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Sufficient conditions for the precompactness of sets in Local Morrey-type spaces

In this paper we give sufficient conditions for the pre-compactness of sets in local Morrey-type spaces $LM_{p\theta,w(\cdot)}(\mathbb{R}^n)$. For $w(r) = r^{-\lambda},\ \theta = \infty,\ 0 \le \lambda \le \frac{n}{p}$ there follows a known result for the Morrey spaces $M_p^{\lambda}(\mathbb{R}^n)$. In the case $\lambda = 0$ this is the well-known Frechet-Kolmogorov theorem. The pre-compactness of sets in Morrey spaces was investigated in the works [1, 2], and in generalized Morrey spaces $M_p^{w(\cdot)}(\mathbb{R}^n)$ in the works [3, 4]. The aim of this paper is to generalize these results to the case of Local Morrey-type spaces $LM_{p\theta,w(\cdot)}(\mathbb{R}^n)$. By using theorem of pre-compactness set in local Morrey-type spaces, compact of operators can be checked in this spaces, since compact operator transfers from bounded set of one space to pre-compact set of another space. In this paper, the conditions of precompactness of sets in local spaces of Morrey type are given in terms of the difference of the function $\lim_{u\to 0}\sup_{f\in S}\|f(\cdot+u)-f(\cdot)\|_{LM_{p\theta,w}} = 0$. Earlier, the necessary and sufficient conditions for precompactness of sets in local spaces of Morrey type were published in [5], which were given in terms of the mean functions $\lim\limits_{\delta\to 0^+}\sup\limits_{f\in S}\|A_\delta f - f\|_{L_p(B(0,R_2)\setminus B(0,R_1))} = 0$.

Keywords: compactness, precompact, Freche-Kolmogorov theorem, local Morrey-type spaces.

The classical Morrey space was introduced in the works of Charles Morrey [6] in 1938 in connection with the investigation of the solution of quasilinear elliptic differential equations. In recent decades questions of the boundedness and compactness of various operators in Morrey-type spaces have been actively studied ([7–9]).

Morrey spaces M_p^{λ} are defined as the set of all functions $f \in L_p^{loc}(\mathbb{R}^n)$, for which , $0 \le \lambda \le \frac{n}{p}$, $0 < p \le \infty$, with a finite quasinorm

$$\|f\|_{M_p^{\lambda}(\mathbb{R}^n)} = \sup_{x\in\mathbb{R}^n,\ r>0} r^{-\lambda}\left(\int\limits_{B(x,r)} |f(y)|^p dy\right)^{\frac{1}{p}} < \infty,$$

where $B(x,r)$ is the open ball in $\mathbb{R}^n$ centered at the point x of radius $r > 0$.

Note that

$$\|f\|_{M_p^{0}(\mathbb{R}^n)} \equiv \|f\|_{L_p(\mathbb{R}^n)}, \|f\|_{M_p^{\frac{n}{p}}(\mathbb{R}^n)} \equiv \|f\|_{L_\infty(\mathbb{R}^n)}.$$

If $\lambda < 0,\ \lambda > \frac{n}{p}$ the space $M_p^{\lambda}(\mathbb{R}^n)$ is trivial, i.e. consists only of functions equivalent to zero on $\mathbb{R}^n$.

According to the well-known Freche-Kolmogorov theorem [10], the set $S \subset L_p(\mathbb{R}^n)$, where $1 \le p < \infty$, is precompact if and only if

$$\sup_{f\in S}\|f\|_{L_p(\mathbb{R}^n)} < \infty; \tag{1}$$

$$\lim_{\delta\to 0^+}\sup_{|h|\le\delta}\|f(\cdot+h)-f(\cdot)\|_{L_p(\mathbb{R}^n)} = 0 \tag{2}$$

and

$$\lim_{R\to\infty}\sup_{f\in S}\|f\|_{L_p({}^cB(0,R))} = 0, \tag{3}$$

where ${}^cB(0,R)$ is the complement of a ball $B(0,R)$.

