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On singular integral equations with variable limits of integration

The wide range of problems of mathematical physics is reduced to a special Volterra integral equation of the second kind or to integral equations with variable limits of integration. Among such problems we can include: boundary value problems for spectrally loaded differential equations [1–4], inverse problems [5, 6], nonlocal problems [7], boundary value problems for domains with moving boundaries as the domain degenerates at the time [8, 9] and others. In the study of integral equations with a variable lower limit of integration, the operational method can not be used directly, since in this case the convolution theorem is not applicable. However, the Laplace transform can be used to study this kind of integral equation by applying the method of model solutions.

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1 Method of model solutions

Consider the operator equation

$$\mathcal{M}[y(x)] = f(x), \quad (1)$$

where \mathcal{M} – some linear (integral) operator; $y(x)$ – sought; $f(x)$ – predetermined function [10].

Let \mathcal{B} – be a certain well-known integral transformation

$$\mathcal{B}\{f(x)\} = \widehat{f}(p),$$

denote by $\psi(x, p)$ – the inverse transformation kernel \mathcal{B}^{-1} , which acts as follows:

$$f(x) = \mathcal{B}^{-1}\{\widehat{f}(p)\} \equiv \int_a^b \widehat{f}(p)\psi(x, p)dp. \quad (2)$$

Here the limits a and b and the path of integration can lie in the complex plane.

Definition. The solution of equation (1), in which the right-hand side is the kernel of some inverse integral transformation, will be called the model solution of this equation.

Supposably $\widehat{y}(x, p)$ – the model solution of the auxiliary problem for equation (1), on the right-hand side of which there is a kernel of the inverse transformation \mathcal{B}^{-1} :

$$\mathcal{M}[\widehat{y}(x, p)] = \psi(x, p). \quad (3)$$

We multiply both sides of the equality (3) by $\widehat{f}(p)$, and integrate with respect to the parameter p within the same limits as in the inverse transformation (2). Since the operator \mathcal{M} does not depend on p , using the equality $\mathcal{B}^{-1} \left\{ \widehat{f}(p) \right\} = f(x)$, will have

$$\mathcal{M} \left\{ \int_a^b \widehat{y}(x, p) \widehat{f}(p) dp \right\} = f(x).$$

The last equality means that the solution of equation (1) for an arbitrary right-hand side $f(x)$ can be written in terms of the solution of the auxiliary equation (3) by the formula:

$$y(x) = \int_a^b \widehat{y}(x, p) \widehat{f}(p) dp. \tag{4}$$

We apply this method to the solution of the second-kind Volterra equation with variable lower limit of integration

$$\varphi(t) - \lambda \int_t^\infty k(t - \tau) \varphi(\tau) d\tau = f(t), \tag{5}$$

which can not be solved by a direct Laplace transform, since the convolution theorem is not applicable here.

We consider an auxiliary equation with exponential right-hand side

$$\varphi(t) - \lambda \int_t^\infty k(t - \tau) \varphi(\tau) d\tau = e^{pt}$$

(function e^{pt} – is the kernel of the inverse Laplace transform, $Re p > 0$).

We seek a solution of this equation in the form $\varphi(t, p) = A \cdot e^{pt}$. As a result, we get

$$\varphi(t, p) = \frac{1}{1 - \lambda \widehat{k}(-p)} \cdot e^{pt}; \quad \widehat{k}(-p) = \int_0^\infty k(-z) \cdot e^{pz} dz.$$

From this, using formula (4), we obtain a solution of equation (5) for an arbitrary right-hand side in the form

$$\varphi(t) = \frac{1}{2\pi i} \cdot \int_{c-i\infty}^{c+i\infty} \frac{\widehat{f}(p)}{1 - \lambda \widehat{k}(-p)} \cdot e^{pt} dp,$$

where $\widehat{f}(p)$ – image of the function $f(t)$ obtained by means of the Laplace transform.

