UDC 510.67

A.R. Yeshkeyev

Ye.A. Buketov Karaganda State University
(E-mail: modth1705@mail.ru)

The property of independence for Jonsson sets

The studies carried out in this article are connected with the description of model-theoretic properties of some, generally speaking, incomplete classes of theories that make a subclass of inductive theories. These theories are well studied both in algebra and in the theory of models. They are called Jonsson's theories. To study these theories there is introduced a new research approach, namely: on the submultitudes of a semantic model of Jonsson's theory there are separated special multitudes that are, firstly, realizations of some existential formula, secondly, the closing of the set gives us the basic set of some existentially closed submodel of the semantic model. Besides, there is developed a technique of studying the central orbital types. It is well known that the perfect Jonsson theory enough comfortable for model-theoretic researches. Practically, in the perfect case, we can say that with the help of semantic method, we can give a specific description of these objects (Jonsson theory and class its existentially closed models). In this article we will give the notion of forking for fragment of fixing Jonsson theory. The nonforking extensions will be the «Mfree» ones. Also we considered for the notion of independence many desirable properties like monotonicity, transitivity, finite basis and symmetry.

Key words: Jonsson's set, forking, algebraic closure, definable closure, central type, orbital type, independence.

Our research interests are connected with the description of model-theoretic properties of some, generally speaking, incomplete classes of theories that make a subclass of inductive theories. These theories are well studied both in algebra and in the theory of models.

As well, we are always dealing with two objects:

1) Jonsson theory [1] and 2) class of its existentially closed models.

It is well known that the perfect Jonsson theory enough comfortable for model-theoretic researches. Practically, in the perfect case, we can say that with the help of semantic method, we can give a specific description of these objects (Jonsson theory and class its existentially closed models).

This allows us to assume that it would be interesting to learn how to allocate in an arbitrary theory its fragment which will Jonsson theory. This approach is not trivial, if only from the fact that any theory set its universal existential consequences, not necessarily Jonsson theories.

On the other hand, for any theory in some special enrichments can always be achieved firstly Jonsson and then its perfect. At least this holds for operations such as skulemization and morlization. In both cases, the class of existentially closed models received Jonsson theories coincides with the class of models of initial theories.

Morlization and skolemization action is applied to the theory under consideration.

This article is invited to the idea of considering a new approach to a a subset of some model, which allows firstly to expand the semantic aspect, and secondly to try to transfer many of the ideas out of technique the of complete theories for Jonsson fragments, which in itself generalizes the considered problems.

We make the following agreements:

1. In this project, we consider only perfect Jonsson theory, complete of existential sentences.
2. In this project, we consider only classes existentially closed models of the theories.
3. In case of the structure, it is assumed that the model of some signature.

Naturally, when we speak of arbitrary signature (language) without the theory, item 1) of the above arrangements is not important.

Let T is Jonsson perfect theory of complete of existential sentences in the language L. We fix its semantic model C, saturated in a very high power κ (in particular κ is much greater than the power of language). We agree that in the future all the considered models $M, N, \ldots$ of theory T will be existentially closed substructures high model C power less than κ. All considered subsets $A, B, C, \ldots$ will be subsets of C power less than κ.

Note one more useful fact, if f is the automorphism of structure C, leaving in place all the elements of the set $A, f \in Aut_A(C)$, then f it obviously transfer to itself each A is definability subset and therefore transforms to itself and all complete types over A, due to saturation of the semantic model C. Conversely, if $\overline{c}, \overline{d} \in C^n$ then $tp(\overline{c}/A) = tp(\overline{d}/A)$ if and only if there exists $f \in Aut_A(C)$ such that $f(\overline{c}) = \overline{d}$.

The saturated model complete n-types over A exactly correspond to orbits n elements under automorphisms fixing A element by element. Since the theory is complete for the existential sentences of language L, it is true for existential types.

Let L is a language, which from that moment supposed countable. Next, let T is Jonsson perfect theory of complete of existential sentences in the language L and its semantic model C. There remains an agreement sets and model of theory T are strictly less power than C.

