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Criterion for the cosemanticness of the Abelian groups in the enriched signature

In the present paper we give a criterion of the cosemanticness relative to the Jonsson spectrum of the model in the class of Abelian groups with a distinguished predicate. This paper is devoted to the study of model-theoretic questions of Abelian groups in the frame of the study of Jonsson theories. Indeed, the paper shows that Abelian groups with the additional condition of the distinguished predicate satisfy conditions of Jonssonness and also the perfectness in the sense of Jonsson theory. It is well known that classical examples from algebra such as fields of fixed characteristic, groups, abelian groups, different classes of rings, Boolean algebras, polygons are examples of algebras whose theories satisfy conditions of Jonssonness. The study of the model-theoretic properties of Jonsson theories in the class of abelian groups is a very urgent problem both in the Model Theory itself and in an universal algebra. The Jonsson theories form a rather wide subclass of the class of all inductive theories. But considered Jonsson theories in general are not complete. The classical Model Theory mainly deals with complete theories and in case of the study of Jonsson theories, there is a deficit of a technical apparatus, which at the present time is developed for studying the model-theoretic properties of complete theories. Therefore, the finding of analogues of such technique for the study of Jonsson theories has practical significance in thegiven research topic. In this paper the signature for one-place predicate was extended. The elements realizing this predicate form an existentially closed submodel of the considering Jonsson theory's some model. In the final analysis, we obtain the main result of this article as a refinement of the well-known W. Szmielew's theorem on the elementary classification of Abelian groups in the frame of the study of Jonsson theories, thereby the generalization of the well-known question of elementary pairs for complete theories was obtained. Also we obtained the Jonsson analogue for the joint embeddability of two models, or in another way the Schröder-Bernstein properties in the frame of the study of the Johnson pairs of Abelian groups' theory.

Keywords: Jonsson theory, model companion, existentially closed model, perfectness, cosemanticness, Jonsson spectrum, Jonsson pair.

This paper is concerned with the study of certain model-theoretic properties of Jonsson theories in the class of Abelian groups. The class of Jonsson theories is wide enough and it is a natural subclass of the class of all inductive theories. The definition of the Jonsson theory is quite natural. Many theories of a well-known and classical algebras are essentially examples of Jonsson theories. Examples include the following algebras: groups, Abelian groups, fields of fixed characteristic, Boolean algebras, many classes of rings, polygons. As a rule, the considered Jonsson theories are not complete and since the classical Model Theory deals mainly with complete theories, in the case of the Jonsson theories there is a deficit of the technical apparatus, which is accordingly developed at the present time for studying the model-theoretic properties of complete theories. Therefore, the finding of analogues of this technique and accordingly, concepts for the study of Jonsson theories, is a very urgent problem.

In the paper [1] was considered the problem related to the concept of cosemantic and Schröder-Bernstein properties for Jonsson theory of Abelian groups. The concept of cosemanticness is a generalization of the concept of elementary equivalence, which is used as an important tool in the study of complete theories. The Schröder-Bernstein property is also related to models of some fixed complete theory. That is, like the property of cosemanticness, this property is a semantic concept, in contrast to the properties of the theory, which we attribute syntactic properties to. By virtue of the theorem about completeness, the duality of syntax and semantics allows us to seek new connections between the model-theoretic properties of theories and their classes of models. In the Jonsson case, because of the incompleteness, it is impossible to directly use this duality, and in this case we resort to the so-called semantic method, the essence of which is the «transfer» of elementary properties of the first order of the elements of Jonsson theory's center to the theory itself. It turned out that the theory of Abelian groups is an example of the perfect Jonsson theory. In this connection, it was possible to find a criterion for the cosemanticness of Abelian groups [1] and Jonsson analog of the Schröder-Bernstein property.

We note how the study of Jonsson theories differs fundamentally from the study of complete theories. The classification of a fixed complete theory and its class of models with respect to certain syntactic and semantic conditions is one of the most important tasks of the classical Model Theory. In Model Theory itself, as noted in the review article by H.J. Keisler, «Foundations of Model Theory» in the reference book, ed. J. Barwise [2], historically there were two directions. In [2] they are called «western» and «eastern» Model Theory, these names are conditional, they are related to the geographical location of the founders of the Model Theory. A. Robinson lived on the east coast of the USA and A. Tarski lived on the western coast.

«Western» Model Theory develops in the traditions of Skolem and Tarski. It was more motivated by problems in number theory, analysis and set theory, and it uses all formulas of first-order logic. In particular, various types of elementary morphisms are considered as morphisms in the «western» theory of models. «Eastern» Model Theory develops in the traditions of Maltsev and Robinson. It was motivated by problems in abstract algebra, where the formulas of theories usually have at most two blocks of quantifiers. It emphasizes the set of quantifier-free formulas and existential formulas. In the «eastern» Model Theory, as a rule, homomorphisms and isomorphisms are considered as morphisms.

Thus, we can see that when dealing with the model-theoretic attributes of the «eastern» Model Theory, we tend to deal with incomplete theories and morphisms between their models, which maximally preserve the properties of Boolean combinations of atomic formulas. As a model at the study of this type of theories, as a rule, we consider a subclass of class of all models of the considering theory, namely, the class of its existentially closed models.

In this paper we extend the signature by a one-place predicate, the essence of which is that the elements realizing this predicate form an existentially closed submodel of some model of considering Jonsson theory. Thus, we can say that we turn to the situation when the considering problem generalizes the problem of elementary pairs.