2 Solution of reference equations

The main task of this paper is to investigate the following *singular* integral equations:

$$\mathbf{K}_\lambda \mu \equiv (I - \lambda \mathbf{K}) \mu \equiv \mu(t) - \lambda \int_0^t \mathcal{K}(t - \tau) \mu(\tau) d\tau = f(t), \quad t \in \mathbb{R}_+; \tag{6}$$

$$\mathbf{K}_\lambda^* \nu \equiv (I - \lambda \mathbf{K}^*) \nu \equiv \nu(t) - \bar{\lambda} \int_t^\infty \mathcal{K}(\tau - t) \nu(\tau) d\tau = g(t), \quad t \in \mathbb{R}_+, \tag{7}$$

where

$$\mathcal{K}(z) = \frac{1}{2\sqrt{\pi} z^{3/2}} \cdot \exp\left(-\frac{1}{4z}\right). \tag{8}$$

It should be noted that the kernel of the adjoint integral equation (7) – the function $\mathcal{K}(\tau - t)$ has the following property:

$$\int_t^{\infty} \mathcal{K}(\tau - t) d\tau = 1. \quad (9)$$

Equation (9) means that the norm of the integral operator acting in the space of summable functions and defined by the kernel $\mathcal{K}^*(\tau - t)$ is equal to one. This essentially distinguishes equation (7) from the Volterra equations of the second kind, for which the solution exists and is unique.

It is obvious that equation (7) is the union integral equation for (6).

We will solve these equations by the operational method [11]. First we investigate equation (7). As noted earlier, the Laplace transform is not directly applicable to this equation. Using the method of model solutions, we obtain

$$\widehat{\nu}(p) \cdot \left[1 - \bar{\lambda} \exp(-\sqrt{-p}) \right] = \widehat{g}(p), \quad \operatorname{Re} p \leq 0, \quad (10)$$

where $\widehat{\nu}(p)$, $\widehat{g}(p)$ – the Laplace transform, or the functions $\nu(t)$ and $g(t)$. Function

$$\widehat{A}^*(p, \bar{\lambda}) = 1 - \bar{\lambda} \cdot \exp(-\sqrt{-p}).$$

We extend analytically to the whole complex plane with a cut along the positive real semiaxis.

We show that the homogeneous integral equation

$$\mathbf{K}_{\lambda}^* \nu \equiv (I - \lambda \mathbf{K}^*) \nu \equiv \nu(t) - \bar{\lambda} \int_t^{\infty} \mathcal{K}(\tau - t) \nu(\tau) d\tau = 0, \quad (11)$$

for some values $\lambda \in \mathbb{C}$ has nonzero solutions. In order to find these non-trivial solutions and determine the corresponding values of λ , it is necessary to clarify the picture of the zeros of the function $\widehat{A}^*(p, \bar{\lambda})$.

Assuming the parameter $\lambda \in \mathbb{C}$ to be given, we find the roots of equation

$$\widehat{\nu}(p) = 1 - \bar{\lambda} \exp(-\sqrt{-p}) = 0, \quad p = s + i\sigma,$$

which for $|\bar{\lambda}| > 1$ have the form:

$$p_k = s_k + i\sigma_k = - \left[\ln^2 |\bar{\lambda}| - (\arg \bar{\lambda} + 2k\pi)^2 \right] - i2(\arg \bar{\lambda} + 2k\pi) \cdot \ln |\bar{\lambda}|, \quad k \in \mathbb{Z}. \quad (12)$$

All the roots (12) are simple and are located on a parabola

$$s = \frac{1}{4 \ln^2 |\bar{\lambda}|} \cdot \sigma^2 - \ln^2 |\bar{\lambda}|. \quad (13)$$

It is clear that the branches of the parabola are facing to the right, and the vertex of the parabola is located at the point $p = -\ln^2 |\bar{\lambda}|$ on the real axis, and depending on the values $|\bar{\lambda}|$ is shifted left or right along the real axis of the complex plane of the variable p .

For $|\bar{\lambda}| < 1$ it is obvious that the function $\widehat{A}^*(p, \bar{\lambda})$ is not zero at any point of the complex plane $p = s + i\sigma$ with a cut along the real positive semiaxis, since $|\exp(-\sqrt{-p})| > 1$.

But if $|\bar{\lambda}| = 1$, then the equation $|\bar{\lambda}| = |\exp(-\sqrt{-p})|$ with respect to the complex variable $\bar{\lambda}$ has a unique solution $\bar{\lambda} = 1$, which corresponds to the value $p = 0$.