Conditions (1)–(3) are equivalent to the union of conditions (1), (3) and

$$\lim_{\delta\to 0^+}\sup_{f\in S}\|A_\delta f - f\|_{L_p(\mathbb{R}^n)} = 0, \tag{4}$$

where for any $\delta > 0$ and $f \in L_1^{loc}(\mathbb{R}^n)$

$$(A_\delta f)(x) = \frac{1}{|B(x,\delta)|} \int\limits_{B(x,\delta)} f(y)dy = \int\limits_{B(0,\delta)} \omega_\delta(y) f(x-y)dy = (\omega_\delta * f)(x),\ x \in \mathbb{R}^n,$$

where $\omega_\delta(x) = \frac{\chi_{B(0,\delta)}(x)}{|B(0,\delta)|}$, $\chi_{(A)}$ is characteristic function of the set $A \subset \mathbb{R}^n$, ${}^cA$ is the complement of the set A, $|A|$ is Lebesgue measure of the set A. Recall that condition (4) follows from condition (2), and condition (2) follows from the set of conditions (4) and (3).

Note also that if $A \subset \mathbb{R}^n$ is bounded set, then for precompactness of the set $S \subset L_p(A)$ it is necessary and sufficient that conditions (1)–(2) are satisfied, where $\mathbb{R}^n$ replaced by a set A.

The questions of the precompactness of sets in Morrey spaces were investigated in the works [1, 2, 5, 6, 11–14], and when $r^{-\lambda} \equiv w(r)$ for generalized spaces Morrey $M_p^{w(\cdot)}(\mathbb{R}^n)$ were investigated in the works [3, 4]. The questions of the precompactness of sets in Banach spaces [15].

The aim of this paper is to generalize this results to the case of general local Morrey-type spaces.

Definition. Let $0 < p, \theta \leq \infty$, and let w be a nonnegative measurable function on $(0,\infty)$. We denote by $LM_{p\theta,w(\cdot)}$ the general local Morrey-type space, the space of all functions $f \in L_p^{loc}(R^n)$ with finite quasi-norm

$$\|f\|_{LM_{p\theta,w(\cdot)}} \equiv \|f\|_{LM_{p\theta,w(\cdot)}(\mathbb{R}^n)} = \left\| w(r) \, \|f\|_{L_p(B(0,r))} \right\|_{L_\theta(0,\infty)}.$$

We denote by Ω_θ the set of all functions that are nonnegative, measurable on $(0,\infty)$, not equivalent 0 and such, that for some $t > 0$

$$\|w(r)\|_{L_\theta(t,\infty)} < \infty.$$

The space $LM_{p\theta,w(\cdot)}$ is non-trivial, that is, it consists not only of functions, equivalent to 0 on $\mathbb{R}^n$, if and only if $w \in \Omega_\theta$ [14].

Theorem. Let $1 \leq p \leq \theta \leq \infty$ and $w \in \Omega_{p\theta}$. Let $S \subset LM_{p\theta,w}(R^n)$ is satisfied:

$$\sup_{f\in S} \|f\|_{LM_{p\theta,w}} < \infty; \tag{5}$$

$$\lim_{u\to 0} \sup_{f\in S} \|f(\cdot + u) - f(\cdot)\|_{LM_{p\theta,w}} = 0 \tag{6}$$

and

$$\lim_{r\to\infty} \sup_{f\in S} \left\| f \chi_{{}^C B(0,r)} \right\|_{LM_{p\theta,w}} = 0. \tag{7}$$

Then the set S is precompact in $LM_{p\theta,w}(R^n)$.

For proof this theorem we need next statements.

For $f \in L_1^{loc}(\mathbb{R}^n)$ and $r > 0$ define

$$(M_r f)(x) = \frac{1}{|B(x,r)|} \int\limits_{B(x,r)} f(y)dy,$$

where $|\ A\ |$ means Lebega spaces $A \subset \mathbb{R}^n$.

Lemma 1. Let $1 \leq p \leq \theta \leq \infty$, $w \in \Omega_\theta$. Then for all $f \in LM_{p\theta}^{w(\cdot)}$ and $r > 0$ next is true:

$$\|M_r f - f\|_{LM_{p\theta,w}} \leq \sup_{u\in B(0,r)} \|f(\cdot + u) - f(\cdot)\|_{LM_{p\theta,w}}.$$

