UDC 512.552

U.K. Turusbekova, G.T. Azieva

Kazakh University of Economics, Finance and International Trade, Astana
(E-mail: umut.t@mail.ru)

Quadratic Poisson algebras on $k\,[x, y, z]$and their automorphisms

One of the important directions in modern mathematics is applications of Poisson structures and to various problems of mathematics and theoretical mechanics. These problems arise in dynamics of a rigid body, the celestial mechanics, the theory of curls, cosmological models. Poisson algebras play a key role in the Hamiltonian mechanics, symplectic geometry and also are central in the study of quantum groups. Note that a development of the theory of Poisson structures in many respects was stimulated by the dynamics of many-dimensional tops since the latter allows to make the abstract statements of many theorems more vivid and substantial. Note also that some important examples of the Lie-Poisson brackets were already known to Jacobi. In his examples the Poisson brackets appeared on a space of the first integrals of the Hamilton equations. Until recently, an algebraic theory of Poisson structures was scarcely studied. At present, Poisson algebras are investigated by the many mathematicians of Russia, France, the USA, Brazil, Argentina, Bulgaria etc. This paper is devoted to the description of the automorphism group of Poisson algebra P on polynomial algebra $k\,[x, y, z]$, such that $\{x, y\} = z^2, \{y, z\} = x^2, \{z, x\} = y^2$. One interesting Poisson relation between the homogeneous algebraically dependent elements is established and is proved that the group of automorphisms Aut_kP of algebra P is generated by automorphisms $\varphi_\alpha = (\alpha x, \alpha y, \alpha z)$, $\alpha \in k^*, \tau = (y, z, x)$ and $\delta = (x, \varepsilon y, \varepsilon^2 z)$, where ε – a solution of an equation $x^2 + x + 1 = 0$.

Key words: quadratic Poisson algebras, automorphisms, polinominal algebras, rational function.

Introduction

One of the important directions in modern mathematics is applications of Poisson structures and to various problems of mathematics and theoretical mechanics. These problems arise in dynamics of a rigid body, the celestial mechanics, the theory of curls, cosmological models. Poisson algebras play a key role in the Hamiltonian mechanics, symplectic geometry and also are central in the study of quantum groups. Note that a development of the theory of Poisson structures in many respects was stimulated by the dynamics of many-dimensional tops since the latter allows to make the abstract statements of many theorems more vivid and substantial.

Recall that a vector space B over a field k endowed with two bilinear operations $x \cdot y$ (*a multiplication*) and $\{x, y\}$ (*a Poisson bracket*) is called *a Poisson algebra* if B is a commutative associative algebra under $x \cdot y$, B is a Lie algebra under $\{x, y\}$, and B satisfies the following identity (the Leibniz identity)

$$\{x \cdot y, z\} = \{x, z\} \cdot y + x \cdot \{y, z\}.$$

It is well known [1–4], that the automorphisms of polynomial algebras $k\,[x, y]$ and free associative algebras $k\,\langle x, y\rangle$ in two variables are products of affine automorphisms

$$\varphi = (\alpha_{11}x + \alpha_{21}y + \beta_1, \alpha_{12}x + \alpha_{22}y + \beta_2)\,, \alpha_{ij}, \beta_j \in k$$

and triangular automorphisms

$$\psi = (\alpha_1 x + f\,(y)\,, \alpha_2 y + \beta_2)\,, \alpha_1, \alpha_2 \in k^*, f\,(y) \in k\,[y]\,, \beta_2 \in k,$$

i.e are *tame*.

It was proved that the known automorphism of Nagata [5, 6]

$$\sigma = (x + (x^2 - yz)z, \quad y + 2(x^2 - yz)x + (x^2 - yz)^2 z, z),$$

of polynomial algebras $k\,[x, y, z]$ in three variables and Anick automorphism [7, 8]

$$\delta = (x + z(xz - zy), \quad y + (xz - zy)z, z),$$

of free associative algebras $k\langle x,y,z\rangle$ in three variables over a field k of characteristic 0 are not tame, i.e are *wild*.

In work [9] is proved that the automorphisms of two-generated free Poisson algebras $k\{x,y\}$ over a field k of characteristic 0 are time. Moreover [1, 4, 9], groups of automorphisms of algebras $k[x,y]$, $k\langle x,y\rangle$, $k\{x,y\}$ are isomorphic, i.e.

$$Aut\,k[x_1,x_2]\cong Aut\,k<x_1,x_2>\cong Aut\,k\{x_1,x_2\}.$$

One of the main problems of affine algebraic geometry (see, for example [10]) is a description of automorphism groups of polynomial algebras in $n\geq 3$ variables.