The lines described by the equation $|\bar{\lambda}| = \exp(|\arg \lambda + 2k\pi|)$, divide the complex plane of the parameter λ into disjoint domains D_m , $m = 0, 1, 2, \dots$, as follows:

$$D_{2n} = \left\{ D_n^{(1)} \cap D_n^{(2)} \right\} \setminus \bigcup_{k=-1}^{2n-1} D_k, \quad D_{-1} = \phi, \quad D_{2n+1} = \left\{ D_n^{(1)} \cup D_n^{(2)} \right\} \setminus \bigcup_{k=0}^{2n} D_k, \quad (14)$$

where

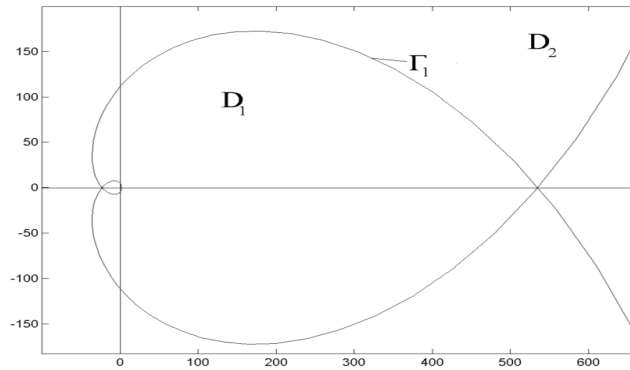
$$D_n^{(1)} = \{ \bar{\lambda} : |\bar{\lambda}| < \exp[(2n+1)\pi - \arg \bar{\lambda}] \}, \quad D_n^{(2)} = \{ \bar{\lambda} : |\bar{\lambda}| < \exp[2n\pi + \arg \bar{\lambda}] \}, \quad n = 0, 1, 2, \dots$$

The outer parts of the boundaries ∂D_m , $m = 0, 1, 2, \dots$, of the domains D_m , $m = 0, 1, 2, \dots$, respectively are denoted by Γ_m , $m = 0, 1, 2, \dots$ (see Picture 1).

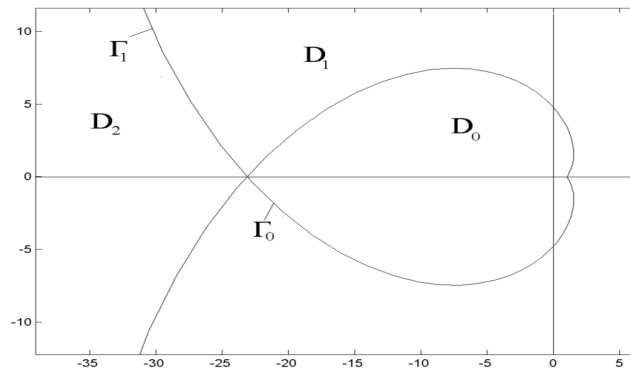
Remark 1. Note that in addition to the domain D_0 (see Picture 2) which has only the outer boundary $\Gamma_0 = \partial D_0$, each of the domains D_m has the boundary ∂D_m , consisting of the outer Γ_m and the inner Γ_{m-1} :

$$\partial D_m = \Gamma_{m-1} \cup \Gamma_m \text{ and } \Gamma_{m-1} \cap \Gamma_m = (-1)^m \exp\{m\pi\},$$

i.e. the outer Γ_m and the inner Γ_{m-1} part of the boundary ∂D_m of the domain D_m have one common point lying on the real axis of the complex plane of the parameter λ .



Picture 1. Plane of the spectral parameter λ (reduced scale)



Picture 2. Plane of the spectral parameter λ (increased scale)

Thus, we get that $\bar{\lambda} \in \Gamma_m$, $m = 0, 1, 2, \dots$, if and only if there exists at least one point p_0 , for which $\widehat{A}^*(p_0, \bar{\lambda}) = 0$.