The Jonsson theories satisfy natural conditions, such as inductive, the joint embedding property and amalgam [2 (def. 6.1, p. 80)]. T. G. Mustafin in the work [3] generalized Jonsson theories and found a connection between the complete theories, Jonsson theories and the generalized Jonsson theories. In the work [4] Yeshkeyev A.R. was continued the study of Jonsson theories concerning the various model-theoretic properties of their companions, including J-stability. In particular, in the frame of the study of Jonsson theories was redefined an important notion such as forcing, which was earlier defined by S. Shelah [5] and is one of the main tools of modern technique of Model Theory in the classification of complete theories. Further A.R.Yeshkeyev were defined new classes of positive Jonsson theories and in the paper [6] were obtained positive Jonsson analogues of F. Weispfenning's work [7] for the positive lattice of the existential formulas of considered theory. The concept of positive Jonsson theories was first considered in the paper [8] and this concept, in a certain sense, was introduced after the appearance of series of I. Ben-Yaakov's works [9-10], as both concepts of theory's positivity from [8] and [9] coincide for the minimal fragment of considered theory. This implies, in particular, non-triviality (not just a generalization of the Jonssonness for generalization) of the concept of positivity in the sense of [8], because, for example, such an important class of mathematical structures as metric spaces is not Jonsson class, but is positive Jonsson in the sense of the works [9-10] and, in particular, in the sense of [8] for the minimal fragment. It should be noted that there are various regular ways of transition from an arbitrary theory to Jonsson theory, which preserves the original class of existentially closed models. One of these methods is the Morleisation of theory [2 (Theorems 2.18, 2.19, p. 63-64, Theorem 6.8′, p. 83)]. Thus, the study of model-theoretic properties of Jonsson theories is an actual problem, both in Model Theory and in universal algebra, and the questions concerning the study of Jonsson theories are exactly the essence of the problem of «eastern» Model Theory.

The study of the model-theoretic properties of complete theories of Abelian groups is a large subsection of model-theoretic algebra. Many classical results have been obtained in this field of research, in particular, the complete classification of Abelian groups up to elementary equivalence was carried out in the work of Polish mathematician W. Szmielew [11].

The following references to the relevant sources will allow the reader to obtain exhaustive information on this classification [11-15].

The concept of an elementary pair was first determined by B. Poizat in [16]. The following stages in the development of study of this concept were noted in works of E. Bouscaren and B. Poizat [17-19]. Further, the history of studying this concept is related to the work [20]. T.G. Mustafin in this paper considered new concepts of stability and showed that one of them in the particular case is the case of an elementary pair. Also to this period are the works of T. Nurmagambetov and B. Poizat [21]. In the future there are papers by E.A. Palyutin [22, 23]. In these papers E.A. Palyutin clarifies the concept of T^*-stability introduced by T.G. Mustafin in [20] with the help of concept of E^*-stability, and in a fairly wide class of primitive-normal theories consider the model-theoretic properties of elementary pairs. Recall that all these works were done in the frame of study of complete theories. It is clear that the transition to generally speaking incomplete theories on the example of Jonsson theories and, in particular, the theory of Abelian groups, is uniquely an actual continuation of above studies.

All the indefinite concepts and results associated with them in this article on Jonsson theories can be found in [24].

In brief we give the main generally accepted notation associated with Abelian groups.

If A is an arbitrary Abelian group, n is an integer, then $nA = \{na : a \in A\}$. It is not difficult to see that both nA and $A[p] = \{a \in A : pa = 0\}$ for a simple p form subgroups of the group A. We say that $n \neq 0$ divides the element a from A if $a = nb$ for some $b \in B$. If there exists $m > 0$ such that $ma = 0$, then the smallest such m is called the order of the element a. Thus, nA is the set of elements of A that are divisible by n and $A[p]$ is the set of all elements of A of order p. The subgroup $(nA)[p]$ is usually denoted by $nA[p]$. The set $T(A)$ of all elements of A of finite order is called the periodic part of A. It is clear that $T(A)$ is a subgroup of A and the factor group $A/_{T(A)}$ is torsion-free, that is group that does not have nonzero elements of finite order. If every element of A has order equal to p^n for some $n \geqslant 1$, then the periodic group A is called a p-group. A group A is called a group of bounded order if $nA = [0]$ for some natural n.

A group A is said to be divisible if for any $a \in A$ and any $n \in \mathbb{Z}\backslash\{0\}$ there exists $b \in A$ such that $a = nb$. If B is a divisible group and at the same time a subgroup of A, then it is called a divisible subgroup of A. A group that does not contain non-zero divisible subgroups is said to be reduced. A subgroup B of a group A is said to be servant if $nA \cap B = nB$ for all $n \in \mathbb{Z}$. We say that a subgroup B of a group A is distinguished in it by a direct summand if there exists a subgroup C of the group A for which $A = B \oplus C$.

A group A is said to be algebraically compact if it is distinguished by a direct summand in every group that contains A as a pure subgroup. These groups have a number of interesting properties. For example, the group A is algebraically compact $\Leftrightarrow$ in it any compatible countable set of equations from any number of unknowns with constants in A is solvable. It is easy to see that ω_1^+-saturated groups are always algebraically compact. The structure of algebraically compact groups is well studied. The following examples of Abelian groups are canonical in the study of their elementary theories.

1. Q – the additive group of rational numbers, called the complete rational group.
2. $Z_p = \left\{\frac{m}{n} : m, n \in Z, (n,p) = 1\right\}$.
3. Z_{p^n} – cyclic group of order p^n.
4. Z_{p^∞} – the multiplicative group of all the roots of equations $x^{p^n} = 1$, $n = 1,\ 2,\ ...$, from the field of complex numbers, called a quasicyclic group of type p^∞, where p is prime number.