A classification of all homogeneous quadratic Poisson brackets in three variables is given in work [11]. Among these algebras the most interesting is the Poisson algebra P on polynomial algebra $k[x,y,z]$, such that

$$\{x,y\}=z^2,\{y,z\}=x^2,\{z,x\}=y^2.$$

This paper is devoted to the description of the automorphism group of Poisson algebra P. In section 2 provided informations necessary, designations and definitions are. One interesting Poisson relation between the homogeneous algebraically dependent elements is established. Further, in section 3 is proved that the group of automorphisms Aut_kP of algebra P is generated by automorphisms $\varphi_\alpha=(\alpha x,\alpha y,\alpha z),\alpha\in k^*,\tau=(y,z,x)$ and $\delta=(x,\varepsilon y,\varepsilon^2 z)$, where ε — a solution of an equation $x^2+x+1=0$.

Results of this work in a short form are explained in [12].

Preliminary information

A vector space P over a field K endowed with two bilinear operations $x\cdot y$ (a multiplication) and $\{x,y\}$ (a Poisson bracket) is called *a Poisson algebra* if P is a commutative associative algebra under $x\cdot y$, P is a Lie algebra under $\{x,y\}$, and P satisfies the following identity

$$\{x,y\cdot z\}=\{x,y\}\cdot z+y\cdot\{x,z\}.$$

There are two important classes of Poisson algebras:

1) Symplectic algebras S_n. For each n the algebra S_n is a polynomial algebra $k[x_1,y_1,\ldots,x_n,y_n]$, endowed with the Poisson bracket defined by $\{x_i,y_j\}=\delta_{ij}$, $\{x_i,x_j\}=0$, $\{y_i,y_j\}=0$, where δ_{ij} is the Kronecker symbol and $1\leq i,j\leq n$;

2) Symmetric Poisson algebras $PS(L)$. Let L be a Lie algebra with a linear basis $e_1,e_2,\ldots,e_k,\ldots$. Then $PS(L)$ is the usual polynomial algebra $K[e_1,e_2,\ldots,e_k,\ldots]$ endowed with the Poisson bracket defined by $\{e_i,e_j\}=[e_i,e_j]$ for all i,j, where $[x,y]$ is the multiplication of the Lie algebra L.

Let is given a Poisson bracket $\{x,y\}$ on polynomial algebra $k[x_1,x_2,...,x_n]$. From Leibniz identity follows that

$$\{f,g\}=\sum_{1\leq i<j\leq n}\left(\frac{\partial f}{\partial x_i}\frac{\partial g}{\partial x_j}-\frac{\partial g}{\partial x_i}\frac{\partial f}{\partial x_j}\right)\{x_i,x_j\},\tag{1}$$

where $f,g\in k[x_1,x_2,...,x_n]$.

For any elements f,g of rational function algebra $k(x_1,x_2,...,x_n)$ we define an element $\{f,g\}$ by a formula (1).

Lemma 1. For any $f,g,h\in k(x_1,x_2,...,x_n)$ the next equations are executed:

(a) $\{f,f\}=0$;

(b) $\{fg,h\}=\{f,h\}g+f\{g,h\}$;

(c)$\{f,\frac{g}{h}\}=\frac{1}{h^2}(\{f,g\}h-g\{f,h\})$;

(d)$\{\{f,g\},h\}+\{\{g,h\},f\}+\{\{h,f\},g\}=0$.

Proof. The statement (a) is trivial.

Let's prove equation (b). Using a formula (1), we get

$$\{fg,h\}=\sum_{1\leq i<j\leq n}\left(\frac{\partial(fg)}{\partial x_i}\frac{\partial h}{\partial x_j}-\frac{\partial h}{\partial x_i}\frac{\partial(fg)}{\partial x_j}\right)\{x_i,x_j\}=$$

$$=\sum_{1\leq i<j\leq n}\left(\left(\frac{\partial f}{\partial x_i}g+f\frac{\partial g}{\partial x_i}\right)\frac{\partial h}{\partial x_j}-\frac{\partial h}{\partial x_i}\left(\frac{\partial f}{\partial x_j}g+f\frac{\partial g}{\partial x_j}\right)\right)\{x_i,x_j\}=$$

$$= \sum_{1\le i<j\le n}\left(\frac{\partial f}{\partial x_i}\frac{\partial h}{\partial x_j}-\frac{\partial h}{\partial x_i}\frac{\partial f}{\partial x_j}\right)\{x_i,x_j\}\,g+f\sum_{1\le i<j\le n}\left(\frac{\partial g}{\partial x_i}\frac{\partial h}{\partial x_j}-\frac{\partial h}{\partial x_i}\frac{\partial g}{\partial x_j}\right)\{x_i,x_j\}=$$