Let $A \subseteq C$. We fix some $n \geq 1$ and consider the family Def^n_A of all A — definable subsets power over C^n. We identify this definable subset of C^n and defining its formula $\varphi(\overline{x}, \overline{a})$, where $\overline{x}, \overline{a}$, $\overline{a}$ – tuple elements of A (two different formulas may define a subset of, but we consider the formula with an accuracy to equivalence in C the obvious sense).

The following approach to the definition of a relational structure of some signature, is well known. It allows to consider only the predicate signatures. For example, in the case of moralization.

Let's start with a definition of the relational structure of the signature of a Jonsson theory. Defining family of definable subsets of the structure, we follow the terminology and notation of [2, 1], but in [1], all definitions are given for complete theories, we will to work with Jonsson theories and their positive generalizations.

The relational structure $M = \langle M, (B_i)_{i\in I}\rangle$ consists of a (non-empty) set M and subsets $(B_i)_{i\in I}$ of $\bigcup_{n>1} M^n$ and each B_i is a subset of some $M^{n_i}, n_i \geq 1$. Add an additional condition that one of the sets B_i is the diagonal of the set M.

All B_i are called atomic subsets M.

Let $M = \langle M, (B_i)_{i\in I}\rangle$ — relational structure. We introduce the concept of a family of definable subsets of structures M, denoted $Def(M)$. It is the least of the family subsets of $\bigcup_{n>1} M^n$ with the following properties.

For each $i \in I$ the inclusion $B_i \in Def(M)$.

The set $Def(M)$ is closed relatively to finite Boolean combinations, i.e. of inclusions $A, B \subseteq M^n$, $A, B \in$ $\in Def(M) \subseteq M^n$, follow that $A \cup B \in Def(M), A \cap B \in Def(M)$ and $M^n \setminus A \in Def(M)$. The set $Def(M)$ is closed relatively Cartesian product, i.e. of inclusions $A, B \in Def(M)$ follow that $A \times B \in Def(M)$. The set $Def(M)$ is closed relatively to the projection, i.e. if $A \subseteq M^{n+m}$, $A \in Def(M)$ $\pi_n(A) \in Def(M)$, π_n the projection of the set A on M^n, $\pi_n(A) \in Def(M)$. The set $Def(M)$ is closed relatively to specialization, i.e. if $A \in Def(M)$, $A \subseteq M^{n+k}$ and $\overline{m} \in M^n$ then $A(\overline{m}) = \{\overline{b} \in M^k(\overline{m}, \overline{b}) \in A\} \in Def(M)$. The set $Def(M)$ is closed relatively to permutation of coordinates, i.e. if $A \in Def(M)$, σ – a permutation of the set $1, ..., n$ then $\sigma(A) = \{(a_{\sigma(1)}, ..., a_{\sigma(n)}) \mid (a_1, ..., a_n) \in A\} \in Def(A)$. We now say that $S \subseteq M^n$ is the atomic subset if

$$S = \{(a_1, ..., a_n) \in M^n \mid M \models \varphi(a_1, ..., a_n, b_1, ..., b_m)\}$$

for some atomic formula $\varphi(x_1, ..., x_n, x_{n+1}, ..., x_{n+m})$ and some $b \in M^m$. We say that a subset S defined with parameters $\overline{b}$ or defined above $\overline{b}$.

We now say that $D \subseteq M^n$ is definable subset L-structure of M, where there are $b \in M^m$ (here $\overline{b}$ may be empty) and a formula $\varphi(x_1, ..., x_n, x_{n+1}, ..., x_{n+m})$ such that

$$D = \{(a_1, ..., a_n) \in M^n \mid M \models \varphi(a_1, ..., a_n, b_1, ..., b_m)\}.$$

If $\overline{b} \subseteq B$, then we say that D is definable with the parameters of B (or above B) or that D is defined formula with the parameters of B. Clearly definable sets in this sense – not that other, as $Def(M)$ a relational structure $\langle M, (A_i)_{i\in I}\rangle$, which A_i taken as a whole all nuclear definable set.

