do not have a saturation criterion in terms of homogeneous-universal as in [2; 299]. Therefore, those Jonsson theories, the semantic model of which is saturated, allocate in a special subclass of class of all Jonsson theories, and such theories are called perfect. We give a definition of perfectness of Jonsson theory.

Definition 5. Jonsson theory T is called a perfect theory, if each a semantic model of theory T is saturated model of T^*.

The first author of this article obtained a result describing the perfect Jonsson theory.

Theorem 1 [1; 158]. Let T is a Jonsson theory. Then the following conditions are equivalent:

1) theory T is perfect;

2) theory T^* is a model companion of theory T.

From the above list of Jonsson theories, the following examples 2)–4), 6), 7) are examples of a perfect Jonsson theory. But, for example, group theory is not such, due to the fact that it does not have a model companion.

Let E_T be a class of all existentially closed models of Jonsson theory T.

This class of models in general case for an arbitrary theory can be empty. But the following result [4; 367] is well known, who says that any inductive theory has a nonempty class of existentially closed models. Since the Jonsson theory is a subclass of class of inductive theories, we can say that E_T is a non-empty class.

In the case of a perfect Jonsson theory, the class of models of center of this theory coincides with E_T. This follows from the following theorem.

Theorem 2 [1; 162]. If T is a perfect Jonsson theory then $E_T = ModT^*$.

Let L is a countable language of first order. Let T is Jonsson theory in the language L and its semantic model is $\mathcal{C}$.

Let us turn to the definition of central concept of this article. Namely, the concept of a hybrid of Jonsson theories. In the beginning, we define a hybrid for two Jonsson theories, and two cases are possible. The first case is a hybrid of Jonsson theories of one signature. The second case is a hybrid of Jonsson theories of different signatures. In the first case, we are talking about a hybrid of the first type, in the second case about a hybrid of the second type. In this article we will deal only with hybrids of the first kind for two Jonsson theories, but it is easy to understand that this concept of a hybrid is generalized to an arbitrary number of Jonsson theories. Consider the case of the first type.

Let T be some Jonsson theory in a fixed language and C its semantic model.

Definition 6. Let $X \subseteq C$. We will say that a set X is $\nabla - cl$-Jonsson subset of C, if X satisfies the following conditions:

1) X is ∇-definable set (this means that there is a formula from ∇, the solution of which in the C is the set X, where $\nabla \subseteq L$, that is ∇ is a view of formula, for example $\exists, \forall, \forall\exists$ and so on.);

2) $cl(X) = M$, $M \in E_T$, where cl is some closure operator defining a pregeometry [5; 289] over C (for example $cl = acl$ or $cl = dcl$).

Lemma 1. Let T be Jonsson theory, E_T be the class of its existentially closed models. Then for any model $A \in E_T$ the theory $Th_{\forall\exists}(A)$ is a Jonsson theory.

Proof can be extract from [1, 4].

Let X_1, X_2 be ∇-cl-Jonsson subset of C, where C is semantic model of theory T.

Let $M_1 = cl(X_1)$, $M_2 = cl(X_2)$, where $M_1, M_2 \in E_T$.

$Th_{\forall\exists}(M_1) = T_1$, $Th_{\forall\exists}(M_2) = T_2$.

C_1 is semantic model of theory T_1, C_2 is semantic model of theory T_2.

We define the essence of the operation of an algebraic construction. Let $\boxdot \in \{\cup, \cap, \times, +, \oplus, \prod_F, \prod_U\}$, where $\cup$ — union, $\cap$ — -intersection; $\times$ — Cartesian product; +-sum and $\oplus$ — direct sum; $\prod_F$ — filtered and $\prod_U$ — ultra-production.

Let $Th_{\forall\exists}(C_1 \boxdot C_2) = H(T_1, T_2)$, where C_1 is semantic model of theory T_1, C_2 is semantic model of theory T_2.

The following definition gives a hybrid of the first type for two Jonsson theories.

Definition 7. A hybrid $H(T_1, T_2)$ of Jonsson theories T_1, T_2 is called the theory $Th_{\forall\exists}(C_1 \boxdot C_2)$, if it is Jonsson. Herewith, the algebraic construction $(C_1 \boxdot C_2)$ is called a semantic hybrid of the theories T_1, T_2.