$$=\{f,h\}\,g+f\,\{g,h\}\,.$$

For the proof of the statement (c) we use equation (b). We have

$$\left\{f,\frac{g}{h}\right\}=\{f,g\}\,\frac{1}{h}+g\left\{f,\frac{1}{h}\right\}=\{f,g\}\,\frac{1}{h}+g\sum_{1\le i<j\le n}\left(\frac{\partial f}{\partial x_i}\frac{\partial\left(\frac{1}{h}\right)}{\partial x_j}-\frac{\partial\left(\frac{1}{h}\right)}{\partial x_i}\frac{\partial f}{\partial x_j}\right)\{x_i,x_j\}=$$

$$=\{f,g\}\,\frac{1}{h}-\frac{g}{h^2}\sum_{1\le i<j\le n}\left(\frac{\partial f}{\partial x_i}\frac{\partial h}{\partial x_j}-\frac{\partial h}{\partial x_i}\frac{\partial f}{\partial x_j}\right)\{x_i,x_j\}=\{f,g\}\,\frac{1}{h}-\frac{g}{h^2}\,\{f,h\}=$$

$$=\frac{1}{h^2}\left(\{f,g\}\,h-g\,\{f,h\}\right).$$

Let's prove the statement (d). For any $a,b,c,d,p,q\in k\,[x_1,x_2,...,x_n]$ we put $f=\frac{a}{b}$, $g=\frac{c}{d}$, $h=\frac{p}{q}$ and consider added $\left\{\left\{\frac{a}{b},\frac{c}{d}\right\},\frac{p}{q}\right\}$ in equation (d). Using statements (a), (b), (c) we get

$$\left\{\left\{\frac{a}{b},\frac{c}{d}\right\},\frac{p}{q}\right\}=\left\{\frac{1}{d^2}\left(\left\{\frac{a}{b},c\right\}d-c\left\{\frac{a}{b},d\right\}\right),\frac{p}{q}\right\}=-\left\{\frac{1}{d}\left\{c,\frac{a}{b}\right\},\frac{p}{q}\right\}+$$

$$+\left\{\frac{c}{d^2}\left\{d,\frac{a}{b}\right\},\frac{p}{q}\right\}=-\left\{\frac{1}{b^2d}\left(\{c,a\}\,b-a\,\{c,b\}\right),\frac{p}{q}\right\}+\left\{\frac{c}{b^2d^2}\left(\{d,a\}\,b-a\,\{d,b\}\right),\frac{p}{q}\right\}=$$

$$=-\frac{1}{q^2}\left(\left\{\frac{1}{b^2d}\left(\{c,a\}\,b-a\,\{c,b\}\right),p\right\}q-p\left\{\frac{1}{b^2d}\left(\{c,a\}\,b-a\,\{c,b\}\right),q\right\}-\right.$$

$$\left.-\left\{\frac{c}{b^2d^2}\left(\{d,a\}\,b-a\,\{d,b\}\right),p\right\}q+p\left\{\frac{c}{b^2d^2}\left(\{d,a\}\,b-a\,\{d,b\}\right),q\right\}\right)=$$

$$=-\frac{1}{q^2}\left(-\frac{1}{b^4d^2}\left(\{p,\{c,a\}\,b-a\,\{c,b\}\}\,b^2dq-\left(\{c,a\}\,b-a\,\{c,b\}\right)\left\{p,b^2d\right\}q\right)+\right.$$

$$+\frac{p}{b^4d^2}\left(\{q,\{c,a\}\,b-a\,\{c,b\}\}\,b^2d-\left(\{c,a\}\,b-a\,\{c,b\}\right)\left\{q,b^2d\right\}\right)+$$

$$+\frac{1}{b^4d^4}\left(\{p,c\left(\{d,a\}\,b-a\,\{d,b\}\right)\}\,b^2d^2q-c\left(\{d,a\}\,b-a\,\{d,b\}\right)\left\{p,b^2d^2\right\}q\right)-$$

$$\left.-\frac{p}{b^4d^4}\left(\{q,c\left(\{d,a\}\,b-a\,\{d,b\}\right)\}\,b^2d^2-c\left(\{d,a\}\,b-a\,\{d,b\}\right)\left\{q,b^2d^2\right\}\right)\right)=$$

$$=\frac{1}{b^4d^4q^2}\left(\{p,\{c,a\}\,b-a\,\{c,b\}\}\,b^2d^3q-d^2\left(\{c,a\}\,b-a\,\{c,b\}\right)\left\{p,b^2d\right\}q-\right.$$

$$-pb^2d^3\,\{q,\{c,a\}\,b-a\,\{c,b\}\}+pd^2\left(\{c,a\}\,b-a\,\{c,b\}\right)\left\{q,b^2d\right\}-$$

$$-\{p,c\left(\{d,a\}\,b-a\,\{d,b\}\right)\}\,b^2d^2q+c\left(\{d,a\}\,b-a\,\{d,b\}\right)\left\{p,b^2d^2\right\}q+$$

$$\left.+p\,\{q,c\left(\{d,a\}\,b-a\,\{d,b\}\right)\}\,b^2d^2-c\left(\{d,a\}\,b-a\,\{d,b\}\right)\left\{q,b^2d^2\right\}\right).$$