Suppose that $|\bar{\lambda}| > 1$. Then, according to (13) the function $\widehat{A}^*(p, \bar{\lambda})$ in the left half-plane can have only a finite number of zeros of the form (12), where

$$-N_1 \leq k \leq N_2, \quad N_1 = \left\lfloor \frac{\ln |\bar{\lambda}| + \arg \bar{\lambda}}{2\pi} \right\rfloor, \quad N_2 = \left\lfloor \frac{\ln |\bar{\lambda}| - \arg \bar{\lambda}}{2\pi} \right\rfloor, \quad (15)$$

(here the symbol $[a]$ denotes the integer part of the number a , whereby the integer part of the negative number is set equal to zero). Indeed, the relations (15) follow from the condition that the real parts of the roots (12) must take negative values, that is $\text{Re}\{p_k\} \leq 0$. Hence, from the inequality $(2\pi k + \arg)^2 < \ln^2 |\lambda|$ follows assertion (15).

Thus, for $|\bar{\lambda}| \geq 1$ the homogeneous equation (11) has a general solution of the form

$$\mu(t) = \sum_{k=-N_1}^{N_2} c_k \cdot e^{p_k t},$$

where c_k — are arbitrary constants, the numbers N_1 and N_2 are determined from the relations (15) (for given λ).

We now find a particular solution of the inhomogeneous equation (7). Suppose that the Laplace transform of $g(t)$ is analytic in the $-\varepsilon < \operatorname{Re} p < \varepsilon$. Then from equation (10) for $\forall \lambda \notin \Gamma_m$ ($m = 0, 1, 2, \dots$) we obtain

$$\widehat{\nu}(p) = \widehat{g}(p) + \lambda \frac{\exp(-\sqrt{-p})}{1 - \bar{\lambda} \exp(-\sqrt{-p})} \cdot \widehat{g}(p).$$

Passing in this relation to the originals we shall have

$$\nu(t) = g(t) + \bar{\lambda} \int_t^{\infty} r_{\lambda-}(t - \tau) g(\tau) d\tau,$$

where

$$r_{\lambda-}(y) = \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} \frac{\exp(-\sqrt{-p})}{1 - \bar{\lambda} \exp(-\sqrt{-p})} \exp(y p) dp. \quad (16)$$

If the roots of equation

$$1 - \bar{\lambda} \exp(-\sqrt{-p}) = 0$$

lie on the imaginary axis, then we will make the integration along the contour, bypassing these points on the left. The integral must be understood in the sense of Cauchy's principal value.

Since $y < 0$, we find the residue of the integrand in (16) along the right cut half-plane

$$\begin{aligned} r_{\lambda-}(y) = & 2 \sum_{k=-\infty}^{-(N_1+1)} \sqrt{-p_k} \cdot \exp(p_k \cdot y) + 2 \sum_{k=N_2+1}^{\infty} \sqrt{-p_k} \cdot \exp(p_k \cdot y) + \\ & + \frac{1}{2\pi(-y)^{3/2}} \sum_{k=1}^{\infty} \frac{m}{\bar{\lambda}^m} \cdot \exp\left(\frac{m^2}{4y}\right), \quad \operatorname{Re} p_k > 0, \quad |\bar{\lambda}| > 1, \quad y \in \mathbb{R}_-, \end{aligned} \quad (17)$$

the numbers N_1, N_2 and the roots p_k are determined from formulas (15) and (12), respectively.

But if $|\bar{\lambda}| \leq 1$, then

$$r_{\lambda-}(y) = \frac{1}{2\pi(-\theta)^{3/2}} \sum_{m=1}^{\infty} m \cdot \lambda^m \cdot \exp\left(\frac{m^2}{4y}\right), \quad y \in \mathbb{R}_-. \quad (18)$$

Consequently, the general solution of the integral equation (7) for $|\lambda| > 1$ has the form

$$\nu(t) = g(t) + \bar{\lambda} \int_t^{\infty} r_{\lambda-}(t - \tau) g(\tau) d\tau + \sum_{k=-N_1}^{N_2} c_k \cdot \exp(p_k t), \quad (19)$$

where $r_{\lambda-}(\theta)$ — is determined from the equality (17), and if $|\lambda| < 1$, then the integral equation (7) has a unique solution

$$\nu(t) = g(t) + \bar{\lambda} \int_t^{\infty} r_{\lambda-}(t - \tau) g(\tau) d\tau, \quad (20)$$

where $r_{\lambda-}(\theta)$ is now found from equation (18), the numbers N_1, N_2, p_k are determined from equalities (15) and (12).