Note the following fact:

Fact 3. In order for the theory $H(T_1, T_2)$ to be Jonsson enough to $(C_1 \boxdot C_2) \in E_T$.

Proof. This follows by Lemma 1.

Let us give an example of a semantic hybrid. Linear space is a special case of a module over a field. A well-known result from linear algebra is related to the dimension of linear space:

$dimV = dimV_1 + dimV_2 - dim(V_1 \cap V_2)$, where V is linear space, a V_1, V_2 are its subspace and $V = V_1 + V_2$.

It is easy to see that this dependence of the dimensions of these linear spaces can be interpreted in the language of R-modules, where R is field and $\nabla - cl$-sets will be considered $\forall\exists$-dcl-sets and $acl = dcl$. Moreover, V will be a semantic hybrid of V_1 and V_2, where the algebraic construction is a direct sum of subspaces, i.e. $\boxdot = \oplus$. This follows from the fact that the theory of modules is a Jonsson theory.

Thus, we note that the above definition of a hybrid of Jonsson theories and their semantic hybrid was defined in the class of fragments of some fixed Jonsson theory. Moreover, we have several parameters regarding this definition:

1) view formulas from $\nabla \subseteq L$;

2) view of closure operator cl;

3) views of algebraic constructions of semantic hybrids.

In the general case, algebraic constructions of a semantic hybrid can be non-closed with respect to the class of models of this given theory. In this connection, it is further assumed that the theory under consideration is closed with respect to the algebraic construction under consideration.

Those, it is necessary to select the following parameter:

4) the closedness of the theory under consideration with respect to an algebraic construction.

By virtue of the fact that the definition of a hybrid is sufficiently general, i.e. it depends on many parameters, we must always specify these parameters to obtain specific results. In this article, we will deal with a convex existentially prime Jonsson theory complete for $\forall\exists$ -sentences. As the closure operator, we will consider the operator dcl and such that it is equal to the algebraic closure, i.e. $acl = dcl$. As an algebraic construction for obtaining a semantic hybrid, we will consider a Cartesian product. The above parameters define a sufficiently wide class Jonsson theories, in particular linear spaces get there. The example of linear spaces was basic for us in the sense of intuition and ideas. Therefore, in order to preserve some internal ideology of linear spaces and at the same time not losing generality, we will deal with a certain subclass of class of all Jonsson theories, which contains the theory of linear spaces and also satisfies the properties of other types of algebras. For this we consider the following definitions.

Definition 8. The inductive theory T is called the existentially prime if: it has a algebraically prime model, the class of its AP (algebraically prime models) denote by AP_T; class E_T non trivial intersects with class AP_T, i.e. $AP_T \bigcap E_T \neq 0$.

Definition 9. The theory is called convex if for any its model A and for any family $\{B_i \mid i \in I\}$ of substructures of A, which are models of the theory T, the intersection $\bigcap_{i \in I} B_i$ is a model of T.

Further, the object of our study will be the class of existentially prime convex $\forall\exists$ - complete Jonsson theories.

In the study of this class of theories, we obtained the following results:

Theorem 3. Let T be perfect convex existentially prime complete for $\forall\exists$-sentences Jonsson theory. X_1, X_2 are $\forall\exists$-dcl-sets of the theory T, where $M_i{=}dcl(X_i) \in E_T$, $T_i = Th_{\forall\exists}(M_i)$ are also perfect convex existentially prime complete for $\forall\exists$-sentences Jonsson theories. C_1, C_2 are their semantic models, respectively. Then, if their hybrid $H(T_1, T_2)$ is a model consistent with T_i, then $H(T_i)$ is a perfect Jonsson theory for $i = 1, 2$.