The family Def^n_A is a Boolean algebra with relative to the usual operations of intersection, union and complement. Full n-type over A the same ultrafilter in this Boolean algebra. The space above the full n-types, denoted $S_n(A)$ is the Stone space corresponding to the Boolean algebra Def^n_A. We introduce in $S_n(A)$ the (normal) topology in which the open base of the set $\langle\varphi(\overline{x}, \overline{a})\rangle = \{p \in S_n(A) \mid \varphi(\overline{x}, \overline{a}) \in p\}$.

We say that the set X — Σ-defined, if it is definitely some existential formula.

a) The set X is called Jonsson in the theory T if it satisfies the following properties:

X is a – Σ definability subset of C;

$dcl(X)$ is the universe of some existentially-closed submodels of C;

b) The set X is called algebraic Jonsson in the theory T, if it satisfies the following properties:

X is a – definability subset of C;

$acl(X)$ is the universe some existentially-closed submodels of C.

We consider countable language L and Jonsson perfect theory of complete of existential sentences in language L and their semantic models C, in this language and other models (classes existentially closed models of the theories).

If M is model theory of T and φ – a formula language L, then we will use the following notation:

$$\varphi(M) = \{m \in M^n \mid M \models \varphi(m)\}.$$

The set S will call 0–definability if it ϕ-definability (definable without parameters).

The set of all complete types over A the denoted by $S(A)$, i.e. $S(A) = \bigcup_{n \geq 1} S_n(A)$.

Saturated models of Jonsson theory (κ – saturation models power k) are uniquely determined by their power. But they can not exist without a certain set-theoretic assumptions, such as the generalized continuum hypothesis. On the other hand, there are different ways to avoid set-theoretic problems of this sort. For example, assume stable or weaken the concept of the semantic model as in [2]. Therefore, we assume that we got rid of all the issues of the existence of the semantic model.

Further, it is convenient to work within the semantic model C of Jonsson theory, containing all others.

In the future, any set of parameters A considered in the subset C. Model M is a subset of C which is the universe of existentially closed substructure. This means that any $L(M)$ – existential formula $\varphi(x)$, true in C and performed on some element of M. Formula parameters in the future always belongs to C and if we write $\models \varphi(c)$ if $C \models \varphi(c)$.

Lemma 1. Definable set of D is definable over set A, if and only if it is invariant relatively to all automorphisms of the model C, leaving in place each element of A. (Let's call them over automorphisms over A).

It follows that the definable closure $dcl(A)$ of the set A, e.a. the set of all elements of the definable over A, coincides with the set of elements that are invariant relatively to all automorphisms over A.

The element b contained in the finite A is definability set, called algebraic over A. It follows that the element b algebraic over A if and only if it has only a finite number of adjoint over A.

The set $acl(A)$ consisting of all elements algebraic over A, will be called the algebraic closure of the set A.

Forking. We give an axiomatic reference forking.

Let M $\exists$ – saturated existentially closed model power k (k enough big cardinal) of Jonsson theory T ($\exists$ – saturation means the saturation relative to existential types). Let A — the class of all subsets M, P – the class of all $\exists$-types (not necessarily complete), let $JNF \subseteq P \times A$ – a binary relation. We impose JNF the following axiom:

Axiom 1. If $(p, A) \in JNF, f : A \to B$ — automorphism M, then $(f(p), f(A)) \in JNF$.

Axiom 2. If $(p, A) \in JNF, q \subseteq p$, then $(q, A) \in JNF$.

Axiom 3. If $A \subseteq B \subseteq C, p \in S^G(C)$, then $(p, A) \in JNF \Leftrightarrow (p, B) \in JNF$ and $(p \mid^- B, A) \in JNF$.

Axiom 4. If $A \subseteq B, dom(p) \subseteq B, (p, A) \in JNF$, then $\exists q \in S^J(B)$, $p \subseteq q$ and $(q, a) \in JNF$

Axiom 5. There is a cardinal κ such that if $A \subseteq B \subseteq C$, $p \in S^G(C)$, $(p, A) \in JNF$ then $\mid \{q \in S^J(C) : p \subseteq q \text{ and } (q, a) \in JNF\} \mid < \kappa$.

Axiom 6. There is a cardinal ρ such that if $\forall p \in P, \forall A \in A$, if $(p, A) \in JNF$, then $\exists A_1 \subseteq A$, $(\mid A_1 \mid < \rho)$ and $p, A_1 \in JNF$.