Remark. The group Z_{p^∞} can be defined as the additive group generated by elements $c_1,\ c_2,\ ...,\ c_n, ...$, where $pc_1 = 0$, $pc_2 = c_1$, ..., $pc_{n+1} = c_n$.

Let A be a model of signature of Abelian groups, where $\sigma_{\text{AG}} = \langle +, -, 0\rangle$.

A formula of the form $\exists x_1 ... \exists x_n \varphi$, where φ is the conjunction of atomic formulas, is called positively primitive (p.p. formula). P.p. formulas express the solvability of finite systems of linear equations of the form $m_1x_1 + m_2x_2 + ... + m_kx_k = 0$. It is not difficult to show that p.p. formulas are closed with respect to the conjunction and suspension of the existence quantifier. One can directly verify that the truth of p.p. formulas is preserved under extensions, cartesian products and homomorphisms of Abelian groups.

The following facts are well known.

Theorem 1. Let A be an arbitrary Abelian group. Each formula of signature σ_{AG} is equivalent regarding $Th(A)$ of the Boolean combination of p.p. formulas.

Sentence 1. Q and Z_{p^∞} are divisible groups.

Theorem 2. Every divisible periodic Abelian group G is a direct sum of quasicyclic groups (possibly on different prime numbers).

It is known that any group A can be decomposed as follows:

$$A = A_d \oplus A_r,$$

where A_d is the single maximal divisible subgroup of A, A_r is a reduced subgroup, i.e. group without non-zero divisible subgroups. Algebraically compact groups are constructed in a certain way from the indecomposable groups Z_{p^∞}, Z_{p^n}, Z and Q, where p is a prime number.

The divisible group A_d has the following decomposition:

$$A_d \cong \oplus_p Z_{p^\infty}^{(\gamma_p)} \oplus Q^{(\delta)},$$

where γ_p (p is a prime number) and δ are arbitrary cardinal numbers, $\oplus_p$ means a direct summation over all simple p, $Z_{p^\infty}^{(\gamma_p)}$ is the direct sum of γ_p -copies of quasicyclic groups Z_{p^∞}, and $Q^{(\delta)}$ is the direct sum of δ-copies of the additive group of rational numbers.

Recall that two models $\mathfrak{A}$ and $\mathfrak{B}$ of the same language $\mathfrak{L}$ of the first order are called *elementarily equivalent* if the same the first-order sentence of language $\mathfrak{L}$ are realized in the above models.

We define the Szmielew's invariants. In future we assume that $\infty < \kappa$ for any cardinal $\kappa \geq \omega$.

For any abelian group A and any simple p:

- $U(p,n;A) = \min\left\{\omega,\ \dim_p \left(p^n A[p] \big/ p^{n+1} A[p] \right)\right\}$;
- $T_f(p;A) = \min\left\{\omega,\ \inf_n \dim_p \left(p^n A \big/ p^{n+1} A \right)\right\}$;
- $D(p;A) = \min\left\{\omega,\ \inf_n \dim_p \left(p^n A[p] \right)\right\}$;
- $Exp(A) = \begin{cases} 0, & \text{if the group } A \text{ is of bounded order;} \\ \infty, & \text{otherwise,} \end{cases}$

where $\dim_p$ is the dimension of the corresponding vector space over the field $Z\big/pZ$.

This is the elementary Szmielew's invariants.

We denote by $W(A)$ the ordered sequence of elementary Szmielew's invariants of group A:

$$W(A) = \langle\langle\langle U(p,n;A) : n \in \omega\rangle,\ T_f(p;A),\ D(p;A) : p = 2,3,...\rangle,\ Exp(A)\rangle\,.$$

Theorem 3. Let A, B be two arbitrary groups. Then the following conditions are equivalent:

1. $A \equiv B$.
2. W(A)=W(B).

We define the standard group of Szmielew A^0 for any group A:

$$A^0 = \oplus_{p,n} Z_{p^n}^{(\alpha^0_{p_n})} \oplus \oplus_p Z_p^{(\beta^0_p)} \oplus \oplus_p Z_{p^\infty}^{(\gamma^0_p)} \oplus Q^{(\delta^0)},$$

where $\alpha^0_{p_n} = \min\left(U(p,n-1;A),\omega\right)$, $\beta^0_p = \min\left(T_f(p;A),\omega\right)$, $\gamma^0_p = \min\left(D(p;A),\omega\right)$, $\delta^0 = \min\left(Exp(A),1\right)$.

Theorem 4. $A \equiv A^0$.

Theorem 5. Any Abelian group can be embedded as a subgroup of a divisible group.

We give some well-known definitions of concepts and results related to the Jonsson theories, which are necessary for studying Abelian groups in the frameof the Jonssonness.

Definition 1. The theory T is called Jonsson if:

1) T has an infinite model;

2) T is inductive, i.e. T is equivalent to the set $\forall\exists$-propositions;

3) T has the joint embedding property (JEP), that is, any two models $\mathfrak{A} \models T$ and $\mathfrak{B} \models T$ are isomorphically embedded in a certain model $\mathfrak{C} \models T$;

4) T has the property of amalgamation (AP), that is, if for any $\mathfrak{A}, \mathfrak{B}, \mathfrak{C} \models T$ such that $f_1 : \mathfrak{A} \to \mathfrak{B}$, $f_2 : \mathfrak{A} \to \mathfrak{C}$ are isomorphic embeddings, exist $\mathfrak{D} \models T$ and isomorphic embeddings $g_1 : \mathfrak{B} \to \mathfrak{D}$, $g_2 : \mathfrak{C} \to \mathfrak{D}$ such that $g_1 f_1 = g_2 f_2$.