Proof. Let $z \in R^n$ and $\rho > 0$. Then by inequality of Gelder

$$\|M_r f - f\|_{L_p(B(z,\rho))} =$$

$$= \left(\int\limits_{B(z,\rho)} \left| \frac{1}{|B(x,r)|} \int\limits_{B(x,r)} f(y)dy - f(x) \right|^p dx \right)^{\frac{1}{p}} =$$

$$= \left(\int\limits_{B(z,\rho)} \left| \frac{1}{|B(x,r)|} \int\limits_{B(x,r)} (f(y) - f(x)) dy \right|^p dx \right)^{\frac{1}{p}} \leq$$

$$\leq \left(\int\limits_{B(z,\rho)} \left(\frac{1}{|B(x,r)|} \int\limits_{B(x,r)} |f(y) - f(x)|^p dy \right) dx \right)^{\frac{1}{p}} = |y = x + u| =$$

$$= \left(\int\limits_{B(z,\rho)} \left(\frac{1}{|B(0,r)|} \int\limits_{B(0,r)} |f(x+u) - f(x)|^p du \right) dx \right)^{\frac{1}{p}} =$$

$$= \left(\frac{1}{|B(0,r)|} \int\limits_{B(0,r)} \left(\int\limits_{B(z,\rho)} |f(x+u) - f(x)|^p dx \right) du \right)^{\frac{1}{p}} =$$

$$= \left(\frac{1}{|B(0,r)|} \int\limits_{B(0,r)} \|f(\cdot + u) - f(\cdot)\|^p_{L_p(B(z,\rho))} du \right)^{\frac{1}{p}}.$$

Next

$$\|M_r f - f\|_{LM_{p\theta,w}} = \left\| w(\rho) \, \|M_r f - f\|_{L_p(B(0,\rho))} \right\|_{L_\theta(0,\infty)} \leq$$

$$\leq \left\| w(\rho) \left(\frac{1}{|B(0,r)|} \int\limits_{B(0,r)} \|f(\cdot + u) - f(\cdot)\|^p_{L_p(B(0,\rho))} du \right)^{\frac{1}{p}} \right\|_{L_\theta(0,\infty)} =$$

$$= \left\| \frac{1}{|B(0,r)|} \int\limits_{B(0,r)} w(\rho)^p \, \|f(\cdot + u) - f(\cdot)\|^p_{L_p(B(0,\rho))} du \right\|^{\frac{1}{p}}_{L_{\frac{\theta}{p}}(0,\infty)}.$$

As this $\frac{\theta}{p} \geq 1$, then using inequality of Minkovskogo for integrals, we get next

$$\|M_r f - f\|_{LM_{p\theta,w}} \leq \left(\frac{1}{|B(0,r)|} \int\limits_{B(0,r)} \left(\int\limits_0^\infty w(\rho)^\theta \, \|f(\cdot + u) - f(\cdot)\|^\theta_{L_p(B(0,\rho))} d\rho \right)^{\frac{p}{\theta}} du \right)^{\frac{1}{p}} =$$

$$= \left(\frac{1}{|B(0,r)|} \int\limits_{B(0,r)} \|f(\cdot + u) - f(\cdot)\|^p_{LM_{p\theta,w}} du \right)^{\frac{1}{p}} \leq$$

$$\leq \sup_{u \in B(0,r)} \|f(\cdot + u) - f(\cdot)\|_{LM_{p\theta,w}}.$$

Lemma 1 is proofed.

Lemma 2. Let $1 \leq p \leq \theta \leq \infty$, $w \in \Omega_{p\theta}$. Then for all $f \in LM^{w(\cdot)}_{p\theta}$ and $r > 0$ next inequality is:

$$\|M_r f\|_{LM_{p\theta,w}} \leq \|f\|_{LM_{p\theta,w}}. \tag{8}$$

Proof. By inequality of Gelder

$$\|M_r f\|_{L_p(B(z,\rho))} = \left(\int\limits_{B(z,\rho)} \left| \frac{1}{|B(x,r)|} \int\limits_{B(x,r)} f(y) dy \right|^p dx \right)^{\frac{1}{p}} \leq$$

$$\leq \left(\int\limits_{B(z,\rho)} \left(\frac{1}{|B(x,r)|} \int\limits_{B(x,r)} |f(y)|^p \, dy \right) dx \right)^{\frac{1}{p}} =$$

$$= (y = x + u)$$

$$= \left(\int\limits_{B(z,\rho)} \left(\frac{1}{|B(0,r)|} \int\limits_{B(0,r)} |f(x+u)|^p \, du \right) dx \right)^{\frac{1}{p}} =$$

$$= \left(\frac{1}{|B(0,r)|} \int\limits_{B(0,r)} \left(\int\limits_{B(z,\rho)} |f(x+u)|^p \, dx \right) du \right)^{\frac{1}{p}} =$$

$$= (x + u = v)$$

$$= \left(\frac{1}{|B(0,r)|} \int\limits_{B(0,r)} \left(\int\limits_{B(z+u,\rho)} |f(v)|^p \, dv \right) du \right)^{\frac{1}{p}} =$$

$$= \left(\frac{1}{|B(0,r)|} \int\limits_{B(0,r)} \|f\|^p_{L_p(B(z+u,\rho))} \, du \right)^{\frac{1}{p}}.$$