Axiom 7. If $p \in S^J(A)$, then $(p, A) \in JNF$.

The classical notion of forking belongs Shelah.

A set of formulas $\{\varphi(\overline{x}, \overline{a_i}) : i < k\}$ are called k — inconsistent for some positive integer k, if every finite subset p of power k is inconsistent, ie. $\models \neg\overline{x}(\varphi(\overline{x}, \overline{a_{i_1}} \wedge ... \wedge \varphi(\overline{x}, \overline{a_{i_k}}))$ for each $i_1 < ... < i_k < k$.

Partial type p divided over a set of relative to $k \in \omega$ if there is a formula $\varphi(\overline{x}, \overline{a})$ and a sequence $\langle \overline{a_i} : i \in \omega \rangle$ such that

1) $p \vdash \varphi(\overline{x}, \overline{a})$;

2) $tp(\overline{a}/A) = tp(\overline{a_i}/A)$ for all i;

3) $\varphi\{(\overline{x}, \overline{a}) : i \in \omega\}$, k – not jointly.

It is also p divided over A if p divided over A relative to some k. In addition, p fork over A to T, if there are formulas $\phi_1(\overline{x}, \overline{a_0}), ..., \phi_n(\overline{x}, \overline{a_n})$ such that:

(i) $p \models \bigvee_{0 \leq i \leq n} \varphi_i(\overline{x}, \overline{a_i})$;

(ii) $\phi_i(\overline{x}, \overline{a_i})$ divided over A for any i.

The following result makes it possible to use all features of forking for complete theories in the class above in this report Jonsson theories.

Theorem 1. Let T perfect Jonsson theory of complete for $\exists$ – sentences. Then the following conditions are equivalent:

• the relation JNF satisfies the axioms 1–7 relative to the theory T;

• T^* stable and for all $p \in P$, $A \in A$ $((p, A) \in JNF \Leftrightarrow p$ not fork over $A)$.

Let T is Jonsson theory, $S^J(X)$ is the set of all full n-types over X, joint with T, for all finite n.

We say that Jonsson theory T is $J - \lambda$ - stable, if for any T existentially closed of model A, for any subset A of the set A, $|X| \geq \lambda \Rightarrow| S^J(X) |\leq \lambda$.

Theorem 2. Let T — complete for existential sentences is perfect Jonsson theory, $\lambda \geq \omega$. Then the following conditions are equivalent:

$T - J - \lambda$-stably;

$T - J - \lambda$-stably, where T^* is center of theory T.

Definition 1. Suppose that A⊆B, $p \in S_n(A)$, $q \in S_n(B)$, and p⊆q. If RM(q) < RM(p), we say that q is a forking extension of p and that q forks over A. If RM(q) = RM(p), we say that q is a nonforking extension of p.

Our first goal is to show that nonforking extensions exist.

Theorem 3. (Existence of nonforking extensions) Suppose that $p \in S_n(A)$ and A⊆B.

i) There is $q \in S_n(B)$ a nonforking extension of p.

ii) There are at most $deg_M(p)$ nonforking extensions of p in $S_n(B)$ and if $\mathcal{M}$ is an $\exists - \aleph_0$ – saturated model with A⊆M, there are exactly $deg_M(p)$ nonforking extensions of p in S $_n$(M). iii) There is at most one $q \in S_n(B)$, a nonforking extension of p with $deg_M(p) = deg_M(q)$. In particular, if $deg_M(p)$=1, then p has a unique nonforking extension in $S_n(B)$.

Independence. The nonforking extensions will be the «free» ones.

Forking as in Theorem 1 can be used to give a notion of independence in $J - \omega$-stable theories.

Definition 2. We say that $\overline{a}$ is independent from B over A if tp($\overline{a}$/A) does not fork over A ∪ B. We write a $\overline{a} \perp_A B$.

This notion of independence has many desirable properties.

Lemma 2 (Monotonicity). If a $\overline{a} \perp_A B$ and C⊆B, then a $\overline{a} \perp_A C$.