Let's define the semantic model. This model plays an important role as a semantic invariant. Such model always exists for any Jonsson theory. In future we will use the so-called semantic method [24] in the study of Jonsson Abelian groups. The essence of this method consists in translating the elementary properties of a fixed complete theory (the center of Jonsson theory) to Jonsson theory itself.

Initially, the concept of semantic model assumed another concept of homogeneity, but to prove the existence of a semantic model it was necessary to add to the axiom of the theory of sets ZF the axiom of the existence of a strongly inaccessible cardinal. To eliminate this axiom, it was necessary to change the definition of homogeneity of the semantic model to an acceptable variant. This was done in the work [25] Y.T. Mustafin. The concept of a universal model does not change. Recall it.

Definition 2. Let $\kappa \geq \omega$. The model $\mathfrak{M}$ of theory T is said to be κ-universal for T if every model T of cardinality is strictly less than κ is isomorphically embedded in $\mathfrak{M}$.

The following definition of κ-homogeneity for the model was introduced in [25].

Definition 3 [25]. Let $\kappa \geq \omega$. The model $\mathfrak{M}$ of theory T is said to be κ-homogeneous for T if for any two models $\mathfrak{A}$ and $\mathfrak{A}_1$ of T, which are submodels of $\mathfrak{M}$, the cardinality is strictly less than κ, and the isomorphism $f : \mathfrak{A} \to \mathfrak{A}_1$, for each extension $\mathfrak{B}$ of the model $\mathfrak{A}$, which is a submodel of $\mathfrak{M}$ and a model T of cardinality strictly less than κ, there exists an extension $\mathfrak{B}_1$ of the model $\mathfrak{A}_1$, which is a submodel of $\mathfrak{M}$, and an isomorphism $g : \mathfrak{B} \to \mathfrak{B}_1$ that extends f.

A homogeneous-universal model for T is a κ-homogeneous-universal model for T of cardinality κ, where $\kappa \geq \omega$.

Theorem 6 [25]. Each Jonsson theory T has a κ^+-homogeneous-universal model of power 2^κ. Conversely, if T is inductive, has an infinite model, and has a ω^+-homogeneous-universal model, then T is Jonsson theory.

Theorem 7 [25]. Let T be Jonsson theory. Two models $\mathfrak{M}$ and $\mathfrak{M}_1$ κ-homogeneous-universal for T are elementary eivalent.

Definition 4 [25]. The semantic model C_T of Jonsson theory T is the ω^+-homogeneous-universal model of theory T.

Sentence 2 [25]. Any two semantic models of Jonsson theory T are elementarily equivalent to each other.

Definition 5 [24]. The semantic completion (center) of Jonsson theory T is the elementary theory T^* of the semantic model C_T of theory T, that is, $T^* = Th(C_T)$.

Let T be some Jonsson theory of fixed signature σ and $\operatorname{Mod} T$ the class of all models of theory T. Consider an arbitrary model A from $\operatorname{Mod} T$. We define the following notion by means of which we are going to distinguish the models of Jonsson theory. Let $JSp(A) = \{T \,|\, T$ be Jonsson theory in the language σ and $A \in \operatorname{Mod} T\}$ and call $JSp(A)$ Jonsson spectrum of model A.

The following definitions 6, 7 belong to T. G. Mustafin.

Definition 6 [24]. We say that Jonsson theory T_1 is cosemantic to Jonsson theory T_2 ($T_1 \bowtie T_2$) if $C_{T_1} = C_{T_2}$, where C_{T_i} is the semantic model of T_i, $i = 1, 2$.

The relation of the cosemanticness on the set of theories is an equivalence relation. Then $JSp(A)/_{\bowtie}$ is the factor-set of Jonsson spectrum of model A with respect to $\bowtie$.

Definition 7 [24]. Jonsson theory of T is called perfect if every semantic model of T is a saturated model of T^*.

Definition 8 [2]. The theory T is called model-complete if for any models $\mathfrak{A}$ and $\mathfrak{B}$ of T, any subsystem $\mathfrak{A} \subseteq \mathfrak{B}$ is an elementary subsystem $\mathfrak{B}$. Equivalently, every isomorphic embedding is an elementary embedding.

Theorem 8 [2]. The theory T is model-complete if and only if theory $T \cup D(\mathfrak{M})$ is complete for any model $\mathfrak{M}$ of theory T.

Definition 9 [2]. Let T, T^* be some L-theories. The theory T^* is called a model completion of theory T if:

(a) T and T^* are mutually model joint, i.e. any model of theory T is embedded in the model of T^* and vice versa;

(b) T^* is model complete theory;

(c) if $\mathfrak{M} \models T$, then $T^* \cup \operatorname{Diagram}(\mathfrak{M})$ is complete theory.

The theory T^* is called a model companion of T if conditions (a) and (b) are satisfied.

Theorem 9 [2]. The theory T has no more than one model companion.

Theorem 10 [2, p. 68, table 1)]. The theory of algebraically closed Abelian groups is a model complement to the theory of Abelian groups.

Theorem 11 [24]. Let T be an arbitrary Jonsson theory, then the following conditions are equivalent:

1) the theory T is perfect;

2) T^* is a model companion of the theory T.