In order for the solution $\nu(t)$, defined by (19), (20), to be summable, it is sufficient that the function $r_{\lambda-}(t - \tau)$ be bounded for any $0 < \tau \leq t < \infty$, so 5 as a function $g_1(t) + \sum_{k=-N_1}^{N_2} c_k \exp(p_k t)$ is an integrable function of t . The function $r_{\lambda-}(t - \tau)$ will be bounded, since $r_{\lambda-}(\theta)$ (17) satisfies the estimate:

$$|r_{\lambda-}(\theta)| \leq C_1 |\theta|^{-1/2} \exp(-\delta_0 |\theta|) + C_2 |\theta|^{-3/2} \exp(-\delta_0 |\theta|^{-1}), \quad \forall \theta \in \mathbb{R}_-, \quad (21)$$

where

$$\delta_0 = \min \left\{ 1/4; [2\pi(N_1 + 1) + \arg \lambda]^2 - \ln^2 |\lambda|; [2\pi(N_2 + 1) + \arg \lambda]^2 - \ln^2 |\lambda| \right\}. \quad (22)$$

The validity of the estimate (21) follows from the relations below. For the second we obtain from (17):

$$\begin{aligned} & \left| \sum_{k=N_2+1}^{\infty} \sqrt{-p_k} \exp(p_k \theta) \right| \leq |\ln \lambda| \sum_{k=N_2+1}^{\infty} |\exp(p_k \theta)| \leq \\ & \leq |\ln \lambda| \sum_{k=N_2+1}^{\infty} \exp\left\{ (2k\pi + \arg \lambda)^2 - \ln^2 |\lambda| \right\} \theta \leq \left\| \begin{array}{l} y = 2k\pi + \arg \lambda \\ a = 2\pi(N_2 + 1) + \ln |\lambda| \end{array} \right\| \leq \\ & \leq |\ln \lambda| \int_a^{\infty} \exp\{(y^2 - \ln^2 |\lambda|)\theta\} dy = |\ln \lambda| \exp\{-\theta \ln^2 |\lambda|\} \int_a^{\infty} \exp\{\theta y^2\} dy = \\ & = \|z = y - a\| = |\ln \lambda| \exp\{-\theta \ln^2 |\lambda|\} \int_0^{\infty} \exp\{\theta(a^2 + z^2 + 2az)\} dz = \\ & = |\ln \lambda| \exp\{-\theta \ln^2 |\lambda| + \theta a^2\} \int_0^{\infty} \exp\{\theta z^2 + \theta 2az\} dz \leq \\ & \leq |\ln \lambda| (-\theta)^{-1/2} \cdot \exp\{\theta(a^2 - \ln^2 |\lambda|)\} \int_0^{\infty} \exp\{-(\sqrt{-\theta}z)^2\} d(\sqrt{-\theta}z) = |\ln \lambda| \frac{\sqrt{\pi}}{2\sqrt{-\theta}} \exp\{\delta_2 \theta\}, \end{aligned}$$

where $\delta_2 = [2\pi(N_2 + 1) + \arg \lambda]^2 - \ln^2 |\lambda| > 0$.

Similarly for the first term we have the inequality:

$$\left| \sum_{k=-\infty}^{-(N_1+1)} \sqrt{-p_k} \exp(p_k \theta) \right| \leq |\ln \lambda| \frac{\sqrt{\pi}}{2\sqrt{-\theta}} \exp\{\delta_1 \theta\},$$

where $\delta_1 = [2\pi(N_1 + 1) + \arg \lambda]^2 - \ln^2 |\lambda| > 0$.