Since $\frac{\theta}{p} \geq 1$, that

$$\|M_r f\|_{LM_{p\theta,w}} \leq \sup_{z \in R^n} \left(\frac{1}{|B(0,r)|} \int\limits_{B(0,r)} \left(\int\limits_0^\infty w(\rho)^\theta \, \|f\|^\theta_{L_p(B(z+u,\rho))} \, d\rho \right)^{\frac{p}{\theta}} du \right)^{\frac{1}{p}} \leq$$

$$\leq \left(\frac{1}{|B(0,r)|} \int\limits_{B(0,r)} \sup_{z \in R^n} \left(\int\limits_0^\infty w(\rho)^\theta \, \|f\|^\theta_{L_p(B(z+u,\rho))} \, d\rho \right)^{\frac{p}{\theta}} du \right)^{\frac{1}{p}} =$$

$$= \sup_{z \in R^n} \left\| \frac{1}{|B(0,r)|} \int\limits_{B(0,r)} \left(w(\rho) \, \|f\|_{L_p(B(z+u,\rho))} \right)^p du \right\|^{\frac{1}{p}}_{L_{\frac{\theta}{p}}(0,\infty)} \leq$$

$$\leq \sup_{z \in R^n} \left(\frac{1}{|B(0,r)|} \int\limits_{B(0,r)} \left\| w(\rho) \, \|f\|_{L_p(B(z+u,\rho))} \right\|^p_{L_\theta(0,\infty)} du \right)^{\frac{1}{p}} \leq$$

$$\leq \left(\frac{1}{|B(0,r)|} \int\limits_{B(0,r)} \left(\sup_{v \in R^n} \left\| w(\rho) \, \|f\|_{L_p(B(v,\rho))} \right\|_{L_\theta(0,\infty)} \right)^p dv \right)^{\frac{1}{p}} = \|f\|_{LM_{p\theta,w}}.$$

For defining $\delta > 0$ by $w_\delta(f, G)$ function f in set $G \subset R^n$:

$$w_\delta(f, G) = \sup_{\substack{x_1, x_2 \in G \\ |x_1 - x_2| \leq \delta}} |f(x_1) - f(x_2)|.$$

$$\|M_r f\|_{LM_{p\theta,w}} \leq \sup_{z \in R^n} \left\| w(\rho) \, \|M_r f\|_{L_p(B(z,\rho))} \right\|_{L_\theta(0,\infty)} \leq$$

$$\leq \sup_{z\in R^n}\left\|w(\rho)\left(\frac{1}{|B(0,r)|}\int\limits_{B(0,r)}\|f\|^p_{L_p(B(z+u,\rho))}\,du\right)^{\frac{1}{p}}\right\|_{L_\theta(0,\infty)}.$$

Lemma 2 is proofed.

Lemma 3. Let $1 \leq p,\theta \leq \infty$, $w \in \Omega_{p\theta}$. Then exists $r_0 > 0$ for every $0 < r \leq r_0$ exists $C_r > 0$, depending from r, n, p, θ, w

1) for every $f \in LM^{w(\cdot)}_{p\theta}$

$$\|M_r f\|_{C(R^n)} \leq C_r \|f\|_{LM_{p\theta,w}}; \tag{9}$$

2) for every $\delta > 0$

$$w_\delta(M_r f; R^n) \leq C_r \sup_{\substack{|u|\leq\delta\\ u\in B(0,\delta)}} \|f(\cdot+u) - f(\cdot)\|_{LM_{p\theta,w}}.$$