Making similar conversions with addeds $\left\{\left\{\frac{c}{d},\frac{p}{q}\right\},\frac{a}{b}\right\}$ and $\left\{\left\{\frac{p}{q},\frac{a}{b}\right\},\frac{c}{d}\right\}$, and summing them up, we get

$$\left\{\left\{\frac{a}{b},\frac{c}{d}\right\},\frac{p}{q}\right\}+\left\{\left\{\frac{c}{d},\frac{p}{q}\right\},\frac{a}{b}\right\}+\left\{\left\{\frac{p}{q},\frac{a}{b}\right\},\frac{c}{d}\right\}=0.$$

Corollary. The bracket $\{\cdot,\cdot\}$ sets up the structure of Poisson algebra on rational function algebra $k\,(x_1,x_2,...,x_n)$.

Lemma 2. Let a,b,c be homogeneous algebraically dependent elements of polynomial algebra $k\,[x_1,x_2,...,x_n,...]$ over a field k of characteristic 0. If $\{\cdot,\cdot\}$ is a Poisson bracket on $k\,[x_1,x_2,...,x_n,...]$, then

$$\deg(a)\,a\,\{b,c\}+\deg(b)\,b\,\{c,a\}+\deg(c)\,c\,\{a,b\}=0.$$

Proof. Let's consider a case when polynomials a, b and c have identical degrees. Then there is a nontrivial homogeneous polynomial $F(x, y, z)$ of degree p such that

$$F(a, b, c) = 0.$$

If $c \neq 0$ then having divided the last equation on c^p we have

$$F\left(\frac{a}{c}, \frac{b}{c}, 1\right) = 0.$$

Since $F(X, Y, 1) \neq 0$ from here we get algebraic dependence of $\frac{a}{c}$ and $\frac{b}{c}$.

By [13], if $f, g \in k(x_1, x_2, ..., x_n, ...)$ are algebraically dependent then for all $i < j$ we have

$$\frac{\partial f}{\partial x_i}\frac{\partial g}{\partial x_j} - \frac{\partial g}{\partial x_i}\frac{\partial f}{\partial x_j} = 0,$$

i.e. $\{f, g\} = 0$. Therefore,

$$\left\{\frac{a}{c}, \frac{b}{c}\right\} = 0.$$

By Lemma 1 (c), we have

$$0 = \left\{\frac{a}{c}, \frac{b}{c}\right\} = \frac{1}{c^2}\left(\left\{\frac{a}{c}, b\right\}c - b\left\{\frac{a}{c}, c\right\}\right) =$$

$$= \frac{1}{c^2}\left(\frac{1}{c^2}\left(\{a, b\}c - a\{c, b\}\right)c - \frac{1}{c^2}\left(\{a, c\}c - a\{c, c\}\right)b\right) = \frac{1}{c^2}\left(\{a, b\} - \frac{a\{c, b\}}{c} - \frac{\{a, c\}b}{c}\right),$$

i.e.

$$a\{b, c\} + b\{c, a\} + c\{a, b\} = 0.$$

Now let $deg(a) = p$, $deg(b) = q$, $deg(c) = r$. Then a^{qr}, b^{pr}, c^{pq} are homogeneous algebraically dependent elements of identical degree. Therefore,

$$a^{qr}\{b^{pr}, c^{pq}\} + b^{pr}\{c^{pq}, a^{qr}\} + c^{pq}\{a^{qr}, b^{pr}\} = 0.$$

Using Leibniz identity from here we get

$$pqr\left(pa\{b, c\} + qb\{c, a\} + rc\{a, b\}\right)a^{qr-1}b^{pr-1}c^{pq-1} = 0.$$

Therefore,

$$pa\{b, c\} + qb\{c, a\} + rc\{a, b\} = 0.$$

Lemma 3. Let p, q and r – pairwise coprime elements of $k[x, y, z]$ and $deg(p) = deg(q) = deg(r)$.
If $p^3 + q^3 + r^3 = 0$, then p, q and r are constants.

Proof. Turning on, if necessary, to algebraic closure of the field k, it is possible to consider what k is algebraically closed. Let's prove the statement of a lemma by induction on $deg(p)$. If $deg(p) = 0$, then $p, q, r \in k^*$.

Let $deg(p) > 0$. Then

$$(-p)^3 = q^3 + r^3 = (q - \gamma_1 r)(q - \gamma_2 r)(q - \gamma_3 r), \tag{2}$$

where $\gamma_1, \gamma_2, \gamma_3$ – roots of an equation $x^3 + 1 = 0$. Note, that multipliers in the right part of this equation are pairwise coprime, since p, q and r are pairwise coprime, by lemma's condition.