The third term in (17) is estimated as follows:

$$\begin{aligned} |\theta|^{-3/2} \sum_{m=1}^{\infty} \frac{m}{\lambda^m} \exp\left(-\frac{m^2}{4|\theta|}\right) &= |\theta|^{-3/2} \exp\left\{-\frac{1}{4|\theta|}\right\} \sum_{m=1}^{\infty} \frac{m}{\lambda^m} \exp\left(-\frac{m^2-1}{4|\theta|}\right) \leq \\ &\leq C |\theta|^{-3/2} \exp\left\{-\frac{1}{4|\theta|}\right\}. \end{aligned}$$

For the representation from (18) for $|\lambda| = 1$ we obtain the estimate:

$$|\theta|^{-3/2} \sum_{m=1}^{\infty} m \exp\left(-\frac{m^2}{4|\theta|}\right) \leq \frac{2}{\sqrt{|\theta|}} \int_1^{\infty} \exp\left(-\frac{y^2}{4|\theta|}\right) d\left(-\frac{y^2}{4|\theta|}\right) = \frac{2}{\sqrt{|\theta|}} \exp\left(-\frac{1}{4|\theta|}\right);$$

and for $|\lambda| < 1$ we have the estimate:

$$|\theta|^{-3/2} \sum_{m=1}^{\infty} m \lambda^m \exp\left(-\frac{m^2}{4|\theta|}\right) \leq C |\theta|^{-3/2} \exp\left(-\frac{1}{4|\theta|}\right).$$

It is easy to verify that (19) is a solution of equation (7) for arbitrary coefficients c_k .ck.

We state our results in the form of the following theorems.

Theorem 1. The values $\lambda \in D_0$ in (14) are regular numbers of the operator (16).

Theorem 2. The set $\mathbb{C} \setminus D_0$ consists of the characteristic numbers of the operator \mathbf{K}^* (7). Moreover, if $\lambda \in D_m \cup \Gamma_{m-1} \setminus \{(-1)^m e^{m\pi}\}$, $m = 1, 2, \dots$, then $\dim \text{Ker}(\mathbf{K}^*) = m$; and the corresponding eigenfunctions have the form

$$\nu_{\lambda k}(t) = \exp(p_k t), \quad k = 1, \dots, m = N_1 + N_2 + 1,$$

where the numbers p_k , N_1 , N_2 are determined from the equalities (12), (15).

Now consider the integral equation (6), which is usually called the *recovery equation* [12]. This name is explained by the fact that such equations arise in the theory of recovery — the section of probability theory, which describes a wide range of phenomena associated with the failure and restoration of the elements of a system. The reconstruction equation is of great importance also in the study of both applied and theoretical problems in reliability theory, queuing theory, in reserve theory, in the theory of branching processes, and so on.

Applying the Laplace transform to (6) and using the convolution theorem in this case, we obtain

$$\widehat{\mu}(p) = \widehat{f}(p) + \frac{\lambda e^{-\sqrt{p}}}{1 - \lambda e^{-\sqrt{p}}} \widehat{f}(p), \quad p = s + i\sigma, \quad \text{Re } p = s > 0.$$

Using the inverse Laplace transform, we have:

$$\mu(p) = f(t) + \lambda \int_0^t r_{\lambda+}(t - \tau) f(\tau) d\tau, \tag{23}$$

where the resolvent $r_{\lambda+}(\theta)$ is defined in terms of the kernel of the original equation (6) by the formula

$$r_{\lambda+}(\theta) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\lambda e^{-\sqrt{p}}}{1 - \lambda e^{-\sqrt{p}}} e^{p\theta} dp, \quad p = s + i, \tag{24}$$

the path of integration is parallel to the imaginary axis of the complex plane to the right of all the singular points of the integrand, that is, to the right of all zeros of the function

$$\widehat{\mathbf{A}}(p, \lambda) = 1 - \lambda \cdot \exp(-\sqrt{p}).$$

The zeros of the function $\widehat{\mathbf{A}}(p, \lambda)$ have the form:

$$p_n = s_n + i\sigma_n = [\ln^2 |\bar{\lambda}| - (\arg \lambda + 2n\pi)^2] + i2(\arg \lambda + 2n\pi) \cdot \ln \lambda, \quad n \in \mathbb{Z}; \tag{25}$$

they are all simple and arranged on a parabola

$$s = -\frac{\sigma^2}{4 \ln^2 |\bar{\lambda}|} + \ln^2 |\bar{\lambda}|,$$

the branches of which are turned to the left, and the vertex is on the real axis at the point

$$s_0 = \ln^2 |\bar{\lambda}|.$$

We note that the function $\widehat{\mathbf{A}}(p, \lambda)$ in the right half-plane can have only a finite the number of zeros of p_n

$$-N_1 \leq n \leq N_2, \quad N_1 = \left\lceil \frac{\ln |\lambda| + \arg \lambda}{2\pi} \right\rceil, \quad N_2 = \left\lceil \frac{\ln |\lambda| - \arg \lambda}{2\pi} \right\rceil \tag{26}$$

(here again the square bracket denotes the integer part of the number). Moreover, their number increases with increasing $|\lambda|$, if $\lambda \in D_m$ (14), then their number is $m = N_1 + N_2 + 1$. Note that if $\lambda \in D_0$, then the function $\widehat{\mathbf{A}}(p, \lambda)$ does not have zeros at all in the complex plane.