Proof. 1. For function $w \in \Omega_{p\theta}$ no equivalent 0, then exists $r_0 > 0$ this, that $\|w\|_{L_\theta(r_0,\infty)} > 0$. Let $0 < r \leq r_0$. $x \in R^n$

$$|M_r f(x)| \leq \frac{1}{|B(x,r)|^{\frac{1}{p}}}\|f\|_{L_p(B(x,r))}.$$

Hence,

$$\|w(\rho)M_r f(x)\|_{L_\theta(r,\infty)} \leq \frac{1}{(v_n r^n)^{\frac{1}{p}}}\left\|w(\rho)\|f\|_{L_p(B(x,r))}\right\|_{L_\theta(r,\infty)},$$

where v_n — volume of unit ball in R^n and

$$|M_r f(x)|\,\|w(\rho)\|_{L_\theta(r,\infty)} \leq \frac{1}{(v_n r^n)^{\frac{1}{p}}}\left\|w(\rho)\|f\|_{L_p(B(x,r))}\right\|_{L_\theta(0,\infty)}.$$

That's why

$$\sup_{x\in R^n}|M_r f(x)| \leq C_r \sup_{x\in R^n}\left\|w(\rho)\|f\|_{L_p(B(x,r))}\right\|_{L_\theta(0,\infty)} = C_r\|f\|_{LM_{p\theta,w}},$$

where $C_r = \left(\|w\|_{L_\theta(r,\infty)}(v_n r^n)^{\frac{1}{p}}\right)^{-1}$.

2. Next for every $x_1, x_2 \in B(0,r)$

$$|(M_r f)(x_1) - (M_r f)(x_2)| = \frac{1}{v_n r^n}\left|\int\limits_{B(x_1,r)} f(y)dy - \int\limits_{B(x_2,r)} f(y)dy\right| =$$

$$= (v_n r^n)^{-1}\left|\int\limits_{B(0,r)} f(z+x_1)dz - \int\limits_{B(0,r)} f(z+x_2)dz\right| \leq$$

$$\leq (v_n r^n)^{-1}\int\limits_{B(0,r)}|f(z+x_1) - f(z+x_2)|\,dz =$$

$$= (v_n r^n)^{-1}\int\limits_{B(x_2,r)}|f(s+x_1-x_2) - f(s)|\,ds \leq$$

$$\leq (v_n r^n)^{-\frac{1}{p}}\|f(\cdot+x_1-x_2) - f(\cdot)\|_{L_p(B(x_2,r))}.$$

That's why, by first step of proof

$$\sup_{\substack{x_1,x_2\in R^n\\ |x_1-x_2|\leq\delta}}|(M_r f)(x_1) - (M_r f)(x_2)| \leq C_r \sup_{\substack{x_1,x_2\in R^n\\ |x_1-x_2|\leq\delta}}\|f(\cdot+x_1-x_2) - f(\cdot)\|_{LM_{p\theta,w}} =$$

$$= C_r \sup_{\substack{|u|\leq\delta\\ u\in B(0,\delta)}}\|f(\cdot+u) - f(\cdot)\|_{LM_{p\theta,w}}.$$

Lemma 3 is proofed.

Lemma 4. Let $1 \le p, \theta \le \infty$, $w \in \Omega_{p\theta}$. Then exists $C > 0$, depending only from n, p, θ, w, that's for every, $r, R > 0$ and for all $f, g \in LM_{p\theta,w}$ next statement

$$\|M_r f - M_r g\|_{LM_{p\theta,w}} \le C\left(1 + R^{\frac{n}{p}}\right)\|M_r f - M_r g\|_{C\left(\overline{B(0,R)}\right)} + \sup_{u\in B(0,R)}\|f(\cdot + u) - f(\cdot)\|_{LM_{p\theta,w}} +$$

$$+ \sup_{u\in B(0,R)}\|g(\cdot + u) - g(\cdot)\|_{LM_{p\theta,w}} + \left\|f\chi_{^cB(0,R)}\right\|_{LM_{p\theta,w}} + \left\|g\chi_{^cB(0,R)}\right\|_{LM_{p\theta,w}}.$$

Proof. Indeed,

$$\|M_r f - M_r g\|_{LM_{p\theta,w}} \le \left\|(M_r f - M_r g)\,\chi_{B(0,R)}\right\|_{LM_{p\theta,w}} + \left\|(M_r f - M_r g)\,\chi_{^cB(0,R)}\right\|_{LM_{p\theta,w}} = I_1 + I_2.$$