Since the left part of equation (2) is a full cube of a polynomial p from here we get

$$\begin{cases} q - \gamma_1 r = a^3 \\ q - \gamma_2 r = b^3 \\ q - \gamma_3 r = c^3 \end{cases} \tag{3}$$

for some pairwise coprime polynomials $a, b, c \in k[x, y, z]$.

Let's choose $\alpha, \beta, \gamma \in k$ such that $\alpha^3 = \gamma_3 - \gamma_2$, $\beta^3 = \gamma_1 - \gamma_3$, $\gamma^3 = \gamma_2 - \gamma_1$. Then $a_1 = \alpha a$, $b_1 = \beta b$, $c_1 = \gamma c$ satisfy an equation

$$a_1^3 + b_1^3 + c_1^3 = 0.$$

Obviously, $deg(a_1) = deg(b_1) = deg(c_1) < deg(p)$. Therefore, according to the assumption of induction a_1, b_1 and c_1 – constants. Then from the system of the equations (3) follows that p, q and r are also constants. This contradiction finishes the proof.

Main results

In this section we study automorphisms of Poisson algebra P on polynomial algebra $k[x,y,z]$ such that $\{x,y\}=z^2,\ \ \{y,z\}=x^2,\ \ \{z,x\}=y^2$.

Recall that $Aut_k P$ denotes automorphism group of Poisson algebra P. The algebra P has automorphisms

$$\varphi_\gamma : x \to \gamma x,\ y \to \gamma y,\ z \to \gamma z,\ \gamma \in k^*;$$

$$\tau : x \to y,\ y \to z,\ z \to x$$

and

$$\delta : x \to x,\ y \to \epsilon y,\ z \to \epsilon^2 z,$$

where ϵ – root of an equation $x^2+x+1=0$.

Theorem. Let k – any field of the characteristic 0 in which the quadratic equation $x^2+x+1=0$ is solvable. Then the automorphism group $Aut_k P$ of Poisson algebra P is generated by automorphisms φ_γ, τ and δ.

Proof. Let σ – any automorphism of algebra P such that

$$\sigma(x)=a,\ \sigma(y)=b,\ \sigma(z)=c.$$

Therefore,

$$\{a,b\}=c^2,\ \ \{b,c\}=a^2,\ \ \{c,a\}=b^2. \tag{4}$$

Then we have

$$2deg(c) \le deg(a)+deg(b);$$

$$2deg(a) \le deg(b)+deg(c);$$

$$2deg(b) \le deg(c)+deg(a).$$

Summing these inequalities up, we get $deg(a)=deg(b)=deg(c)$.

Suppose that $deg(a) \ge 2$. Let's consider the leading homogeneous parts $\overline{a}, \overline{b}$ and $\overline{c}$ of polynomials a, b and c, respectively. Since σ – automorphism of polynomial algebra $k[x,y,z]$ then the elements $\overline{a}, \overline{b}$ and $\overline{c}$ are algebraically dependent and

$$\left\{\overline{a},\overline{b}\right\}=\overline{c}^2,\ \ \left\{\overline{b},\overline{c}\right\}=\overline{a}^2,\ \ \{\overline{c},\overline{a}\}=\overline{b}^2. \tag{5}$$

By Lemma 2 we get

$$\overline{a}^3+\overline{b}^3+\overline{c}^3=0. \tag{6}$$

If $\left(\overline{a},\overline{b},\overline{c}\right)=p$, where $p \in k[x,y,z]$, then there are the homogeneous polynomials $a_1, b_1, c_1 \in$ $\in k[x,y,z]$ such that

$$\overline{a}=p\cdot a_1,\ \overline{b}=p\cdot b_1,\ \overline{c}=p\cdot c_1$$

and $(a_1,b_1,c_1)=1$. From equality (6) follows that

$$a_1^3+b_1^3+c_1^3=0.$$

Therefore by Lemma 3 the elements a_1, b_1 and c_1 are constants, that contradicts (5).

Thus σ is affine automorphism, i.e.

$$\sigma:\ \begin{array}{l} x \to l_1+\lambda_1; \\ y \to l_2+\lambda_2; \\ z \to l_3+\lambda_3\ , \end{array}$$

where l_i – linear parts of automorphism σ and $\lambda_i \in k$, $1 \le i \le 3$. Let's write

$$\sigma:\ \begin{array}{l} x \to t_1=\alpha_{11}x+\alpha_{21}y+\alpha_{31}z+\lambda_1; \\ y \to t_2=\alpha_{12}x+\alpha_{22}y+\alpha_{32}z+\lambda_2; \\ z \to t_3=\alpha_{13}x+\alpha_{23}y+\alpha_{33}z+\lambda_3\ , \end{array}$$

where

$$J(\sigma)=\begin{bmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{bmatrix}=\mathtt{A}=(\alpha_{ij})$$

and $det\,(\mathtt{A}) \neq 0$.