Lemma 3 (Transitivity). a $\overline{a} \perp_A \overline{b}, \overline{c}$ if and only if a $\overline{a} \perp_A \overline{b}$ and $\overline{a} \perp_{A,\overline{b}} \overline{c}$.

Lemma 4 (Finite Basis). a $\overline{a} \perp_A B$ if and only if a $\overline{a} \perp_A B_0$ for all finite $B \subseteq B_0$.

Lemma 5 (Symmetry). If a $\overline{a} \perp_A \overline{b}$, then $\overline{b} \perp_A \overline{a}$.

Corollary 1. $\overline{a}, \overline{b} \perp_A C$ if and only if $\overline{a} \perp_A C$ and $\overline{b} \perp_{A,\overline{a}} C$.

Symmetry also gives an easy proof that no type forks when it is extended to the algebraic closure.

Corollary 2. For any $\overline{a}, \overline{a} \perp_A acl(A)$.

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А.Р. Ешкеев

Йонсондық жиындар үшін тәуелсіздік қасиеті

Мақалада жүргізілген зерттеулер индуктивті теорияның ішкі класы болатын, жалпы айтканда, толық емес кластар теориясының модельді-теоретикалық қасиеттерін сипаттаумен байланысты. Бұл теориялар алгебрада және модельдер теориясында да кеңінен қарастырылған. Мұндай теориялар йонсондық деп аталады. Осы теорияларды зерттеу үшін жаңа әдіс-тәсілдер енгізілген. Йонсондық теорияның семантикалық модельдер жиынында айрықша жиындар қарастырылды, олар, біріншіден, кейбір экзистенциалдық формулаларды жүзеге асыу болып табылады, екіншіден, жиындардың тұйықталуы бізге семантикалық модельдің экзистенциалды тұйықталуының ішкі моделінің негізгі жиынын береді. Сонымен қатар орталық орбиталдық типтерді зерттеу үшін техника дамиды. Кемел йонсондық теориялар модельді-теоретикалық зерттеу үшін қолайлы екені жақсы белгілі. Практика жүзінде кемелділік жағдайында семантикалық тәсіл көмегімен жоғарыда айтылған нысандардың анықтамаларын бере аламыз. Ягни, олар йонсондық теориялар және оның экзистенционалды-тұйық модельдер класы. Біздің ғылыми қызығушылығымыз, жалпы айтқанда, индуктивті теориялардың ішкі кластары болатын теориялардың толық емес кластарын кейбір модельді-теоретикалық қасиеттермен сипаттауға байланысты. Бұл теориялар алгебрада және модельдер теориясында кеңінен зерттелді. Мақалада йонсондық теорияның фрагменті үшін форкинг ұғымын келтірді. Форкинг болмаса, онда кеңейтулер бос болады. Сонымен қатар біз тәуелсіздік ұғымы үшін транзитивтілік, монотондылық, үзіліссіздік және симметрия сияқты көптеген маңызды қасиеттерді қарастырдық.

А.Р. Ешкеев

Свойство независимости для йонсоновских множеств

Исследования, проведенные в статье, связаны с описанием теоретико-модельных свойств некоторых, вообще говоря, неполных классов теорий, которые являются подклассом индуктивных теорий. Эти теории, хорошо изучаемые и в алгебре, и в теории моделей, называются йонсоновскими. Для изучения этих теорий вводится новый подход исследования. А именно, на подмножествах семантической модели йонсоновской теории выделяются особые множества, которые являются, во-первых, реализациями некоторой экзистенциальной формулы, во-вторых, замыкание этих множеств дает нам основное множество некоторой экзистенциально замкнутой подмодели семантической модели. Помимо этого развивается техника для изучения центральных орбитальных типов. Хорошо известно, что совершенные йонсоновские теории достаточно удобны для теоретико-модельных исследований. Практически, в случае совершенности, мы можем утверждать, что с помощью семантического метода дается определенное описание указанных выше объектов (йонсоновской теории и классом ее экзистенциально-замкнутых моделей). В этой статье рассмотрены понятия форкинга для фрагмента фиксируемой йонсоновской теории и независимости, а также многие полезные свойства, такие как транзитивность, монотонность, непрерывность и симметрия.

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