Next

$$I_1 = \sup_{x\in R^n}\left\|w(\rho)\,\|M_r f - M_r g\|_{L_p(B(x,\rho)\cap B(0,R))}\right\|_{L_\theta(0,\infty)} \le$$

$$\le \sup_{x\in R^n}\left\|w(\rho)\,\|M_r f - M_r g\|_{L_p(B(x,\rho)\cap B(0,R))}\right\|_{L_\theta(0,1)} +$$

$$+ \sup_{x\in R^n}\left\|w(\rho)\,\|M_r f - M_r g\|_{L_p(B(x,\rho)\cap B(0,R))}\right\|_{L_\theta(1,\infty)} \le$$

$$\le \|M_r f - M_r g\|_{C(\overline{B(0,R)})} \times$$

$$\times\left(\left\|w(\rho)\,(v_n\rho^n)^{\frac{1}{p}}\right\|_{L_\theta(0,1)} + \left\|w(\rho)\,(v_n R^n)^{\frac{1}{p}}\right\|_{L_\theta(1,\infty)}\right) \le$$

$$\le C\left(1 + R^{\frac{n}{p}}\right)\|M_r f - M_r g\|_{C(\overline{B(0,R)})},$$

where

$$C = v_n^{\frac{1}{p}}\left(\left\|w(\rho)\rho^{\frac{n}{p}}\right\|_{L_\theta(0,1)} + \|w(\rho)\|_{L_\theta(1,\infty)}\right) < \infty,$$

since $w \in \Omega_{p\theta}$.

By Lemma 1

$$I_2 \le \|M_r f - f\|_{LM_{p\theta,w}} + \left\|(f - g)\chi_{^cB(0,R)}\right\|_{LM_{p\theta,w}} + \|M_r g - g\|_{LM_{p\theta,w}} \le$$

$$\le \sup_{u\in B(0,r)}\|f(\cdot + u) - f(\cdot)\|_{LM_{p\theta,w}} + \sup_{u\in B(0,r)}\|g(\cdot + u) - g(\cdot)\|_{LM_{p\theta,w}} +$$

$$+ \left\|f\chi_{^cB(0,R)}\right\|_{LM_{p\theta,w}} + \left\|g\chi_{^cB(0,R)}\right\|_{LM_{p\theta,w}}.$$

Lemma 4 is proofed.

Lemma 5. Let $1 \le p, \theta \le \infty$, $w \in \Omega_{p\theta}$. Then for every $r, R > 0$and for every $f, g \in LM_{p\theta,w}$

$$\|f - g\|_{LM_{p\theta,w}} \le C\left(1 + R^{\frac{n}{p}}\right)\|M_r f - M_r g\|_{C(\overline{B(0,R)})} +$$

$$+2\sup_{u\in B(0,r)}\|f(\cdot + u) - f(\cdot)\|_{LM_{p\theta,w}} + 2\sup_{u\in B(0,r)}\|g(\cdot + u) - g(\cdot)\|_{LM_{p\theta,w}} +$$

$$+ \left\|f\chi_{^cB(0,R)}\right\|_{LM_{p\theta,w}} + \left\|g\chi_{^cB(0,R)}\right\|_{LM_{p\theta,w}},$$

where $C > 0$ is like in Lemma 4.

Proof. Enough to notice,that

$$\|f - g\|_{LM_{p\theta,w}} \le \|M_r f - f\|_{LM_{p\theta,w}} + \|M_r f - M_r g\|_{LM_{p\theta,w}} + \|M_r g - g\|_{LM_{p\theta,w}}$$

and is used Lemma 1 and 4

Proof of theorem. Let$S \subset LM_{p\theta}^{w(\cdot)}$ and let conditionals be done (5)–(7).

Step 1. Let $0 < r < r_0$, where r_0 defined in Lemma 3, and $R > 0$ fixed. By using inequality (7) and conditions (1), followed, next

$$\sup_{f\in S} \|M_r f\|_{C(\overline{B(0,R)})} < \infty.$$

Apart from (8) and conditions (6), followed, next

$$\lim_{u\to 0} \sup_{f\in S} \|M_r f(\cdot + u) - M_r f(\cdot)\|_{C(\overline{B(0,R)})} = 0.$$

