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Syntactic similarity of definable closures of Jonsson sets

In the framework of the classification of the Jonsson theories concept of interpretability and admissibility in the language of the semantic triple of the Jonsson theory was considered. A description of the syntactic and semantic similarity of perfect fragments of Jonsson subsets of the semantic model of the existential-prime convex Jonsson theory was obtained. Some model-theoretic properties for Jonsson's theories are considered. Such theories, as group theory, the theory of Abelian groups, the theory of Boolean algebra, the theory of ordered sets, the theory of polygons, and many others satisfies Jonsson's properties.

Keywords: Jonsson theory, perfect Jonsson theory, semantic model, Jonsson set, fragment of Jonsson set, syntactic and semantic similarity, existentially prime model.

The study of Jonsson theories is one of the interesting problems of the classical model theory. In the works [1, 2] you can find the main aspects of this type of research. One of the important concepts of model theory is the concept of definability (interpretability) of one algebraic system in another. It is said that the algebraic system $\mathfrak{B} =< B, R_i, i \in I >$ is definable on $\mathfrak{A} =< A, P_j, j \in J >$, if exists such formular relations $\Phi_i\ i \in I$ in language $\mathfrak{A}$ that $< A;\ \Phi_i,\ i \in I >$ is an isomorphic $< B;\ R_i,\ i \in I >$. In the course of the development of model theory, this notion was generalized, and the most general (at present) definition can be formulated as follows. If $\mathfrak{A}$ algebraic system, $n < \omega$, $B \subseteq A^n$, λ is the cardinal then B is called τ_λ-subset if exists such n-type $p(x_1, ..., x_n)$ over $\emptyset$ of language of system $\mathfrak{A}$, such $|p(x_1, ..., x_n)| < \lambda$ and B consists of all n of A^n, realising $p(x_1, ..., x_n)$ in $\mathfrak{A}$. Obviously, τ_λ-subsets are invariant relatively automorphisms. Therefore, τ_λ it can be considered as a way of isolating a certain class of invariant subsets of algebraic systems. If $\mathfrak{A}$, $\mathfrak{B}$ - algebraic systems $G = Aut($ A), then we say that $\mathfrak{B}$ is τ_λ – interpreted in $\mathfrak{A}$, if $\mathfrak{B}$, is τ_λ – interpreted in pure pair (A, G). If $\lambda = \omega$ then usually instead of τ_λ – interpretability says formally (or elementarily) interpreted (definable). The problem of interpretability can be considered through other similar concepts, for example, syntactic and semantic similarity.

Definition 1. A theory T is called Jonsson if:
1) the theory T has an infinite model;
2) the theory T is inductive;
3) the theory T has the joint embedding property JEP;
4) the theory T has the amalgamation property AP.

Definition 2. The set X is said to be Jonsson in theory T, if it satisfied to following properties:
1) X is definable subset of C, where C is semantic model of theory T;
2) $dcl(X)$ is support of some existentially closed submodel C, where $dcl(X)$ is definable closure of X.

Definition 3. We say that all $\forall\exists$ consequences of an arbitrary theory form the Jonsson fragment if the deductive closure of these $\forall\exists$ consequences is the Jonsson theory.

Let T is an arbitrary Jonsson theory, then $E(T) = \cup_{n<\omega} E_n(T)$, where $E_n(T)$ is a lattice of $\exists$ formulas with n free variables, T^* is a center of Jonsson theory T, i.e. $T^* = Th(C)$, where C is semantic model of Jonsson theory T in the sense of [3].

Definition 4 [4]. Let T_1 and T_2 are Jonsson theories. We say, that T_1 and T_2 are Jonsson's syntactically similar, if exists a bijection $f : E(T_1) \longrightarrow E(T_2)$ such that:
1) restriction f to $E_n(T_1)$ is an isomorphism of lattices $E_n(T_1)$ and $E_n(T_2)$, $n < \omega$;
2) $f(\exists v_{n+1}\varphi) = \exists v_{n+1} f(\varphi)$, $\varphi \in E_{n+1}(T)$, $n < \omega$;
3) $f(v_1 = v_2) = (v_1 = v_2)$.

We would like to give some examples of syntactic similarity of certain algebraic examples. For this, we recall the basic definitions associated with these examples following denotions from B. Poizat [5].