Let us calculate the integral (24). We continue the integrand analytically on the whole complex plane with a cut along the negative real axis. Then, according to the theory of residues, we get:

$$r_{\lambda+}(\theta) = 2 \sum_{n=-\infty}^{+\infty} \sqrt{p_n} \cdot \exp(p_n \cdot \theta) + \frac{1}{2\sqrt{\pi}\theta^{3/2}} \sum_{m=1}^{\infty} \frac{m}{\lambda_m} \cdot \exp\left(-\frac{m^2}{4\theta}\right);$$

$$\operatorname{Re} p_n > 0, \quad |\lambda| > 1, \quad \theta \in \mathbf{R}_+;$$

$$r_{\lambda+}(\theta) = \frac{1}{2\sqrt{\pi}\theta^{3/2}} \sum_{m=1}^{\infty} m\lambda^m \cdot \exp\left(-\frac{m^2}{4\theta}\right), \quad |\lambda| \leq 1, \quad \theta \in \mathbf{R}_+, \quad (27)$$

where the numbers p_n are found from (25).

In order that the function $\mu(t)$ defined by (23) be essentially bounded, it is necessary and sufficient that the following conditions are satisfied:

$$\int_0^{\infty} f(t) \cdot \exp(-p_k t) dt = 0, \quad -N_1 \leq k \leq N_2 \quad (28)$$

where the numbers N_1, N_2, N_1, N_2 are determined from the equalities (26), p_k from the equality (25).

Indeed, in this case the resolvent of the integral equation (6) for $|\lambda| > 1$ will have the form (6) and at it will have the form $|\lambda| > 1$

$$r_{\lambda+}(\theta) = 2 \sum_{n=-\infty}^{-(N_1+1)} \sqrt{p_n} \cdot \exp(p_n \cdot \theta) + 2 \sum_{n=N_2+1}^{\infty} \sqrt{p_n} \cdot \exp(p_n \cdot \theta) + \frac{1}{2\sqrt{\pi}\theta^{3/2}} \sum_{m=1}^{\infty} \frac{m}{\lambda^m} \cdot \exp\left(-\frac{m^2}{4\theta}\right), \operatorname{Re} p_k > 0, \quad \theta \in \mathbf{R}_+, \quad (29)$$

and will be a summable function, since it has the estimate

$$|r_{\lambda+}| \leq C \cdot |\theta|^{-\frac{1}{2}} \cdot \exp(-\delta_0|\theta|) + C_2 \cdot |\theta|^{-\frac{3}{2}} \cdot \exp(-\delta_0|\theta|^{-1}) \quad \forall \theta \in \mathbf{R}_+,$$

in which the constant δ_0 determines their equalities (22).

Thus it is fair

Theorem 3. If $\lambda \in D_0$, then the inhomogeneous equation (6) is unconditionally uniquely solvable; if $\lambda \in \mathbb{C} \setminus D_0$, $\lambda \in D_m$, then for the unique solvability of (6) it is necessary and sufficient that m -solvability conditions (28) be satisfied. The conditions (28) mean that the free term of the integral equation (6) must be orthogonal to the solutions of the homogeneous conjugate integral equation (7).

The validity of these statements, as well as of conditions (28), can also be shown in the following way. The image of the solution of the integral equation (6) is defined by

$$\widehat{\mu}(p) = \frac{\widehat{f}(p)}{1 - \lambda e^{-\sqrt{p}}}. \quad (30)$$

The following options are possible.