Hence, by theorem Askoli-Arcellas' set $S_r = \{M_r f : f \in S\}$ pre-compactness in $C(\overline{B(0,R)})$, or, is the same,set S_r completely limited ,then for all $\varepsilon > 0$ exists $m \in \mathbb{N}$, $f_1, ..., f_m \in S$ (depending from ε, r and R) that, for all $f \in S$

$$\min_{j=1,\ldots,m} \|M_r f - M_r f_j\|_{C(\overline{B(0,R)})} < \varepsilon.$$

Step 2. Let $\{\varphi_1, ..., \varphi_m\}$ any bounded set S. By using inequality from lemma 5 for any $f \in S$ and for any $j = 1, ..., m$

$$\|f - \varphi_j\|_{LM^w_{p\theta}} \le C(1 + R^{\frac{n}{p}}) \|M_r f - M\varphi_j\|_{C(\overline{B(0,R)})} +$$

$$+2 \sup_{u\in B(0,r)} \|f(\cdot + u) - f(\cdot)\|_{LM^{w(\cdot)}_{p\theta}} + 2 \sup_{u\in B(0,r)} \|\varphi_j(\cdot + u) - \varphi_j(\cdot)\|_{LM^{w(\cdot)}_{p\theta}} +$$

$$+ \left\|f\chi_{^cB(0,R)}\right\|_{LM^{w(\cdot)}_{p\theta}} + \left\|\varphi_j\chi_{^cB(0,R)}\right\|_{LM^{w(\cdot)}_{p\theta}} \le$$

$$\le C(1 + \mathbb{R}^{\frac{n}{p}}) \|M_r f - M_r\varphi_j\|_{C(\overline{B(0,R)})} + 4 \sup_{u\in B(0,r)} \sup_{g\in S} \|g(\cdot + u) - g(\cdot)\|_{LM^w_{p\theta}} + 2 \sup_{g\in S} \left\|g\chi_{^cB(0,R)}\right\|_{LM^w_{p\theta}}.$$

Hence,

$$\min_{j=1,\ldots,m} \|f - \varphi_j\|_{LM^w_{p\theta}} \le C(1 + \mathbb{R}^{\frac{n}{p}}) \min_{j=1,\ldots,m} \|M_r f - M_r\varphi_j\|_{C(\overline{B(0,R)})} +$$

$$+4 \sup_{u\in B(0,r)} \sup_{g\in S} \|g(\cdot + u) - g(\cdot)\|_{LM^w_{p\theta}} + 2 \sup_{g\in S} \left\|g\chi_{^cB(0,R)}\right\|_{LM^w_{p\theta}}.$$

Step 3. Let $\varepsilon > 0$. The first , By using conditions (7), we take $R(\varepsilon) > 0$, that

$$\sup_{g\in S} \left\|g\chi_{^cB(0,R(\varepsilon))}\right\|_{LM^w_{p\theta}} < \frac{\varepsilon}{6}.$$

Next by using conditions (6), we take $r(\varepsilon)$, that

$$\sup_{u\in B(0,r(\varepsilon))} \sup_{g\in S} \|g(\cdot + u) - g(\cdot)\|_{LM^w_{p\theta}} < \frac{\varepsilon}{12}.$$

Because set pre-compactness $S_{r(\varepsilon)}$ в $C(\overline{B(0, R(\varepsilon))})$ exists $m(\varepsilon) \in \mathbb{N}$ and $f_{1,\varepsilon}, ..., f_{m(\varepsilon),\varepsilon} \in S$, than for any $f \in S$

$$\min_{j=1,\ldots,m(\varepsilon)} \left\|M_{r(\varepsilon)} f - M_{r(\varepsilon)} f_{j,\varepsilon}\right\|_{C(\overline{B(0,R(\varepsilon))})} < \frac{\varepsilon}{3C(1 + R(\varepsilon)^{\frac{n}{p}})}.$$

By using inequality (9) с $\varphi_j = f_{j,\varepsilon}, j = 1, ..., m(\varepsilon)$, for any $f \in S$

$$\min_{j=1,\ldots,m(\varepsilon)} \|f - f_{j,\varepsilon}\|_{LM^w_{p\theta}} < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

Then S pre-compactness set in $LM^w_{p\theta}$,the proofed theorem.