A Boolean ring is an associative ring with identity, in which $x^2 = x$ for any x is called a Boolean ring; then, we have also $(\mathrm{x + y})^2 = x^2 + xy + yx + y^2 = x + xy + yx + y$ and besides $(x + y)^2 = x + y$; therefore

$xy + yx = 0$ for an arbitrary x, y; $x^2 + x^2 = 0$ means, $x + x = 0$, for any x or $x = -x$; hence the Boolean ring has characteristic 2 and, since $xy = -yx = yx$, it is commutative.

To axiomatize this concept, we introduce a language containing two symbols of constants 0 and 1, two symbols of binary relations + and ·

We write down some universal axioms, expressing, that A is the Boolean ring, without forgetting thus $0 \neq 1$. In the Boolean ring we will define two binary operations $\wedge$ and $\vee$, and unary operation $\neg$ as follows: $x \wedge y = x \cdot y$; $x \vee y = x + y + xy$; $\neg x = 1 + x$.

It is easy to verify, that the following are true for all x, y and z:

– (de Morgan's laws or duality): $\neg(\neg x) = x$, $\neg(x \wedge y) = \neg x \vee \neg y \neg(x \vee y) = \neg x \wedge \neg y$;

– $x \vee x = x \wedge x = x$;

– (associativity $\wedge$): $(x \wedge y) \wedge z = x \wedge (y \wedge z)$;

– (associativity $\vee$): $(x \vee y) \vee z = x \vee (y \vee z)$;

– (distributivity $\wedge$ over $\vee$):$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$;

– (distributivity $\vee$ over $\wedge$): $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$;

– (commutativity$\wedge$ over $\vee$): $x \wedge y = y \wedge x$,$x \vee y = y \vee x$;

– $x \wedge \neg x = 0, x \vee \neg x = 1$;

– $x \wedge 0 = 0, x \vee 0 = \text{x}, x \wedge 1 = x, x \vee 1 = 1$;

– $0 \neq 1, \neg 0 = 1, \neg 1 = 0$.

A structure in language $0, 1, \neg, \wedge, \vee$ satisfying to these universal axioms is called a Boolean algebra.

Fact 1 [5]. In each Boolean ring one can interpret a certain Boolean algebra.

Proof. With the Boolean ring A we have connected some Boolean algebra $b(A)$; the converse is also true: $x \cdot y = x \wedge y$, $x + y = (x \vee y) \wedge (\neg x \vee \neg y)$, then we receive the Boolean ring $a(B)$; and besides $a(b(A)) = A$, $b(a(B)) = B$. Thus we see, that up to a language, the Boolean ring and Boolean algebras have the same structures, the Boolean ring canonically is transformed into a Boolean algebra and vice versa, transformations in both directions are carried out using quantifier free formulas.

The following example connects Boolean algebras with Abelian groups. In work [6], conditions were found for the cosemanticness of Abelian groups.

Fact 2 [7]. In each Boolean algebra one can interpret an Abelian group.

Proof. In Boolean algebra A we suppose $a + b = (a \wedge b') \vee (a' \wedge b)$.

$[A, +]$ is Abelian group, in which each not unit element has an order 2.

The element 0 is group unit in G, and each element x is reciprocal to itself: $x + x = 0$ for all x $\in$ A.

We state the obtained results.

Let's denote through T_{BA}, T_{BR}, T_{AG} accordingly theories in their signatures (they are different) of Boolean algebras, Boolean rings and Abelian groups.

Lemma 1. T_{BA}, T_{BR}, T_{AG} are examples of Jonsson theories.

Proof. T_{BA} and T_{BR} from [4], T_{AG} from [6].

Theorem 1. Theories T_{BA} and T_{BR} are syntactically similar, and mutually interpreted among themselves, as for complete theories and for Jonsson theories.

Proof. Follows from the fact 1.

Theorem 2. Theory T_{BA} is interpreted in theory T_{AG}, as for complete theories and for the Jonsson theories.

Proof. Follows from Fact 2 and Theorem 1.

Let L be a countable first-order language and T is some inductive theory in this language, E_T and AP_T are denoting correspondingly the following classes of this theory: class of all existentially closed models and class of all algebraically prime models.

Definition 5. The inductive theory T called existential-prime (EP), if it has a algebraically prime model and $AP_T \cap E_T \neq \emptyset$.

Definition 6. The theory T is called convex (C) if for any model $\mathfrak{A}$ and any family $\{\mathfrak{B}_i | i \in I\}$ of its substructures, which are models of the theory T, the intersection $\bigcap_{i \in I} \mathfrak{B}_i$ is a model theory T. It is assumed that this intersection is not empty. If this intersection is never empty, then the theory is called the strongly convex (SC).