To find coefficients $\alpha_{ij} \quad (1 \le i, j \le 3)$ it is enough to substitute values t_1, t_2 and t_3 in ratios (4) and to compare coefficients at the corresponding degrees x, y and z. We have

$$\{\alpha_{11}x + \alpha_{21}y + \alpha_{31}z + \lambda_1, \alpha_{12}x + \alpha_{22}y + \alpha_{32}z + \lambda_2\} =$$
$$= (\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21})\, z^2 + (\alpha_{12}\alpha_{31} - \alpha_{11}\alpha_{32})\, y^2 + (\alpha_{21}\alpha_{32} - \alpha_{22}\alpha_{31})\, x^2 =$$
$$= (\alpha_{13}x + \alpha_{23}y + \alpha_{33}z + \lambda_3)^2 .$$

From here $\alpha_{13} \cdot \alpha_{23} = 0,\ \ \alpha_{13} \cdot \alpha_{33} = 0,\ \ \alpha_{23} \cdot \alpha_{33} = 0$ and $\lambda_3 = 0$. 3 cases are possible:

1) If $\alpha_{33} \neq 0$ then $\alpha_{23} = 0$ and $\alpha_{13} = 0$, i.e. $t_3 = \alpha_{33}z$.
Substituting value t_3 in the second equation of ratios (4), we get

$$\{\alpha_{12}x + \alpha_{22}y + \alpha_{32}z + \lambda_2, \alpha_{33}z\} = -\alpha_{12}\alpha_{33}y^2 + \alpha_{22}\alpha_{33}x^2 = (\alpha_{11}x + \alpha_{21}y + \alpha_{31}z + \lambda_1)^2 .$$

From here $\alpha_{11} \cdot \alpha_{21} = 0,\ \ \alpha_{11} \cdot \alpha_{31} = 0,\ \ \alpha_{21} \cdot \alpha_{31} = 0,\ \ \alpha_{31} = 0$ and $\lambda_1 = 0$.

At $\alpha_{11} \neq 0$ we have

$$t_1 = \alpha_{11}x,\ t_2 = \alpha_{22}y + \alpha_{32}z + \lambda_2.$$

The third equation of ratios (4) gives

$$\{\alpha_{33}z, \alpha_{11}x\} = (\alpha_{22}y + \alpha_{32}z + \lambda_2)^2 ,$$

wherefrom we get $\alpha_{32} = 0$ and $\lambda_2 = 0$. Therefore,

$$t_1 = \alpha_{11}x,\ t_2 = \alpha_{22}y,\ t_3 = \alpha_{33}z,$$

where $\alpha_{11} \cdot \alpha_{33} = \alpha_{22}^2$, $\alpha_{11} \cdot \alpha_{22} = \alpha_{33}^2$ and $\alpha_{22} \cdot \alpha_{33} = \alpha_{11}^2$. From here follows $\alpha_{11} = \frac{\alpha_{22}^2}{\alpha_{33}}$. Then $\alpha_{22}^3 - \alpha_{33}^3 = 0$. Therefore, $\alpha_{22} = \alpha_{33}$, $\alpha_{22} = \epsilon\alpha_{33}$ or $\alpha_{22} = \epsilon^2\alpha_{33}$, where ε – root of an equation $x^3 - 1 = 0$. Thus, we have

$$\sigma_1 : x \to \alpha_{33}x,\ y \to \alpha_{33}y,\ z \to \alpha_{33}z;$$
$$\sigma_2 : x \to \alpha_{33}\epsilon^2x,\ y \to \alpha_{33}\epsilon y,\ z \to \alpha_{33}z;$$
$$\sigma_3 : x \to \alpha_{33}\epsilon x,\ y \to \alpha_{33}\epsilon^2 y,\ z \to \alpha_{33}z.$$

2) If $\alpha_{23} \neq 0$ then $\alpha_{13} = 0$ and $\alpha_{33} = 0$. Similar reasonings give

$$\sigma_4 : x \to \alpha_{23}z,\ y \to \alpha_{23}x,\ z \to \alpha_{23}y;$$
$$\sigma_5 : x \to \alpha_{23}\epsilon z,\ y \to \alpha_{23}\epsilon^2x,\ z \to \alpha_{23}y;$$
$$\sigma_6 : x \to \alpha_{23}\epsilon^2z,\ y \to \alpha_{23}\epsilon x,\ z \to \alpha_{23}y.$$

3) If $\alpha_{13} \neq 0$ then $\alpha_{23} = 0$ and $\alpha_{33} = 0$. In this case we have

$$\sigma_7 : x \to \alpha_{13}y,\ y \to \alpha_{13}z,\ z \to \alpha_{13}x;$$
$$\sigma_8 : x \to \alpha_{13}\epsilon y,\ y \to \alpha_{13}\epsilon^2z,\ z \to \alpha_{13}x;$$
$$\sigma_9 : x \to \alpha_{13}\epsilon^2y,\ y \to \alpha_{13}\epsilon z,\ z \to \alpha_{13}x.$$