Proof. Suppose the contrary. Then, since the hybrid $H(T_1, T_2)$ is a Jonsson theory and has a semantic model M, by the assumption not perfectness of this hybrid $H(T_1, T_2)$, the considered semantic model M will not be saturated in its power. And this means that there is such $X \subseteq M$ and such type $p \in S_1(X)$, which is not realized in M, more precisely in $(M, m)_{m \in X}$. By virtue of the consistency of type p, this type is realized in some elementary extension $M' \succ M$. By virtue of the Jonssonness of hybrid $H(T_1, T_2)$ and model consistency with T_i, $i = 1, 2$, there is a model $A_i \in ModT_i$, $i = 1, 2$, such that M' is a submodel of A. A in turn, is embedded in the semantic model C_i, $i \in 1, 2$, but C_i is a saturated model of the theory T_i, $i \in 1, 2$. By virtue of an isomorphic embedding, suppose f from M' to A, $f(X) \subseteq A$ and since the type of p is realized in M' it will be realized in C_i. But $C_i \in E_{T_i}$ and since T_i are existentially prime convex theories, there exists a countable model $N_i \in E_{T_i}$, in which the type p will be realized. By virtue of convexity, the model N_i will be a nuclear model, i.e. it is algebraically prime embedded in other models from $ModT_i$ exactly one time. But by virtue of the model consistency of T_i with the hybrid $H(T_i)$, N_i is isomorphically embedded in some model from $ModH(T_i)$, i.e. by the map g. Since T_i are perfect theories, their center is model-complete, i.e. any monomorphism is elementary between the models of this center. And such, by virtue of perfection, are all the models from E_{T_i}. Then the above isomorphism g will be elementary, i.e. type p is realized in a countable submodel of model M. We got a contradiction with the assumption of imperfection.

Theorem 4. Let T, T_1, T_2 satisfy the conditions of Theorem 3 and T_1, T_2 be ω-categorical. Then their hybrid $H(T_1, T_2)$ is also a ω-categorical Jonsson theory.

Proof. We note that, by virtue of the above Theorem 3, the hybrid $H(T_1, T_2)$ will be a perfect Jonsson theory. Suppose the contrary, i.e. the hybrid $H(T_i)$ is not a ω-categorical Jonsson theory. Let A and B be two countable models from $ModH(T_i)$ that are not isomorphic to each other. Then there are A' and B' countable models from $E_{H(T_i)}$ such that A is isomorphically embedded into A', and B is isomorphically embedded into B'. This follows from the fact that in any inductive theory any model is isomorphically embedded in some existentially closed model of this theory. But the theories of T_i are mutually model consistent with $H(T_i)$ by virtue of the condition of the theorem. Then A' and B' are isomorphically embedded in some countable model $D \in E_{T_i}$, but as T_i are convex theories, then the image of A' and the image of B' in the model D intersects non-empty. Let this intersection be a model R. By virtue of the above existential primeness and countable categoricity of T_i, since $R \in E_{T_i}$ it follows that in $R \models \varphi(x) \wedge \neg\varphi(x)$, where in $A' \models \varphi(x)$, and in $B' \models \neg\varphi(x)$. But this is not true, as T_i are ω-categorical by condition. Consequently, we obtain a contradiction with the assumption of non-ω-categoricity $H(T_i)$.

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Йонсондық теориялардың гибридтерінің қасиеттері

Мақаланың негізгі мақсаты — йонсондық теориялардың гибридтерінің компаньондарының модельдітеоретикалық қасиеттерін зерттеу. Негізгі нысандары йонсондық гибридтер мен олардың модельдер класы болып табылады. Осындай теориялардың әртүрлі байланыстары қарастырылды. Ол байланыстарды тереңірек түсіну және алғашқы теориямен байланыстыру үшін қаралған фрагменттің семантикалық модельдерінің арнайы алгебралық құрылымдары анықталды және осы негізде бұл фрагменттердің гибридтері белгілі болды. Жалпы мұндай алгебралық құрылымдар семантикалық гибридтер деп аталады.

Кілт сөздер: йонсондық теория, семантикалық модель, гибрид, экзистенционалды жай, модельді компаньон.

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Свойства гибридов йонсоновских теорий

В статье основной задачей является рассмотрение различных связей между йонсоновскими теориями. Для того чтобы более глубоко понять эти связи и, в конечном итоге, связь с самой первоначальной теорией и были определены специальные алгебраические конструкции семантических моделей рассматриваемых фрагментов. Кроме того, были определены гибриды этих фрагментов, такие алгебраические конструкции называются семантическими гибридами.

Ключевые слова: йонсоновская теория, семантическая модель, гибрид, экзистенциально простая, модельный компаньон.

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