An inductive theory is called an existentially prime strongly convex theory if it satisfies the above definitions simultaneously and is denoted by $EPSC$.

Let X be the Jonsson set in the theory T and M is existentially closed submodel semantic model $\mathfrak{C}$, where $dcl(X) = M$. Then let $Th_{\forall\exists}(M) = Fr(X)$, $Fr(X)$ is Jonsson fragment of Jonsson set X. Let A_1 and A_2

are Jonsson subset of the semantic model the some of Jonsson $EPSC$-theory. Where $Fr(A_1)$ and $Fr(A_2)$ are fragments of Jonsson sets A_1 and A_2.

In the work [8] was obtained the result on syntactically similarity in the frame of EPPCJ theories in some enrichment. The class of EPPCJ theories is the subclass of class of all Jonsson theories. Now we are considering the following result for Jonsson theoreis without any enrichment. We have the following result.

Let T be $\exists$ -complete perfect Jonsson theory and $Fr(A_1)$ and $Fr(A_2)$ be fragments of A_1 and A_2 correspondingly, where A_1 and A_2 are Jonsson subset of the semantic model for theory T.

Even if given theory is $\exists$ -complete perfect Jonsson theory, its fragments can be not perfect.So we will be demand perfectness for fragments of the following theorem.

Theorem 3. Let $Fr(A_1)$ and $Fr(A_2)$ are $\exists$-complete perfect Jonsson theories. Then following conditions are equivalent:

1) $Fr(A_1)$ and $Fr(A_2)$ are J-syntactically similar as Jonsson theories [9];

2) $(Fr(A_1))^*$ and $(Fr(A_2))^*$ are syntactically similar as the complete theories [9], where $(Fr(A_1))^*$ and $(Fr(A_2))^*$ respectively be the centers of fragments of considered sets A_1, A_2.

Proof. We also need the following facts.

Fact 3 [10]. For any complete for the existential sentences Jonsson's theory T the following conditions are equivalent:

1) T is perfect;

2) T^* model-complete.

Fact 4 [10]. For any complete for the existential sentences Jonsson's theory T the following conditions are equivalent: are equivalent:

1) T^* model-complete;

2) For each $n < \omega$, $E_n(T)$ is a Boolean algebra, where $E_n(T)$ is a lattice of existential formulas with n free variables.

We note that by the perfectness of $Fr(A_1)$ and $Fr(A_2)$ implies that $(Fr(A_1))^*$ and $(Fr(A_2))^*$ are J Jonsson's theory.

We will show 1) $\Rightarrow$ 2). We have that for every $n < \omega$, $E_n(Fr(A_1))$ is an isomorphic to $E_n(Fr(A_2))$. Let this is an isomorphism of f_{1n}.By the hypothesis of the theorem and facts 3, 4 for every $n < \omega$, $E_n(Fr(A_1))$ and $E_n(Fr(A_2))$ are a Boolean algebras. But due to the perfection $Fr(A_1)$ and $Fr(A_2)$ follow that $(Fr(A_1))^*$ and $(Fr(A_2))^*$ are model-complete by virtue of fact 3, and so for each $n < \omega$, for any formula $\varphi(\bar{x})$ of $F_n((Fr(A_1))^*)$ by Corollary 1 there is a formula $\psi(\bar{x})$ of $E_n((Fr(A_1))^*)$ so that in $(Fr(A_1))^* \models \varphi \leftrightarrow \psi$. Because the theory of $Fr(A_1)$ is complete for existential sentences and $E_n(Fr(A_1)) \subseteq E_n((Fr(A_1))^*)$ (as $Fr(A_1) \subseteq (Fr(A_1))^*$), follow that $E_n(Fr(A_1)) = E_n((Fr(A_1))^*)$. Due to the fact that theory of $Fr(A_2)$ is complete for existential proposals and $E_n(Fr(A_2)) \subseteq E_n((Fr(A_2))^*)$ (as $Fr(A_2) \subseteq (Fr(A_2))^*$), follow that $E_n(Fr(A_2)) = E_n((Fr(A_2))^*)$. For each $n < \omega$, for each $\varphi_1(\bar{x})$ of $F_n((Fr(A_1))^*)$, we define the following mapping between the $F_n((Fr(A_1))^*)$ and $F_n((Fr(A_2))^*)$: $f_{2n}(\varphi_1(\bar{x})) = f_{1n}(\psi_1(\bar{x}))$, where $(Fr(A_1))^* \models \psi_1 \leftrightarrow \varphi_1$, $\psi_1 \in E_n(Fr(A_1))$. It is easy to understand, that by virtue of properties of f_{1n} and above what has been said, f_{2n} is a bijection, an isomorphism between $F_n((Fr(A_1))^*)$ and $F_n((Fr(A_2))^*)$. Consequently, $(Fr(A_1))^*$ and $(Fr(A_2))^*$ are syntactically similar (in the sense of [6]).