Thus, we got all possible automorphisms of algebra P. From here it is easy to conclude that

$$\sigma_1 = \varphi_{\alpha_{33}},\ \sigma_2 = \tau^2\delta^2\tau\varphi_{\alpha_{33}},\ \sigma_3 = \tau^2\delta\tau\varphi_{\alpha_{33}};$$
$$\sigma_4 = \tau^2\varphi_{\alpha_{23}},\ \sigma_5 = \tau\delta\tau\varphi_{\alpha_{23}},\ \sigma_6 = \tau\delta^2\tau\varphi_{\alpha_{23}};$$
$$\sigma_7 = \tau\varphi_{\alpha_{13}},\ \sigma_8 = \delta\tau\varphi_{\alpha_{13}},\ \sigma_9 = \delta^2\tau\varphi_{\alpha_{13}}.$$

Therefore the automorphism group of Poisson algebra is generated by automorphisms φ_α, τ and δ.

References

1 *Czerniakiewicz A.G.* Automorphisms of a free associative algebra of rank 2, I, II, Trans. Amer. Math. Soc. — 1971. — Vol. 160. — P. 393–401. — 1972. — Vol. 171. — P. 309–315.

2 *Jung H.W.E.* Uber ganze birationale Transformationen der Ebene // Journal reine angew. Math. — 1942. — Vol. 184. — P. 161–174.

3 *Van der Kulk W.* On polynomial rings in two variables // Nieuw Archief voor Wiskunde. — 1953. — No. (3)1. — P. 33–41.

4 *Макар-Лиманов Л.Г.* Об автоморфизмах свободной алгебры с двумя образующими // Функциональный анализ и его приложения. — 1970. — Т. 4. — № 3. — С. 107–108. // Functional. Anal. Appl. — 1970. — № 4. — С. 262, 263.

5 *Shestakov I.P., Umirbaev U.U.* The Nagata automorphism is wild // Proc. Natl. Acad. Sci. USA 100 journal. — 2003. — Vol. 22. — P. 12561–12563.

6 *Shestakov I.P., Umirbaev U.U.* Tame and wild automorphisms of rings of polynomials in three variables // Journal of the American Mathematical Society. — 2004. — Vol. 17. — P. 197–227.

7 *Umirbaev U.U.* Defining relations for tame automorphism groups of polynomial rings and wild automorphisms of free associative algebras. Dokl. Akad. Nauk, 2006. — No. 3(407). — P. 319–324. English translation: Doklady Mathematics. — 2006. — No. 2(73). — P. 229–233.

8 *Umirbaev U.U.* The Anick automorphism of free associative algebras // Journal Reine Angew. Math. — 2007. — Vol. 605. — P. 165–178.

9 *Makar-Limanov L., Turusbekova U., Umirbaev U.* Automorphisms and derivations of free Poisson algebras in two variables of // Algebra Journal. — 2009. — P. 3318–3330.

10 *Van den Essen A.* Polynomial automorphisms and the Jacobian conjecture, Progress in Mat., 190. — Birkhauser verlag, Basel, 2000.

11 *Donin J., Makar-Limanov L.* Quantization of quadratic Poisson brackets on a polynomial algebra of three variables // Journal of Pure and Applied Algebra. — 1998. — Vol. 129. — P. 247–261.

12 *Turusbekova U.* The automorphism group of a Poisson quadratic algebra on C[x,y,z]: abstracts of the Conference «Algebras, Representations and Applications». — Maresias, Brazil, 2007. — P. 70.

13 *Shestakov I.P., Umirbaev U.U.* Poisson brackets and two generated subalgebras of rings of polynomials // Journal of the American Mathematical Society. — 2004. — Vol. 17. — P. 181–196.

Ү.Қ. Тұрысбекова, Г.Т. Азиева

$k[x, y, z]$ көпмүшеліктер алгебрасындағы квадраттық Пуассон алгебралары және олардың автоморфизмдері