Let E_T be the class of all existentially closed models of theory T.

Theorem 12 [24]. If Jonsson theory T is perfect, then $E_T = \operatorname{Mod} T^*$, where $T^* = Th(C_T)$.

Let T be Jonsson theory, $S^J(X)$ be the set of all existential complete n-types over X that are compatible with T for any finite n, where $X \subset C$.

Definition 10 [24]. We say that Jonsson theory T J-λ-stable if for any T-existentially closed model $\mathfrak{A}$ for any subset X of A, $|X| \leq \lambda \Rightarrow |S^J(X)| \leq \lambda$.

Theorem 13. Let T be a perfect Jonsson theory complete for $\exists$-propositions, $\lambda \geq \omega$. Then the following conditions are equivalent:

1) T is J-λ-stable;

2) T^* is λ-stable, where T^* is center of Jonsson theory T.

Proof. Follow from Theorem 2.1 from [26].

We consider the language L_P obtained by adding the one-place predicate $P(x)$ to the language L. Denote by T_P the theory obtained by adding to T axioms, which state that the interpretation of P is also a model of theory T. We can say that P is interpreted as an existentially closed substructure, i.e. for each quantifier formula φ in L the following is true: $(\forall \bar{x})\,[P(\bar{x}) \wedge (\exists \bar{y})\varphi(\bar{x}, \bar{y}) \rightarrow (\exists \bar{z})P(\bar{z}) \wedge \varphi(\bar{x}, \bar{z})]$, where $P(\bar{x})$ means $P(x_1) \wedge \ldots \wedge P(x_n)$.

The model of theory T_P is called Jonsson pair (J-pair) of models of T. We denote this pair (N, M), where M is the interpretation of the predicate $P(\bar{x})$. In this pair we call N a large model, and M a small model.

We denote by $T_{P_{\mathrm{AG}}}$ the theory of Jonsson pairs of Abelian groups' theory. Let M be the class of models of $T_{P_{\mathrm{AG}}}$.

The following lemmas are necessary to prove the sentence 3.

Lemma 1. If $\mathfrak{A}, \mathfrak{B} \in M$; $\mathfrak{C} \in M$ and $\mathfrak{C} \subseteq \mathfrak{A}$, $\mathfrak{C} \subseteq \mathfrak{B}$; $|\mathfrak{C}| = |\mathfrak{A}| \cap |\mathfrak{B}|$, then there exists a system $\mathfrak{D} \in M$ such that $\mathfrak{A} \subseteq \mathfrak{D}$, $\mathfrak{B} \subseteq \mathfrak{D}$.

Proof. Let A and B be Abelian groups and $A \cap B = C$, $C \subseteq A$, $C \subseteq B$. Let $A \underset{C}{\times} B$ be the free product of groups A and B with the amalgamated subgroup C. Then $A \subseteq A \underset{C}{\times} B$ and $B \subseteq A \underset{C}{\times} B$.

Lemma 2. If M is an abstract and satisfies the lemma 1, then M satisfies the amalgamation property (AP).

Proof. The proof can be extracted from Theorems A, B of the paper [27], but we need only take into account the new definition of homogeneity.

Sentence 3. The theory $T_{P_{\mathrm{AG}}}$ is the perfect Jonsson theory.

Proof. We first show that $T_{P_{\mathrm{AG}}}$ is Jonsson theory. $T_{P_{\mathrm{AG}}}$ has an infinite model. It is inductive, because the union of an increasing chain of Abelian groups is Abelian group. That is the conditions (1) and (2) from the definition 1 are satisfied.

If A and B are two J-pairs of theory $T_{P_{\mathrm{AG}}}$, then their direct product $A \times B$ is Abelian group. The set of elements $< a, e^B >$, where $a \in A$, e^B is the unit element of B, is a subgroup of $A \times B$ isomorphic to A. Similarly, the set of elements $< e^A, b >$, where $b \in B$, e^A is the unit element of A, is a subgroup of $A \times B$ isomorphic to B. Thus condition (3) is satisfied.

Let us verify the satisfaction of condition (4). As the class of Abelian groups is abstract that is is closed with respect to isomorphisms, then according to the lemma 1 and the lemma 2, the class of Abelian groups has amalgamation property (AP). Thus $T_{P_{\mathrm{AG}}}$ is Jonsson theory.

Perfection follows from Sentence 3 of [1], since due to the perfection of theory of Abelian groups, the semantic model of this theory is saturated in its power. Consider the semantic model (N, M) of theory $T_{P_{\mathrm{AG}}}$. Let the realization of predicate P by the small model M be an existentially closed submodel of the large model N. All types over this model are realized in the large model N by virtue of Sentence 3 from [1]. There are no other new types, thus, the J-pair (N, M) is saturated in its power.

The next notion was considered by J. Goodrick in [28] and there he denoted it as a Schröder-Bernstein (SB) property.

Definition 11. The theory T admits the property SB if for any two mutually elementary embeddable models of the theory T it follows that they are isomorphic.

But J. Goodrick notes that this property was first considered for ω-stable theories by T.A. Nurmagambetov. In the works [29, 30]. In particular, T.A. Nurmagambetov was obtained the following result with respect to the property SB.

Theorem 1.2 from [29] If T is ω-stable theory, then T has an SB property if and only if T of bounded dimension.

J. Goodrick in the paper [28] received a description of the SB property for a classifiable (superstable, with NDOP and NOTOP) theory with a limited dimension. In particular, in work [31] J.Goodrick and M.Laskovsky described the property SB for weakly minimal theories.