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Д.Т. Матин, Ж.Ж. Байтуякова, А.Н. Әділханов, Б.О. Бостанов

Локалды Морри типтес кеңістігінде жиынның компактылы болуының жеткілікті шарттары

Мақалада локалды Морри типтес кеңістігінде $LM_{p\theta}^{w(\cdot)}(\mathbb{R}^n)$ жиындардың компактылығының жеткілікті шарттары келтірілген. Берілген теоремадан $\theta = \infty$ жағдайында жалпыланған Морри кеңістігіндегі $M_p^{w(\cdot)}$ нәтиже шығады, ал $w(r) = r^{-\lambda}$, $\theta = \infty 0 \leq \lambda \leq \frac{n}{p}$ жағдайындайы үшін Морри кеңістігінің белгілі теоремасы шығады, ал $\lambda = 0$ жағдайында бұл белгілі Фреше-Колмогоров теоремасы. Морри кеңістігіндегі жиындардың компактты болуының шарттары — [1, 2], ал жалпыланған Морри кеңістігі $M_p^{w(\cdot)}(\mathbb{R}^n)$ үшін [3, 4] жұмыстарында дәлелденген. Берілген мақаланың мақсаты осы нәтижелерді локалды Морри кеңістігі $LM_{p\theta,w(\cdot)}(\mathbb{R}^n)$ үшін жиындардың компактты болуының шарттарын жалпылау болып табылады. Локалды Морри кеңістігіндегі жиындардың компактты болуының шарттарын пайдаланып, осы кеңістігіндегі операторлардың компактты болу шарттарын тексеруге болады. Себебі оператор бір кеңістіктегі шенелген жиынды келесі кеңістіктегі компактылы жиынға аударады. Авторлар локалды Морри типтес кеңістігінде жиындардың компактылығының жеткілікті шарттары функциялардың айырымы терминінде $\lim_{u\to 0}\sup_{f\in S}\|f(\cdot+u)-f(\cdot)\|_{LM_{p\theta,w}}=0$ келтірген. [5] жұмыста локалды Морри типтес кеңістігінде жиындардың компактылығының қажетті және жеткілікті шарттары алынып, олар функциялардың орта мәні терминінде $\lim_{u\to 0}\sup_{f\in S}\|f(\cdot+u)-f(\cdot)\|_{LM_{p\theta,w}}=0$ келтірілген.

Кілт сөздер: компактылық, компактылық алды, Фреше-Колмагоров теоремасы, локалды Морри типтес кеңістігі.

Д.Т. Матин, Ж.Ж. Байтуякова, А.Н. Адильханов, Б.О. Бостанов

Достаточные условия предкомпактности множеств в локальных пространствах типа Морри

В статье приведены достаточные условия предкомпактности множеств в локальных пространствах типа Морри $LM_{p\theta}^{w(\cdot)}(\mathbb{R}^n)$. Из доказанной теоремы в случае $\theta=\infty$ вытекает результат для обобщенного пространства $M_p^{w(\cdot)}$, а при $w(r)=r^{-\lambda},\ \theta=\infty,\ 0\le\lambda\le\frac{n}{p}$ — известный результат для пространства Морри $M_p^{\lambda}(\mathbb{R}^n)$, а в случае $\lambda=0$ – это хорошо известная теорема Фреше-Колмогорова. Условия предкомпактности множеств в пространствах Морри были доказаны в работах [1, 2], а в случае обобщенных пространств Морри $M_p^{w(\cdot)}(\mathbb{R}^n)$ — в [3, 4]. Цель статьи — обобщение результатов предкомпактности множеств для локальных пространств типа Морри $LM_{p\theta,w(\cdot)}(\mathbb{R}^n)$. Используя теорему предкомпактности множеств в локальных пространствах типа Морри, можно проверить условия компактности операторов в этих пространствах, так как компактный оператор переводит ограниченное множество одного пространства в компактное множество другого пространства. В этой работе условия предкомпактности множеств в локальных пространствах типа Морри даны в терминах разности функции $\lim_{u\to 0}\sup_{f\in S}\|f(\cdot+u)-f(\cdot)\|_{LM_{p\theta,w}}=0$. Ранее [5] были опубликованы необходимые и достаточные условия предкомпактности множеств в локальных пространствах типа Морри, которые были приведены в терминах средних функций $\lim\limits_{\delta\to 0^+}\sup\limits_{f\in S}\|A_\delta f-f\|_{L_p(B(0,R_2)\setminus B(0,R_1))}=0$.

Ключевые слова: компактность, предкомпактность, теорема Фреше-Колмагорова, локальные пространства типа Морри.

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