Қазіргі заманғы математиканың өзекті бағыттарының бірі Пуассон құрылымдарын математика және теориялық механиканың әр түрлі проблемаларына қолдану болып табылады. Бұл есептер қатты денелер динамикасында, аспан механикасында, космологиялық модельдерде кездеседі. Пуассон алгебралары Гамильтон механикасында, симплектикалық геометрияда, сонымен қатар кванттық топтарды зерттеуде маңызды рөл атқарады. Ли–Пуассон жақшаларының кейбір маңызды мысалдары Якобиге белгілі болғанын ескерте кетейік. Оның мысалдарында Пуассон жақшалары Гамильтон теңдеулерінің бірінші интегралдары кеңістігінде пайда болған. Соңғы уақытқа дейін Пуассон құрылымдарының алгебралық теориясы аз зерттелген. Қазіргі уақытта Пуассон алгебраларын Ресей, Франция, АҚШ, Бразилия, Аргентина, Болгария және т.б. елдердің көптеген математиктері зерттеуде. Мақала $k[x, y, z]$ көпмүшеліктер алгебрасында $\{x, y\} = z^2, \{y, z\} = x^2, \{z, x\} = y^2$ болатындай P Пуассон алгебрасының автоморфизмдері тобын сипаттауға арналған. Біртекті алгебралық тәуелді элементтер арасында бір пуассондық арақатынас тағайындалды, сонымен қатар P алгебрасының $Aut_k P$ автоморфизмдері тобы $\varphi_\alpha = (\alpha x, \alpha y, \alpha z)$, $\alpha \in k^*, \tau = (y, z, x)$ және $\delta = (x, \varepsilon y, \varepsilon^2 z)$ автоморфизмдерінен туындайтыны дәлелденді, мұндағы $\varepsilon - x^2 + x + 1 = 0$ теңдеуінің түбірі.

У.К. Турусбекова, Г.Т. Азиева

Квадратичные алгебры Пуассона на $k[x, y, z]$ и их автоморфизмы

Одним из актуальных направлений в современной математике являются приложения пуассоновых структур к различным проблемам математики и теоретической механики. Эти задачи возникают в динамике твердого тела, небесной механике, теории вихрей, космологических моделях. Алгебры Пуассона играют ключевую роль в гамильтоновой механике, симплектической геометрии и также являются центральными в изучении квантовых групп. Отметим, что само развитие теории пуассоновых структур во многом было стимулировано динамикой многомерных волчков, так как последняя позволяет сделать абстрактные формулировки многих теорем более наглядными и содержательными. Заметим также, что некоторые важные примеры скобок Ли–Пуассона были известны еще Якоби. В его примерах скобки Пуассона возникли на пространстве первых интегралов уравнений Гамильтона. До последнего времени алгебраическая теория пуассоновых структур была мало изучена. В настоящее время алгебры Пуассона исследуются многими математиками России, Франции, США, Бразилии, Аргентины, Болгарии и т.д. Настоящая работа посвящена описанию группы автоморфизмов алгебры Пуассона P на алгебре многочленов $k[x, y, z]$ такой, что $\{x, y\} = z^2, \{y, z\} = x^2, \{z, x\} = y^2$. Установлено одно интересное пуассоново соотношение между однородными алгебраически зависимыми элементами и доказано, что группа автоморфизмов $Aut_k P$ алгебры P порождается автоморфизмами $\varphi_\alpha = (\alpha x, \alpha y, \alpha z), \alpha \in k^*, \tau = (y, z, x)$ и $\delta = (x, \varepsilon y, \varepsilon^2 z)$, где ε – корень уравнения $x^2 + x + 1 = 0$.

References

1 Czerniakiewicz A.G. *Trans. Amer. Math. Soc.,* 1971, 160, p. 393–401

2 Jung H.W.E. *Journal reine angew. Math.,* 1942, 184, p. 161–174.

3 Van der Kulk W. *Nieuw Archief voor Wiskunde.,* 1953, (3)1, p. 33–41.

4 Makar-Limanov L.G. *Functional. Anal. Appl.,* 1970, 4, p. 262, 263.

5 Shestakov I.P., Umirbaev U.U. *Proc. Natl. Acad. Sci. USA 100 Journal,* 2003, 22, p. 12561–12563.

6 Shestakov I.P., Umirbaev U.U. *Journal of the American Mathematical Society,* 2004, 17, p. 197–227.

7 Umirbaev U.U. *Defining relations for tame automorphism groups of polynomial rings and wild automorphisms of free associative algebras. Dokl. Akad. Nauk,* 2006, 3(407), p. 229–233.

8 Umirbaev U.U. *Journal Reine Angew. Math.,* 2007, 605, p. 165–178.

9 Makar-Limanov L., Turusbekova U., Umirbaev U. *Algebra Journal,* 2009, p. 3318–3330.

10 Van den Essen A. *Polynomial automorphisms and the Jacobian conjecture, Progress in Mat., 190,* Birkhauser verlag, Basel, 2000.

11 Donin J., Makar-Limanov L. *Journal of Pure and Applied Algebra,* 1998, 129, p. 247–261.

12 Turusbekova U. *The automorphism group of a Poisson quadratic algebra on C[x,y,z]: abstracts of the Conference «Algebras, Representations and Applications»,* Maresias, Brazil, 2007, p. 70.

13 Shestakov I.P., Umirbaev U.U. *Journal of the American Mathematical Societym,* 2004, 17, p. 181–196.