We show 2) $\Rightarrow$ 1). Is trivial, since $F_n((Fr(A_1))^*)$ an isomorphic to $F_n((Fr(A_2))^*)$ for each $n < \omega$, and by the hypothesis of the theorem and the facts 3, 4 this an isomorphism extends to all subalgebras.

All concepts that are not defined in this article can be extracted from [4].

Lemma 2. Any two cosemantic Jonsson's theories are J — semantically similar.

Proof. Follows from the definitions.

Lemma 3. If two perfect $\exists$ — complete of Jonsson's theories are J — syntactically similar, then they are J — semantically similar.

Proof. It follows from [9, Prop. 1] and above what was said.

Definition 7. A property (or a notion) of theories (or models, or elements of models) is called semantic if and if it is invariant relative to semantic similarity.

Let us recall the definition of polygon.

Definition 8. By polygon over monoid S we mean a structure with only unary functions $\langle A; f_{\alpha:\alpha \in S}\rangle$ such that:

(i) $f_e(a) = a, \forall a \in A$, where e is the unit of S;

(ii)$f_{\alpha\beta}(a) = f_\alpha(f_\beta(a)), \forall \alpha, \beta \in S, \forall a \in A$.

And now we can formulate the main result of this job.

Theorem 4. For each $\exists$ - complete perfect Jonsson J theory there exists a J syntactically similar $\exists$ - complete perfect Jonsson's J theory of polygons, such that its center is model complete.

Proof. It follows from theorems [11, Th.7, Th.8] and [9, Th.4, Th.5].

As we know from [9] the following Proposition 1 is true for any complete theory, so we will be intrested such properties from this Proposition 1 in the frame of Jonsson theories and to research the notion of semantic similarity of Jonsson's theories. Recall the content of the Proposition 1.

Proposition 1 [9]. The following properties and notions are semantic:

(1) type;

(2) forking;

(3) λ-stability;

(4) Lascar rank;

(5) Strong type;

(6) Morley sequence;

(7) Orthogonality, regularity of types;

(8) $I(\aleph_\alpha, T)$ — the spectrum function.

Finally we can note that all above properties from Proposition 1 will be semantic also in the frame of Jonsson theory. The proof trivial follows from Proposition 1 using Jonsson analogues of Proposition' main notions.

References

1 Yeshkeyev A.R. On Jonsson stability and some of its generalizations / A.R. Yeshkeyev // Journal of Mathematical Sciences. — 2010. – Vol. 166. — No 5. — P. 646–654.

2 Yeshkeyev A.R. The structure of lattices of positive existential formulae of (Δ - PJ)-theories / A.R. Yeshkeyev // Scienceasia. — 2013. — Vol. 39. — P. 19–24 .

3 Мустафин Т.Г. Введение в прикладную теорию моделей / Т.Г. Мустафин, Т.А. Нурмагамбетов. — Караганда: Изд-во КарГУ, 1987. — 94 с.

4 Ешкеев А.Р. Йонсоновские теории и их классы моделей: монография / А.Р. Ешкеев, М.Т. Касыметова. — Караганда: Изд-во КарГУ, 2016. — 370 с.

5 Poizat B. A Course in Model Theory / B.Poizat. — Springer-Verlag New York, lnc., 2000. — P. 445.

6 Ешкеев А.Р. JSp-косемантичность и JSB-свойство абелевых групп [Электронный ресурс] / А.Р. Ешкеев, О.И. Ульбрихт // Siberian Electronic Mathematical Reports — 2016. — Вып. 13. — С. 861–874. — Режим доступа: http://semr.math.nsc.ru.

7 Биркгоф Г. Современная прикладная алгебра / Г.Биркгоф, Т.Барти. — М.: Мир, 1976. — С. 400.

8 Yeshkeyev A.R. The properties of central types with respect to enrichment by Jonsson set / A.R. Yeshkeyev // Bulletin of the Karaganda University. Mathematics Series. — 2017. — No. 1(85). — P. 36–40.

9 Mustafin T.G. On similarities of complete theories / T.G.Mustafin // Logic Colloquium '90. Proceedings of the Annual European Summer Meeting of the Association for Symbolic Logic. — Helsinki, 1990. — P. 259–265.