Later J.Goodrick [32] found necessary and sufficient conditions for the theory of abelian groups to admit the SB property. Namely, he proved the following theorem:

Theorem 3.8 from [32] If G is Abelian group, then the following conditions are equivalent:

1. $Th(G, +)$ has the Schröder-Bernstein property.
2. $Th(G, +)$ is ω-stable.
3. G is the direct sum of a divisible group and a group with torsion of a bounded exponent.
4. $Th(G, +)$ is superstable, and if $(\overline{G}, +) \equiv (G, +)$ is saturated, then every map in $Aut(\overline{G}/\overline{G}^0)$ is unipotent.

We redefined this concept for Jonsson theories and is denoted as JSB, namely: Jonsson theory T has JSB property if for any two existentially closed models $\mathfrak{A}$ and $\mathfrak{B}$ of theory T from the fact that they are mutually isomorphically embedded into each other it follows that they are isomorphic.

The following result is Jonsson analog of Theorem 3.8 from the paper [32] in the enriched language, namely:

Theorem 14. Let $T_{P_{\mathrm{AG}}}$ be the theory of Jonsson pairs of Abelian groups' theory, then the following conditions are equivalent:

1) $T_{P_{\mathrm{AG}}}$ is J-ω-stable;

2) $T^*_{P_{\mathrm{AG}}}$ is ω-stable;

3) $T_{P_{\mathrm{AG}}}$ has the JSB property.

Proof. By the sentence 3, the equivalence of items (1) and (2) follows from theorem 13, since in theorem 13 we use $\exists$-completeness and then, by the theorem 1, we can apply the theorem 13.

We will show from (3) in (2). By virtue of the sentence 3, the theory of Jonsson pairs of Abelian groups' theory is a perfect Jonsson theory. Then by virtue of the criterion of perfectness 11 and theorem 12 we have that $E_{T_{P_{\mathrm{A}}}} = \mathrm{Mod}\, T^*_{P_{\mathrm{AG}}}$. Consequently, any model of the center of theory $T_{P_{\mathrm{AG}}}$ is existentially closed. By the theorem 11 $T^*_{P_{\mathrm{AG}}}$ is the center of theory $T_{P_{\mathrm{AG}}}$ is a model companion of theory $T_{P_{\mathrm{AG}}}$. Hence $T^*_{P_{\mathrm{AG}}}$ is the model complete theory and any embedding is elementary. It remains to apply Theorem 3.8 from [32].

We will show from (2) in (3). Let the center $T^*_{P_{\mathrm{AG}}}$ be ω -stable. Since $T_{P_{\mathrm{AG}}} \subset T^*_{P}$ it follows that $\mathrm{Mod}\, T^*_{P_{\mathrm{AG}}} \subset \mathrm{Mod}\, T_{P_{\mathrm{AG}}}$. In view of Theorem 3.8 from [32] $T^*_{P_{\mathrm{AG}}}$ admits SB. But by virtue of Theorem 11 and the perfectness of theory $T_{P_{\mathrm{AG}}}$ $\mathrm{Mod}\, T^*_{P_{\mathrm{AG}}} = E_{T_{P_{\mathrm{AG}}}}$. And this completes the proof, because the Josnson property of JSB is defined only for models from $E_{T_{P_{\mathrm{AG}}}}$.

The following result concerning Jonsson Abelian groups is an analog of W. Szmielew's theorem on the elementary classification of Abelian groups. We know that for any Abelian group G there exists a standard group G^0 such that $G \equiv G^0$, and in this case

$$G^0 = \oplus_{p,n} Z_{p^n}^{(\alpha^0_{p^n})} \oplus \oplus_p Z_p^{(\beta^0_p)} \oplus \oplus_p Z_{p^\infty}^{(\gamma^0_p)} \oplus Q^{(\delta^0)}.$$

Denote by $JSp(A)$ the Jonsson spectrum of Abelian group A, where

$$JSp(A)=\{T_{P_{\mathrm{AG}}} \,|\, T_{P_{\mathrm{A\Gamma}}} \text{ is Jonsson theory in the language } \sigma_{P_{\mathrm{A\Gamma}}} \text{ and } A \in \mathrm{Mod}\, T_{P_{\mathrm{A\Gamma}}}\}.$$

The following result gives a description of semantic model of Abelian groups' Jonsson theory.

Theorem 15. Let $T_{P_{\mathrm{AG}}}$ be Jonsson theory of Abelian groups in the language $\sigma_{P_{\mathrm{AG}}}$, then its center $C_{T_{P_{\mathrm{AG}}}} \in E_{T_{P_{\mathrm{AG}}}}$, while $C_{T_{P_{\mathrm{AG}}}}$ is a divisible group and its standard Szmielew group is representable as $\oplus_p \mathbb{Z}_{p^\infty}^{(\alpha_p)} \oplus \mathbb{Q}^{(\beta)}$, where $\alpha_p, \beta \in \omega^+$, $2^\omega = \left|C_{T_{P_{\mathrm{AG}}}}\right|$.

Proof. It follows from the theorems 5 and 2 and the fact that any Jonsson theory has a semantic model that is a ω^+-homogeneous-universal model.

Sentence 4. There exists a continuum of imperfect subclasses of the class of all Abelian groups.