1. The function $\widehat{\mathbf{A}}(p, \lambda) = 1 - \lambda \cdot \exp(-\sqrt{p})$ does not have zeros in the right half-plane (this means that $|\lambda| < 1$ and $\lambda \in D_0$ (14)). In this case, the equation for any right-hand member $f(t)$ has a unique solution that is expressed in terms of the resolvent $r_{\lambda+}(\theta)$, defined by formula (27)

$$\mu(t) = f(t) + \lambda \int_0^t r_{\lambda+}(t - \tau) f(\tau) d\tau, \quad t \in \mathbf{R}_+. \quad (31)$$

2. The function $\widehat{f}(p)$ vanishes at the points p_n , $N_1 \leq n \leq N_2$ from (25), that is, in the zeros of the function $\widehat{\mathbf{A}}(p, \lambda)$ located in the right half-plane. In this case, the function (30) again will not have poles in the region $\operatorname{Re} p > 0$, so equation (6) also has a unique solution of the form (31), but the resolvent $r_{\lambda+}(\theta)$ is now determined from (29). The condition $\widehat{f}(p_n) = 0$, $N_1 \leq n \leq N_2$, on the inversion of the function $\widehat{f}(p)$ to zero at the points $p = p_n$ is equivalent to the following conditions

$$\int_0^{\infty} f(t) \cdot e^{-p_n t} dt = 0; \quad N_1 \leq n \leq N_2.$$

So we have proved the following statement.

Lemma. On the complex plane \mathbb{C} there are no characteristic numbers of the operator \mathbf{K} (9).

Thus, it follows from the results obtained that the solutions of the integral equations (10) and (9) are determined by expressions

$$\nu_\lambda(t) = g(t) + \bar{\lambda} \int_t^\infty r_{\lambda-}(t-\tau)g(\tau)d\tau + \sum_{k=-N_1}^{N_2} c_k \cdot \exp(p_k t), \quad t \in \mathbf{R}_+,$$

where the numbers p_k, N_1, N_2 are determined from the equalities (12), (15),

$$\mu_\lambda(t) = f(t) + \lambda \int_0^t r_{\lambda+}(t-\tau)f(\tau)d\tau, \quad t \in \mathbf{R}_+,$$

and satisfy the conditions

$$\nu_\lambda(t) \in L_1(\mathbb{R}_+), \quad \mu_\lambda(t) \in L_\infty(\mathbb{R}_+).$$

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Айнымалы шекті интегралды ерекше интегралдық теңдеулер жайлы

Математикалық физика есептерінің кең аумағы Вольтерраның екінші текті арнайы интегралдық теңдеуіне, немесе айнымалы шекті интегралды интегралдық теңдеулерге, келтірілді. Мұндай есептердің ішінде келесі есептерді атап өтуге болады: спектралды жүктелген дифференциалдық теңдеулер үшін шекаралық есептер [1–4], кері есептер [5, 6], локалды емес есептер [7], жылжымалы шекаралы облыстар үшін шекаралық есептер [8, 9] т.б. Төменгі айнымалы шекті интегралданатын интегралдық теңдеулерді оқып-үйрену кезінде жұмыс әдісі, свертка теоремасын пайдалануға болмайтындықтан, бірден қолданылмайды. Алайда мұндай интегралдық теңдеулерді оқып-үйрену үшін модельдік шешу әдісін қолдана отырып, Лаплас түрлендіруін қолдануға болады.

Кілт сөздер: модельді шешу, интегралдық оператор, спектр, резольвента, сипаттамалық сандар, меншікті функциялар.

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Об особых интегральных уравнениях с переменными пределами интегрирования

Широкий спектр задач математической физики сводится к специальному интегральному уравнению Вольтерра второго рода или к интегральным уравнениям с переменными пределами интегрирования. Среди таких задач можно выделить: краевые задачи для спектрально нагруженных дифференциальных уравнений [1–4], обратные задачи [5, 6], нелокальные задачи [7], краевые задачи для областей с движущимися границами, когда область вырождается [8, 9] и др. При изучении интегральных уравнений с переменным нижним пределом интегрирования рабочий метод не может быть использован непосредственно, так как в этом случае неприменима теорема свертки. Однако для изучения такого интегрального уравнения можно использовать преобразование Лапласа путем применения метода модельных решений.

Ключевые слова: модельное решение, интегральный оператор, спектр, резольвента, характеристические числа, собственные функции.

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