10 Yeshkeyev A.R. Jonsson's Theories / A.R. Yeshkeyev. — Karaganda: Izdatelstvo KarHU, 2009.

11 Yeshkeyev A.R. The Properties of Positive Jonsson's Theories and Their Models / A.R. Yeshkeyev // International Journal of Mathematics and Computation. — 2014. — Vol. 22.1. — P. 161–171.

Г.А. Уркен

Йонсондық жиындардың анықталған тұйықтамалардың синтаксистік ұқсастылығы

Йонсондық теорияның аясында интерпретациялау және рұқсатылық ұғымы йонсондық теорияның семантикалық үштік тілінде қарастырылды. Экзистенционалды-жай дөңес йонсондық теорияның семантикалық моделінің кемел йонсондық ішкі жиындар фрагментінің синтаксистік және семантикалық ұқсастылық сипаттамасы алынды. Йонсондық теорияның кейбір теория-модельдік қасиеттері зерттелді. Йонсондық теорияның қасиеттерді группалар теориясы абелдік группалар теориясы, бульдік алгебра теориясы, реттелген группалар теориясы, полигондар теориясы және тағы басқа теориялар қанағаттандырады.

Кілт сөздер: йонсондық теория, кемел йонсондық теория, семантикалық модель, йонсондық жиын, йонсондық жиынның фрагменті, семантикалық және синтаксистік ұқсастылық, экзистенционалдыжай модель.

Г.А. Уркен

Синтаксическое подобие определимых замыканий йонсоновских множеств

В рамках классификации йонсоновских теорий рассмотрено понятие интерпретируемости и допустимости на языке семантической тройки йонсоновской теории. Получено описание синтаксического и семантического подобий совершенных фрагментов йонсоновских подмножеств семантической модели экзистенциально-простой выпуклой йонсоновской теории. Рассмотрены некоторые теоретикомодельные свойства для йонсоновских теорий. Йонсоновским свойствам удовлетворяют такие теории, как теория групп, теория абелевых групп, теория булевой алгебры, теория упорядоченных множеств, теория полигонов и другие.

Ключевые слова: йонсоновская теория, совершенная йонсоновская теория, семантическая модель, йонсоновское множество, фрагмент йонсоновского множества, семантическое и синтаксическое подобие, экзистенциально-простая модель.

References

1 Yeshkeyev, A.R. (2010). On Jonsson stability and some of its generalizations. *Journal of Mathematical Sciences, Vol. 166,* 5, 646–654.

2 Yeshkeyev, A.R. (2013). The structure of lattices of positive existential formulae of (Δ-PJ)-theories. *Scienceasia, Vol. 39,* 19–24.

3 Mustafin, T.G., & Nurmagambetov, T.A. (1987). Vvedeniie v prikladnuiu teoriiu modelei [Introduction to applied Model Theory]. Karaganda: Izdatelstvo KarHU [in Russian].

4 Yeshkeyev, A.R., & Kasymetova, M.T. (2016). *Ionsonovskie teorii i ikh klassy modelei [Jonsson theories and their classes of models].* Karaganda: Izdatelstvo KarHU [in Russian].

5 Poizat, B. (2000). Course in Model Theory. Springer-Verlag New York, lnc.

6 Yeshkeyev, A.R., & Ulbricht, O.I. (2016). *JSp*-kosemantichnost i JSB-svoistvo abelevykh hrupp [*JSp*-kosemanticness and JSB-property of Abelian groups]. *Siberian Electronic Mathematical Reports, Vol. 13,* 861–874. Retrieved from http://semr.math.nsc.ru [in Russian].

7 Birkgof, G., & Barti, T. (1976). *Sovremennaia prikladnaia alhebra [Modern Applied Algebra].* Moscow: Mir [in Russian].

8 Yeshkeyev, A.R. (2017).The properties of central types with respect to enrichment by Jonsson set. *Bulletin of the Karaganda University. Mathematics Series, 1(85),* 36–40.

9 Mustafin, T.G. (1990). On similarities of complete theories. *Logic Colloquium '90. Proceedings of the Annual European Summer Meeting of the Association for Symbolic Logic.* Helsinki.

10 Yeshkeyev, A.R. (2009). Jonsson's Theories. Karaganda: Izdatelstvo KarHU.

11 Yeshkeyev, A.R. (2014). The Properties of Positive Jonsson's Theories and Their Models. *International Journal of Mathematics and Computation, Vol. 22.1,* 161–171.