Proof. In [33] this sentence was proved using the old definition of semantic model. For a new definition 4 of semantic model, we can repeat the proof from [33] considering only the power estimates of semantic model. To prove this fact, it suffices to show that not elementary equivalent semantic models of imperfect Jonsson theories of Abelian groups will be a continuum. From Theorem 15 the semantic model of any Abelian group will be the direct sum of the corresponding number of groups' copies of two kinds: $\mathbb{Z}_{p^\infty}$ and Q. In the imperfect case, only Q can be absent since Q can not be a universal model. The number of copies of $\mathbb{Z}_{p^\infty}$ can be any subset of ω.

We call the pair $(\alpha_p, \beta)^A_{C_{[T_{P_{\mathrm{AG}}}]}}$ Jonsson invariant of Abelian group A if the standard group of Szmielew's group A is representable in the form $\oplus_p \mathbb{Z}_{p^\infty}^{(\alpha_p)} \oplus \mathbb{Q}^{(\beta)}$, where $C_{[T_{P_{\mathrm{AG}}}]}$ is semantic model of $[T_{P_{\mathrm{AG}}}] \in JSp(A)/_{\bowtie}$.

We give the following definitions of concepts, which are specified in the frame of the study of Jonsson theories, the definition of elementary equivalence for complete theories.

Let A and B be models of the same signature.

Definition 12. We will say that the model A is Jonsson elementary equivalent to the model B ($A \underset{J}{\equiv} B$) if $JSp(A) = JSp(B)$.

Lemma 3. $\forall A, B \in \mathrm{Mod}\, \sigma_{P_{\mathrm{AG}}}$ $JSp(A) \cap JSp(B) \neq \emptyset$.

Proof. This is true because at least $T_{P_{\mathrm{AG}}} \in JSp(A) \cap JSp(B)$.

Definition 13. We say that the model A is JSp-cosemantic to model B ($A \underset{JSp}{\bowtie} B$), if

$$JSp(A)/_{\bowtie} = JSp(B)/_{\bowtie}.$$

Lemma 4. $A \underset{JSp}{\bowtie} B \Leftrightarrow JSp(A) \cap JSp(B) = JSp(A) \cup JSp(B)$.

Proof. It follows from the definition.

It is easy to understand that the concepts introduced in the definitions 12 and 13 generalize the notion of elementary equivalence. In the following lemma, by virtue of the sentence 3, we can note that the following is true:

Lemma 5. Let A and B be arbitrary abelian groups, then

$$A \equiv B \Rightarrow A \underset{J}{\equiv} B \Rightarrow A \underset{JSp}{\bowtie} B.$$

Proof. It follows from the definition.

The following result is the Jonsson analog of a well-known W. Szmielew's theorem on the elementary classification of Abelian groups and is a corollary of Theorem 15 and Lemma 5.

We define the following set $\left\{(\alpha_p, \beta)^A_{C_{[T_{P_{\mathrm{AG}}}]}} : [T_{P_{\mathrm{AG}}}] \in JSp(A)/_{\bowtie}, \text{for all simple } p\right\}$ as Jonsson invariant of the factor-set $JSp(A)/_{\bowtie}$ and denote it by $JInv(JSp(A)/_{\bowtie})$.

Theorem 16. Let $A, B \in \mathrm{Mod}\, \sigma_{P_{\mathrm{AG}}}$, $A = \langle M_1, N_1\rangle$, $B = \langle M_2, N_2\rangle$, then the following conditions are equivalent:

1) $A \underset{JSp}{\bowtie} B$;

2) $JInv(JSp(A)/_{\bowtie}) = JInv(JSp(B)/_{\bowtie})$.

Proof. From (2) to (1). If (2) is satisfied, this means that the standard Szmielew's groups for A and B coincide, then by lemma 5 it follows that $A \underset{JSp}{\bowtie} B$, i. e. (1) is satisfied.

Suppose that (1) holds, then $JSp(A)/_{\bowtie} = JSp(B)/_{\bowtie}$. Assume the contrary, i.e.

$$JInv(JSp(A)/_{\bowtie}) \neq JInv(JSp(B)/_{\bowtie}).$$

Then there exists

$$(\alpha_p, \beta)^A_{C_{[T_{P_{\mathrm{AG}}}]}} \in JInv(JSp(A)/_{\bowtie}) \text{ and } (\alpha_p, \beta)^A_{C_{[T_{P_{\mathrm{AG}}}]}} \notin JInv(JSp(B)/_{\bowtie}).$$

Therefore, for each class $[T'_{P_{\mathrm{AG}}}] \in JSp(B)/_{\bowtie}$ we have $(\alpha_p, \beta)^A_{C_{[T_{P_{\mathrm{AG}}}]}} \neq (\alpha_p, \beta)^B_{C_{[T'_{P_{\mathrm{AG}}}]}}$, i. e. there is not a single Jonsson theory of the group B, the semantic model of which is $C_{[T_{P_{\mathrm{AG}}}]}$. But it is known that any Jonsson theory is uniquely determined by its semantic model ([3], Theorem 2.2 at $\alpha = 0$). It follows that there is Jonsson theory $T_{P_{\mathrm{AG}}}$, which is determined by Jonsson invariant $(\alpha_p, \beta)^A_{C_{[T_{P_{\mathrm{AG}}}]}}$ and $T_{P_{\mathrm{AG}}} \notin JSp(B)$. And this contradicts condition (1), so our assumption is incorrect.

The main result of this paper is the content of theorem 16. In this theorem we obtained Jonsson analogue in the extended signature by the one-place predicate of a well-known theorem of Polish mathematician W. Szmielew on the elementary classification of abelian groups. Interpretation of the predicate symbol in Jonsson pair $\langle M, N\rangle$ is an existentially closed submodel of M in the large model$N$. Such a statement of the problem is a generalized Jonsson generalization of a well-known problem on elementary pairs for the complete theories [17]. On the other hand, in the frame of the study of stability in the sense of [20] and [22] in connection with theorem 14 presented in this paper, there is an obvious relationship. Thus, the enrichment of signature with a single predicate can be considered in the frame of the classification of Jonsson theories and their classes of existentially closed models. As the obtained results (Theorems 14 and 16) show, the model-theoretic properties of Jonsson pair are closely related to the model-theoretic properties of the center of the considering Jonsson theory. Since the center is a complete theory and originally Jonsson theory (the theory of abelian groups) is an example of perfect Jonsson theory, we can conclude that in the case of enriching of this theory's language by the one-place predicate, the basic properties obtained in the work [1] are also preserved, as before enrichment.

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Байытылған сигнатурада абельдік группалардың косемантикалық критериі

Мақалада белгіленген предикаты бар абельдік группалар класында модельдің йонсондық спектріне қатысты косемантикалық критерийі ұсынылды. Йонсондық теориялардың модельдерін зерттеу аясында абельдік группалардың модельді-теоретикалық сұрақтары зерттелді. Жұмыста йонсондылық шарты абельдік группалардың белгіленген предикаттың қосымша шартын қанағаттандырады, сонымен қатар йонсондық теория мағынасында кемелділігі көрсетілген. Алгебрадан классикалық мысалдары болып келетін бекітілген сипаттамамен өрістер, группалар, абельдік группалар, сақинаның әртүрлі кластары, бульдік алгебра, полигондар алгебраның және йонсондық шартты қанағаттандыратын теорияның мысалдары болып табылады. Йонсондық теориялар барлық индуктивті теориялардың кластарының жеткілікті кең ішкі класын құрайды. Бірақ, қарастырып отырған йонсондық теориялар жалпы айтқанда, толық емес болып табылады. Классикалық модельдер теориясы негізінен толық теориямен жұмыс жасайды, ал казіргі уақытта толық теориялардың модельді-теоретикалық қасиеттерін зерттеу үшін дамыған, бірақ йонсондық теорияларды зерттеу барысында техникалық аппараттың жетіспеулігі орын алады. Сондықтан йонсондық теорияларды зерттеу техникасысының аналогын табу бұл зерттеу тақырыбында практикалық маңыздылығы бар. Осы мақалада сигнатура бірорынды предикатқа кеңейтілді. Осы предикатты құрайтын элементтер қарастырып отырған йонсондық теорияның кейбір модельдерінің экзистенционалды-тұйық ішкі моделін құрайды. Сонымен біз осы мақаланың басты нәтижесін В. Шмелеваның йонсондық теорияларын зерттеу шеңберінде абельдік группалардың элементарлық классификациясы бойынша танымал теоремасы ретінде нақтылап, осылайша толық теориялар үшін элементарлық қосарлары жайлы белгілі сұрақтың жалпылауын аламыз. Сонымен қатар екі модельдің үйлесімді енуінің йонсондық аналогі, баскаша айтқанда, абельдік группалар теорияларының йонсондық қосарлары зерттеу аясында Шрёдер-Бернштейн қасиеттері алынды.

Кілт сөздер: йонсондық теория, модельді компаньон, экзистенционалды-тұйық модель, кемелділік, косеманттылық, йонсондық спектр, йонсондық қосар.

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Критерий косемантичности абелевых групп в обогащённой сигнатуре

В статье рассмотрен критерий косемантичности относительно йонсоновского спектра модели в классе абелевых групп с выделенным предикатом. Изучены некоторые теоретико-модельные свойства абелевых групп в рамках их исследования, как моделей йонсоновских теорий. Нами получено, что абелевы группы с дополнительным условием выделенного предиката удовлетворяют условиям йонсоновости, а также совершенности в смысле йонсоновской теории. Известно, что классические примеры из алгебры, такие как поля фиксированной характеристики, группы, абелевы группы, различные классы колец, булевы алгебры, полигоны, являются примерами алгебр, теории которых удовлетворяют условиям йонсоновости. Изучение теоретико-модельных свойств йонсоновских теорий в классе абелевых групп является весьма актуальной задачей как в самой теории моделей, так и в универсальной алгебре. Йонсоновские теории образуют достаточно широкий подкласс класса всех индуктивных теорий. Но рассматриваемые йонсоновские теории, вообще говоря, не являются полными. Классическая теория моделей в основном имеет дело с полными теориями, а в случае изучения йонсоновских теорий существует дефицит технического аппарата, который в данное время развит для изучения теоретико-модельных свойств полных теорий. Поэтому нахождение аналогов такой техники для изучения йонсоновских теорий имеет практическую значимость в данной теме исследования. Авторами была расширена сигнатура на один одноместный предикат. Элементы, реализующие этот предикат, образуют экзистенциально-замкнутую подмодель некоторой модели рассматриваемой йонсоновской теории. В конечном итоге получен основной результат данной статьи как уточнение хорошо известной теоремы В. Шмелёвой об элементарной классификации абелевых групп в рамках изучения йонсоновских теорий. Тем самым найдены обобщение известного вопроса об элементарных парах для полных теорий, а также йонсоновский аналог для совместной вложимости двух моделей, или, по-другому, свойства Шрёдера-Бернштейна, в рамках изучения йонсоновских пар теории абелевых групп.

Ключевые слова: йонсоновская теория, модельный компаньон, экзистенциально-замкнутая модель, совершенность, косемантичность, йонсоновский спектр, йонсоновская пара.

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